

9. Differential calculus - Limits and continuity

Important Notes:

* Left-hand Limit :- $f(x_0^-) = \lim_{x \to x_0^-} f(x) = l_1$

* Right-hand Limit :- $f(x_0^+) = \lim_{x \to x_0^+} f(x) = l_2$

* $\lim_{x \to x_0} f(x) = L$ exists if the following hold:

  i) $\lim_{x \to x_0^+} f(x)$ exists.

  ii) $\lim_{x \to x_0^-} f(x)$ exists

  iii) $\lim_{x \to x_0^-} f(x) = \lim_{x \to x_0^+} f(x) = L$

Exercise: 9.1

I) In problems 1–6, complete the table using calculator and use the result to estimate the limit.

1) $\lim_{x \to 2} \frac{x-2}{x^2-x-2}$

x	1.9	1.99	1.999	2.001	2.01	2.1
$f(x)$						

Soln: $\lim_{x \to 2} f(x) = \lim_{x \to 2} \frac{x-2}{x^2-x-2}$.

$$\lim_{x \to 2} \frac{x-2}{x^2-x-2} = \lim_{x \to 2} \frac{\cancel{x-2}}{(x+1)\cancel{(x-2)}} = \lim_{x \to 2} \frac{1}{x+1}.$$

$$\begin{array}{r|rr} 1 & & -2 \\ & & -2 \\ \hline & -1 & \end{array}$$

x	1.9	1.99	1.999	2.001	2.01	2.1
$f(x)$	0.3448275	0.3344482	0.333444	0.333222	0.33223	0.32258064

$$\therefore \lim_{x \to 2} \frac{x-2}{x^2-x-2} = 0.3333333 = 0.\overline{3}$$

$$\boxed{\lim_{x \to 2} \frac{x-2}{x^2-x-2} = 0.\overline{3}}$$

2) $\lim_{x \to 2} \frac{x-2}{x^2-4}$

Soln:

$$\lim_{x\to 2}\frac{x-2}{x^2-4} = \lim_{x\to 2}\frac{(x-2)}{(x+2)(x-2)} = \lim_{x\to 2}\frac{1}{x+2}$$

x	1.9	1.99	1.999	2.001	2.01	2.1
$f(x)$	0.25641	0.250627	0.250063	0.24994	0.24938	0.2439

$$\therefore \boxed{\lim_{x\to 2}\frac{x-2}{x^2-4} = 0.25}$$

3. $\lim_{x\to 0}\frac{\sqrt{x+3}-\sqrt{3}}{x}$

Soln:

$$\lim_{x\to 0} f(x) = \lim_{x\to 0}\frac{\sqrt{x+3}-\sqrt{3}}{x}$$

x	−0.1	−0.01	−0.001	0.001	0.01	0.1
$f(x)$	0.29112	0.288916.	0.288699	0.28865	0.2884	0.2863

$$\therefore \boxed{\lim_{x\to 0}\frac{\sqrt{x+3}-\sqrt{3}}{\sqrt{x}} = 0.288}$$

4) $\lim_{x\to -3}\frac{\sqrt{1-x}-2}{x+3}$

Soln:

$$\lim_{x\to -3} f(x) = \lim_{x\to -3}\frac{\sqrt{1-x}-2}{x+3}$$

x	−3.1	−3.01	−3.001	−2.999	−2.99	−2.9
$f(x)$	−0.24846	−0.24984	−0.24998	−0.25001	−0.25016	−0.2516

$$\therefore \boxed{\lim_{x\to -3}\frac{\sqrt{1-x}-2}{x+3} = -0.25}$$

5) $\lim_{x\to 0}\frac{\sin x}{x}$

Soln:

$$\lim_{x\to 0} f(x) = \lim_{x\to 0}\frac{\sin x}{x}$$

x	-0.1	-0.01	-0.001	0.001	0.01	0.1
$f(x)=$	0.99833	0.99998	0.9999998	0.999998	0.99998	0.99833

$\therefore \boxed{\lim_{x\to 0} \frac{\sin x}{x} = 1}$

b) $\lim_{x\to 0} \frac{\cos x - 1}{x}$ $\qquad \lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{\cos x - 1}{x}$.

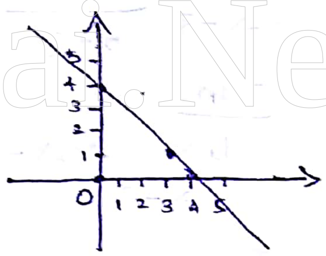
soln:

x	-0.1	-0.01	-0.001	0.0001	0.01	0.1
$f(x)$	0.049958	0.004999	0.0000004	-0.0000000499	-0.00004999	-0.00499

$\therefore \boxed{\lim_{x\to 0} \frac{\cos x - 1}{x} = 0}$

In exercise problems 7-15, use the graph to find the limits (if it exists). If the limit does not exist, explain why?

7. $\lim_{x\to 3} (4-3)$



soln:

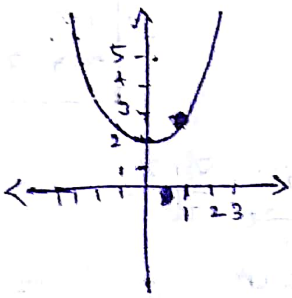
$\lim_{x\to 3^-} f(x) = \lim_{x\to 3^-} (4-x) = 4-3 = 1$

$\lim_{x\to 3^+} f(x) = \lim_{x\to 3^+} (4-x) = 4-3 = 1$

$\lim_{x\to 3} f(x) = \lim_{x\to 3} (4-x) = 4-3 = 1$

$\therefore \lim_{x\to 3} (4-x)$ exists. $\boxed{\lim_{x\to 3} (4-x) = 1}$

8. $\lim_{x\to 1} (x^2+2)$



Soln:

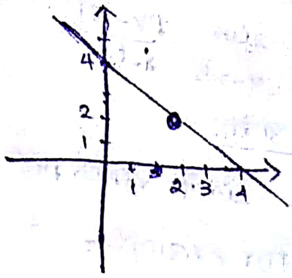
$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} (x^2+2) = (1)^2+2 = 3$

$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} (x^2+2) = (1)^2+2 = 3$

$\lim_{x\to 1} f(x) = \lim_{x\to 1} (x^2+2) = 1+2 = 3$

$\therefore \lim_{x\to 1} (x^2+2)$ exists. $\boxed{\lim_{x\to 1} (x^2+2) = 3}$

9. $\lim_{x\to 2} f(x)$ where $f(x)=\begin{cases} 4-x, & x\neq 2 \\ 0, & x=2 \end{cases}$



Soln:

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-}(4-x) = 4-2 = 2$$

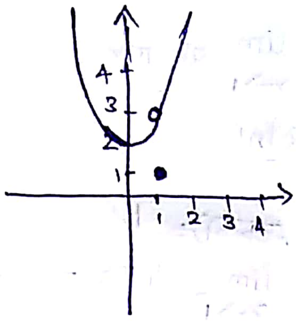
$$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+}(4-x) = 4-2 = 2$$

$$\boxed{\lim_{x\to 2} f(x) = 2}$$

$\therefore \lim_{x\to 2} f(x)$ exist.

