SURA'S

Mathematics

12th Standard

VOLUME - II

Based on the New Syllabus and New Textbook for the year 2019-20

Salient Features

- Prepared as per the New Textbook for the year 2019 - 20.
- Exhaustive Additional Questions & Answers in all chapters.
- Sura's Model Question Paper 1 to 3 with answers

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CHAPTER 7

APPLICATIONS OF DIFFERENTIAL CALCULUS

MUST KNOW DEFINITIONS

- The **tangent line** to a plane curve at a given point is the straight line that just touches the curve at that point.
- The **normal** at a point on the curve is the straight line which is perpendicular to the tangent at the that point.

Intermediate value theorem :

- If f is continuous on $[a, b]$ and c is any number between $f(a)$ and $f(b)$, then there is atleast one number x in $[a, b]$ such that $f(x) = c$.

Rolle's theorem :

- Let $f(x)$ be continuous on $[a,b]$ and differentiable in (a, b). If $f(a) = f(b)$ then there is atleast one point $c \in (a, b)$ where $f'(c) = 0$.

Langrange's mean value theorem :

- Let $f(x)$ be continuous in $[a, b]$ and differentiable in (a, b). Then there exists atleast one point $c \in (a, b)$ such that $f'(c) = \dfrac{f(b)-f(a)}{b-a}$
- If $f(x)$ is continuous in $[a, b]$ and differentiable in (a, b) and if $f'(x) > 0$ $\forall x \in (a, b)$ then for $x_1, x_2 \in [a, b]$ such that $x_1 < x_2$ we have $f(x_1) < f(x_2)$
- Let $\lim\limits_{x\to\alpha} g(x)$ exists and let it be L and let $f(x)$ be continuous at $x = \text{L}$.

  Then $\lim\limits_{x\to\alpha} f(g(x)) = f\left(\lim\limits_{x\to\alpha} g(x)\right)$
- A stationary point $(x_0, f(x_0))$ of a differentiable function $f(x)$ is where $f'(x_0) = 0$
- A critical point $(x_0, f(x_0))$ of a function $f(x)$ is where $f'(x_0) = 0$ or does not exist.
- If $f(x)$ is continuous on a closed interval $[a, b]$, then f has both an absolute maximum and an absolute minimum on $[a, b]$.

Fermat's theorem :

- If $f(x)$ has a relative extrema at $x = c$ then c is critical number.

IMPORTANT FORMULA TO REMEMBER

- Slope or Gradient of a curve $\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = f'(x)$ [Limit of Newton quotient]
- Angle between two curves $\tan\theta = \left|\frac{m_1 - m_2}{1+m_1m_2}\right|$
- If two lines are parallel, then $m_1 = m_2$.
- If two lines are perpendicular, then $m_1\ m_2 = -1$
- **Taylor's Series :**

$$f(x) = \sum_{n=0}^{\infty}\frac{f^{(n)}(a)}{n!}(x-a)^n = f(a) + \frac{f'(a)}{1!}(x-a) + \ldots \frac{f^n(a)(x-a)^n}{n!} + \ldots\ldots$$

- **Maclaurin's Series :**

$$f(x) = \sum_{n=0}^{\infty}\frac{f^n(0)}{n!}x^n = f(0) + \frac{f'(0)}{1!}x + \ldots + \frac{f^n(0)}{n!}x^n + \ldots$$

- ***l*' Hôpital's Rule :**

$\lim_{x\to a} f(x) = 0 = \lim_{x\to a} g(x)$, then $\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}$

- $\lim_{x\to a} f(x) = \pm\infty = \lim_{x\to a} g(x)$, then $\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}$
- A function $f(x)$ is said to be an increasing function in an interval I if $a < b \Rightarrow f(a) \le f(b)\ \forall\ a, b \in$ I
- A function $f(x)$ is said to be an decreasing function in an interval I if $a < b \Rightarrow f(a) \ge f(b)\ \forall\ a, b \in$ I
- If $\frac{d}{dx}(f(x)) \ge 0\ \forall\ x \in (a,b)$ then $f(x)$ is an increasing function in (a,b)
- If $\frac{d}{dx}(f(x)) > 0\ \forall\ x \in (a,b)$ then $f(x)$ is strictly increasing in (a,b)
- If $\frac{d}{dx}(f(x)) \le 0\ \forall\ x \in (a,b)$ then $f(x)$ is decreasing in (a,b)
- If $\frac{d}{dx}(f(x)) < 0\ \forall\ x \in (a,b)$ then $f(x)$ is strictly decreasing in (a,b)
- **First derivative test**

(i) If $f(x)$ changes from negative to positive at c, then $f(x)$ has a local minimum $f(c)$

(ii) If $f'(x)$ changes from positive to negative at c, then $f(x)$ has a local maximum $f(c)$

(iii) If $f'(x)$ is positive or negative on both sides of c, then $f(x)$ is neither a local minimum nor a local maximum.

- **Text for Concavity :**
  (i) If $f''(x) > 0$ on an open interval I, then $f(x)$ is concave up on I.
  (ii) If $f''(x) < 0$ on an open interval I, then $f(x)$ is concave down on I.
- **Text for points of inflection :**
  (i) If $f''(c)$ exists and $f''(c)$ changes sign when passing through $x = c$, then the point $(c, f(c))$ is a point of inflection of the graph of f.
  (ii) If $f''(c)$ exists at the point of inflection, then $f''(c) = 0$.
- **Second derivative test :**
  Suppose that c is a critical point at which $f'(c) = 0$, that $f'(x)$ exists in a neighbourhood of c and that $f'(c)$ exists.
  Then f has a relative maximum value at c if $f''(c) < 0$ and a relative minimum value at c if $f''(c) > 0$. If $f''(c) = 0$, the test is not informative.
- Symmetric with respect to y - axis if $f(x, y) = f(-x, y)\ \forall\ x, y$
- Symmetric with respect to x - axis if $f(x, y) = f(x, -y)\ \forall\ x, y$
- Symmetric with respect to origin if $f(x, y) = f(-x, -y)\ \forall\ x, y$
- **Horizontal Asymptote :**
  $y = L$ is said to be horizontal asymptote if $\lim\limits_{x\to\infty} f(x) = L$ or $\lim\limits_{x\to-\infty} f(x) = L$
- **Vertical Asymptote :**
  $x = a$ is said to be vertical asymptote if $\lim\limits_{x\to a^-} f(x) = \pm\infty$ or $\lim\limits_{x\to a^+} f(x) = \pm\infty$
- **Slant Asymptote :**
  A slant (oblique) asymptote occurs when the polynomial in the numerator is a higher degree than the polynomial in the denominator.

EXERCISE 7.1

1. A point moves along a straight line in such a way that after t seconds its distance from the origin is $s = 2t^2 + 3t$ meters.

(i) Find the average velocity of the points between $t = 3$ and $t = 6$ seconds.

(ii) Find the instantaneous velocities at $t = 3$ and $t = 6$ seconds.

Sol. **(i)** Given $s = 2t^2 + 3t$

$s(3) = 2 \times 3^2 + 3(3) = 2 \times 9 + 9 = 27$ m ... (1)

$s(6) = 2 \times 6^2 + 3(6) = 72 + 18 = 90$ m ... (2)

Average velocity $= \dfrac{s(6) - s(3)}{6-3} = \dfrac{90-27}{3} = \dfrac{63}{3} = 21$ m/s

(ii) Instantaneous Velocity $V(t) = \dfrac{ds}{dt} = 4t + 3$

Instantaneous Velocity at $t = 3$
$= V(3) = 15$ m/sec [From (1)]

Instantaneous Velocity at $t = 6$
$= V(6) = 27$ m/sec [From (2)]

$\Rightarrow \log x = \dfrac{x^3}{24} \Rightarrow \quad x = e^{\frac{x^3}{24}}$

$\therefore$ It has no Y - intercept.

5. $f'(x) = \dfrac{3x^2}{24} - \dfrac{1}{x} = \dfrac{x^2}{8} - \dfrac{1}{x}$

$f'(x) = 0 \Rightarrow \dfrac{x^2}{8} = \dfrac{1}{x} \Rightarrow x^3 = 8 \Rightarrow x = 2$

$\therefore$ The possible intervals are (0, 2) (2,∞)

Interval	(0, 2)	(2, ∞)
Sign of $f'(x)$	Say $x = 1$, $f'(x) = \dfrac{1}{8} - \dfrac{1}{2}$ $= \dfrac{-3}{8}$ – ve	Say $x = 3$, $f'(x) = \dfrac{9}{8} - \dfrac{1}{3}$ $= \dfrac{19}{24}$ + ve
Monotonicity	Decreasing	Increasing

$\therefore f(x)$ is decreasing in (0, 2) and increasing in (2,∞)

6. Since $f'(x)$ changes its sign from negative to positive, it has local minimum at $x = 2$.

$\therefore f(2) = \dfrac{8}{24} - \log 2$

$= \dfrac{1}{3} - \log 2 = 0.33 - 0.30 = 0.03$

7. $f''(x) = 0 \Rightarrow e^{-x} - e^{-2x} = 0 \Rightarrow e^{-x} = e^{-2x}$

$\Rightarrow \dfrac{1}{e^x} = \dfrac{1}{e^{2x}} \Rightarrow \dfrac{e^{2x}}{e^x} = 1 \Rightarrow e^x = 1$

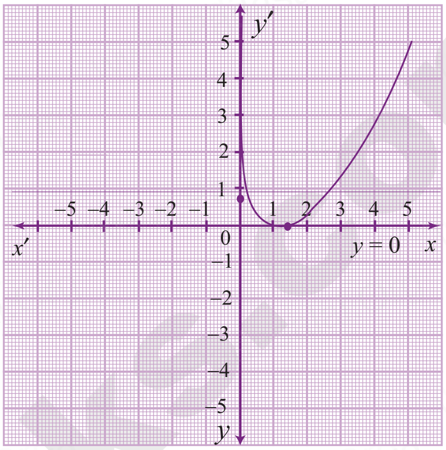
$\Rightarrow e^x = e^0 \Rightarrow x = 0$

$\therefore$ The possible interval of concavity are (–∞, 0) (0, ∞)

Interval	(–∞, 0)	(0, ∞)
Sign of $f''(x)$	Say $x = 1$, $f'(x) = \dfrac{e^1}{(1+e^1)^3}$ $(e^1 - e^2) = \dfrac{2.72}{(3.72)^3}$ $= (2{,}72 - 7.39)$ $= -$ ve	Say $x = +1$, $f''(x) = \dfrac{e^{-1}}{(1+e^{-1})^3}$ $[e^{-1} - e^{-2}]$ $= \dfrac{0.37}{(1.37)^3}$ $(0.37 - 0.14)$ $= +$ ve
Concavity	Concave down	Concave up

8. Since $y = 0$, does not exist, it has a horizontal asymptote at $y = 0$.

9. Since $f''(x)$ changes its sign from – ve to + ve, it has point of inflection $(0, f(0)) = (0, \infty)$.



EXERCISE 7.10

Choose the Correct or the most suitable answer from the given four alternatives :

1. The volume of a sphere is increasing in volume at the rate of 3 πcm³ / sec . The rate of change of its radius when radius is $\frac{1}{2}$ cm

(1) 3 cm/s (2) 2 cm/s

(3) 1 cm/s (4) $\frac{1}{2}$ cm/s

[Ans. (1) 3 cm/s]

Hint : $\dfrac{dv}{dt} = 3\pi \text{ cm}^3/\text{sec}, \dfrac{dr}{dt} = \ ?\ , r = \dfrac{1}{2}$

$v = \dfrac{4}{3}\pi r^3$

$\Rightarrow \dfrac{dv}{dt} = \dfrac{4}{3}\pi \times 3r^2 \dfrac{dr}{dt}$

$\Rightarrow 3\pi = \dfrac{4}{3}\pi \times 3\left(\dfrac{1}{2}\right)^2 \cdot \dfrac{dr}{dt}$

$\Rightarrow 3 = \dfrac{dr}{dt}$

2. A balloon rises straight up at 10 m/s. An observer is 40 m away from the spot where the balloon left the ground. Find the rate of change of the balloon's angle of elevation in radian per second when the balloon is 30 metres above the ground.

ADDITIONAL QUESTIONS

1 MARK

I. Choose the Correct or the most suitable answer from the given four alternatives :

1. If a particle moves in a straight line according to $s = t^3 - 6t^2 - 15t$, the time interval during which the velocity is negative and acceleration is positive is

(1) $2 < t < 5$ (2) $2 \le t \le 5$
(3) $t \ge 2$ (4) $t \le 2$

[Ans: (1) $2 < t < 5$]

Hint :

$$V = \frac{ds}{dt} = 3t^2 - 12t - 15$$
$$V = 0$$
$$\Rightarrow 3t^2 - 12t - 15 = 0$$
$$\Rightarrow 3[t^2 - 4t - 5] = 0$$
$$\Rightarrow 3(t-5)(t+1) = 0$$
$$\therefore v = 3(t+5)(t+1)$$

Velocity is negative in [0, 5)

$$\text{Acceleration} = \frac{dv}{dt} = 6t - 12$$
$$\text{Acceleration} = 6(t-2)$$

Acceleration is positive in $(2, \infty)$

Time interval (2,5).

2. The law of linear motion of a particle is given by $s = \frac{1}{3}t^3 - 16t$, the acceleration at the time when the velocity vanishes is

(1) 4 (2) 6 (3) 2 (4) 8

[Ans: (4) 8]

Hint :

$$s = \frac{1}{3}t^3 - 16t$$
$$\frac{ds}{dt} = t^2 - 16\ ;\ t^2 - 16 = 0$$
$$\Rightarrow t = \pm 4$$
$$\frac{d^2s}{dt^2} = 2t$$

3. If the rate of increase of $s = x^3 - 5x^2 + 5x + 8$ is twice the rate of increase of x, then one value of x is

(1) $\frac{3}{5}$ (2) $\frac{10}{3}$ (3) $\frac{3}{10}$ (4) $\frac{1}{3}$

[Ans: (4) $\frac{1}{3}$]

Hint :

$$s = x^3 - 5x^2 + 5x + 8$$
$$\frac{ds}{dt} = 2\frac{dx}{dt}$$
$$\frac{ds}{dt} = (3x^2 - 10x + 5)\frac{dx}{dt}$$
$$(3x^2 - 10x + 5)\frac{dx}{dt} = 2\frac{dx}{dt}$$
$$2 = 3x^2 - 10x + 5$$
$$3x^2 - 10x + 3 = 0$$
$$(3x-1)(x-3) = 0$$
$$x = \frac{1}{3} \text{ or } 3$$

4. The point on the curve $y = x^2$ is the tangent parallel to X - axis is

(1) (1,1) (2) (2,2) (3) (4,4) (4) (0, 0)

[Ans: (4) (0, 0)]

Hint :

$$y = x^2$$
$$\frac{dy}{dx} = 2x \Rightarrow 2x = 0$$
$$\Rightarrow x = 0 \text{ and } (0,0)$$

5. The equation of the tangent to the curve $y = x^2 - 4x + 2$ at (4, 2) is

(1) $x + 4y + 12 = 0$ (2) $4x + y + 12 = 0$
(3) $4x - y - 14 = 0$ (4) $x + 4y - 12 = 0$

[Ans: (3) $4x - y - 14 = 0$]

Hint :

$$\frac{dy}{dx} = 2x - 4$$
$$m = \left(\frac{dy}{dx}\right)_{(4,2)} = 2(4) - 4 = 4$$
$$y = 4x + c$$
$$2 = 4(4) + c$$
$$c = 2 - 16 = -14$$
$$y = 4x - 14$$

6. The function $f(x) = x^9 + 3x^7 + 64$ is increasing on ....

(1) R (2) $(-\infty, 0)$
(3) $(0, \infty)$ (4) None of these

[Ans: (1) R]

7. If $x + y = 8$, then the maximum value of xy is..................

(1) 8 (2) 16 (3) 20 (4) 24

[Ans: (2) 16]

8. The curve $y = e^x$ is ....

(1) convex (2) concave
(3) convex upwards (4) concave upwards

[Ans: (4) concave upwards]

9. $\lim\limits_{x\to 0} \dfrac{x}{\tan x}$ is ....

(1) 1 (2) –1 (3) 0 (4) ∞

[Ans: (1) 1]

10. The statement "If f has a local extremum at c and if $f'(c)$ exists then $f'(c) = 0$" is ....

(1) the extreme value theorem
(2) Fermats' theorem
(3) Law of mean
(4) Rolle's theorem

[Ans: (2) Fermats' theorem]

III. Match the following :

1.

	List - I		List - II
i.	$f'(x)$ changes from + ve to – ve at c	a)	neither a local minimum nor a local maximum
ii.	$f'(x)$ changes from – ve to + ve at c	b)	local maximum at c
iii.	$f'(x)$ + ve on both sides of c	c)	local minimum at c
iv.	$f'(x)$ – ve on both sides of c	d)	neither a local maximum nor a local minimum

The Correct match is

	(i)	(ii)	(iii)	(iv)
(1)	b	c	d	a
(2)	a	b	c	d
(3)	b	d	c	a
(4)	b	c	a	d

[Ans : (1) i – b ii – c iii – d iv – a]

2.

	List - I		List - II
i.	$f'(c) = \dfrac{f(b) - f(a)}{b - a}$	a)	Rolles' theorem
ii.	$f'(c) = 0$	b)	Maclaurin's series
iii.	$\sum\limits_{n=0}^{\infty} \dfrac{f^n(a)}{n!}(x-a)^n$	c)	Lagranges mean value theorem
iv.	$\sum\limits_{n=0}^{\infty} \dfrac{f^n(0)}{n!}.x^n$	d)	Taylor's series

The Correct match is

	(i)	(ii)	(iii)	(iv)
(1)	b	c	d	a
(2)	c	a	d	b
(3)	c	a	b	d
(4)	c	d	a	b

[Ans : (2) i – c ii – a iii – d iv – b]

3.