Important Note: Limit value and function value is different.

10. ~~Soln~~ $\lim_{x\to 1} f(x)$ where $f(x)=\begin{cases} x^2+2, & x\neq 1 \\ 1, & x=1 \end{cases}$



Soln:

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-}(x^2+2) = 1+2 = 3$$

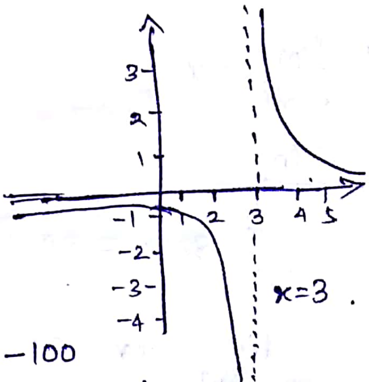
$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+}(x^2+2) = 1+2 = 3$$

$$\therefore \boxed{\lim_{x\to 1^-} f(x) = 3 = \lim_{x\to 1^+} f(x)}$$

$$\therefore \boxed{\lim_{x\to 1} f(x) = 3}$$

$\therefore \lim_{x\to 1} f(x)$ is exists.

11. $\lim_{x\to 3} \frac{1}{x-3}$



Soln:

limit does not exist.

For example,

$$f(2.99) = \frac{1}{2.99-3} = \frac{1}{-0.01} = \frac{-1}{0.01}\times\frac{100}{100} = -100$$

$$f(3.01) = \frac{1}{(3.01)-3} = \frac{1}{0.01}\times\frac{100}{100} = 100.$$

$\therefore$ limit does not exist . . . .

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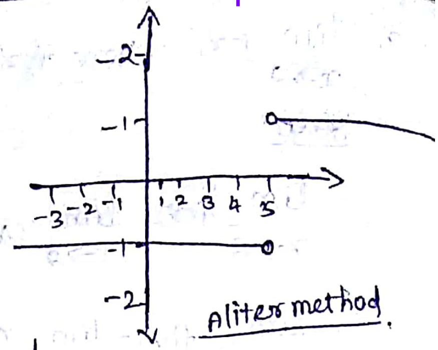
12. $\lim\limits_{x\to 5} \dfrac{|x-5|}{x-5}$



Soln:

limit does not exist.

For example,

$$f(4.99) = \frac{|4.99-5|}{4.99-5} = \frac{|-0.01|}{-0.01} = -1$$

Aliter method

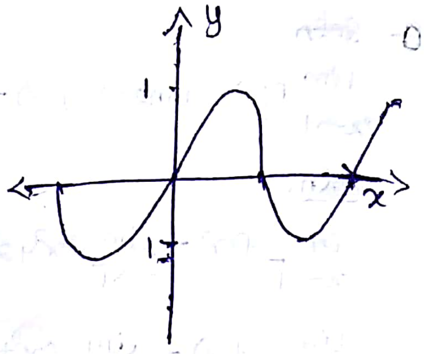
(or) $f(5^-) = \dfrac{-(x-5)}{x-5} = -1$

$$f(5.01) = \frac{|5.01-5|}{5.01-5} = \frac{|0.01|}{0.01} = 1$$

(or) $f(5^+) = \dfrac{(x-5)}{x-5} = 1$.

$\therefore$ limit does not exist....

13. $\lim\limits_{x\to 1} \sin\pi x$



Soln:

$$\lim_{x\to 1^-} \sin\pi x = \sin\pi(1) = 0$$

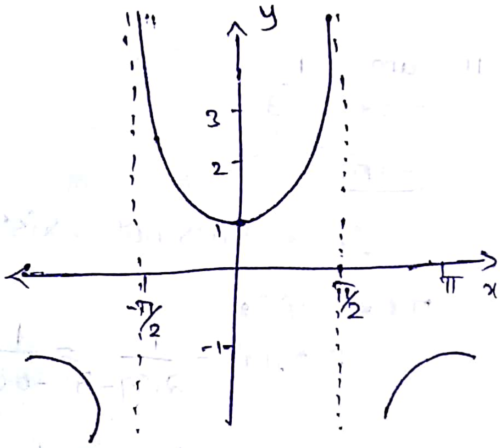
$$\lim_{x\to 1^+} \sin\pi x = \sin\pi(1) = 0.$$

$$\therefore \boxed{\lim_{x\to 1^-} \sin\pi x = 0 = \lim_{x\to 1^+} \sin\pi x}$$

$\therefore$ limit exists.

$$\therefore \boxed{\lim_{x\to 1} \sin\pi x = 0}$$

14. $\lim\limits_{x\to 0} \sec x$



Soln:

$$\lim_{x\to 0^-} \sec x = \sec 0 = 1$$

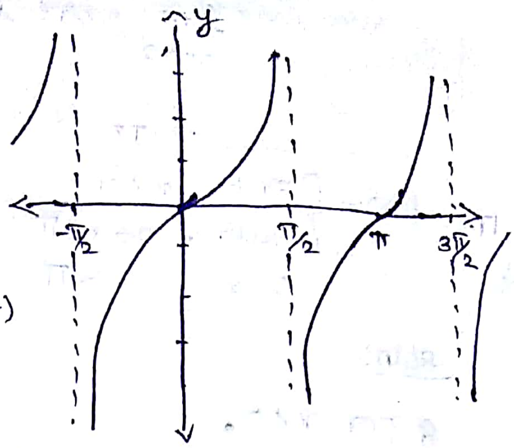
$$\lim_{x\to 0^+} \sec x = \sec 0 = 1$$

$$\boxed{\lim_{x\to 0^-} \sec x = 1 = \lim_{x\to 0^+} \sec x}$$

$\therefore$ limit exists.

$$\boxed{\lim_{x\to 0} \sec x = 1}$$

15. $\lim_{x \to \frac{\pi}{2}}$



Soln:

$\lim_{x \to \frac{\pi}{2}^-} f(x) = \lim_{x \to \frac{\pi}{2}^-} \tan x$

= large positive value (∞) (I^{st} quadrant)

$\lim_{x \to \frac{\pi}{2}^+} f(x) = \lim_{x \to \frac{\pi}{2}^+} \tan x$

= large negative value $(-\infty)$ (II^{nd} quadrant).

∴ limit does not exist.

16. Sketch the graph of f, then identify the values of x_0 for which $\lim_{x \to x_0} f(x)$ exists.

$$f(x) = \begin{cases} x^2, & x \leq 2 \\ 8-2x, & 2 < x < 4 \\ 4, & x \geq 4 \end{cases}$$

Soln:

For $x \leq 2$,

If $x=2$, then $f(2) = 2^2 = 4$

If $x=1$, then $f(1) = 1^2 = 1$

If $x=0$, then $f(0) = 0 = 0$.

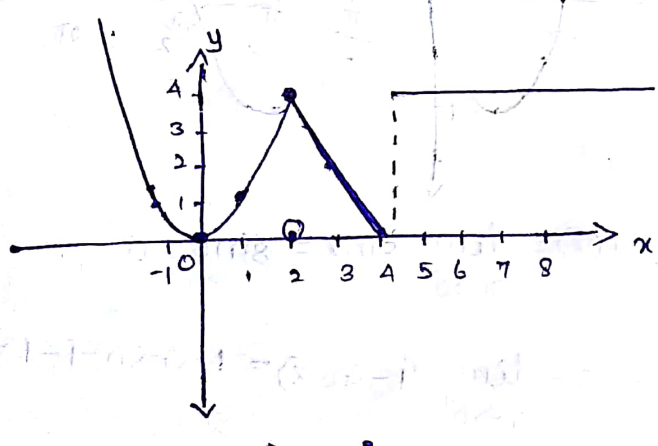
If $x=-1$, then $f(-1) = (-1)^2 = 1$.

For $2 < x < 4$,

If $x = 3$, then $f(3) = 8-2(3) = 8-6 = 2$

Rough work:
$f(2^+) = 8-2(2) = 8-4 = 4$
$f(4^-) = 8-2(4) = 8-8 = 0$.

Graph:



$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} x^2 = (2)^2 = 4$

$\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} 8-2x = 8-2(2) = 8-4 = 4$

 $\lim_{x\to 2} f(x) = 4$

$\therefore$ It is exists.

$\lim_{x\to 4^-} f(x) = 0$

$\lim_{x\to 4^+} f(x) = 4$

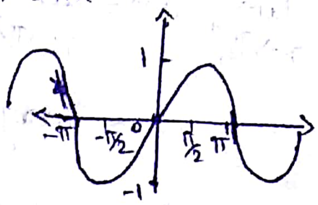
$\therefore$ limit does not exists.

17. $f(x) = \begin{cases} \sin x, & x < 0 \\ 1-\cos x, & 0 \le x \le \pi \\ \cos x, & x > \pi \end{cases}$

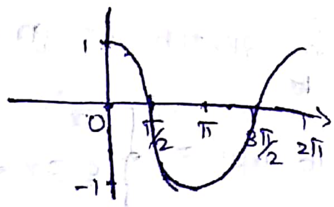
Soln:

Rough diagram:

$f(x) = \sin x$



$f(x) = \cos x$



For $x < 0$,

If $x = -\pi/2$, then $f(-\pi/2) = \sin(-\pi/2) = -1$

If $x = -\pi$, then $f(-\pi) = \sin(-\pi) = 0$

For $0 \le x \le \pi$,

If $x = 0$, then $f(0) = 1 - \cos 0 = 1 - 1 = 0$

If $x = \pi/2$, then $f(\pi/2) = 1 - \cos\pi/2 = 1 - 0 = 1$

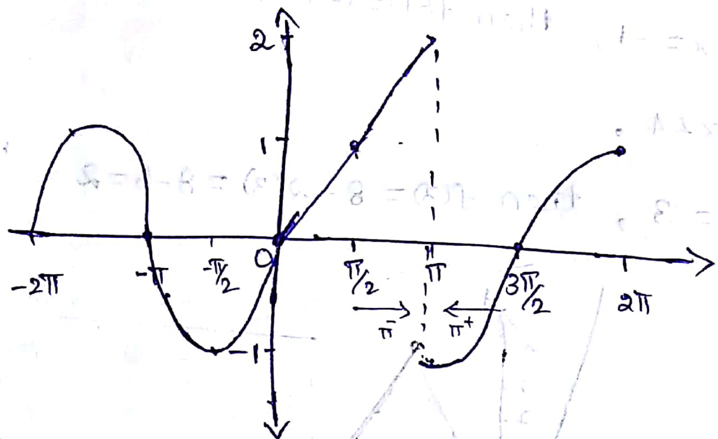
If $x = \pi$, then $f(\pi) = 1 - \cos\pi = 1 + 1 = 2$. $(\because \cos\pi = -1)$

For $x > \pi$,

If $x = 3\pi/2$, then $f(3\pi/2) = \cos 3\pi/2 = 0$.

If $x = 2\pi$, then $f(2\pi) = \cos 2\pi = 1$

Graph:



$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \sin x = \sin 0 = 0$.

$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} (1-\cos x) = 1 - \cos 0 = 1 - 1 = 0$.

$\therefore \boxed{\lim_{x\to 0^-} f(x) = 0 = \lim_{x\to 0^+} f(x)}$

$\therefore \boxed{\lim_{x\to 0} f(x) = 0}$

It is exists.

$\lim_{x\to\pi^-} f(x) = \lim_{x\to\pi^-} (1-\cos x) = 1-\cos\pi = 1-(-1) = 2$

$\lim_{x\to\pi^+} f(x) = \lim_{x\to\pi^+} \cos x = \cos\pi = -1$

$\therefore \lim_{x\to\pi} f(x)$ does not exists.

19. Write a brief description of the meaning of the notatation $\lim_{x\to 8} f(x) = 25$.

Soln:

$\lim_{x\to 8} f(x) = 25$, x is approaches the value 8.