	List - I		List - II
i.	Rate of change in vertical length with respect to horizontal length	a)	Steepness of a hill side
ii.	Rate of displacement with respect to time	b)	acceleration
iii.	Rate of change in velocity with respect to time	c)	velocity
iv.	Rate of change in elevation with respect to linear distance	d)	slope

The Correct match is

	(i)	(ii)	(iii)	(iv)
(1)	a	b	c	d
(2)	d	c	a	b
(3)	d	c	b	a
(4)	b	c	d	a

[Ans : (3) i – d ii – c iii – b iv – a]

4. For the function $f(x) = \tan x$

	List - I		List - II
i.	$\sec^2 x$	a)	$f''(x)$
ii.	$2\sec^2 x \tan x$	b)	$f'''(0)$
iii.	2	c)	$f'(x)$
iv.	16	d)	$f^{v}(0)$

The Correct match is

	(i)	(ii)	(iii)	(iv)
(1)	b	c	d	a
(2)	c	a	d	b
(3)	b	d	c	a
(4)	c	a	b	d

[Ans : (4) i – c ii – a iii – b 4 – d]

IV. Choose the incorrect statement :

1. **Which of the following statement is incorrect?**
   (1) Initial velocity means velocity at $t = 0$.
   (2) Initial acceleration means, acceleration at $t = 0$.
   (3) If the motion is upward, at the maximum height the velocity is not zero.
   (4) If the motion is horizontal, $u = 0$ when the particle comes to rest.

   [Ans: (3) If the motion is upward, at the maximum height the velocity is not zero]

2. **Identify the incorrect statement.**
   (1) Every constant function is an increasing function.
   (2) Every constant function is a decreasing function.
   (3) Every identity function is an increasing function.
   (4) Every polynomial function is continuous

   [Ans: (2) Every constant function is a decreasing function.]

3. **Identify the false statement**
   (1) All the stationary numbers are critical numbers.
   (2) At the stationary point, the first derivative is zero.
   (3) At critical numbers, the first derivative does not exist.
   (4) All the critical numbers are stationary numbers.

   [Ans: (4) All the critical numbers are stationary numbers.]

2 MARKS

1. **A particle moves in a line so that $x = \sqrt{t}$. Show that the acceleration is negative and proportional to the cube of the velocity.**

Sol.
$$x = \sqrt{t}$$
$$V = \frac{dx}{dt} = \frac{1}{2}t^{-\frac{1}{2}} \quad ...(1)$$
$$\text{Acceleration} = \frac{d^2x}{dt^2} = \frac{1}{2}\left(-\frac{1}{2}t^{-\frac{3}{2}}\right) = \frac{-t^{-\frac{3}{2}}}{4}$$

$\therefore$ Acceleration is negative.

$$\text{Acceleration} = -\frac{1}{4}\left(t^{-\frac{1}{2}}\right)^3$$
$$= -2\left(\frac{1}{2}t^{-\frac{1}{2}}\right)^3 = 2V^3$$

[using (1)]

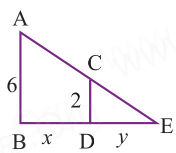
Hence, acceleration is negative proportional to the cube of the velocity.

2. **A man 2 m high walks at a uniform speed of 5 km/ hr away from a lamp post 6 m high. Find the rate at which the length of his shadow increases?**

Sol. Let AB be the lamp post. Let the man CD be at distance x m from lamp post and y m be the length of his shadow at any time t.

$$\text{Given } \frac{dx}{dt} = 5 \text{ km / hr} = 5000 \text{ m/hr.}$$

ΔABE and CDE are similar



$$\therefore \frac{DE}{CD} = \frac{BE}{AB}$$
$$\Rightarrow \frac{y}{2} = \frac{x+y}{6}$$
$$\Rightarrow 6y = 2x + 2y$$
$$\Rightarrow 4y = 2x$$
$$\Rightarrow y = \frac{x}{2}$$
$$\Rightarrow \frac{dy}{dt} = \frac{1}{2}\frac{dx}{dt} = \frac{1}{2}(5000)$$
$$= 2500 \text{ m / hr} = 2.5 \text{ km/hr}$$

3. At what point on the curve $y = x^2$ on [–2, 2] is the tangent parallel to X - axis?

Sol. $y = x^2$ is continuous on [–2, 2] and differentiable on [–2, 2]

$$f(a) = f(-2) = (-2)^2 = 4$$
$$f(b) = f(2) = 2^2 = 4$$
$$\therefore f(a) = f(b)$$

Since the tangent is parallel to X - axis, $f'(c) = 0$

$$\Rightarrow 2c = 0$$
$$\Rightarrow c = 0$$
$$\therefore \text{When } c = 0, y = 0$$

∴ AE (0,0) the tangent is parallel to X - axis.

4. Find the maximum and minimum values of $f(x) = |x + 3|\ \forall\ x \in \mathbb{R}$.

Sol.
$$f(x) = |x+3|\ \forall\ x \in \mathbb{R}$$
$$\text{Now, } |x+3| \geq 0\ \forall\ x \in \mathbb{R}$$
$$\Rightarrow f(x) \geq 0\ \forall\ x \in \mathbb{R}$$

So, the minimum value of $f(x)$ is 0

Also, $f(x) = |x + 3|$ does not have the maximum value.

5. Find the intervals of increasing and decreasing function for $f(x) = x^3 + 2x^2 - 1$.

Sol.
$$f(x) = x^3 + 2x^2 - 1$$
$$f'(x) = 3x^2 + 4x = 0$$
$$\Rightarrow x(3x+4) = 0$$
$$\Rightarrow x = 0 \text{ or } -\frac{4}{3}$$

The possible intervals are $\left(-\infty, -\frac{4}{3}\right)\left(-\frac{4}{3}, 0\right)$ and $(0, \infty)$.

Interval	$\left(-\infty, -\frac{4}{3}\right)$	$\left(-\frac{4}{3}, 0\right)$	$(0, \infty)$
Sign of $f'(x)$	Say $x = -2$ $3(-2)^2 + 4(-2) = 4$ +ve	Say $x = -1$ $3(-1)^2 + 4(-1) = -1$ –ve	Say $x = 1$ $3(1)^2 + 4(1) = 7$ +ve
Monotonicity	Strictly increasing	Strictly decreasing	Strictly increasing

3 MARKS

1. Prove that $\frac{x}{1+x} < \log(1+x)$ for $x > 0$.

Sol.
$$\text{Let } f(x) = \log(1+x) - \frac{x}{1+x}$$
$$f'(x) = \frac{1}{1+x} - \frac{(1+x)-x}{(1+x)^2}$$
$$= \frac{1}{1+x} - \frac{1}{(1+x)^2}$$
$$= \frac{x}{(1+x)^2} > 0 \text{ for } x > 0$$

$\therefore f(x)$ is strictly increasing in $(0, \infty)$

$$\therefore x > 0 \Rightarrow f(x) > f(0)$$
$$\Rightarrow \log(1+x) - \frac{x}{1+x} > 0$$
$$\Rightarrow \log(1+x) > \frac{x}{1+x}$$
$$\Rightarrow \frac{x}{1+x} < \log(1+x)$$

2. Find the equation of normal to the cure $y = \sin^2 x$ at $\left(\frac{\pi}{3}, \frac{3}{4}\right)$.

Sol.
$$y = \sin^2 x$$
$$\frac{dy}{dx} = 2\sin x\cos x = \sin 2x$$
$$\therefore m = \left(\frac{dy}{dx}\right)_{\left(\frac{\pi}{3}, \frac{3}{4}\right)} = \sin\frac{2\pi}{3} = \frac{\sqrt{3}}{2}$$

$$\therefore \text{Slope of the normal} = -\frac{1}{m} = -\frac{2}{\sqrt{3}}$$

$$\therefore \text{Equation of normal is } y - y_1 = -\frac{1}{m}(x - x_1)$$

$$\Rightarrow y - \frac{3}{4} = -\frac{2}{\sqrt{3}}\left(x - \frac{\pi}{3}\right)$$

$$\Rightarrow 12\sqrt{3}y - 9\sqrt{3} = -24x + 8\pi \text{ [Multiply } 12\sqrt{3}\text{]}$$

$$\therefore 24x + 12\sqrt{3}\,y = 8\pi + 9\sqrt{3}.$$

3. Verify LMV theorem for $f(x) = x^3 - 2x^2 - x + 3$ in [0,1]

Sol.
$$f(x) = x^3 - 2x^2 - x + 3$$
$$f'(x) = 3x^2 - 4x - 1$$
$$f'(c) = 3c^2 - 4c - 1$$
$$f(a) = f(0) = 3$$

$f(b) = f(1) = 1^3 - 2 - 1 + 3 = 1$

Then, if atleast one $C \in (0, 1)$ such that

$f'(c) = \dfrac{f(b) - f(a)}{b - a}$

$\Rightarrow 3c^2 - 4c - 1 = \dfrac{1-3}{1-0} = -2$

$\Rightarrow 3c^2 - 4c + 1 = 0$

$\Rightarrow c = \dfrac{4 \pm \sqrt{16 - 4(3)}}{2(3)}$

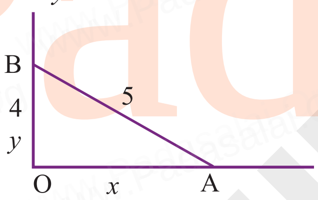
$\Rightarrow c = \dfrac{4 \pm 2}{6} = \dfrac{6}{6}$ or $\dfrac{2}{6}$

$\Rightarrow c = 1$ or $\dfrac{1}{3}$

4. The ends of a rod AB which is 5 m long moves along two grooves OX, OY which at the right angles. If A moves at a constant speed of $\frac{1}{2}$ m/sec, what is the speed of B, when it is 4m from O?

Sol. Let $OA = x$ m, $OB = y$ m

Then $x^2 + y^2 = 25$



Differentiating, $2x\dfrac{dx}{dt} + 2y\dfrac{dy}{dt} = 0$

$\Rightarrow \dfrac{dy}{dt} = -\dfrac{x}{y}\dfrac{dx}{dt}$

When $\dfrac{dx}{dt} = \dfrac{1}{2}, \dfrac{dy}{dt} = \dfrac{-x}{2y}$

When $y = 4, x^2 = 25 - y^2$

$\Rightarrow x = \sqrt{25-16} = 3$

Thus $\dfrac{dy}{dt} = -\dfrac{3}{2 \times 4} = \dfrac{-3}{8}$

5. A ball is thrown vertically upwards, moves according to the law $s = 13.8\,t - 4.9\,t^2$ where s is in metres and t is in seconds.

(i) Find the acceleration at $t = 1$

(ii) Find velocity at $t = 1$

(iii) Find the maximum height reached by the ball?

Sol. $s = 13.8\,t - 4.9\,t^2$

$v = \dfrac{ds}{dt} = 13.8 - 4.9\,(2t)$

$= 13.8 - 9.8t$

When $t = 1$, $v = 13.8 - 9.8(1)$

$= 4$ m / sec.

Acceleration $= \dfrac{d^2s}{dt^2} = -9.8$ m/sec^2

At maximum height, $v = 0$

$\therefore 13.8 - 9.8\,t = 0$

$\Rightarrow 13.8 = 9.8t$

$\Rightarrow t = \dfrac{13.8}{9.8} = 1.40$ sec

At $t = 1.4$ sec,

distance $(s) = 13.8\,(1.40) - 4.9\,(1.40)^2$

$= 19.32 - 9.604 = 9.716$ m

5 MARKS

1. If Rolle's theorem holds for $f(x) = x^3 + bx^2 + ax + 5$ on [1, 3] with $c = \left(2 + \frac{1}{\sqrt{3}}\right)$ find the values of a and b.

Sol. Given $f(x) = x^3 + bx^2 + ax + b$

Given that Rolle's theorem holds for $c = 2 + \dfrac{1}{\sqrt{3}}$

$\Rightarrow f'(c) = 0$

$\Rightarrow 3c^2 + 2bc + a = 0$

$\Rightarrow c = \dfrac{-2b \pm \sqrt{4b^2 - 4(3)a}}{6}$

$= \dfrac{-2b \pm \sqrt{4b^2 - 12a}}{6}$

$\Rightarrow 2 + \dfrac{1}{\sqrt{3}} = 2\left(\dfrac{(-b) \pm \sqrt{b^2 - 3a}}{6}\right)$

$= \dfrac{-b \pm \sqrt{b^2 - 3a}}{3}$

$= \left(\dfrac{-b}{3}\right) \pm \dfrac{\sqrt{b^2 - 3a}}{3}$

$\Rightarrow \dfrac{-b}{3} = 2 \Rightarrow -b = 6 \Rightarrow b = -6$

Also, $\dfrac{\sqrt{b^2 - 3a}}{3} = \dfrac{1}{\sqrt{3}}$

CHAPTER 8

DIFFERENTIALS AND PARTIAL DERIVATIVES

MUST KNOW DEFINITIONS

- **Linear approximation :**

  Let $f : (a, b) \to \mathbb{R}$ be a differentiable function and $x_0 \in (a,b)$

  $L(x) = f(x_0) + f'(x_0)(x - x_0) \; \forall \; x \in (a,b)$

- $\dfrac{df}{dx} = \lim\limits_{\Delta x \to 0} \dfrac{f(x+\Delta x) - f(x)}{\Delta x} = f'(x_0)$

- $df = f'(x)\,\Delta x$

- **Continuity :** Let $A = \{(x, y) / a < x < b, c < y < d\} \subset R^2$, $F: \mathbb{A} \to \mathbb{R}$

  F is continuous at (u, v) if

  a) F is defined at (u, v)

  b) $\lim\limits_{(x,y)\to(u,v)} F(x,y) = L$ exists

  c) $L = F(u, v)$

- **Clairaut's Theorem :** Let $A = \{(x, y) / a < x < b, c < y < d\} \subset R^2$, $F: \mathbb{A} \to \mathbb{R}$

  If f_{xy} and f_{yx} exist in A are continuous in A, then $f_{xy} = f_{gx}$ in A

- **Laplace's Equation :** Let $A = \{(x, y) / a < x < b, c < y < d\} \subset R^2$, $F: \mathbb{A} \to \mathbb{R}$. A function $U : \mathbb{A} \to \mathbb{R}^2$ is said to be harmonic in A if it satisfies $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 y}{\partial y^2} = 0 \; \forall \; (x, y) \in \mathbb{A}$

- F is a homogeneous function on A, if there exists *a* constant P such that $F(\lambda x, \lambda y) = \lambda^p F(x, y) \; \forall \; \lambda \in \mathbb{R}$ such that $(\lambda x, \lambda y) \in \mathbb{A}$. This constant *p* is called degree of F.

IMPORTANT FORMULA TO REMEMBER

- Absolute error = Actual Value – Approximate value
- Relative error = $\dfrac{\text{Actual value – Approximate value}}{\text{Acutal Value}}$
- Percentage error = Relative error × 100
- $df = f'(x)\,\Delta x$
- Linear approximation of F at $(x_0, y_0, z_0) \in \mathbb{A}$ is

$$F(x, y, z) = F(x_0, y_0, z_0) + \left.\frac{\partial F}{\partial x}\right|_{(x_0,y_0,z_0)} (x - x_0) + \left.\frac{\partial F}{\partial y}\right|_{(x_0,y_0,z_0)} (y - y_0) + \left.\frac{\partial F}{\partial z}\right|_{(x_0,y_0,z_0)} (z - z_0)$$

- Differential of F is

$$dF = \frac{\partial F}{\partial x}(x, y, z)\,dx + \frac{\partial F}{\partial y}(x, y, z)\,dy + \frac{\partial F}{\partial z}(x, y, z)\,dz$$

Where $dx = \Delta x$, $dy = \Delta y$ and $dz = \Delta z$.

- If $w(x, y)$ is a function of two variables x, y then $\dfrac{\partial w}{dt} = \dfrac{\partial w}{\partial x}\cdot\dfrac{dx}{dt} + \dfrac{\partial w}{\partial y}\cdot\dfrac{dy}{dx}$
- If $w(x, y)$ is a function of two variables (x, y) then $\dfrac{\partial w}{\partial s} = \dfrac{\partial w}{\partial x}\cdot\dfrac{\partial x}{\partial s} + \dfrac{\partial w}{\partial y}\cdot\dfrac{\partial y}{\partial s}$
- **Euler's theorem :** If F is having continuous partial derivatives and homogeneous on A with degree p, then

$$x.\frac{\partial F}{\partial x}(x, y, z) + y\frac{\partial F}{\partial y}(x, y, z) + z\frac{\partial F}{\partial z}(x, y, z) = p\,F(x, y, z)\ \forall\ (x, y, z) \in \mathbb{B}.$$

EXERCISE 8.1

1. Let $f(x) = \sqrt[3]{x}$. Find the linear approximation at $x = 27$. Use the linear approximation to approximate $\sqrt[3]{27.2}$.

Sol.

Given $f(x) = \sqrt[3]{x}$

Let $x_0 = 27$ and $\Delta x = 0.2$

We know $L(x) = f(x_0) + f'(x_0)(x - x_0)\ \forall\ x \in (a,b)$

$\therefore \sqrt[3]{27.2} = f(27) + f'(27)(0.2)$ ...(1)

Now $f(27) = \sqrt[3]{27} = 3$

$$f'(x) = \frac{1}{3}x^{\frac{1}{3}-1} = \frac{1}{3}x^{\frac{2}{3}} = \frac{1}{3x^{\frac{2}{3}}}$$

$$\therefore f'(27) = \frac{1}{3(27)^{\frac{2}{3}}} = \frac{1}{3(3^3)^{\frac{2}{3}}}\ \frac{1}{3(3^2)} = \frac{1}{27}$$

$\therefore$ (1) becomes,

$$\sqrt[3]{27.2} = 3 + \frac{1}{27}(0.2)$$

$$= 3 + .0074 = 3.0074$$

$$\therefore \sqrt[3]{27.2} = 3.0074$$

EXERCISE 8.8

Choose the Correct or the most suitable answer from the given four alternatives :

1. **A circular template has a radius of 10 cm. The measurement of radius has an approximate error of 0.02 cm. Then the percentage error in calculating area of this template is**

(1) 0.2% (2) 0.4%
(3) 0.04% (4) 0.08%

[Ans.(2) 0.4%]

Hint :
Area of circle $= \pi r^2$
Approximate area $= 2\pi r\, dr$
$= 2\pi (10)(0.02)$
$[\because r = 10, dr = 0.02]$
Percentage error $= \dfrac{2\not\pi(\not{10})(0.02)}{\not\pi(10^{\not 2})} \times 100$
$= \dfrac{0.04}{10} \times 100 = 0.4\%$

2. **The percentage error of fifth root of 31 is approximately how many times the percentage error in 31?**

(1) $\frac{1}{31}$ (2) $\frac{1}{5}$ (3) 5 (4) 31

[Ans. (2) $\frac{1}{5}$]

Hint : Percentage error of 5th root of 31 is $\frac{1}{5}$ times the percentage error in 31.

3. **If $u(x, y) = e^{x^2+y^2}$, then $\frac{\partial u}{\partial x}$ is equal to**

(1) $e^{x^2+y^2}$ (2) $2xu$ (3) x^2u (4) y^2u

[Ans. (2) $2xu$]

Hint :
$u(x, y) = e^{x^2+y^2}$
$\dfrac{\partial u}{\partial x} = e^{x^2+y^2}(2x) = 2xu$

4. **If $v(x, y) = \log(e^x + e^y)$, then $\frac{\partial v}{\partial x} + \frac{\partial v}{\partial y}$ is equal to**

(1) $e^x + e^y$ (2) $\dfrac{1}{e^x+e^y}$
(3) 2 (4) 1

[Ans. (4) 1]

Hint :
$v(x, y) = \log(e^x + e^y)$
$\dfrac{\partial v}{\partial x} = \dfrac{1}{e^x+e^y}(e^x)$
$\dfrac{\partial v}{\partial y} = \dfrac{1}{e^x+e^y}(e^y)$
$\therefore \dfrac{\partial v}{\partial x} + \dfrac{\partial v}{\partial y} = \dfrac{e^x+e^y}{e^x+e^y} = 1$

5. **If $w(x, y) = x^y, x > 0$, then $\frac{\partial w}{\partial x}$ is equal to**

(1) $x^y \log x$ (2) $y \log x$
(3) yx^{y-1} (4) $x \log y$

[Ans. (3) yx^{y-1}]

Hint :
$w(x, y) = x^y$
$\dfrac{\partial w}{\partial x} = yx^{y-1}$ (1)

6. **If $f(x, y) = e^{xy}$, then $\frac{\partial^2 f}{\partial x \partial y}$ is equal to**

(1) xye^{xy} (2) $(1+xy)e^{xy}$
(3) $(1+y)e^{xy}$ (4) $(1+x)e^{xy}$

[Ans. (2) $(1+xy)e^{xy}$]

Hint :
$f(x, y) = e^{xy}$
$\dfrac{\partial f}{\partial y} = e^{xy}(x) = xe^{xy}$
$\dfrac{\partial^2 f}{\partial x \partial y} = \dfrac{\partial}{\partial x}\left(\dfrac{\partial f}{\partial y}\right)$
$= x.\, e^{xy}(y) + e^{xy}(1)$
$= e^{xy}(1 + xy)$

7. **If we measure the side of a cube to be 4 cm with an error of 0.1 cm, then the error in our calculation of the volume is**

(1) 0.4 cu.cm (2) 0.45 cu.cm
(3) 2 cu.cm (4) 4.8 cu.cm

[Ans. (4) 4.8 cu.cm]

Hint :
$V = a^3$
$\Rightarrow$ error in volume $= 3a^2\, da = 3(4)^2(0.1)$
$= 48(0.1) = 4.8$ cu.cm

8. **The change in the surface area $S = 6x^2$ of a cube when the edge length varies from x_0 to $x_0 + dx$ is**

(1) $12x_0 + dx$ (2) $12x_0\, dx$
(3) $6x_0\, dx$ (4) $6x_0 + dx$

[Ans. (2) $12x_0\, dx$]

Hint :
$s = 6x^2$
Change in surface area $= 12\, x dx = 12x_0\, dx$

$$\frac{\partial w}{\partial z} = x^2(-1) + y^2(1) + 2z(x-y)$$
$$= -x^2 + y^2 + 2xz - 2yz$$
$$\therefore \frac{\partial w}{\partial x} + \frac{\partial w}{\partial y} + \frac{\partial w}{\partial z} = 0$$

15. If $f(x, y, z) = xy + yz + zx$, then $f_x - f_z$ is equal to

(1) $z - x$ (2) $y - z$
(3) $x - z$ (4) $y - x$

[Ans. (1) $z - x$]

Hint :
$$f(x, y, z) = xy + yz + zx$$
$$f_x = \frac{\partial f}{\partial x} = y + 0 + z = y + z$$
$$f_z = \frac{\partial f}{\partial z} = 0 + y + x = y + x$$
$$\therefore f_x - f_z = y + z - y - x = z - x$$

ADDITIONAL QUESTIONS

1 MARK

I. Choose the Correct or the most suitable answer from the given four alternatives :

1. If $y = x^4 - 10$ and if x changes from 2 to 1.99, the approximate change in y is

(1) -32 (2) -0.32
(3) -10 (4) 10 [Ans: (2) -0.32]

Hint :
$$dy = 4x^3\,dx$$
$$= 4(2)^3(-0.01) = -0.32$$

2. If the radius of the sphere is measured as 9 cm with an error of 0.03 cm, the approximate error in calculating its volume is

(1) 9.72 cm^3 (2) 0.972 cm^3
(3) 0.972π cm^3 (4) 9.72π cm^3

[Ans: (4) 9.72π cm^3]

3. If $\log_e 4 = 1.3868$, then $\log_e 4.01$ =

(1) 1.3968 (2) 1.3898
(3) 1.3893 (4) none

[Ans: (3) 1.3893]

Hint :
$$y(x) = f(x) + f(x_0)(x - x_0)$$
$$\log_e 4.01 = 1.3863 + \frac{1}{4}(0.01)$$
$$= 1.3893$$

4. If $u = \log\sqrt{x^2 + y^2}$, then $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$ is

(1) $\sqrt{x^2 + y^2}$ (2) 0
(3) u (4) $2u$ [Ans: (2) 0]

5. If $u = x^y + y^x$ then $u_x + u_y$ at $x = y = 1$ is

(1) 0 (2) 2 (3) 1 (4) ∞

[Ans: (2) 2]

Hint :
$$u_x + u_y = yx^{y-1} + y^x \log y - x^y \log x + xy^{x-1}$$
$$\text{At } x = y = 1$$
$$u_x + u_y = 1 + 0 + 0 + 1 = 2$$

6. If $u = (x - y)^4 + (y - z)^4 + (z - x)^4$ then $\sum \frac{\partial u}{\partial x}$ =

(1) 4 (2) 1 (3) 0 (4) -4

[Ans: (3) 0]

Hint :
$$\frac{\partial u}{\partial x} = 4(x-y)^3 + 4(z-x)^3(-1)$$
$$= 4(x-y)^3 - 4(z-x)^3$$
$$\frac{\partial u}{\partial y} = 4(x-y)^3(-1) + 4(y-z)^3$$
$$= -4(x-y)^3 + 4(y-z)^3$$
$$\frac{\partial u}{\partial z} = 4(x-y)^3(-1) + 4(y-z)^3$$
$$= -4(y-z)^3 + 4(z-x)^3$$
$$\sum \frac{dy}{dx} = 0$$

7. If $f(x, y, z) = \sin(xy) + \sin(yz) + \sin(zx)$ then f_{xx} is

(1) $-y^2 \sin(xy) + z^2 \cos(xz)$
(2) $y^2 \sin(xy) - z^2 \cos(xz)$
(3) $y^2 \sin(xy) + z^2 \cos(xz)$
(4) $-y^2 \sin(xy) - z^2 \cos(xz)$

[Ans: (4) $-y^2 \sin(xy) - z^2 \cos(xz)$]

5 MARKS

1. If $u = \tan^{-1}\left(\dfrac{x^3+y^3}{x-y}\right)$

Prove that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y}$ sin 2u.

Sol.
$$\text{Given } u = \tan^{-1}\left(\frac{x^3+y^3}{x-y}\right)$$
$$\Rightarrow \tan u = \frac{x^3+y^3}{x-y} \text{ and}$$
$$\text{let } f = \tan u$$
$$\therefore f(x,y) = \frac{x^3+y^3}{x-y}$$
$$f(tx, ty) = \frac{t^3x^3+t^3y^3}{tx-ty} = \frac{t^3\left(x^3+y^3\right)}{t(x-y)}$$
$$= t^2 f(x,y)$$

$\therefore f(x,y)$ is a homogeneous function of degree 2.

$\therefore$ By Euler's theorem,
$$x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y} = 2f$$
$$\Rightarrow x.\frac{\partial}{\partial x}(\tan u)+y.\frac{\partial}{\partial y}(\tan u) = 2\tan u \qquad [\because f = \tan u]$$
$$\Rightarrow x.\sec^2 u\frac{\partial u}{\partial x}+y\sec^2 u\frac{\partial u}{\partial y} = 2\tan u$$

Dividing by $\sec^2 u$ we get,
$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y} = \frac{2\sin u}{\sec^2 u} = \frac{2\sin u}{\dfrac{1}{\cos^2 u}}$$
$$= 2\sin u\cos u = \sin 2u$$

Hence proved.

2. Find $\dfrac{\partial f}{\partial x}, \dfrac{\partial f}{\partial y}, \dfrac{\partial^2 f}{\partial x^2}, \dfrac{\partial^2 f}{\partial y^2}$ at $x = 2, y = 3$ if $f(x, y) = 2x^2 + 3y^2 - 8xy$

Sol.
$$\text{Given } f(x,y) = 2x^2+3y^2-8xy$$
$$\frac{\partial f}{\partial x} = 4x-8y$$
$$\left(\frac{\partial f}{\partial x}\right)_{(2,3)} = 4(2)-8(3)$$
$$= 8-24 = -16$$
$$\frac{\partial f}{\partial y} = 6y-8x$$
$$\left(\frac{\partial f}{\partial y}\right)_{(2,3)} = 6(3)-8(2)$$
$$= 18-16 = 2$$
$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = 4$$
$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = 6$$

3. Using differential find the approximate value of cos 61 ; if it is given that sin 60º = 0.86603 and 1º = 0.01745 radians.

Sol.
$$\text{Let } f(x) = \cos x,\ x_0 = 60^\circ,\ dx = 1^\circ$$
$$f(x_0) = \cos 60^\circ = \frac{1}{2} = 0.5$$
$$f'(x) = -\sin x\, dx$$
$$f'(x_0) = -\sin x_0\, dx$$
$$f'(60) = -\sin 60^\circ(1^\circ)$$
$$= -(0.86603)(0.01745)$$
$$= -0.0154$$
$$\therefore f(x) = f(x_0)+f'(x_0)\,dx$$
$$f(61) = 0.5-0.0154$$
$$\therefore \tan 46^\circ = f(x_0)+f'(x_0)\,dx$$
$$\cos 61^\circ = 0.4849$$

4. If $V = \log r$ and $r^2 = x^2 + y^2 + z^2$, then prove that $\dfrac{\partial^2 V}{\partial x^2}+\dfrac{\partial^2 V}{\partial y^2}+\dfrac{\partial^2 V}{\partial z^2} = \dfrac{1}{r^2}$.

Sol.
$$\text{Given } r^2 = x^2+y^2+z^2$$
$$\log r^2 = \log(x^2+y^2+z^2)$$
$$\Rightarrow 2\log r = \log(x^2+y^2+z^2)$$
$$\therefore 2V = \log(x^2+y^2+z^2)\ [\because V = \log r]$$
$$\Rightarrow V = \frac{1}{2}\log(x^2+y^2+z^2)$$
$$\frac{\partial V}{\partial x} = \frac{1}{2}\frac{2x}{x^2+y^2+z^2} = \frac{x}{x^2+y^2+z^2}$$
$$\frac{\partial^2 V}{\partial x^2} = \frac{\left(x^2+y^2+z^2\right)(1)-x(2x)}{\left(x^2+y^2+z^2\right)^2}$$
$$= \frac{y^2+z^2-x^2}{\left(x^2+y^2+z^2\right)^2}$$

|||^{ly} $$\frac{\partial^2 V}{\partial y^2} = \frac{z^2+x^2-y^2}{(x^2+y^2+z^2)^2}$$

$$\frac{\partial^2 V}{\partial z^2} = \frac{x^2+y^2-z^2}{(x^2+y^2+z^2)^2}$$

$$\therefore \frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}+\frac{\partial^2 V}{\partial z^2} = \frac{y^2+z^2-x^2+z^2+x^2-y^2+x^2+y^2-z^2}{(x^2+y^2+z^2)^2}$$

$$= \frac{x^2+y^2+z^2}{(x^2+y^2+z^2)^2} = \frac{1}{x^2+y^2+z^2}$$

$$= \frac{1}{r^2}$$

Hence proved.

5. If $z = f(x-cy) + F(x+cy)$ where f and F are any two functions and c is a constant, show that $c^2\frac{\partial^2 z}{\partial x^2} = \frac{\partial^2 z}{\partial y^2}$.

Sol. Given $z = f(x-cy) + F(x+cy)$

$$\frac{\partial z}{\partial x} = f'(x-cy)(1) + F'(x+cy)(1)$$

$$= f'(x-cy) + F'(x+cy)$$

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)$$

$$= f''(x-cy) + F''(x+cy) \quad \text{... (1)}$$

$$\frac{\partial z}{\partial y} = f'(x-cy)(-c) + F'(x+cy)(c)$$

$$\frac{\partial^2 z}{\partial y^2} = f''(x-cy)(c^2) + F''(x+cy)(c^2)$$

$$= c^2[f''(x-cy) + F''(x+cy)] \quad \text{... (2)}$$

$$\frac{\partial^2 z}{\partial y^2} = c^2 . \frac{\partial^2 z}{\partial x^2} \quad \text{[using (1)]}$$

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CHAPTER 9

APPLICATIONS OF INTEGRATION

MUST KNOW DEFINITIONS

- First fundamental Theorem of Integral Calculus $F(x) = \int_a^x f(u)du$, $a < x < b$ then $\frac{d}{dx}(F(x)) = f(x)$
- Second fundamental Theorem of Integral Calculus $\int_a^b f(x)dx = F(b) - F(a)$
- $\int_0^\infty e^{-x}x^{n-1}dx$ is the gamma integral $\Gamma(n)$

IMPORTANT FORMULA TO REMEMBER

- **Right end rule for Riemann Integral:**

$$\int_a^b f(x)dx = \lim_{\substack{n\to\infty \\ \max(x_i - x_{i-1})\to 0}} \sum_{i=1}^{n} f(x_i)(x_i - x_{i-1})$$

- **Left end rule for Riemann Integral:**

$$\int_a^b f(x)dx = \lim_{\substack{n\to\infty \\ \max(x_i - x_{i-1})\to 0}} \sum_{i=1}^{n} f(x_{i-1})(x_i - x_{i-1})$$

- **Mid point rule for Riemann Integral:**

$$\int_a^b f(x)dx = \lim_{\substack{n\to\infty \\ \max(x_i - x_{i-1})\to 0}} \sum_{i=1}^{n} f\left(\frac{x_{i-1}+x_i}{2}\right)(x_i - x_{i-1})$$

EXERCISE 9.1

1. Find an approximate value of $\int_1^{1.5} x\,dx$ by applying the left-end rule with the partition {1.1, 1.2, 1.3, 1.4, 1.5}.

Sol. To find $\int_1^{1.5} x\,dx$ in {1.1, 1.2, 1.3, 1.4, 1.5}

Here $a = 1$,

$b = 1.5, n = 5, f(x) = x$

$\therefore h = \Delta x = \frac{b-a}{n} = \frac{1.5-1}{5} = \frac{0.5}{5} = 0.1$

The left end rule for Riemann sum with equal width Δx is

$$\int_a^b x\,dx = [f(x_0)+f(x_1)+.....+f(x_{(n-1)})]\Delta x$$
$$= [f(1)+f(1.1)+f(1.2)+f(1.3)+f(1.4)](0.1)$$
$$= (1+1.1+1.2+1.3+1.4)(0.1)$$
$$= 6(0.1)$$
$$\therefore \int_1^{1.5} x\,dx = 0.6$$

2. Find an approximate value of $\int_1^{1.5} x^2dx$ by applying the right-end rule with the partition {1.1, 1.2, 1.3, 1.4, 1.5}.

Sol. Here $a = 1, b = 1.5$,

$h = \Delta x = \frac{b-a}{n} = \frac{1.5-1}{5} = \frac{0.5}{5} = 0.1$

The right end rule for Riemann sum with equal width Δx is

$$S = [f(x_1)+f(x_2)+f(x_3)+f(x_4)+f(x_5)]\Delta x$$
$$= [f(1.1)+f(1.2)+f(1.3)+f(1.4)+f(1.5)](0.1)$$
$$= [(1.1)^2+(1.2)^2+(1.3)^2+(1.4)^2+(1.5)^2](0.1)$$
$$= (1.21+1.44+1.69+1.96+2.25)(0.1)$$
$$= (8.55)(0.1)$$
$$\therefore \int_1^{1.5} x^2\,dx = 0.855$$

3. Find an approximate value of $\int_1^{1.5}(2-x)dx$ by applying the mid-point rule with the partition {1.1, 1.2, 1.3, 1.4, 1.5}.