$\lim_{x\to 8^-} f(x) = 25 = \lim_{x\to 8^+} f(x)$

$\boxed{L\cdot H\cdot L = R\cdot H\cdot L}$

$\boxed{f(8^-) = f(8^+) = 25}$

20. If $f(2)=4$, can you conclude anything about the limit of $f(x)$ as x approaches 2?

Soln:

Given $f(2)=4$ (function value).

$\lim_{x\to 2} f(x) = ?$

$\lim_{x\to 2^-} f(x)$ & $\lim_{x\to 2^+} f(x)$

The limit value is different..

$\therefore$ NO.

Rough work:
Function value is different as limit value.

21. If the limit of $f(x)$ as x approaches 2 is 4, can you conclude anything about $f(2)$? Explain reasoning?

Soln:

$\lim_{x\to 2} f(x) = 4$

$f(2) = ?$

$f(2)$ cannot be concluded.

For example,

$\lim_{x\to 2} f(x)$, where $f(x) = \begin{cases} 4-x & , x \neq 2 \\ 0 & , x=2 \end{cases}$

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} (4-x) = 4-2 = 2$$

$$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} (4-x) = 4-2 = 2$$

$$\lim_{x\to 2} f(x) = 2.$$

$\therefore$ $f(2)$ cannot be concluded.

22. Evaluate $\lim_{x\to 3} \frac{x^2-9}{x+3}$ if it exists by finding $f(3^-)$ and $f(3^+)$.

Soln: [Roughwork
$\lim_{x\to 3} f(x) = \lim_{x\to 3} \frac{x^2-9}{x-3} = \frac{3^2-9}{3-3} = \frac{0}{0}$ [Indeterminate form]].

$$f(3^-) = \lim_{x\to 3^-} f(x) = \lim_{x\to 3^-} \frac{(x+3)\cancel{(x-3)}}{\cancel{(x-3)}} = 3+3 = 6$$

$$f(3^+) = \lim_{x\to 3^+} f(x) = \lim_{x\to 3^+} \frac{(x+3)\cancel{(x-3)}}{\cancel{x-3}} = 3+3 = 6$$

$$\boxed{\lim_{x\to 3^-} f(x) = 6 = \lim_{x\to 3^+} f(x)}$$

$\therefore$ $\lim_{x\to 3} f(x)$ = exists.

23. Verify the existence of $\lim_{x\to 1} f(x)$, where $f(x) = \begin{cases} \frac{|x-1|}{x-1}, & \text{for } x \neq 1 \\ 0, & \text{for } x=1. \end{cases}$

Soln:

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} \frac{|x-1|}{x-1} = \frac{-\cancel{(x-1)}}{\cancel{(x-1)}} = -1$$

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} \frac{|x-1|}{x-1} = \frac{\cancel{x-1}}{\cancel{x-1}} = 1$$

$$\therefore \boxed{\lim_{x\to 1^-} f(x) \neq \lim_{x\to 1^+} f(x)}$$

$\therefore$ Limit does not exists.

Important Notes:

* $\lim_{x\to x_0} c f(x) = c \lim_{x\to x_0} f(x)$

* $\lim_{x\to x_0} [f(x) \pm g(x)] = \lim_{x\to x_0} f(x) \pm \lim_{x\to x_0} g(x)$

* $\lim_{x\to x_0} [f(x) \cdot g(x)] = \lim_{x\to x_0} f(x) \cdot \lim_{x\to x_0} g(x)$

* $\lim_{x\to x_0} \frac{f(x)}{g(x)} = \frac{\lim_{x\to x_0} f(x)}{\lim_{x\to x_0} g(x)}$, provided $\lim_{x\to 0} g(x) \neq 0$.

* $\lim_{x\to a} \frac{x^n - a^n}{x-a} = na^{n-1}$

Exercise: 9.2

Evaluate the following limits:

1) $\lim_{x\to 2} \frac{x^4-16}{x-2}$

Soln:

$$\lim_{x\to 2} \frac{x^4-16}{x-2} = \lim_{x\to 2} \frac{x^4-2^4}{x-2} = 4(2)^{4-1} = 4(2)^3 = 4\times 8 = 32$$

$$\therefore \boxed{\lim_{x\to 2} \frac{x^4-16}{x-2} = 32}$$

2) $\lim_{x\to 1} \frac{x^m-1}{x^n-1}$, m and n are integers.

Soln:

$$\lim_{x\to 1} \frac{x^m-1}{x^n-1} = \lim_{x\to 1} \frac{\frac{x^m-1}{x-1}}{\frac{x^n-1}{x-1}}$$ (Dividing both Numerator & Denominator) $(x-1)$.

$$= \frac{\lim_{x\to 1} \frac{x^m-1^m}{x-1}}{\lim_{x\to 1} \frac{x^n-1^n}{x-1}} = \frac{m(1)^{m-1}}{n(1)^{n-1}} = \frac{m}{n}$$

$$\therefore \boxed{\lim_{x\to 1} \frac{x^m-1}{x^n-1} = \frac{m}{n}}.$$

3) $\lim_{\sqrt{x}\to 3} \frac{x^2-81}{\sqrt{x}-3}$

Soln:

$$\lim_{\sqrt{x}\to 3} \frac{x^2-81}{\sqrt{x}-3} = \lim_{\sqrt{x}\to 3} \frac{(\sqrt{x})^4-3^4}{\sqrt{x}-3}$$

$$= 4(3)^{4-1} = 4(3)^3 = 4(27) = 108$$

$$\boxed{\lim_{\sqrt{x}\to 3} \frac{x^2-81}{\sqrt{x}-3} = 108}$$

4. $\lim_{h\to 0} \frac{\sqrt{x+h}-\sqrt{x}}{h}, \; x>0$.

Soln:

$$\lim_{h\to 0} \frac{\sqrt{x+h}-\sqrt{x}}{h} = \lim_{h\to 0} \frac{\sqrt{x+h}-\sqrt{x}}{h} \times \frac{\sqrt{x+h}+\sqrt{x}}{\sqrt{x+h}+\sqrt{x}}$$

$$= \lim_{h\to 0} \frac{(\sqrt{x+h})^2-(\sqrt{x})^2}{h(\sqrt{x+h}+\sqrt{x})} = \lim_{h\to 0} \frac{\cancel{x}+h-\cancel{x}}{h(\sqrt{x+h}+\sqrt{x})}$$

$$= \lim_{h\to 0} \frac{\cancel{h}}{\cancel{h}(\sqrt{x+h}+\sqrt{x})} = \frac{1}{\sqrt{x}+\sqrt{x}} = \frac{1}{2\sqrt{x}}$$

$$\therefore \boxed{\lim_{h\to 0} \frac{\sqrt{x+h}-\sqrt{x}}{h} = \frac{1}{2\sqrt{x}}}$$

5. $\lim_{x\to 5} \frac{\sqrt{x+4}-3}{x-5}$

Soln:

$$\lim_{x\to 5} \frac{\sqrt{x+4}-3}{x-5} = \lim_{x\to 5} \frac{\sqrt{x+4}-3}{x-5} \times \frac{\sqrt{x+4}+3}{\sqrt{x+4}+3}$$

$$= \lim_{x\to 5} \frac{(\sqrt{x+4})^2-3^2}{(x-5)(\sqrt{x+4}+3)}$$

$$= \lim_{x\to 5} \frac{x+4-9}{(x-5)(\sqrt{x+4}+3)} = \lim_{x\to 5} \frac{\cancel{(x-5)}}{\cancel{(x-5)}(\sqrt{x+4}+3)}$$

$$= \lim_{x\to 5} \frac{1}{(\sqrt{x+4}+3)} = \frac{1}{\sqrt{5+4}+3} = \frac{1}{\sqrt{9}+3} = \frac{1}{3+3} = 1/6$$

$$\boxed{\lim_{x\to 5} \frac{\sqrt{x+4}-3}{x-5} = \frac{1}{6}}$$

6. $\lim_{x\to 2} \frac{\frac{1}{x}-\frac{1}{2}}{x-2}$

Soln:

$$\lim_{x\to 2} \frac{\frac{1}{x}-\frac{1}{2}}{x-2} = \lim_{x\to 2} \frac{\frac{2-x}{2x}}{x-2}$$

$$= \lim_{x\to 2} \frac{\cancel{2-x}}{2x \times (-(\cancel{2-x}))} = \lim_{x\to 2} \left(-\frac{1}{2x}\right) = \frac{-1}{2\times 2} = -\frac{1}{4}.$$

$$\therefore \boxed{\lim_{x\to 2} \frac{\frac{1}{x}-\frac{1}{2}}{x-2} = -\frac{1}{4}}$$

7. $\displaystyle \lim_{x\to 1} \frac{\sqrt{x}-x^2}{1-\sqrt{x}}$

Soln:

$$\lim_{x\to 1} \frac{\sqrt{x}-x^2}{1-\sqrt{x}} = \lim_{x\to 1} \frac{\sqrt{x}-x^2}{1-\sqrt{x}} \times \frac{1+\sqrt{x}}{1+\sqrt{x}}$$

$$= \lim_{x\to 1} \frac{(\sqrt{x}-x^2)(1+\sqrt{x})}{(1^2-(\sqrt{x})^2)} = \lim_{x\to 1} \frac{\sqrt{x}+x-x^2-x^2\sqrt{x}}{1-x}$$

$$= \lim_{x\to 1} \frac{\sqrt{x}-x^2\sqrt{x}+x-x^2}{1-x}$$

$$= \lim_{x\to 1} \frac{\sqrt{x}(1-x^2)+x(1-x)}{1-x}$$

$$= \lim_{x\to 1} \frac{\sqrt{x}(1+x)(1-x)+x(1-x)}{(1-x)}$$

$$= \lim_{x\to 1} \frac{\cancel{(1-x)}\left[\sqrt{x}(1+x)+x\right]}{\cancel{(1-x)}}$$

$$= \lim_{x\to 1} \sqrt{x}(1+x)+x.$$

$$= \sqrt{1}(1+1)+1 = 2+1 = 3.$$

$$\therefore \boxed{\lim_{x\to 1} \frac{\sqrt{x}-x^2}{1-\sqrt{x}} = 3}$$

8. $\displaystyle \lim_{x\to 0} \frac{\sqrt{x^2+1}-1}{\sqrt{x^2+16}-4}$

Soln:

$$\lim_{x\to 0} \frac{\sqrt{x^2+1}-1}{\sqrt{x^2+16}-4} = \lim_{x\to 0} \frac{\sqrt{x^2+1}-1}{\sqrt{x^2+16}-4} \times \frac{\sqrt{x^2+16}+4}{\sqrt{x^2+16}+4} \times \frac{\sqrt{x^2+1}+1}{\sqrt{x^2+1}+1}$$

$$= \lim_{x\to 0} \frac{\sqrt{x^2+16}+4}{\sqrt{x^2+1}+1} \times \frac{(\sqrt{x^2+1})^2-(1)^2}{(\sqrt{x^2+16})^2-4^2} = \lim_{x\to 0} \frac{\sqrt{x^2+16}+4}{\sqrt{x^2+1}+1} \times \frac{x^2+1-1}{x^2+16-\cancel{16}}$$

$$= \lim_{x\to 0} \frac{\sqrt{x^2+16}+4}{\sqrt{x^2+1}+1} \times \frac{\cancel{x^2}}{\cancel{x^2}}$$

$$= \frac{\sqrt{0+16}+4}{\sqrt{0+1}+1} = \frac{4+4}{2} = 4.$$

$$\therefore \boxed{\lim_{x\to 0} \frac{\sqrt{x^2+1}-1}{\sqrt{x^2+16}-4} = 4}$$

9. $\lim_{x\to 0} \frac{\sqrt{1+x}-1}{x}$

Soln:

$$\lim_{x\to 0} \frac{\sqrt{1+x}-1}{x} = \lim_{x\to 0} \frac{\sqrt{1+x}-1}{x} \times \frac{\sqrt{1+x}+1}{\sqrt{1+x}+1}$$

$$= \lim_{x\to 0} \frac{(\sqrt{1+x})^2-1^2}{x(\sqrt{1+x}+1)} = \lim_{x\to 0} \frac{1+x-1}{x(\sqrt{1+x}+1)}$$

$$= \lim_{x\to 0} \frac{\cancel{x}}{\cancel{x}(\sqrt{1+x}+1)} = \frac{1}{\sqrt{1+0}+1} = \frac{1}{1+1} = \frac{1}{2}$$

$$\therefore \boxed{\lim_{x\to 0} \frac{\sqrt{1+x}-1}{x} = \frac{1}{2}}$$

10. $\lim_{x\to 1} \frac{\sqrt[3]{7+x^3}-\sqrt{3+x^2}}{x-1}$

Soln:

$$\lim_{x\to 1} \frac{\sqrt[3]{7+x^3}-\sqrt{3+x^2}}{x-1} = \lim_{x\to 1} \frac{(7+x^3)^{1/3}-(3+x^2)^{1/2}}{x-1}$$

$$= \lim_{x\to 1} \frac{(7+x^3)^{1/3}-2+2-(3+x^2)^{1/2}}{x-1}$$ (∵ Add & subtract 2.)