Sol. Here $a = 1, b = 1.5, n = 5$ and $f(x) = 2-x$

$h = \Delta x = \frac{b-a}{n} = \frac{1.5-1}{5} = \frac{0.5}{5} = 0.1$

Mid-point rule for Riemann sum with equal width Δx is

$$S = \left[f\left(\frac{x_0+x_1}{2}\right)+f\left(\frac{x_1+x_2}{2}\right)+f\left(\frac{x_2+x_3}{2}\right)+f\left(\frac{x_3+x_4}{2}\right)+f\left(\frac{x_4+x_5}{2}\right)\right]\Delta x$$
$$= \left[f\left(\frac{1+1.1}{2}\right)+f\left(\frac{1.1+1.2}{2}\right)+f\left(\frac{1.2+1.3}{2}\right)+f\left(\frac{1.3+1.4}{2}\right)+f\left(\frac{1.4+1.5}{2}\right)\right]0.1$$
$$= [f(1.05)+f(1.15)+f(1.25)+f(1.35)+f(1.45)](0.1)$$
$$= [0.95+0.85+0.75+0.65+0.55)](0.1)$$
$$= [(2-1.05)+(2-1.15)+(2-1.25)+(2-1.35)+(2-1.45)](0.1)$$
$$= (3.75)(0.1) = 0.375 \quad \therefore \int_1^{1.5}(2-x)dx = 0.375$$

EXERCISE 9.2

1. Evaluate the following integrals as the limits of sums

(i) $\int_0^1 (5x+4)dx$ **(ii)** $\int_1^2 (4x^2-1)dx$

Sol. (i) $\int_0^1 (5x+4)dx$

Here $a = 0, b = 1, f(x) = 5x+4$

$$\therefore f\left(a+(b-a)\tfrac{r}{n}\right) = f\left(0+1\left(\tfrac{r}{n}\right)\right) = f\left(\tfrac{r}{n}\right) = 5\left(\tfrac{r}{n}\right)+4$$

$$\int_a^b f(x)dx = \lim_{n\to\infty}\frac{b-a}{n}\sum_{r=1}^{n} f\left(a+(b-a)\tfrac{r}{n}\right)$$

EXERCISE 9.10

Choose the correct or the most suitable answer from the given four alternatives :

1. The value of $\int_0^{\frac{2}{3}} \frac{dx}{\sqrt{4-9x^2}}$ is

(1) $\frac{\pi}{6}$ (2) $\frac{\pi}{2}$ (3) $\frac{\pi}{4}$ (4) π

[Ans. (1) $\frac{\pi}{6}$]

Hint : $\frac{1}{3}\int_0^{\frac{2}{3}} \frac{dx}{\sqrt{\frac{4}{9}-x^2}} = \frac{1}{3}\int_0^{\frac{2}{3}} \frac{dx}{\sqrt{\left(\frac{2}{3}\right)^2-x^2}}$

$$= \frac{1}{3}\sin^{-1}\left(\frac{x}{\frac{2}{3}}\right)_0^{\frac{2}{3}}$$

$$= \frac{1}{3}\left[\sin^{-1}\left(\frac{\frac{2}{3}}{\frac{2}{3}}\right) - \sin^{-1}(0)\right]$$

$$= \frac{1}{3}\sin^{-1}(1)$$

$$= \frac{1}{3}\times\frac{\pi}{2} = \frac{\pi}{6}$$

2. The value of $\int_{-1}^{2}|x|\,dx$

(1) $\frac{1}{2}$ (2) $\frac{3}{2}$ (3) $\frac{5}{2}$ (4) $\frac{7}{2}$

[Ans. (3) $\frac{5}{3}$]

Hint : $\int_{-1}^{2}|x|\,dx = \int_{-1}^{0} -x\,dx + \int_0^2 x\,dx$

$$= \int_0^{-1} x dx + \int_0^2 x\,dx = \left(\frac{x^2}{2}\right)_0^{-1} + \left(\frac{x^2}{2}\right)_0^2$$

$$= \frac{1}{2} + 2 = \frac{5}{2}$$

3. For any value of $n\in\mathbb{Z}$, $\int_0^{\pi} e^{\cos^2 x}\cos^3\left[(2n+1)x\right]dx$ is

(1) $\frac{\pi}{2}$ (2) π (3) 0 (4) 2

[Ans. (3) 0]

Hint : $\int_0^{\pi} e^{\cos^2 x}\cos^3\left[(2n+1)x\right]dx$

Let $e^{\cos^2 x}\cos^3[(2n+1)x]$

$$f(-x) = e^{\cos(-x)^2}\cos^3[(2n+1)(-x)]$$
$$= e^{\cos x^2} - \cos^3[(2n+1)x]$$
$$= -f(x)$$

$\therefore f(x)$ is an odd functions.

4. The value of $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^2 x\cos x\,dx$ is

(1) $\frac{3}{2}$ (2) $\frac{1}{2}$ (3) 0 (4) $\frac{2}{3}$

[Ans. (4) $\frac{2}{3}$]

Hint : $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^2 x\cos x\,dx = \text{I}$

put $\sin x = t \Rightarrow \cos x\,dx = dt$

x	$-\frac{\pi}{2}$	$\frac{\pi}{2}$
t	-1	1

$$\text{I} = \int_{-1}^{1} t^2 dt = \left[\frac{t^3}{3}\right]_{-1}^{1} = \frac{1}{3} - \left(\frac{1}{3}\right) = \frac{2}{3}$$

5. The value of $\int_{-4}^{4}\left[\tan^{-1}\left(\frac{x^2}{x^4+1}\right) + \tan^{-1}\left(\frac{x^4+1}{x^2}\right)\right]dx$ is

(1) π (2) 2π (3) 3π (4) 4π

[Ans. (4) 4π]

Hint : $\int_{-4}^{4} \tan^{-1}\left(\frac{x^2}{x^4+1}\right) + \tan^{-1}\left(\frac{x^4+1}{x^2}\right)dx$

$$= \int_{-4}^{4}\left[\tan^{-1}\left(\frac{x^2}{x^4+1}\right) + \cot^{-1}\left(\frac{x^2}{x^4+1}\right)\right]dx$$

$[\because \tan^{-1}(x) = \cot^{-1}(\frac{1}{x})]$

$$= \int_{-4}^{4}\frac{\pi}{2}dx \quad [\because \tan^{-1}(x) + \cot^{-1}(x) = \frac{\pi}{2}]$$

$$= \frac{\pi}{2}[x]_{-4}^{4} = \frac{\pi}{2}[4-(-4)]$$

$$= 8\frac{\pi}{2} = 4\pi$$

ADDITIONAL QUESTIONS

1 MARK

I. Choose the Correct or the most suitable answer from the given four alternatives :

1. **The value of $\int_0^{\frac{\pi}{2}} \frac{dx}{1+\tan x}$ is**

(1) π (2) $\frac{\pi}{2}$ (3) $\frac{\pi}{4}$ (4) 0

[Ans: (3) $\frac{\pi}{4}$]

2. **The value of $\int_{\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{\frac{1-\cos 2x}{2x}}dx$ is**

(1) $\frac{1}{2}$ (2) 2 (3) 0 (4) 1

[Ans: (2) 2]

3. **$\int_1^{\sqrt{3}} \frac{dx}{1+x^2}$ is**

(1) $\frac{\pi}{3}$ (2) $\frac{\pi}{6}$ (3) $\frac{\pi}{12}$ (4) $-\frac{\pi}{6}$

[Ans: (3) $\frac{\pi}{12}$]

Hint : $\int_1^{\sqrt{3}} \frac{dx}{1+x^2} = \left[\tan^{-1} x\right]_1^{\sqrt{3}}$

$= \tan^{-1}\sqrt{3} - \tan^{-1} 1$

$= \frac{\pi}{3} - \frac{\pi}{4} = \frac{4\pi - 3\pi}{12} = \frac{\pi}{12}$

4. **If $\int_0^{2a} f(x)dx = 2\int_0^{a} f(x)$ then**

(1) $f(2a - x) = -f(x)$

(2) $f(2a - x) = f(x)$

(3) $f(x)$ is odd

(4) $f(x)$ is even

[Ans: (2) $f(2a - x) = f(x)$]

5. **The value of $\int_{-\pi}^{\pi} \sin^3 x \cos^3 x\, dx$ is**

(1) 0 (2) π (3) 2π (4) 4π

[Ans: (1) 0]

Hint : $f(x) = \sin^3 x \cos^3 x$

$f(-x) = \sin^3(-x)\cos^3(-x)$

$= -\sin^3 x \cos^3 x = -f(x)$

$\therefore f(x)$ is an odd function

6. **The area enclosed by the curve $y = \frac{x^2}{2}$, the x - axis and the lines $x = 1, x = 3$ is**

(1) 4 (2) $8\frac{2}{3}$ (3) 13 (4) $4\frac{1}{3}$

[Ans: (4) $4\frac{1}{3}$]

Hint : Area $= \int_1^3 \frac{x^2}{2}dx = \frac{1}{2}\left[\frac{x^3}{3}\right]_1^3 = \frac{1}{2}\left[9 - \frac{1}{3}\right]$

$= \frac{1}{2}\left[\frac{27-1}{3}\right] = \frac{26}{6} = 4\frac{1}{3}$

7. **The area bounded by the parabola $y = x^2$ and the line $y = 2x$ is**

(1) $\frac{4}{3}$ (2) $\frac{2}{3}$ (3) $\frac{51}{3}$ (4) $\frac{20}{3}$

[Ans: (1) $\frac{4}{3}$]

8. **The ratio of the volumes generated by revolving the ellipse $\frac{x^2}{9} + \frac{y^2}{4} = 1$ about major and minor axes is**

(1) 4:9 (2) 9:4 (3) 2:3 (4) 3:2

[Ans: (3) 2:3]

9. **$\int_0^{\infty} e^{-mx} x^7\, dx$ is**

(1) $\frac{\lfloor m}{7^m}$ (2) $\frac{\lfloor 7}{m^7}$ (3) $\frac{\lfloor m}{7^{m+1}}$ (4) $\frac{\lfloor 7}{m^8}$

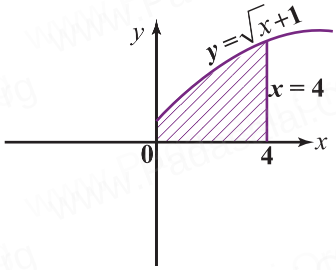
[Ans: (4) $\frac{\lfloor 7}{m^8}$]

10. **If $\int_0^{a} f(x)dx + \int_0^{a} f(2a - x)\, dx =$**

(1) $\int_0^a f(x)\,dx$ (2) $2\int_0^a f(x)\,dx$

(3) $\int_0^{2a} f(x)dx$ (4) $\int_0^{2a} f(a-x)dx$

[Ans: (3) $\int_0^{2a} f(x)dx$]



Required area

$$\int_0^4 y\,dx = \int_0^4 \left(\sqrt{x}+1\right)dx$$

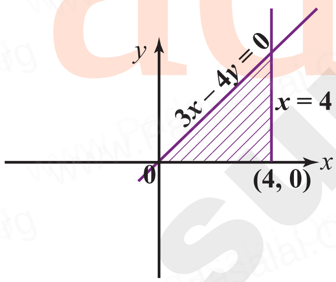
$$= \left[\frac{2}{3}x^{\frac{3}{2}} + x\right]_0^4 = \frac{2}{3}(4)^{\frac{3}{2}} + 4$$

$$= \frac{2}{3}(4)\sqrt{4} + 4 = \frac{16}{3} + 4$$

$$= \frac{16+12}{3} = \frac{28}{3} \text{ sq.units}$$

5. Find the volume of the solid obtained by revolving the area of the triangle whose sides are $x = 4$, $y = 0$ and $3x - 4y = 0$ about x - axis

Sol. Limits are from 0 to 4



$$3x - 4y = 0 \Rightarrow 4y = 3x$$

$$\Rightarrow \quad y = \frac{3}{4}x$$

$$\therefore \text{ Required volume} = \pi\int_0^4 y^2\,dx$$

$$= \pi\int_0^4 \frac{9}{16}x^2\,dx$$

$$= \frac{9\pi}{16}\left[\frac{x^3}{3}\right]_0^4 = \frac{9\pi}{16} \times \frac{64}{3}$$

$$\therefore V = 12\pi \text{ Cubic units}$$

3 MARKS

1. Evaluate $\displaystyle\int_0^{\frac{\pi}{2}} \frac{\sqrt{\cot x}}{\sqrt{\cot x}+\sqrt{\tan x}}\,dx$

Sol. Let $I = \displaystyle\int_0^{\frac{\pi}{2}} \frac{\sqrt{\cot x}}{\sqrt{\cot x}+\sqrt{\tan x}}\,dx$ ...(1)

By the property, $\displaystyle\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$,

$$\therefore I = \int_0^{\frac{\pi}{2}} \frac{\sqrt{\cot\left(\frac{\pi}{2}-x\right)}}{\sqrt{\cot\left(\frac{\pi}{2}-x\right)}+\sqrt{\tan\left(\frac{\pi}{2}-x\right)}}\,dx$$

$$= \int_0^{\frac{\pi}{2}} \frac{\sqrt{\tan x}}{\sqrt{\tan x}+\sqrt{\cot x}}\,dx \quad ...(2)$$

$$(1) + (2) \Rightarrow 2I = \int_0^{\frac{\pi}{2}} \frac{\sqrt{\cot x}+\sqrt{\tan x}}{\sqrt{\cot x}+\sqrt{\tan x}}\,dx$$

$$= \int_0^{\pi/2} dx = [x]_0^{\frac{\pi}{2}} = \frac{\pi}{2} - 0 = \frac{\pi}{2}$$

$$\therefore I = \frac{\pi}{4}$$

2. Evaluate $\displaystyle\int_{-1}^{1} \log\left(\frac{2-x}{2+x}\right)dx$

Sol. Let $f(x) = \log\left(\frac{2-x}{2+x}\right)$

$$f(-x) = \log\left(\frac{2-(-x)}{2+(-x)}\right) = \log\left(\frac{2+x}{2-x}\right)$$

$$= \log\left[\left(\frac{2-x}{2+x}\right)\right]^{-1} = -\log\left(\frac{2-x}{2+x}\right)$$

$$= -f(x)$$

$\therefore f(x)$ is an odd function of x.

$$\therefore \int_{-1}^{1} \log\left(\frac{2-x}{2+x}\right)dx = 0$$

3. Evaluate $\int_0^{\frac{\pi}{2}} \frac{\sin x}{9+\cos^2 x} dx$

Sol. Let I $= \int_0^{\frac{\pi}{2}} \frac{\sin x}{9+\cos^2 x} dx$

x	0	$\frac{\pi}{2}$
t	1	0

Put $t = \cos x \Rightarrow -\sin x\, dx = dt \Rightarrow \sin x\, dx = -dt$

$$\therefore \text{ I } = \int_1^0 -\frac{dt}{3^2+t^2} = \int_0^1 \frac{dt}{3^2+t^2} = \frac{1}{3}\left[\tan^{-1}\left(\frac{t}{3}\right)\right]_0^1$$

$$= \frac{1}{3}\left[\tan^{-1}\left(\frac{1}{3}\right) - \tan^{-1}(0)\right]$$

$$= \frac{1}{3}\tan^{-1}\left(\frac{1}{3}\right) \qquad [\because \tan^{-1}(0) = 0]$$

4. Evaluate $\int_0^1 \sqrt{9-4x^2}\, dx$.

Sol. $\int_0^1 \sqrt{9-4x^2}\,dx = \int_0^1 2\sqrt{\left(\frac{3}{2}\right)^2 - x^2}\,dx$

$$= 2\left[\frac{x}{2}\sqrt{\left(\frac{3}{2}\right)^2 - x^2} + \frac{9}{8}\sin^{-1}\left(\frac{x}{\frac{3}{2}}\right)\right]_0^1$$

$$\left[\because \sqrt{a^2-x^2} = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)\right]$$

$$= 2\left[\frac{1}{2}\sqrt{\frac{9}{4}-1} + \frac{9}{8}\sin^{-1}\left(\frac{2}{3}\right) - 0\right]$$

$$= 2\left[\frac{1}{2}\sqrt{\frac{5}{2}} + \frac{9}{8}\sin^{-1}\left(\frac{2}{3}\right)\right]$$

$$= \frac{\sqrt{5}}{2} + \frac{9}{4}\sin^{-1}\left(\frac{2}{3}\right)$$

5. Evaluate $\int_0^1 xe^{-2x} dx$

Sol. $u = x;$ $\qquad v = e^{-2x}$

$u' = 1;$ $\qquad v_1 = \frac{e^{-2x}}{-2}$

$\qquad v_2 = \frac{e^{-2x}}{4}$

Bernoulli's formula

$\int uv\,dx = uv^{(1)} - u'v^{(2)}$

$$\therefore \int_0^1 xe^{-2x}dx = \left[x\left(\frac{e^{-2x}}{-2}\right) - 1\left(\frac{e^{-2x}}{4}\right)\right]_0^1$$

$$= -\left[\frac{x}{2}e^{-2x} + \frac{e^{-2x}}{4}\right]_0^1$$

$$= -e^{-2x}\left[\frac{x}{2}+\frac{1}{4}\right]_0^1$$

$$= -e^{-2}\left(\frac{1}{2}+\frac{1}{4}\right) + e^0\left(0+\frac{1}{4}\right)$$

$$= -e^{-2}\left(\frac{3}{4}\right) + \frac{1}{4} = \frac{1}{4}\left(1-3e^{-2}\right)$$

5 MARKS

1. Prove that $\int_0^{\frac{\pi}{2}} \sin 2x \log(\tan x)\, dx = 0$

Sol. Let I $= \int_0^{\pi/2} \sin 2x \log(\tan x) dx$ ...(1)

Using the property $\int_0^a f(x)\,dx = \int_0^a f(a-x)dx$ we get

$$\text{I} = \int_0^{\pi/2} \sin 2\left(\frac{\pi}{2}-x\right) \log\tan\left(\frac{\pi}{2}-x\right) dx$$

$$= \int_0^{\pi/2} \sin(\pi - 2x) \log\cot x\, dx$$

$$= \int_0^{\pi/2} \sin 2x \log\left(\frac{1}{\tan x}\right) dx$$

$$= \int_0^{\pi/2} \sin 2x (\log 1 - \log\tan x)\, dx$$

$[\because \log 1 = 0]$

$$= -\int_0^{\pi/2} \sin 2x \log(\tan x)\, dx \qquad ...(2)$$

$\therefore (1) + (2) \rightarrow$

$$2\text{I} = \int_0^{\pi/2} \sin 2x \log(\tan x)\,dx - \int_0^{\pi/2} \sin 2x \log(\tan x)\,dx = 0$$

$\therefore$ I $= 0$

$\therefore$ The limit is from $t = 0$ to $t = \sqrt{3}$

$$\therefore \text{ Volume} = \pi \int_0^{\sqrt{3}} y^2 dx = \pi \int_0^{\sqrt{3}} \left(t - \frac{t^3}{3}\right)^2 (2t\,dt)$$

$[\because x = t^2 \Rightarrow dx = 2t\,dt]$

$$= 2\pi \int_0^{\sqrt{3}} \left(\frac{3t - t^3}{3}\right)^2 t\,dt$$

$$= \frac{2\pi}{9} \int_0^{\sqrt{3}} (9t^2 - 6t^4 + t^6)t\,dt$$

$$= \frac{2\pi}{9} \int_0^{\sqrt{3}} (9t^3 - 6t^5 + t^7)\,dt$$

$$= \frac{2\pi}{9} \left[\frac{9t^4}{4} - \frac{6t^6}{6} + \frac{t^8}{8}\right]_0^{\sqrt{3}}$$

$$= \frac{2\pi}{9} \left[\frac{81}{4} - 27 + \frac{81}{8} - 0\right]$$

$$\text{Volume} = \frac{2\pi}{9}\left(\frac{27}{8}\right) = \frac{3\pi}{4} \text{ cubic units}$$

❁❁❁❁❁

CHAPTER 10

ORDINARY DIFFERENTIAL EQUATIONS

MUST KNOW DEFINITIONS

- The order of a differential equation is the highest **order** derivative present in the differential equation.
- The integral power of the highest order derivative appears is called the **degree** of the differential equation.
- If a differential equation contains only ordinary derivatives of one or more functions with respect to a single independent variable, it is an **ordinary differential equation** (ODE).
- An equation involving only partial derivatives of one or more functions of two or more independent variables is a **partial differential equation** (PDE)
- If $g(x) = 0$ in $a_n(x)y^{(n)} + a_{n-1}(x)\,y^{n-1} + \ldots + a_1(x)y + a_0 y = g(x)$ then it is said to be **homogeneous.**
- The solution which contains as many arbitrary constants as the order of the differential equation is called the general solution.
- Integrating factor I.F $= e^{\int pdx}$ in $\frac{dy}{dx} + \text{P}y = \text{Q}$.
- Solution of $\frac{dy}{dx} + \text{P}y = \text{Q}$ is $ye^{\int pdx} = \int \text{Q}e^{\int pdx}dx + \text{C}$

- **A first order differential equation** of the form $\frac{dy}{dx} + \text{P}y = \text{Q}$ where P and Q are functions of x only. Here no product of y and its derivative $\frac{dy}{dx}$ occurs and the dependent variable y and its derivative with respect to independent variable x occur only in the first degree.
- The solution of the given differential equation (1) is given by $ye^{\int \text{P}dx} = \int \text{Q}e^{\int \text{P}dx}dx + \text{C}$. Here $e^{\int \text{P}dx}$ is known as the integrating factor (I.F.)
- A first order differential equation of the form $\frac{dx}{dy} + \text{P}x = \text{Q}$, where P and Q are functions of y only. Here no product of x and its derivative $\frac{dx}{dy}$ occurs and the dependent variable x and its derivative with respect to independent variable y occur only in the first degree. In this case, the solution is given by $xe^{\int \text{P}dy} = \int \text{Q}e^{\int \text{P}dy}dy + \text{C}$.
- If x denotes the amount of the quantity present at time t, then the instantaneous rate at which the quantity changes at time t is $\frac{dx}{dt}$
- This leads to a differential equation of the form $\frac{dx}{dt} = f(x,t)$.

EXERCISE 10.1

1. For each of the following differential equations, determine its order, degree (if exists)

(i) $\frac{dy}{dx} + xy = \cot x$

(ii) $\left(\frac{d^3y}{dx^3}\right)^{\frac{2}{3}} - 3\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 4 = 0$

(iii) $\left(\frac{d^2y}{dx^2}\right)^2 + \left(\frac{dy}{dx}\right)^2 = x\sin\left(\frac{d^2y}{dx^2}\right)$

(iv) $\sqrt{\frac{dy}{dx}} - 4\frac{dy}{dx} - 7x = 0$

(v) $y\left(\frac{dy}{dx}\right) = \frac{x}{\left(\frac{dy}{dx}\right) + \left(\frac{dy}{dx}\right)^3}$

(vi) $x^2\frac{d^2y}{dx^2} + \left[1 + \left(\frac{dy}{dx}\right)^2\right]^{\frac{1}{2}} = 0$

(vii) $\left(\frac{d^2y}{dx^2}\right)^3 = \sqrt{1 + \left(\frac{dy}{dx}\right)}$

(viii) $\frac{d^2y}{dx^2} = xy + \cos\left(\frac{dy}{dx}\right)$

(ix) $\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + \int y\,dx = x^3$

(x) $x = e^{xy\left(\frac{dy}{dx}\right)}$

Sol. (i) $\frac{dy}{dx} + xy = \cot x$

Given differential equation is

$$\frac{dy}{dx} + xy = \cot x.$$

The highest derivative is 1 and its power is 1 order 1, degree 1.

6. Show that $y = ae^{-3x} + b$, where a and b are arbitrary constants, is a solution of the differential equation $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} = 0$.

Sol. Consider $y = ae^{-3x} + b$

Differentiating with respect to 'x' we get,

$$\frac{dy}{dx} = -3ae^{-3x}$$

$$\Rightarrow \quad \frac{y'}{e^{-3x}} = -3a$$

Differentiating again with respect to 'x' we get,

$$\frac{e^{-3x}(y'') - y'(-3)e^{-3x}}{e^{-6x}} = 0$$

$$\Rightarrow \quad y''e^{-3x} + 3y'e^{-3x} = 0$$

$$\Rightarrow \quad e^{-3x}(y'' + 3y'') = 0$$

$$y'' + 3y' = 0 \quad [\because e^{-3x} \neq 0]$$

$\therefore y = ae^{-3x} + b$ is a solution of $y'' + 3y'' = 0$

7. Show that the differential equation representing the family of curves $y^2 = 2a\left(x + a^{\frac{2}{3}}\right)$ where a is a positive parameter, is $\left(y^2 - 2xy\frac{dy}{dx}\right)^3 = 8\left(y\frac{dy}{dx}\right)^5$

Sol. Consider $y^2 = 2a\left(x + a^{\frac{2}{3}}\right)$

$$\Rightarrow \quad y^2 = 2ax + 2a^{\frac{2}{3}}$$

Differentiating with respect to 'x' we get,

$$2yy' = 2a(1) \Rightarrow a = yy' \quad ...(2)$$

Substituting (2) in (1) we get,

$$y^2 = 2x(yy') + 2(yy')^{\frac{5}{3}}$$

$$\Rightarrow \quad y^2 - 2xyy' = (yy')^{\frac{2}{3}}$$

Taking power 3 both sides we get,

$$(y^2 - 2xyy')^3 = 2^3(yy')^{\frac{5}{3}\times 3}$$

$$\Rightarrow \quad (y^2 - 2xyy')^3 = 8(yy')^5$$

$$y^2 = 2a(x + a^{\frac{2}{3}})$$

is a solution of

$$\left(y^2 - 2xy\frac{dy}{dx}\right)^3 = 8\left(y\frac{dy}{dx}\right)^5$$

8. Show that $y = a\cos bx$ is a solution of the differential equation $\frac{d^2y}{dx^2} + b^2y = 0$.

Sol. Consider $y = a\cos bx \quad ...(1)$

Differentiating with respect to 'x' we get,

$$y' = -a\sin bx(b) = -ab\sin(bx)$$

Differentiating again with respect to 'x' we get,

$$y'' = -ab\cos(bx)(b)$$

$$= -ab^2\cos(bx)$$

$$\Rightarrow \quad y'' = -b^2[a\cos bx]$$

$$\Rightarrow \quad y'' = -b^2y \text{ [using (1)]}$$

$$\Rightarrow \quad y'' + b^2y = 0$$

$\therefore y = a\cos bx$ is a solution of $\frac{d^2y}{dx^2} + b^2y = 0$.

EXERCISE 10.5

1. If F is the constant force generated by the motor of an automobile of mass M, its velocity V is given by $M\frac{dV}{dt} = F - kV$, where k is a constant. Express V in terms of t given that V = 0 when $t = 0$.

Sol. Given equation is $m\frac{dv}{dt} = F - kv$ on separating the variables, we get

$$\frac{dv}{F - kv} = \frac{dt}{m}$$

$$\int \frac{dv}{F - kv} = \int \frac{dt}{m}$$

$$\Rightarrow \quad \frac{\log(F - kv)}{-k} = \frac{t}{m} + \log c$$

$$\Rightarrow \quad \log(F - kv) = e^{\frac{-kt}{m}} + \log c.$$

$$\Rightarrow \quad \log(F - kv) - \log c = e^{\frac{-kt}{m}}$$

$$\Rightarrow \quad \log\left(\frac{F - kv}{c}\right) = e^{\frac{-kt}{m}}$$

$$\Rightarrow \quad \frac{F - kv}{e} = e^{\frac{-kt}{m}}$$

$$\Rightarrow \quad \frac{F - kv}{c} = \frac{1}{e^{\frac{kt}{m}}}$$

When $t = 10$, $T = 65$

$$(2) \Rightarrow \quad 65 - T_m = (100 - T_m)e^{10K} = (100 - T_m)(e^{5K})^2 = (100 - T_m)\left(\frac{80 - T_m}{100 - T_m}\right)^2 \quad \text{[using (2)]}$$

$$\Rightarrow \quad 65 - T_m = \frac{(80 - T_m)^2}{100 - T_m}$$

$$\Rightarrow 6500 - 65T_m - 100T_m + T_m^2 = 6400 + T_m^2 - 160T_m$$

$$\Rightarrow \quad 6500 - 6400 = 165T_m - 160T_m$$

$$\Rightarrow \quad 100 = 5T_m$$

$$\Rightarrow \quad T_m = \frac{100}{5} = 20°C$$

Hence the temperature of the kitchen is 20°C

10. A tank initially contains 50 litres of pure water. Starting at time $t = 0$ a brine containing with 2 grams of dissolved salt per litre flows into the tank at the rate of 3 litres per minute. The mixture is kept uniform by stirring and the well-stirred mixture simultaneously flows out of the tank at the same rate. Find the amount of salt present in the tank at any time $t > 0$.

Sol. Let $x(t)$ denote the amount of salt in the tank at time t.

Its rate of change is

$$\frac{dx}{dt} = \text{inflow rate} - \text{outflow rate}$$

Now, 2 gram time 3 litres per minutes is

inflow rate = 6 grams of salt. (3×2 = 6)

The out flow of salt is $\frac{3}{50}$ times $x = \frac{3x}{50}$

$$\therefore \frac{dx}{dt} = 6 - \frac{3x}{50} = \frac{300 - 3x}{50} = -\frac{3(x-100)}{50}$$

$$\Rightarrow \quad \frac{dx}{x - 100} = -\frac{3}{50}\, dt$$

$$\Rightarrow \quad \int \frac{dx}{x - 100} = -\frac{3}{50}\int dt$$

$$\Rightarrow \quad \log(x - 100) = -\frac{3}{50}t + \log C$$

$$\Rightarrow \log(x - 100) - \log C = -\frac{3}{50}t$$

$$\Rightarrow \log\left(\frac{x-100}{C}\right) = -\frac{3}{50}t$$

$$\Rightarrow \quad \frac{x-100}{C} = e^{-\frac{3t}{50}}$$

$$\Rightarrow \quad x - 100 = Ce^{-\frac{3t}{50}} \quad ...(1)$$

When $t = 0$, $x = 0$

[Since initial water was pure without any salt]

$$\Rightarrow \quad 0 - 100 = Ce^0$$

$$\Rightarrow \quad C = -100$$

(1) becomes $x - 100 = -100\, e^{-\frac{3t}{50}}$

$$\Rightarrow \quad x = 100 - 100\, e^{-\frac{3t}{50}}$$

$$\Rightarrow \quad x = 100(1 - e^{-\frac{3t}{50}})$$

Hence the amount of salt in the tank at time t is

$$x = 100(1 - e^{-\frac{3t}{50}})$$

EXERCISE 10.9

1. The order and degree of the differential equation are respectively

(1) 2, 3 (2) 3, 3 (3) 2, 6 (4) 2, 4

[Ans. (1) 2, 3]

Hint : $\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^{\frac{1}{3}} + x^{\frac{1}{4}} = 0$

$$\Rightarrow \quad \frac{d^2y}{dx^2} + x^{\frac{1}{4}} = \left(\frac{dy}{dx}\right)^{\frac{1}{3}}$$

$$\Rightarrow \quad \left(\frac{d^2y}{dx^2} + x^{\frac{1}{4}}\right)^3 = -\left(\frac{dy}{dx}\right)^1$$

Order 2, degree 3

2. The differential equation representing the family of curves $y = A\cos(x + B)$, where A and B are parameters, is

(1) $\frac{d^2y}{dx^2} - y = 0$ (2) $\frac{d^2y}{dx^2} + y = 0$

(3) $\frac{d^2y}{dx^2} = 0$ (4) $\frac{d^2x}{dy^2} = 0$

[Ans. (2) $\frac{d^2y}{dx^2} + y = 0$]

Hint :

$$\frac{dy}{dx} = -A\sin(x + B)$$

$$\frac{d^2y}{dx^2} = -A\cos(x + B) = -y$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} + y = 0$$

25. The slope at any point of a curve $y = f(x)$ is given by $\frac{dy}{dx} = 3x^2$ and it passes through (–1,1). Then the equation of the curve is

(1) $y = x^3 + 2$ (2) $y = 3x^2 + 4$
(3) $y = 3x^3 + 4$ (4) $y = x^3 + 5$

[Ans. (1) $y = x^3 + 2$]

Hint :
$$\frac{dy}{dx} = 3x^2$$
$$\Rightarrow dy = 3x^2dx$$
$$\Rightarrow y = \frac{3x^3}{3} + c$$
$$\Rightarrow y = x^3 + c$$
Since this passes through (–1, 1), $1 = (-1)^3 + c$
$\Rightarrow 1 = -1 + c \Rightarrow c = 2$
$\therefore$ Equation of the curve is $y = x^3 + 2$

ADDITIONAL QUESTIONS

1 MARK

I. Choose the Correct or the most suitable answer from the given four alternatives :

1. The order and degree of the differential equation $\left[\left(\frac{d^2y}{dx^2}\right)+\left(\frac{dy}{dx}\right)^2\right]^{\frac{1}{2}} = \frac{d^3y}{dx^3}$ are

(1) 1, 2 (2) 2, 1 (3) 3, 2 (4) 2, 3

[Ans. (3) 3, 2]

Hint : $\left(\frac{d^2y}{dx^2}\right)+\left(\frac{dy}{dx}\right)^2 = \left(\frac{d^3y}{dx^3}\right)^2$

Order 3 ; Degree 2

2. The differential equation of the family of parabolas $y^2 = 4ax$ is

(1) $2y = x\left(\frac{dy}{dx}\right)$ (2) $y = 2x\left(\frac{dy}{dx}\right)$
(3) $y = 2x^2\left(\frac{dy}{dx}\right)$ (4) $y^2 = 2x\left(\frac{dy}{dx}\right)$

[Ans. (2) $y = 2x\left(\frac{dy}{dx}\right)$]

Hint :
$$y^2 = 4ax$$
$$2y\frac{dy}{dx} = 4a$$
$$\frac{y}{2}\frac{dy}{dx} = a$$
$$\therefore y^2 = 4\frac{y}{2}\frac{dy}{dx}x ; \quad = y = 2x\left(\frac{dy}{dx}\right)$$

3. The solution of $\sec^2x \tan y\, dx + \sec^2y \tan x\, dy = 0$ is

(1) $\tan x + \tan y = c$ (2) $\sec x + \sec y = c$
(3) $\tan x \tan y = c$ (4) $\sec x - \sec y = c$

[Ans. (3) $\tan x \tan y = c$]

4. The solution of $(x^2 - ay)dx = (ax - y^2)\, dy$ is

(1) $y = x^2 + y^2 - a(x + y)$
(2) $x^2 - y^2 + x - ay = 0$
(3) $x^3 + y^3 = 3ayx + c$
(4) $(x^2 - ay)(ax - y^2) = 0$

[Ans. (3) $x^3 + y^3 = 3ayx + c$]

5. The transformation $y = vx$ reduces $\frac{dy}{dx} = \frac{x+y}{3x}$ to

(1) $\frac{3av}{4v+1} = \frac{dx}{x}$ (2) $\frac{3dv}{v+1} = \frac{dx}{x}$
(3) $2x\frac{dv}{dx} = v$ (4) $\frac{3dv}{1-2v} = \frac{dx}{x}$

[Ans. (4) $\frac{3dv}{1-2v} = \frac{dx}{x}$]

Hint :
$$\frac{dy}{dx} = \frac{x+y}{3x}$$
$$\frac{3dv}{1-20} = \frac{x}{da}$$

6. The I.F. of $\operatorname{cosec} x \frac{dy}{dx} + y \sec^2x = 0$ is

(1) $e^{\sec x}$ (2) $e^{\tan x}$
(3) $e^{\sec x \tan x}$ (3) $e^{\sec^2 x}$

[Ans. (1) $e^{\sec x}$]

2 MARKS

1. Form the differential equation satisfied by are the straight lines in my-plane.

Sol. Equation of family of straight lines in my plane is $y = mx - c$ where m and c are arbitrary constraints.

Differentiating, $y' = m$

Differentiating again, $y'' = 0$, is the required differential equation.

2. A curve passing through the origin has its slope e^x. Find the equation of the curve.

Sol.
$$\text{Given slope} = \frac{dy}{dx} = e^x$$
$$\Rightarrow dy = e^x\,dx$$
$$\int dy = \int e^x dx$$
$$\Rightarrow y = e^x + c$$

Since the curve passes through (0, 0),
$$0 = e^0 + c$$
$$\Rightarrow 0 = 1 + c$$
$$\Rightarrow c = -1$$

$y = e^x - 1$ is the required equation of the curve.