$$= \lim_{x\to 1} \frac{(7+x^3)^{1/3}-2}{x-1} + \lim_{x\to 1} \frac{2-(3+x^2)^{1/2}}{x-1}$$

$$= \lim_{x\to 1} \frac{(7+x^3)^{1/3}-8^{1/3}}{x-1} - \lim_{x\to 1} \frac{(3+x^2)^{1/2}-4^{1/2}}{x-1}$$

$= \lim_{x\to1} \frac{(7+x^3)^{1/3}-8^{1/3}}{x-1} \times \frac{(7+x^3)-8}{(7+x^3)-8} - \lim_{x\to1} \frac{(3+x^2)^{1/2}-4^{1/2}}{x-1} \times \frac{(3+x^2)-4}{(3+x^2)-4}$

$= \lim_{x\to1} \frac{(7+x^3)^{1/3}-8^{1/3}}{(7+x^3)-8} \times \frac{(7+x^3)-8}{x-1} - \lim_{x\to1} \frac{(3+x^2)^{1/2}-4^{1/2}}{(3+x^2)-4} \times \frac{(3+x^2)-4}{x-1}$

$= \lim_{x\to1} \left(\frac{1}{3}\right) 8^{1/3-1} \times \frac{x^3-1^3}{x-1} - \lim_{x\to1} \left(\frac{1}{2}\right)(4)^{1/2-1} \times \frac{x^2-1^2}{x-1}$

$= \frac{1}{3}(2^3)^{-2/3} \times 3(1)^{3-1} - \left(\frac{1}{2}\right)(2^2)^{-1/2} \times 2(1)^{2-1}$ $\left(\because \lim_{x\to a} \frac{x^n-a^n}{x-a} = na^{n-1}\right)$

$= \frac{1}{\cancel{3}\times4} \times \cancel{3} - \frac{1}{\cancel{4}_2} \times \cancel{2}^1 = \frac{1}{4} - \frac{1}{2} = -\frac{1}{4}.$

$x\to1$
$x^3\to1^3$
$7+x^3\to7+1$
$7+x^3\to8$

$x\to1$
$x^2\to1^2$
$3+x^2\to3+1$
$3+x^2\to4$

$$\boxed{\lim_{x\to1} \frac{\sqrt[3]{7+x^3}-\sqrt{3+x^2}}{x-1} = -\frac{1}{4}}$$

11. $\lim_{x\to2} \frac{2-\sqrt{x+2}}{\sqrt[3]{2}-\sqrt[3]{4-x}}$

Soln:

$\lim_{x\to2} \frac{2-\sqrt{x+2}}{\sqrt[3]{2}-\sqrt[3]{4-x}} = \lim_{x\to2} \frac{(x+2)^{1/2}-2}{(4-x)^{1/3}-2^{1/3}}$

$= \lim_{x\to2} \frac{(x+2)^{1/2}-4^{1/2}}{(4-x)^{1/3}-2^{1/3}}$

$= \lim_{x\to2} \frac{\frac{(x+2)^{1/2}-4^{1/2}}{(x+2)-4} \times (x+2)-4}{\frac{(4-x)^{1/3}-2^{1/3}}{(4-x)-2} \times (4-x)-2}$

Numerator:

Let $y=x+2$, when $x\to2$ as $y\to4$

Denominator:

Let $z=4-x$, when $x\to2$ as $z\to2$

$= \frac{\lim_{y\to4} \frac{y^{1/2}-4^{1/2}}{y-4} \times (\cancel{x-2})^{-1}}{\lim_{z\to2} \frac{z^{1/3}-2^{1/3}}{z-2} \times (\cancel{2-x})}$

$$= -1\left(\frac{\frac{1}{2}(4)^{\frac{1}{2}-1}}{\frac{1}{3}(2)^{\frac{1}{3}-1}}\right) = -1\left(\frac{\frac{1}{2}(2^2)^{-\frac{1}{2}}}{\frac{1}{3}(2)^{-\frac{2}{3}}}\right)$$

$$= -\frac{3}{2}\times\frac{1}{2}\times 2^{\frac{2}{3}} = \frac{-3}{2^2}\times 2^{\frac{2}{3}} = \frac{-3}{4}\times(4^{\frac{1}{2}})^{\frac{2}{3}}$$

$$\boxed{\lim_{x\to 2}\frac{2-\sqrt{x+2}}{\sqrt[3]{2}-\sqrt[3]{4-x}} = \frac{-3}{4}\sqrt[3]{4}}$$

12. $\lim\limits_{x\to 0}\dfrac{\sqrt{1+x^2}-1}{x}$

<u>Soln:</u>

$$\lim_{x\to 0}\frac{\sqrt{1+x^2}-1}{x} = \lim_{x\to 0}\frac{\sqrt{1+x^2}-1}{x}\times\frac{\sqrt{1+x^2}+1}{\sqrt{1+x^2}+1}$$

$$= \lim_{x\to 0}\frac{(\sqrt{1+x^2})^2-1}{x(\sqrt{1+x^2}+1)} = \lim_{x\to 0}\frac{1+x^2-1}{x(\sqrt{1+x^2}+1)}$$

$$= \lim_{x\to 0}\frac{x^{\cancel{2}}}{\cancel{x}(\sqrt{1+x^2}+1)} = \frac{0}{\sqrt{1+0}+1} = \frac{0}{2} = 0$$

$$\therefore \boxed{\lim_{x\to 0}\frac{\sqrt{1+x^2}-1}{x} = 0}$$

13. $\lim\limits_{x\to 0}\dfrac{\sqrt{1-x}-1}{x^2}$

<u>Soln:</u>

$$\lim_{x\to 0}\frac{\sqrt{1-x}-1}{x^2} = \lim_{x\to 0}\frac{\sqrt{1-x}-1}{x^2}\times\frac{\sqrt{1-x}+1}{\sqrt{1-x}+1}$$

$$= \lim_{x\to 0}\frac{(\sqrt{1-x})^2-1^2}{x^2(\sqrt{1-x}+1)}$$

$$= \lim_{x\to 0}\frac{(1-x)-1}{x^2(\sqrt{1-x}+1)} = \lim_{x\to 0}\frac{-\cancel{x}}{x^{\cancel{2}}(\sqrt{1-x}+1)}$$

$$= \lim_{x\to 0}\frac{-1}{x(\sqrt{1-x}+1)}$$

$$= \frac{-1}{0(\sqrt{1-0}+1)} = \frac{-1}{0} = -\infty$$

$\therefore$ limit does not exist.