3. Solve: $\frac{dy}{dx} = 1 + e^{x-y}$

Sol.
$$\text{Given } \frac{dy}{dx} = 1 + e^{x-y} \quad ...(1)$$
$$\text{putting } x - y = z \Rightarrow 1 - \frac{dy}{dx} = \frac{dz}{dx}$$
$$\Rightarrow \frac{dy}{dx} = 1 - \frac{dz}{dx}$$

∴ (1) becomes,
$$1 - \frac{dz}{dx} = 1 + e^z$$
$$\Rightarrow -\frac{dz}{dx} = e^z$$
$$\Rightarrow -\frac{dz}{e^z} = dx$$
$$-\int e^{-z}dz = \int dx$$
$$\Rightarrow \frac{e^{-z}}{-1} = x + c$$
$$e^{-z} = x + c$$
$$\Rightarrow e^{y-x} = x + c$$

4. Solve: $x\frac{dy}{dx} = x + y$

Sol.
$$\text{Given } x\frac{dy}{dx} = x + y$$
$$\frac{dy}{dx} = \frac{x+y}{x} \quad ...(1)$$

This is a homogeneous differential equation
$$\text{put } y = vx$$
$$\Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$$

∴ (1) becomes,
$$v + x\frac{dv}{dx} = \frac{x + vx}{x} = 1 + v$$
$$\Rightarrow x\frac{dv}{dx} = 1 + v - v = 1$$
$$\Rightarrow dv = \frac{dx}{x}$$
$$\Rightarrow \int dv = \int \frac{dx}{x}$$
$$\Rightarrow v = \log x + c$$
$$\Rightarrow \frac{y}{x} = \log x + c \quad [\because v = \frac{y}{x}]$$

5. Solve: $\frac{dy}{dx} + y = e^{-x}$

Sol. This is a linear differential equation
$$\text{Here P} = 1, \text{Q} = e^{-x}$$
$$\therefore \int p\,dx = \int 1 \cdot dx = x$$
$$\text{I.F.} = e^{\int pdx} = e^x$$

The solution is
$$y e^{\int p\,dx} = \int Qe^{\int p\,dx} dx + c$$
$$\Rightarrow y\,e^x = \int e^{-x} \cdot e^x dx + c = \int dx + c$$
$$\Rightarrow y\,e^x = x + c$$

3 MARKS

1. Solve $\frac{dy}{dx} + \frac{y^2}{x^2} = \frac{y}{x}$

Sol.
$$\text{Given } \frac{dy}{dx} + \frac{y^2}{x^2} = \frac{y}{x}$$
$$\Rightarrow \frac{dy}{dx} = \frac{y}{x} - \frac{y^2}{x^2} \quad ...(1)$$

$$\Rightarrow \quad ye^x = \int \cos x \cdot e^x dx + c$$

$$\Rightarrow \quad ye^x = \frac{e^x}{2}(\cos x + \sin x) + c$$

$$\Rightarrow \quad y = \frac{1}{2}(\cos x + \sin x) + ce^{-x}$$

$$\because \int e^{ax} \cos bx\, dx = \frac{e^{ax}}{a^2+b^2}[a \cos bx + b \sin ax]$$

5 MARKS

1. **Solve: $(1 + e^{2x})\, dy + (1 + y^2)e^x\, dx = 0$ when $y(0) = 1$**

Sol. Given $(1 + e^{2x})\, dy + (1 + y^2)e^x\, dx = 0$

$$\frac{dy}{1+y^2} = \frac{-e^x}{1+e^{2x}}dx$$

$$\int \frac{dy}{1+y^2} = -\int \frac{e^x dx}{1+(e^x)^2}$$

$$\tan^{-1}(y) = -\tan^{-1}(e^x) + c\,;$$

$$y(0) = 1 \Rightarrow \text{when } x = 0, y = 1;$$

$$t = e^x \Rightarrow dt = e^x\, dx$$

$$\int \frac{dt}{1+t^2} = \tan^{-1}(t)$$

$$= \tan^{-1}(e^x)$$

$\therefore \tan^{-1}(1) + \tan^{-1}(e^0) = c\,;$

$$\Rightarrow \quad \frac{\pi}{4} + \frac{\pi}{4} = c \Rightarrow c = \frac{\pi}{2}$$

$\therefore$ (1) becomes, $\tan^{-1}(y) + \tan^{-1}(e^x) = \dfrac{\pi}{2}$ which is the required solution.

2. **A population grows at the rate of 2% per year. How long does it take for the population to double?**

Sol. Let P_0 be the initial population and the population after t year be P.

$$\text{Given } \frac{dp}{dt} = \frac{2p}{100} \Rightarrow \frac{dp}{dt} = \frac{p}{50}$$

$$\Rightarrow \quad \frac{dp}{p} = \frac{dt}{50} \Rightarrow \int \frac{dp}{p} = \int \frac{dt}{50}$$

$$\Rightarrow \quad \log p = \frac{t}{50} + c \quad \text{...(1)}$$

$$\text{when } t = 0, p = 0$$

$$\Rightarrow \quad \log p_0 = 0 + c \Rightarrow c = \log p_0. \quad \text{....(1)}$$

$\therefore$ (1) becomes, $\log p = \dfrac{t}{50} + \log P_0.$

$$\Rightarrow \log P - \log P_0 = \frac{t}{50}$$

$$\Rightarrow \quad \log\left(\frac{P}{p_0}\right) = \frac{t}{50}$$

$$\Rightarrow \quad t = 50 \log\left(\frac{P}{p_0}\right)$$

$$\text{when } p = 2p_0,\ t = 50 \log\left(\frac{2P_0}{P_0}\right) = 50 \log 2$$

$$= 50(0.3) = 15 \text{ years}$$

Hence the population doubles in 15 years.

3. **The surface area of a balloon being inflated changes at a constant rate. If initially, its radius 3 units and after 2 seconds it is 5 units, find the radius after t seconds.**

Sol. Let s be the surface area of the balloon after t sec.

$$\Rightarrow \quad s = 4\pi r^2$$

$$\frac{ds}{dt} = 8\pi r \cdot \frac{dr}{dt}$$

$$\text{Given that } \frac{ds}{dt} = \text{constant} = k$$

$$\therefore\ 8\pi r\frac{dr}{dt} = k$$

$$\Rightarrow \quad 8\pi r\, dr = k\, dt$$

$$\Rightarrow \quad \int 8\pi r\, dr = k\int dt$$

$$\Rightarrow \quad 8\pi \cdot \frac{r^2}{2} = kt + c$$

$$\Rightarrow \quad 4\pi r^2 = kt + c \quad \text{...(1)}$$

$$\text{when } t = 0,\ r = 3$$

$$\Rightarrow \quad 4\pi(32) = 36(0) + c$$

$$\Rightarrow \quad c = 36\pi$$

$$\text{when } t = 2,\ r = 5$$

$$\Rightarrow \quad 4\pi(52) = 25(k) + 36$$

$$\Rightarrow \quad 100\pi = 2k + 36\pi$$

$$\Rightarrow \quad k = 32\pi$$

$$\text{(1) becomes, } 4\pi r^2 = 32\pi t + 36\pi$$

$$\Rightarrow \quad r^2 = 8t + 9$$

$$\Rightarrow \quad r = \sqrt{8t+9}$$

CHAPTER 11

PROBABILITY DISTRIBUTIONS

MUST KNOW DEFINITIONS

- A random variable. X is defined on a sample space S into the real numbers R is called **discrete random variable** if the range of X is countable.
- If X is a discrete random variable then the function
  $f(x_k) = P(x = x_k)$ for $k = 1, 2, 3 \ldots n$ is called the **probability mass function** of X. (p. m. f.)
- The **Cumulative distribution** function $F(x)$ of a discrete random variable X such that $x_1 < x_2 < \ldots$ with p. m. f. $f(x_i)$ is
- $F(x) = P(X \leq x) = \sum_{x_i \leq x} f(x_i), x \in \mathbb{R}$.
- Let S be a sample space and let a random variable $X = S \rightarrow \mathbb{R}$ that takes on any value in a set I or R. Then X is called a **continuous random variable** if $P(X = x) = 0$ for every x in I.
- Let x be a random variable associated with a Bernoulli trial by defining it as X (success) = 1 and X (failure) = 0 such that
- $f(x) = \begin{cases} p & if \quad x = 1 \\ q = 1 - p & if \quad x = 0 \end{cases}$ where $0 < p < 1$.
- X is called a **Bernoulli random variable** and $f(x)$ is called the **Bernoulli distribution.**
- The binomial random variable X, equals the number of successes with probability p for a success and $q = 1 - p$ for a failure in n independent traits, has a binomial distribution
- The p.m. f. of X is $f(x) = nC_x p^x q^{n-x}, x = 0, 1, 2, \ldots n$.

Properties of mathematical expectation and variance.

(i) E $(ax + b) = a$ E $(x) + b$ where a and b are constrants.

(ii) Var $(x) = E(x^2) - [E(x)]^2$

(iii) Var $(ax + b) = a^2$ Var (x).

One point distribution

✦ A random variable X has a one-point distribution if F x_0 such that p. m. f. $f(x)$ is $f(x) = p(X = x_0) = 1$

Two point distribution

✦ A random variable X has a two point distribution if F two value $x1$ and $x2$ such that

✦ $f(x) = \begin{cases} p & \text{for} \quad x = x_1 \\ 1-p & \text{for} \quad x = x_2 \end{cases}, 0 < p < 1$

✦ $F(x) = \begin{cases} 0 & \text{if} \quad x < x_1 \\ p & \text{if} \quad x_1 \le x < x_2 \\ 1 & \text{if} \quad x \ge x_2 \end{cases}$

For Bernoulli's distribution

✦ $\mu = p$ and $\sigma^2 = pq$

For Binomial distribution

✦ $\mu = np$ and $\sigma^2 = np(1-p)$.

EXERCISE 11.1

1. Suppose X is the number of tails occurred when three fair coins are tossed once simultaneously. Find the values of the random variable X and number of points in its inverse images.

Sol. When 3 fair coins are tossed, sample space
s = {HHH, HHT, THH, HTH, HTT, THT, TTH, TTT}

Let X denote the number of tarits occured.

X (no tail) = {HHH} = 1
X (1 tail) = {HHT, THT, HTH} = 3
X (2 tails) = {HTT, THT, TTH} = 3
X (3 tails) = {TTT} = 1

∴ X takes the values 0, 1, 2, 3.

Values of random variable X	0	1	2	3	Total
Number of elements in inverse images	1	3	3	1	8

2. In a pack of 52 playing cards, two cards are drawn at random simultaneously. If the number of black cards drawn is a random variable, find the values of the random variable and number of points in its inverse images.

Sol. Sample space s = {26 black cards, 26 red cards}.

Let X denote the number of black cards drawn.

X (no black card) = 26 × 25 = 650
Taking 2 cards from red]

X (1 black card) = 26 × 26 = 676 [1 black card & 1 red card]

X (2 black cards) = 26 × 25 = 650 [Taking 2 cards from black]

∴ X takes the values 0, 1, 2.

Values of random variable X	0	1	2	Total
Number of elements in inverse images	650	676	650	1976

$$= \int_0^\infty x \cdot 16x \cdot e^{-4x} dx$$

$$= 16\int_0^\infty x^2 \cdot e^{-4x} dx = 16 \times \frac{2!}{4^3}$$

$$\left[\because \int_0^\infty x^n e^{-ax} dx = \frac{n!}{a^{n+1}}\right]$$

$$= 16 \times \frac{2}{64} = \frac{1}{2}$$

$$E(x^2) = \int_0^\infty x^2 \cdot f(x) dx$$

$$= \int_0^\infty x^2 \cdot 16x\, e^{-4x} dx$$

$$= 16\int_0^\infty x^3 e^{-4x} dx = 16 \times \frac{3!}{4^4}$$

$$= \frac{16 \times 3 \times 2}{{}_8 16 \times 16} = \frac{3}{8}$$

$$\therefore \text{Var(X)} = E(X^2) - [E(x)]^2$$

$$= \frac{3}{8} - \left(\frac{1}{2}\right)^2 = \frac{3}{8} - \frac{1}{4}$$

$$= \frac{3-2}{8} = \frac{1}{8}$$

$$\therefore \text{Var(X)} = \frac{1}{8}$$

8. A lottery with 600 tickets gives one prize of ₹200, four prizes of ₹100, and six prizes of ₹ 50. If the ticket costs is ₹ 2, find the expected winning amount of a ticket.

Sol. $P(X = k) = \binom{n}{k} p^k (1-p)^{n-k}$,

$$E(X) = 200 \times \frac{1}{600} + 100 \times \frac{4}{600} + 50 \times \frac{6}{600} - 2 \times \frac{600}{600}$$

$$= \frac{200}{600} + \frac{400}{600} + \frac{300}{600} - 2$$

$$= \frac{900}{600} - 2 = \frac{3}{2} - 2 = \frac{3-4}{2}$$

$$= \frac{-1}{2} = ₹ -0.50$$

EXERCISE 11.5

1. Compute P(X = k) for the binomial distribution, B(n, p) where

(i) $n = 6, p = \frac{1}{3}, k = 3$

(ii) $n = 10, p = \frac{1}{5}, k = 4$

(iii) $n = 9, p = \frac{1}{2}, k = 7$

Sol. Given $n = 6, p = \frac{1}{3}, k = 3$

The binomial distribution

$$P(X = k) = \binom{n}{k} p^k (1-p)^{n-k}, \quad n = 0, 1, 2, \ldots n$$

$$\therefore P(X = k) = \binom{n}{k} p^k (1-p)^{n-k}, \quad n = 0, 1, 2, \ldots n$$

$$P(X = 3) = \binom{6}{3}\left(\frac{1}{3}\right)^3 (1-p)^{6-3}$$

$$= \binom{6}{3}\left(\frac{1}{3}\right)^3\left(\frac{2}{3}\right)^3$$

$$= \frac{6 \times 5 \times 4}{3 \times 2 \times 1} \times \frac{1}{27} \times \frac{8}{27}$$

$$P(X = 3) = \frac{160}{729}$$

(ii) $$P(X = 10) = \binom{10}{4}\left(\frac{1}{5}\right)^4\left(1-\frac{1}{5}\right)^{10-4}$$

$$= \binom{10}{4}\left(\frac{1}{5}\right)^4\left(\frac{4}{5}\right)^6 = \frac{10 \times 9^3 \times 8 \times 7}{4 \times 3 \times 2 \times 1} \times \left(\frac{1}{5}\right)^4\left(\frac{4}{5}\right)^6$$

$$= 210\left(\frac{1}{5}\right)^4\left(\frac{4}{5}\right)^6$$

Hint :
$$k + 2k + 3k + 4k + 5k = 1$$
$$\Rightarrow 15k = 1$$
$$\Rightarrow k = \frac{1}{15}$$
$$E(X) - 2k - 2k + 0 + 4k + 10k$$
$$= 10k$$
$$= \frac{10}{15} = \frac{2}{3}$$

18. Let X have a Bernoulli distribution with mean 0.4, then the variance of (2X – 3) is

(1) 0.24 (2) 0.48 (3) 0.6 (4) 0.96

[Ans. (4) 0.96]

Hint : For a Bernoulli distribution,
$$\mu = p \text{ and } \sigma^2 = pq$$
$$\text{Given } \mu = p = 0.4$$
$$\Rightarrow q = 0.6$$
$$\text{Variance (X)} = pq$$
$$= (0.4)(0.6)$$
$$= 0.24$$
$$\therefore \text{Var}(2x - 3) = 4$$
$$\text{Var(X)} = 4(0.24)$$
$$= 0.96$$

19. If in 6 trials, X is a binomial variate which follows the relation 9P(X = 4) = P(X = 2), then the probability of success is

(1) 0.125 (2) 0.25 (3) 0.375 (4) 0.75

[Ans. (2) 0.25]

Hint :
$$9\,P(X = 4) = P(X = 2)$$
$$\Rightarrow 9[6C_4 p^4 q^2] = 6C_2 p^2 q^4$$
$$\Rightarrow 9[6C_2 p^4 q^2] = 6C_2 p^2 q^4$$
$$\Rightarrow 9p^2 = q^2$$
$$\Rightarrow 9p^2 = (1 - p)^2$$
$$\Rightarrow 9p^2 = 1 + p^2 - 2p$$
$$\Rightarrow 8p^2 + 2p - 1 = 0$$
$$\Rightarrow (2p + 1)(4p - 1) = 0$$
$$\Rightarrow p = \frac{-1}{2},$$
$$p = \frac{1}{4}$$
$$\Rightarrow p = \frac{1}{4} = 0.25$$

20. A computer salesperson knows from his past experience that he sells computers to one in every twenty customers who enter the showroom. What is the probability that he will sell a computer to exactly two of the next three customers?

(1) $\frac{57}{20^3}$ (2) $\frac{57}{20^2}$ (3) $\frac{19^3}{20^3}$ (4) $\frac{57}{20}$

[Ans. (1) $\frac{57}{20^3}$]

Hint : Give $p = \frac{1}{20}$, $q = \frac{19}{20}$ and $n = 3$

$$P(X = 2) = 3C_2\left(\frac{1}{20}\right)^2\left(\frac{19}{20}\right)^1 = \frac{3 \times 19}{20} = \frac{57}{20^3}$$

ADDITIONAL QUESTIONS

1 MARK

I. Choose the Correct or the most suitable answer from the given four alternatives :

1. If F(x) is the probability distribution function then F(–∞) is

(1) 1 (2) 2 (3) ∞ (4) 0

[Ans. (4) 0]

Hint : $F(\infty) = 1$ and $F(-\infty) = 0$

2. If F(x) is the probability distribution function, then F(∞) is

(1) 1 (2) 2 (3) ∞ (4) 0

[Ans. (1) 1]

Hint : $F(\infty) = 1$ and $F(-\infty) = 0$

3. If $F(x) = \frac{1}{2}$, $E(x^2) = \frac{1}{4}$ then var(x) is

(1) 0 (2) $\frac{1}{4}$ (3) $\frac{1}{2}$ (4) 1

[Ans. (1) 0]

Hint :
$$\text{Var (X)} = E(X^2) - (E(X))^2$$
$$= \frac{1}{4} - \frac{1}{4} = 0$$

3 MARKS

1. The p.d.f. of X is given by

X	–2	2	5
P(X = x)	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{2}$

then find $4E(X^2) – var(2X)$

Sol. $E(X) = \Sigma xp(x)$

$= -2\left(\frac{1}{4}\right)+2\left(\frac{1}{4}\right)+5\left(\frac{1}{2}\right)$

$= \frac{5}{2}$

$4.E(X^2) – Var(2X) = 4E(X^2)–4[E(X^2)–(E(X_1)^2)]$

$= 4\,E(X^2)-4\,E(X^2)+4[E(X)]^2$

$= 4\,[E(X)]^2$

$= 4\left(\frac{5}{2}\right)^2 = 4\left(\frac{25}{4}\right) = 25.$

2. If a continuous random variable X has the p.d.f. $f(x) = 4x(x – 1)^3$, then find $P(1 \le X \le 2)$.

Sol. Given $f(x) = 4k\,(x – 1)^3,\ 1 \le x \le 3.$

Since $f(x)$ is a p.d.f $\int_1^3 f(x)dx = 1$

$\Rightarrow \int_1^3 4k(x-1)^3\,dx = 1$

$\Rightarrow 4k\left[\frac{(x-1)^4}{4}\right]_1^3 = 1$

$\Rightarrow k[2^4 – 0^4] = 1$

$\Rightarrow 16k = 1$

$\Rightarrow k = \frac{1}{16}$

$\therefore P(1 \le X \le 2) = \int_{+1}^{2} f(x)dx$

$= \int_1^2 \frac{1}{4}(x-1)^3\,dx$

$= \frac{1}{4}\left[\frac{(x-1)^4}{4}\right]_1^2$

$= \frac{1}{16}(1^4 – 0^4) = \frac{1}{16}$

3. A player tosses two unbiased coins. He wins ₹ 5 if two heads appear, ₹ 2 if one head appear and ₹ 1 if no head appear. Find the expected amount to win.

Sol. $P(H) = P(T) = \frac{1}{2}$

$P(X = 5) = P(HH) = \frac{1}{2}\times\frac{1}{2} = \frac{1}{4}$

$P(X = 2) = P(HT) + P(TH)$

$= \frac{1}{2}\times\frac{1}{2}+\frac{1}{2}\times\frac{1}{2} = \frac{1}{2}$

$P(X = 1) = P(TT) = \frac{1}{2}\times\frac{1}{2} = \frac{1}{4}$

Hence, the probability distribution function is

X	1	2	5
P(X = x)	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$

$\therefore E(x) = 1\times\frac{1}{4}+2\times\frac{1}{2}+5\times\frac{1}{4}$

$= \frac{1}{4}+1+\frac{5}{4}$

$= \frac{1+4+5}{4} = \frac{10}{4} = 2.50$

Hence, the expected money to win is ₹ 2.50.

4. The probability that an event A happens in one treat of an experiment is 0.4. Three independent treats of the experiment are performed. Find the probability that the event A happens atleast once?

Sol. Given $p = 0.4,\ n = 3$

$q = 1 – 0.4 = 0.6$

$P(X \ge 1) = P(X = 1) + P(X = 2) + P(X = 3)$

$= 3C_1(0.4)^1(0.6)^2 + 3C_2(0.4)^2(0.6)^1 + 3C_3(0.4)^3(0.6)^0$

$= 3\times\left(\frac{4}{10}\right)\left(\frac{36}{100}\right) + 3\left(\frac{16}{100}\right)\left(\frac{6}{10}\right)+\frac{64}{100}$

$= \frac{1}{1000}(432 + 288 + 64) = \frac{784}{1000} = 0.784.$

CHAPTER 12

DISCRETE MATHEMATICS

MUST KNOW DEFINITIONS

- A **binary operation** * on S is defined as follows: $\forall\ a, b \in$ S, $a * b$ is unique and $a * b \in$ S
- A binary operation * defined by *: S × S → S; $(a, b) = a * \in$ S must always lie in the given set and not in the complement of it. Then S is closed w.r.t. to *.
- A binary operation * defined on a non empty set S is said to satisy the **commutative property** if $a * b = b * a\ \forall\ a, b \in$ S
- If $a * (b * c) = (a * b) * c\ \forall\ a, b \in$ S, then S is said to satisfy the **associative property**.
- An element $e \in$ S is said to satisfy the **identity element** of s if $\forall\ a \in$ S, $a * e, = e * a = a$.
- If for every $a \in$ S, there exists b in S such that $a * b = b * a = e$ then $b \in$ S is said to be the inverse element of a.
- In an algebraic structure, the identity element and the inverse of an element must be unique.
- A Boolean matrix is a real matrix whose entries are either 0 or 1.
- Joint of A and B, $A \vee B = [a_{ij}] \vee [b_{ij}] = [a_{ij} \vee b_{ij}] = [c_{ij}]$ where $c_{ij} = \begin{cases} 1 & \text{if either } a_{ij} = 1 \text{ or } b_{ij} = 1 \\ 0 & \text{if both } a_{ij} = 0,\ b_{ij} = 0 \end{cases}$
- Meet of A and B, $A \vee B = [a_{ij} \wedge b_{ij}] = [c_{ij}]$ where $c_{ij} = \begin{cases} 1 & \text{if both } a_{ij} = 1, b_{ij} = 1 \\ 0 & \text{if either } a_{ij} = 0, b_{ij} = 0 \end{cases}$
- **Addition moduls** n

  Let $a, b \in \mathbb{Z}n$. Then $a +_n b$ = the remainder of $a + b$ on division by n.
- Multiplication moduls n

  Let $a, b \in \mathbb{Z}n$. Then $a \times_n b$ = the remainder of $a \times b$ on division by n.
- A statement is said to be a tautology (II) if its truth value is always T irrespective of the truth values of its compound statements.
- A statement is a **contradiction** ($\mathbb{F}$) if its truth value is always F irrespective of the truth value of its compound statements.
- A dual is obtained by replacing II by $\mathbb{F}$ and $\mathbb{F}$ by II

- Any two compound statements A and B are said to be logically equivalent if the columns corresponding to A and B in the truth table have identical truth values. (A ≡ B) or A ⇔ B.
- Laws of equivalence.

1. Idempotent laws: (i) $p \vee q \equiv p$ (ii) $p \wedge q = p$
2. Commutative laws: (i) $p \vee q = q \vee p$ (ii) $p \wedge q = q \wedge p$
3. Associative laws: (i) $p \vee (q \vee r) = (p \vee q) \vee r$ (ii) $p \wedge (q \wedge r) = (p \wedge q) \wedge r$
4. Distributive laws: (i) $p \vee (q \wedge r) = (p \vee q) \wedge (p \vee r)$ (ii) $p \wedge (q \vee r) = (p \wedge q) \vee (p \wedge r)$
5. Identity laws: (i) $p \vee \mathbb{T} = p$ and $p \vee \mathbb{F} = p$ (ii) $p \wedge \mathbb{T} = p$ and $p \wedge \mathbb{F} = p$
6. Component laws: (i) $p \vee \sim p = \mathbb{T}$ and $p \wedge \sim p = \mathbb{F}$ (ii) $\sim \mathbb{T} = \mathbb{F}$ and $\sim \mathbb{F} = \mathbb{T}$
7. Involution law (Double negation law) $\sim(\sim p) = p$
8. DeMorgan's law: (i) $\sim (p \wedge q) = \sim p \vee \sim q$ (ii) $\sim(p \vee q) = \sim p \wedge \sim q$
9. Absorption laws: (i) $p \vee (p \wedge q) \equiv p$ (ii) $p \wedge (p \vee q) \equiv p$

EXERCISE 12.1

1. Determine whether * is a binary operation on the sets given below.

(i) $a*b = a.|b|$ on $\mathbb{R}$.

(ii) $a*b$ = min (a, b) on A ={1,2,3,4,5}

(iii) $(a*b) = a\sqrt{b}$ is binary on $\mathbb{R}$.

Sol. (i) **$a*b = a\,.\,|b|$ on $\mathbb{R}$.**

Given $a * b = a \cdot |b|$ on $\mathbb{R}$.

Let $a, b \in \mathbb{R}$.

Then $a * b = a\,.\,|b| \in \mathbb{R}$.

Since $a \cdot |b| = ab$ if $b > 0$

$= -ab$ if $b < 0$

$\therefore a * (b) = a \cdot |b| \in \mathbb{R}$

* is a binary operation on $\mathbb{R}$.

(ii) **$a*b$ = min (a, b) on A = {1, 2, 3, 4, 5}**

Let $a, b \in$ A

Then min (a, b) = a or b and $a, b \in$ A

$\therefore a * b$ = min $(a, b) \in$ A

∴ * is a binary operation on R.

(iii) **$(a*b) = a\sqrt{b}$ is binary on $\mathbb{R}$.**

Let $a, b \in \mathbb{R}$

[∴ square root of negative numbers does not belong to $\mathbb{R}$]

$a * b = a\sqrt{b} \notin \mathbb{R}$ if $b < 0$

* is not a binary operation on $\mathbb{R}$.

2. On $\mathbb{Z}$, define ⊗ by $(m \otimes n) = m^n + n^m : \forall m, n \in \mathbb{Z}$. Is ⊗ binary on $\mathbb{Z}$?

Sol. Given $m \otimes n = m_n + n_m \; \forall \, m, n \in \mathbb{Z}$

Let $m, n \in \mathbb{Z}$

Consider $m = -3, n = 2$

$\therefore m \otimes n = (-3)^2 + 2^{-3}$

$= 9 + \frac{1}{8} = \frac{72+1}{8} = \frac{73}{8} \notin \mathbb{Z}$

∴ ⊗ is not a binary operation on $\mathbb{Z}$.

3. Let * be defined on $\mathbb{R}$ by $(a*b) = a + b + ab - 7$. Is * binary on $\mathbb{R}$? If so, find $3 * \left(\frac{-7}{15}\right)$.

Sol. Given $a * b = a + b + ab - 7$

$3 * \left(\frac{-7}{15}\right) = 3 - \frac{7}{15} + \cancel{3}^1\left(\frac{-7}{\cancel{15}_5}\right) - 7$

[Here $a = 3$, $b = \frac{-7}{15}$]

$= 3 - \frac{7}{15} - \frac{21}{15} - 7$

$= \frac{45 - 7 - 21 - 105}{15} = \frac{-88}{15}$

$\therefore 3 * \left(\frac{-7}{15}\right) = \frac{-88}{15}$

13. Using truth table check whether the statements $\neg(p \vee q) \vee (\neg p \wedge q)$ and $\neg p$ are logically equivalent.

Sol. $\sim(p \vee q) \vee (\sim p \wedge q)$ and $\sim p$.

p	q	$p \vee q$	$\sim(p \vee q)$	$\sim p$	$\sim p \wedge q$	$\sim(p \vee q) \vee (\sim p \wedge q)$
T	T	T	F	F	F	F
T	F	T	F	F	F	F
F	T	T	F	T	T	T
F	F	F	T	T	F	T

The entries in column (5) and column (7) are identical.

$\therefore \sim(p \vee q) \vee (\sim p \wedge q)$ and $\sim p$ are logically equivalent..

14. Prove $p \to (q \to r) \equiv (p \wedge q) \to r$ without using truth table.

Sol. Prove that $p \to (q \to r) = (p \wedge q) \to r$ without using truth table. From example 12.17 we know that $p \to q = \sim p \vee q$ ...(1)

Consider LHS $= p \to (q \to r)$

$= p \to (\sim q \vee r)$ [using (1)]

$= \sim p \vee (\sim q \vee r)$ [again using (1)]

$= (\sim p \vee \sim q) \vee r$ [using associative property]

$= \sim(p \wedge q) \vee r$ [using Demorgan's law]

$= (p \wedge q) \to$ r [using (1)]

= RHS

Hence proved.

15. Prove that $p \to (\neg q \vee r) \equiv \neg p \vee (\neg q \vee r)$ using truth table.

Sol.

p	q	r	$\sim q$	$\sim q \vee r$	$p \to (\sim q \vee r)$	$\sim p$	$\sim p \vee (\sim q \vee r)$
T	T	T	F	T	T	F	T
T	T	F	F	F	F	F	F
T	F	T	T	T	T	F	T
T	F	F	T	T	T	F	T
F	T	T	F	T	T	T	T
F	T	F	F	F	T	T	T
F	F	T	T	T	T	T	T
F	F	F	T	T	T	T	T

The entries in column (6) and column (8) are identical.

$\therefore p \to (\sim q \vee \sim r) = \sim p \vee (\sim q \vee r)$

EXERCISE 12.3

Choose the Correct or the most suitable answer from the given four alternatives :

1. A binary operation on a set S is a function from

(1) $S \to S$ (2) $(S \times S) \to S$

(3) $S \to (S \times S)$ (4) $(S \times S) \to (S \times S)$

[Ans. (2) $(S \times S) \to S$]

2. Subtraction is not a binary operation in

(1) $\mathbb{R}$ (2) $\mathbb{Z}$ (3) $\mathbb{N}$ (4) $\mathbb{Q}$

[Ans. (3) $\mathbb{N}$]

3. Which one of the following is a binary operation on $\mathbb{N}$?

(1) Subtraction (2) Multiplication

(3) Division (4) All the above

[Ans. (2) Multiplication]

4. In the set $\mathbb{R}$ of real numbers '$*$' is defined as follows. Which one of the following is not a binary operation on $\mathbb{R}$?

(1) $a*b = \min(a.b)$ (2) $a*b = \max(a,b)$

(3) $a*b = a$ (4) $a*b = a^b$

[Ans. (4) $a*b = a^b$]

Hint : Let $a = 0, b = 0, \Rightarrow 0^0 \notin$ R. (4) $a * b = ab$

5. The operation $*$ defined by $a*b = \dfrac{ab}{7}$ is not a binary operation on

(1) $\mathbb{Q}^+$ (2) $\mathbb{Z}$ (3) $\mathbb{R}$ (4) $\mathbb{C}$

[Ans. (3) $\mathbb{Z}$]

Hint : Let $a = 3, b = 2$

$\Rightarrow a * b = 3 * 2 = \dfrac{3(2)}{7} = \dfrac{6}{7} \notin z.$

6. In the set $\mathbb{Q}$ define $a \odot b = a + b + ab$. For what value of y, $3 \odot (y \odot 5) = 7$?

(1) $y = \dfrac{2}{3}$ (2) $y = \dfrac{-2}{3}$

(3) $y = \dfrac{-3}{2}$ (4) $y = 4$

[Ans. (2) $y = \dfrac{-2}{3}$]

Hint : $\otimes b = a + b + ab$

$\therefore 3 \otimes (y \otimes 5) = 7 \Rightarrow 3 \otimes (y + 5 + 5y) = 7$

ADDITIONAL QUESTIONS

1 MARK

I. Choose the Correct or the most suitable answer from the given four alternatives :

1. The binary operation * defined on a set s is said to be commutative if

(1) $a * b \in S \ \forall \ a, b \in S$
(2) $a * b = b * a \ \forall \ a, b \in S$
(3) $(a * b) * c = a * (b * c) \ \forall \ a, b \in S$
(4) $a * b = e \ \forall \ a, b \in S$

[Ans. (2) $a * b = b * a \ \forall \ a, b \in S$]

Hint : $a \times b \in s \ a \ \forall \ b \in s$ - closure

2. If * is defined by $a * b = a^2 + b^2 + ab + 1$, then $(2 * 3) * 2$ is

(1) 20 (2) 40 (3) 400 (4) 445

[Ans. (4) 445]

Hint : $(2 * 3) * 2 = (2^2 + 3^2 + 2 \times 3 + 1)*2$
$= (4 + 9 + 6 + 1) * 2 = 20 * 2$
$= 20^2 + 2^2 + 20 \times 2 + 1$
$= 400 + 4 + 40 + 1 = 445$

3. The number of binary operations that can be defined on a set of 3 elements is

(1) 3^2 (2) 3^3 (3) 3^9 (4) 3^1

[Ans. (1) 3^2]

4. The identity element of $\left\{ \begin{pmatrix} x & x \\ x & x \end{pmatrix} \middle| x \in \mathbb{R}, x \neq 0 \right\}$ under matrix multiplication is

(1) $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ (2) $\begin{pmatrix} \frac{1}{4x} & \frac{1}{4x} \\ \frac{1}{4x} & \frac{1}{4x} \end{pmatrix}$

(3) $\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$ (4) $\begin{pmatrix} \frac{1}{2x} & \frac{1}{2x} \\ \frac{1}{2x} & \frac{1}{2x} \end{pmatrix}$

[Ans. (3) $\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$]

Hint : $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ not belongs to S.

5. Which one of the following is not a statement?

(1) $2 + 3 = 5$
(2) How beautiful is this flower?
(3) Delhi is the capital of Tamil Nadu
(4) A triangle has found angles.

[Ans. (2) How beautiful is this flower?]

Hint : (2) is a question

6. Which of the following is a tautology?

(1) $p \vee q$ (2) $p \wedge q$
(3) $q \vee \sim q$ (4) $q \wedge \sim q$

[Ans. (3) $q \wedge \sim q$]

7. Which of the following is a contradiction?

(1) $p \vee q$ (2) $p \wedge q$
(3) $q \vee \sim q$ (4) $q \wedge \sim q$

[Ans. (4) $q \wedge \sim q$]

8. The identity element in the group $\{R - \{1\}, x\}$ where $a * b = a + b - ab$ is

(a) 0 (b) 1
(c) $\frac{1}{a-1}$ (d) $\frac{a}{a-1}$

[Ans.(1) 0]

Hint : $a * 0 = a + 0 - a\,(0) = a$

9. Define * on $\mathbb{Z}$ by $a * b = a + b + 1 \ \forall a, b \in \mathbb{Z}$. Then the identity element of z is

(1) 1 (2) 0 (3) 1 (4) –1

[Ans. (4) –1]

Hint : $a * c = \not{a} + c + 1 = \not{a}$
$c = -1$

10. A binary operation * is defined on the set of positive rational numbers Q^+ by $a * b = \frac{ab}{4}$. Then $3 * \left(\frac{1}{5} * \frac{1}{2}\right)$ is

(1) $\frac{3}{160}$ (2) $\frac{5}{160}$ (3) $\frac{3}{10}$ (4) $\frac{3}{40}$

[Ans. (1) $\frac{3}{160}$]

Hint : $3 * \left(\frac{1}{5} * \frac{1}{2}\right) = 3 * \left(\frac{\frac{1}{5} \times \frac{1}{2}}{4}\right)$

$= 3 * \frac{1}{40} = \frac{3 \times \frac{1}{40}}{4} = \frac{3}{160}$

2. Show that $p \vee (q \wedge r)$ is a contingency.

Sol.

p	r	q	$q \wedge r$	$p \vee (q \wedge r)$
T	T	T	T	T
T	F	F	F	T
T	T	F	F	T
T	F	F	F	T
F	T	T	T	T
F	F	T	F	F
F	T	F	F	F
F	F	F	F	F

$\therefore p \vee (q \wedge r)$ is a contingency.