 $\left[\frac{0}{0}\right]$ form.

14) $\lim\limits_{x\to5} \frac{\sqrt{x-1}-2}{x-5}$

Soln:

$$\lim_{x\to5} \frac{\sqrt{x-1}-2}{x-5} = \lim_{x\to5} \frac{\sqrt{x-1}-2}{x-5} \times \frac{\sqrt{x-1}+2}{\sqrt{x-1}+2}$$

$$= \lim_{x\to5} \frac{(\sqrt{x-1})^2-2^2}{(x-5)(\sqrt{x-1}+2)} = \lim_{x\to5} \frac{x-1-4}{(x-5)(\sqrt{x-1}+2)}$$

$$= \lim_{x\to5} \frac{x-5}{(x-5)(\sqrt{x-1}+2)} = \lim_{x\to5} \frac{1}{(\sqrt{x-1}+2)}$$

$$= \frac{1}{\sqrt{5-1}+2} = \frac{1}{\sqrt{4}+2} = \frac{1}{2+2} = \frac{1}{4}.$$

$$\therefore \boxed{\lim_{x\to5} \frac{\sqrt{x-1}-2}{x-5} = \frac{1}{4}}$$

15) $\lim\limits_{x\to a} \frac{\sqrt{x-b}-\sqrt{a-b}}{x^2-a^2}$ $(a>b)$.

Soln:

$$\lim_{x\to a} \frac{\sqrt{x-b}-\sqrt{a-b}}{x^2-a^2} = \lim_{x\to a} \frac{\sqrt{x-b}-\sqrt{a-b}}{x^2-a^2} \times \frac{\sqrt{x-b}+\sqrt{a-b}}{\sqrt{x-b}+\sqrt{a-b}}.$$

$$= \lim_{x\to a} \frac{(\sqrt{x-b})^2-(\sqrt{a-b})^2}{(x^2-a^2)(\sqrt{x-b}+\sqrt{a-b})}$$

$$= \lim_{x\to a} \frac{x-\cancel{b}-a+\cancel{b}}{(x+a)(x-a)(\sqrt{x-b}+\sqrt{a-b})}$$

$$= \lim_{x\to a} \frac{1}{(x+a)(\sqrt{x-b}+\sqrt{a-b})}$$

$$= \frac{1}{(a+a)(\sqrt{a-b}+\sqrt{a-b})} = \frac{1}{2a\times2\sqrt{a-b}} = \frac{1}{4a\sqrt{a-b}}.$$

$$\therefore \boxed{\lim_{x\to a} \frac{\sqrt{x-b}-\sqrt{a-b}}{x^2-a^2} = \frac{1}{4a\sqrt{a-b}}}$$

Exercise: 9.3

1) a) Find the left and right limits of $f(x) = \frac{x^2-4}{(x^2+4x+4)(x+3)}$ at $x=-2$

b) $f(x) = \tan x$ at $x = \pi/2$

Soln:

a) Soln:

L.H.L R.H.L

-2^- -2 -2^+

$$\lim_{x\to -2^-} f(x) = \lim_{x\to -2^-} \frac{x^2-4}{(x^2+4x+4)(x+3)}$$

$$= \lim_{x\to -2^-} \frac{\cancel{(x+2)}(x-2)}{(x+2)^{\cancel{2}}(x+3)} = \frac{-2-2}{(-2+2)(-2+3)} = \infty$$

$\therefore f(-2) \to \infty$ as $x \to -2^-$

Rough work

$x = -2.1$

$\frac{-2.1-2}{(-2.1+2)(-2.1+3)}$ = positive

$= \frac{-}{-} = \infty$.

$$\lim_{x\to -2^+} f(x) = \lim_{x\to -2^+} \frac{x^2-4}{(x^2+4x+4)(x+3)}$$

$$= \lim_{x\to -2^+} \frac{\cancel{(x+2)}(x-2)}{(x+2)^{\cancel{2}}(x+3)} = \frac{-2-2}{(-2+2)(-2+3)} = \frac{-4}{0} = -\infty.$$

$\therefore f(-2) \to -\infty$ as $x \to -2^+$

Rough work

$x = 1.9$

$\frac{1.9-2}{(1.9+2)(1.9+3)} = \frac{(-)}{(+)(+)} = -\infty.$

b) Soln:

$$\lim_{x\to \pi/2^-} f(x) = \lim_{x\to \pi/2^-} \tan x = \tan \pi/2 = \infty$$

$\therefore f(\pi/2) \to \infty$ as $x \to \pi/2^-$ (I quadrant)

$\pi/2^-$ $\pi/2$ $\pi/2^+$

$$\lim_{x\to \pi/2^+} f(x) = \lim_{x\to \pi/2^+} \tan x = \tan \pi/2 = -\infty$$

(II quadrant).

$\therefore f(\pi/2) \to -\infty$ as $x \to \pi/2^+$

Evaluate the following limits:

2) $\lim_{x\to 3} \frac{x^2-9}{x^2(x^2-6x+9)}$

Soln:

$$\lim_{x\to 3} \frac{x^2-9}{x^2(x^2-6x+9)} = \lim_{x\to 3} \frac{(x+3)\cancel{(x-3)}}{x^2(x-3)^{\cancel{2}}} = \lim_{x\to 3} \frac{x+3}{x^2(x-3)}.$$

$\lim_{x \to 3^-} \frac{x+3}{x^2(x-3)} = \frac{6}{9(3-3)} = \frac{6}{0} = -\infty$

$x = 2.9$

$\frac{2.9+3}{(2.9)^2(2.9-3)} = \frac{+}{-} = -\infty$.