3. In the set of integers under the operation * defined by $a * b = a + b - 1$. Find the identity element.

Sol. Let a be any element and e be the identity element.

The $a * e = e * a = a$

$a * e = a \Rightarrow a + e - 1 = a \Rightarrow e - 1 = 0 \Rightarrow e = 1$

$\therefore$ The identity element is 1.

4. Let S be the set of positive rational numbers and is defined by $a * b = \frac{ab}{2}$. Then find the identity element and the inverse of 2.

Sol. Let $a \in$ S and e be the identity element.

Then $a * e = a \Rightarrow \frac{ae}{2} = a$

$\Rightarrow ae = 2a \Rightarrow e = 2.$

Let a^{-1} be the inverse of $\frac{1}{2}$

Then $\frac{1}{2} * a^{-1} = e \Rightarrow \frac{1}{2} * a^{-1} = e$

$\Rightarrow \frac{a^{-1}}{2}\frac{a^{-1}}{2} = 2 \Rightarrow a^{-1} = 8$

5. Let G = {1, w, w^2) where w is a complex cube root of unity. Then find the universe of w^2. Under usual multiplication.

Sol. Clearly 1 is the identity element of G $w^2 . a^{-1} = e \Rightarrow w^2 . a^{-1} = a^{-1} = w$

Since $w^2 . w = w^3 = 1$

Inverse of w^2 is w.

3 MARKS

1. Write the truth value for each of the following statements.

(1) $3 + 5 = 8$ and $\sqrt{2}$ is an irrational number.

(2) 5 is a positive integer or a square is a rectangle.

(3) Chennai is not in Tamilnadu.

Sol. (1) p: $3 + 5 = 8$

q: $\sqrt{2}$ is an irrational number.

T and T = T

(2) p: 5 is a positive integer.

q: A square is a rectangle.

T or F = T

(3) p: Chennai is not in Tamilnadu

F.

2. In (z, *) where * is defined by $a * b = ab$, prove that * is not a binary operation on z.

Sol. Let $a = 2, b = -1$

$\therefore a * b = a^b \Rightarrow 2 * -1 = 2^{-1} = \frac{1}{2} \notin z$

* is not a binary operation on z.

3. Let G = {1, i, –1, –i} under thebinary operation multiplication. Find the inverse of all the elements.

Sol. Clearly 1 is the identity element of (G_1).

Inverse of 1 is 1 [$\therefore$ (1) (1) = 1]

Inverse of i is $-i$ [$\therefore$ $(i)(-i) = -i^2 = 1$]

Inverse of –1 is –1 [$\therefore$ (–1) (–1) = 1]

Inver of i is i [$\therefore$ $(-i)$ (i) $= -i^2 = 1$]

4. In (z, *) where * is defined as $a * b = a + b + 2$. Verify the commutative and associative axiom.

Sol. (a) Let $a, b \in 2$.

The $a * b = a + b + z = b + a + z = b * a$

(z, *) has commutative property.

(b) Let $a, b, c \in$ G

$(a * b) * c = (a + b + 2) * c = a + b + 2 + c + 2$

$= a + b + c + 4$

$\therefore$ z is associate under *.

5. Construct the truth table for $(\sim p) \vee (q \wedge r)$

Sol.

p	q	r	$\sim p$	$q \wedge r$	$(\sim p) \vee (q \wedge r)$
T	T	T	F	T	T
T	F	F	F	F	F
T	F	T	F	F	F
T	F	F	F	F	F
F	T	T	T	T	T
F	T	F	T	F	T
F	F	T	T	F	T
F	F	F	T	F	T

5 MARKS

1. Show that $\sim(p \wedge q) \equiv (\sim p) \vee (\sim q)$

p	q	$\sim p$	$\sim q$	$(p \wedge q)$	$\sim(p \wedge q)$	$(\sim p) \vee (\sim q)$
T	T	F	F	T	F	F
T	F	F	T	F	T	T
F	T	T	F	F	T	T
F	F	T	T	F	T	T

The entries in column (6) and column (7) are equal

$\therefore \sim(p \wedge q) \equiv (\sim p) \vee (\sim q)$

2. Construct the truth table for $(p \wedge q) \vee r$.

p	q	r	$(p \wedge q)$	$(p \wedge q) \vee r$
T	T	T	T	T
T	F	F	F	F
T	T	T	T	T
T	F	F	F	F
F	T	T	F	T
F	F	F	F	F
F	T	T	F	T
F	F	F	F	F

3. Verify $(p \wedge \sim p) \wedge (\sim q \wedge p)$ is a tautlogy, contradiction or contingency.

p	q	$\sim p$	$p \wedge \sim p)$	$\sim q$	$(\sim q) \wedge p$	$(p \wedge \sim p) \wedge (\sim q \wedge p)$
T	T	F	F	F	F	F
T	F	F	F	T	T	F
F	T	T	F	F	F	F
F	F	T	F	T	F	F

Since the entries in the last column are F, $(p \wedge \sim p) \wedge (\sim q \wedge p)$ is a contradiction.

4. Let S be a non-empty set and 0 be a binary operation on s defined by x 0 $y = x$; $x, y \in$ S. Determine whether 0 is commutative and association.

Sol. Given s is a non-empty set and $x \; 0 \; y = x, x, y \in s$

$y_0 x = y$

$x_0 u \neq y_0 x \Rightarrow 0$ is not commutative.

Now, $x_0 \, (y_0 z) = x_0 y = x$

and $(x_0 y) \; 0 \; z = \mathrm{x}_0 \, z = x$

$x_0 \, (y_0 z) = (y_0 z) \; 0 \; z$

0 is associative

5. In (ℕ, *) where * is defined by $x * y$ = max (x, y) check the closure axion and identity anion.

Sol. Let $x, y \in \mathbb{N}$.

$$x * y = \max(x, y) = \begin{cases} x & \text{if} \quad x \geq y \\ y & \text{if} \quad x < y \end{cases}$$

$x * y \in \mathbb{N}$.

N is closed under *.

For $1 \in \mathbb{N}$, $x * 1 = \max(x, 1) = x \; \forall x \in \mathbb{N}$.

$\therefore$ 1 is that identity element.

(ℕ, *) has identity element.

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Register Number | | | | | | |

12th Std

SURA's MODEL QUESTION PAPER - 1
Mathematics
(English Version)

Time allowed : 2.30 hours] [Maximum Marks: 90

Instructions: (1) Check the question paper for fairness of printing. If there is any lack of fairness, inform the Hall supervisor immediately.

(2) Use **Blue** or **Black** ink to write and underline and pencil to draw diagrams.

PART - I

(20 × 1 = 20)

Note: (1) Answer all the questions.

(2) Choose the most appropriate answer from the given four alternatives and write the option code and the corresponding answer.

1. **$A = \begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix}$, B = adj A and C = 3A, then $\frac{|\text{adj } B|}{|C|}$ =**

(1) $\frac{1}{3}$ (2) $\frac{1}{9}$ (3) $\frac{1}{4}$ (4) 1

2. **$A = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$ and A (adj A) = $\begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}$, then = k**

(1) 0 (2) sin θ (3) cos θ (4) 1

3. **A balloon rises straight up at 10 m/s. An observer is 40 m away from the spot where the balloon left the ground. Find the rate of change of the balloon's angle of elevation in radian per second when the balloon is 30 metres above the ground.**

(1) $\frac{3}{25}$ radians/sec (2) $\frac{4}{25}$ radians/sec (3) $\frac{1}{5}$ radians/sec (4) $\frac{1}{3}$ radians/sec

4. **The maximum value of the function $x^2 e^{-2x}$, $x > 0$ is**

(1) $\frac{1}{e}$ (2) $\frac{1}{2e}$ (3) $\frac{1}{e^2}$ (4) $\frac{4}{e^4}$

5. **The area of the triangle formed by the complex numbers z, iz, and $z + iz$ in the Argand's diagram is**

(1) $\frac{1}{2}|z|^2$ (2) |z|2 (3) $\frac{3}{2}|z|^2$ (4) 2|z|2

6. **If $|z_1| = 1$, $|z_2| = 2$, $|z_3| = 3$ and $|9\, z_1z_2 + 4z_1\, z_3| + z_2\, z_3 = 12$, then the value of $|z_1 + z_2 + z_3|$ is**

(1) 1 (2) 2 (3) 3 (4) 4

7. **If $z = x + iy$ is a complex number such that $|z + 2| = |z - 2|$, then the locus of z is**

(1) real axis (2) imaginary axis (3) ellipse (4) circle

Case (ii)

When $y = -\frac{5}{2} \Rightarrow x + \frac{1}{x} = -\frac{5}{2}$

$\Rightarrow \quad \frac{x^2+1}{x} = \frac{-5}{2}$

$\Rightarrow \quad 2x^2 + 2 + 5x = 0$

$\Rightarrow \quad 2x^2 + 5x + 2 = 0$

$\Rightarrow \quad (x + 2)(2x + 1) = 0$

$\Rightarrow \quad x = -2, \frac{-1}{2}$

$\therefore$ The roots are $3, \frac{1}{3}, -2, \frac{-1}{2}$.

Register Number

12th Std

SURA's MODEL QUESTION PAPER - 2

Mathematics

(Tamil & English Version)

Time allowed : 2.30 hours] [Maximum Marks: 90

Instructions: (1) Check the question paper for fairness of printing. If there is any lack of fairness, inform the Hall supervisor immediately.

(2) Use **Blue** or **Black** ink to write and underline and pencil to draw diagrams.

PART - I (20 × 1 = 20)

Note: (1) Answer all the questions.

(2) Choose the most appropriate answer from the given four alternatives and write the option code and the corresponding answer.

1. **If $A = \begin{bmatrix} 2 & 0 \\ 1 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 4 \\ 2 & 0 \end{bmatrix}$ then |adj (AB)| =**

(1) –40 (2) –80 (3) –60 (4) –20

2. **Find the point on the curve $6y = x^3 + 2$ at which y-coordinate changes 8 times as fast as x-coordinate is**

(1) (4,11) (2) (4,–11) (3) (–4,11) (4) (–4,–11)

3. **If $0 \le \theta \le \pi$ and the system of equations $x + (\sin\theta)y - (\cos\theta)z = 0$, $(\cos\theta)x - y + z = 0$, $(\sin\theta)x + y - z = 0$ has a non-trivial solution then θ is**

(1) $\frac{2\pi}{3}$ (2) $\frac{3\pi}{4}$ (3) $\frac{5\pi}{6}$ (4) $\frac{\pi}{4}$

4. **The principal argument of $(\sin 40° + i\cos 40°)^5$ is**

(1) –110° (2) –70° (3) 70° (4) 110°

5. **The number given by the Mean value theorem for the function $\frac{1}{x}$, $x \in [1, 9]$ is**

(1) 2 (2) 2.5 (3) 3 (4) 3.5

6. **The value of $\left(\frac{1+\sqrt{3}i}{1-\sqrt{3}i}\right)^{10}$ is**

(1) cis $\frac{2\pi}{3}$ (2) cis $\frac{4\pi}{3}$ (3) –cis $\frac{2\pi}{3}$ (4) –cis $\frac{4\pi}{3}$

7. **If $f(x, y) = e^{xy}$, then $\frac{\partial^2 f}{\partial x \partial y}$ is equal to**

(1) xye^{xy} (2) $(1 + xy)e^{xy}$ (3) $(1 + y)e^{xy}$ (4) $(1 + x)e^{xy}$

8. **If $x^3 + 12x^2 + 10ax + 1999$ definitely has a positive zero, if and only if**

(1) $a \ge 0$ (2) $a > 0$ (3) $a < 0$ (4) $a \le 0$

44. (a) Find the principal value of $\sec^{-1}\left(\frac{2}{\sqrt{3}}\right)$

(OR)

(b) Evaluate : $\int_0^{\frac{\pi}{\sqrt{2}}} \frac{dx}{5+4\sin^2 x}$

45. (a) Find the equations of the tangent and normal to hyperbola $12x^2 - 9y^2 = 108$ at $\theta = \frac{\pi}{3}$. (Hint: use parametric form)

(OR)

(b) Solve the differential equation: $(x^3 + y^3)\, dy - x^2 y dx = 0$

46. (a) Using vector method, prove that if the diagonals of a parallelogram are equal, then it is a rectangle.

(OR)

(b) The probability density function of X is $f(x) = \begin{cases} x & 0 < x < 1 \\ 2-x & 1 \le x < 2 \\ 0 & \text{otherwise} \end{cases}$

Find

(i) $P(0.2 \le X < 0.6)$ (ii) $P(1.2 \le X < 1.8)$ (iii) $P(0.5 \le X < 1.5)$

47. (a) Verify whether the following compound propositions are tautologies or contradictions or contingency

(i) $(p\wedge q)\wedge\neg(p\vee q)$ (ii) $((p\vee q)\wedge\neg p)\rightarrow q$ (iii) $(p\rightarrow q)\leftrightarrow(\neg p\rightarrow q)$

(iv) $((p\rightarrow q)\wedge(q\rightarrow r))\rightarrow(p\rightarrow r)$

(OR)

(b) Parabolic cable of a 60m portion of the roadbed of a suspension bridge are positioned as shown below. Vertical Cables are to be spaced every 6m along this portion of the roadbed. Calculate the lengths of first two of these vertical cables from the vertex

ANSWERS

SECTION - I

1. (2) −80	2. (1) (4,11)	3. (4) $\frac{\pi}{4}$
4. (1) −110°	5. (3) 3	6. (1) cis $\frac{2\pi}{3}$
7. (2) $(1 + xy)e^{xy}$	8. (3) $a < 0$	9. (4) $\frac{2}{3}$
10. (2) $\frac{1}{\sqrt{5}}$	11. (4) 2	12. (4) $\frac{x}{\sqrt{1+x^2}}$
13. (3) $\frac{2}{\sqrt{3}}$	14. (3) $\sqrt{10}$	15. (1) 2, 3
16. (4) 40	17. (1) $\frac{11}{243}$	18. (3) 2, 1
19. (3) i – b ii – c iii – d iv – a	20. (3) (iii), (iv)	

Register Number

12th Std

SURA's MODEL QUESTION PAPER - 3
Mathematics
(English Version)

Time allowed : 2.30 hours] [Maximum Marks: 90

Instructions: (1) Check the question paper for fairness of printing. If there is any lack of fairness, inform the Hall supervisor immediately.
(2) Use **Blue** or **Black** ink to write and underline and pencil to draw diagrams.

PART - I (20 × 1 = 20)

Note: (1) Answer all the questions.
(2) Choose the most appropriate answer from the given four alternatives and write the option code and the corresponding answer.

1. **If $P = \begin{vmatrix} 1 & x & 0 \\ 1 & 3 & 0 \\ 2 & 4 & -2 \end{vmatrix}$ is the adjoint of 3 × 3 matrix A and |A| = 4, then x is**

(1) 15 (2) 12 (3) 14 (4) 11

2. **The volume of a sphere is increasing in volume at the rate of 3 πcm³ / sec . The rate of change of its radius when radius is $\frac{1}{2}$ cm**

(1) 3 cm/s (2) 2 cm/s (3) 1 cm/s (4) $\frac{1}{2}$ cm/s

3. **If $A = \begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}$, then adj (adj A) is**

(1) $\begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}$ (2) $\begin{bmatrix} 6 & -6 & 8 \\ 4 & -6 & 8 \\ 0 & -2 & 2 \end{bmatrix}$ (3) $\begin{bmatrix} -3 & 3 & -4 \\ -2 & 3 & -4 \\ 0 & 1 & -1 \end{bmatrix}$ (4) $\begin{bmatrix} 3 & -3 & 4 \\ 0 & -1 & 1 \\ 2 & -3 & 4 \end{bmatrix}$

4. **If $v(x, y) = \log(e^x + e^y)$, then $\frac{\partial v}{\partial x} + \frac{\partial v}{\partial y}$ is equal to**

(1) $e^x + e^y$ (2) $\frac{1}{e^x + e^y}$ (3) 2 (4) 1

5. **If $\frac{z-1}{z+1}$ is purely imaginary, then $|z|$ is**

(1) $\frac{1}{2}$ (2) 1 (3) 2 (4) 32

6. **If $u(x, y) = x^2 + 3xy + y - 2019$, then $\left.\frac{\partial u}{\partial x}\right|_{(4,-5)}$ is equal to**

(1) −4 (2) −3 (3) −7 (4) 13

ANSWERS

SECTION - I

1. (4) 11

2. (1) 3 cm/s

3. (1) $\begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}$

4. (4) 1

5. (2) 1

6. (3) −7

7. (4) – 4

8. (2) $\frac{5}{4}$

9. (3) 2

10. (1) $|\alpha| \le \frac{1}{\sqrt{2}}$

11. (1) $\frac{\pi}{2}$

12. (1) $\frac{\pi}{2}$

13. (2) unique solution

14. (3) 1, 1

15. (2) $2(a^2 + b^2)$

16. (2) mean exists but variance does not exist

17. (1) $\frac{1}{\sqrt{2}}$

18. (4) 5.004

19. (3) 45°

20. (2) $\frac{d^2y}{dx^2} + 9y$