$\lim_{x \to 3^+} \frac{x+3}{x^2(x-3)} = \frac{3+3}{9(3-3)} = \frac{6}{0} = \infty$

$x = 3.1$

$\frac{3.1+3}{(3.1)^2(3.1-3)} = \frac{+}{+} = \infty$.

3) $\lim_{x \to \infty} \frac{3}{x-2} - \frac{2x+11}{x^2+x-6}$.

Soln:

$\lim_{x \to \infty} \frac{3}{x-2} - \frac{(2x+11)}{x^2+x-6} = \lim_{x \to \infty} \frac{3}{(x-2)} - \frac{(2x+11)}{(x-2)(x+3)}$

$= \lim_{x \to \infty} \frac{3(x+3) - 2x - 11}{(x-2)(x+3)} = \lim_{x \to \infty} \frac{3x+9-2x-11}{(x-2)(x+3)}$

$= \lim_{x \to \infty} \frac{(\cancel{x-2})}{(\cancel{x-2})(x+3)} = \lim_{x \to \infty} \frac{1}{x+3}$ $\cancel{= \frac{1}{\infty+3}}$

$= 0$ $\cancel{= 0}$

$\therefore f(x) \to 0$ as $x \to \infty$.

4) $\lim_{x \to \infty} \frac{x^3+x}{x^4-3x^2+1}$

Soln:

$\lim_{x \to \infty} f(x) = \lim_{x \to \infty} \frac{x^3+x}{x^4-3x^2+1} = \lim_{x \to \infty} \frac{x^3\left(1+\frac{x}{x^3}\right)}{x^4\left(1-\frac{3x^2}{x^4}+\frac{1}{x^4}\right)}$

$= \lim_{x \to \infty} \frac{\left[1+\frac{1}{x^2}\right]}{x\left[1-\frac{3}{x^2}+\frac{1}{x^4}\right)}$

$= \frac{1+\frac{1}{0}}{\infty(1-0+0)} = \frac{1}{\infty} = 0$.

$\boxed{\lim_{x \to \infty} \frac{x^3+x}{x^4-3x^2+1} = 0}$

5) $\lim_{x \to \infty} \frac{x^4-5x}{x^2-3x+1}$

Soln:

$\lim_{x \to \infty} \frac{x^4-5x}{x^2-3x+1} = \lim_{x \to \infty} \frac{x^4\left(1-\frac{5x}{x^4}\right)}{x^2\left(1-\frac{3x}{x^2}+\frac{1}{x^2}\right)}$

$$= \lim_{x\to\infty} \frac{x^2(1-5/x^3)}{[1-3/x+1/x^2]}$$

$$= \frac{\infty^2(1-0)}{(1-0+0)} = \infty^2 = \infty$$

$\therefore f(x) \to \infty$ as $x \to \infty$.

6. $\lim_{x\to\infty} \frac{1+x-3x^3}{1+x^2+3x^3}$

Soln:

$$\lim_{x\to\infty} \frac{1+x-3x^3}{1+x^2+3x^3} = \lim_{x\to\infty} \frac{\cancel{x^3}\left(\frac{1}{x^3}+\frac{x}{x^3}-\frac{3x^3}{x^3}\right)}{\cancel{x^3}\left(1/x^3+\frac{x^2}{x^3}+3\right)}$$

$$= \lim_{x\to\infty} \frac{\frac{1}{x^3}+\frac{1}{x^2}-3}{\frac{1}{x^3}+\frac{1}{x}+3}$$

$$= \frac{0+0-3}{0+0+3} = -1$$

$$\therefore \boxed{\lim_{x\to\infty} \frac{1+x-3x^3}{1+x^2+3x^3} = -1}$$

Roughwork

$\left(\because \frac{1}{0} = \infty\right)$.

$\left(\frac{1}{\infty} = 0\right)$

7. $\lim_{x\to\infty}\left(\frac{x^3}{2x^2-1} - \frac{x^2}{2x+1}\right)$

Soln:

$$\lim_{x\to\infty}\left(\frac{x^3}{2x^2-1} - \frac{x^2}{2x+1}\right) = \lim_{x\to\infty} \frac{x^3(2x+1)-x^2(2x^2-1)}{(2x^2-1)(2x+1)}$$

$$= \lim_{x\to\infty} \frac{2x^4+x^3-2x^4+x^2}{(2x^2-1)(2x+1)} = \lim_{x\to\infty} \frac{x^3+x^2}{4x^3+2x^2-2x-1}$$

$$= \lim_{x\to\infty} \frac{\cancel{x^3}\left(1+\frac{x^2}{x^3}\right)}{\cancel{x^3}\left(4+2/x-2/x^2-1/x^3\right)}$$

$$= \frac{1+0}{4+0-0-0} = \frac{1}{4}$$

$$\therefore \boxed{\lim_{x\to\infty}\left(\frac{x^3}{2x^2-1} - \frac{x^2}{2x+1}\right) = 1/4}$$

8. Show that,

i) $\lim\limits_{n\to\infty} \dfrac{1+2+3+\cdots+n}{3n^2+7n+2} = \dfrac{1}{6}$

Soln:

$L.H.S = \lim\limits_{n\to\infty} \dfrac{1+2+3+\cdots+n}{3n^2+7n+2}$

$1+2+\cdots+n = \dfrac{n(n+1)}{2}$

$= \lim\limits_{n\to\infty} \dfrac{\frac{n(n+1)}{2}}{(3n+1)(n+2)}$ (or) $= \lim\limits_{n\to\infty} \dfrac{\frac{n(n+1)}{2}}{3n^2+7n+2}$

$= \lim\limits_{n\to\infty} \dfrac{n(n+1)}{2(n+2)(3n+1)}$ $= \lim\limits_{n\to\infty} \dfrac{n^2+n}{2(3n^2+7n+2)}$

$= \lim\limits_{n\to\infty} \dfrac{n^2(1+1/n)}{2(3n^2+7n+2)}$ $= \lim\limits_{n\to\infty} \dfrac{n^2(1+1/n)}{2n^2(3+7/n+2/n^2)}$

$= \lim\limits_{n\to\infty} \dfrac{\cancel{n^2}(1+1/n)}{2\cancel{n^2}(3+7/n+2/n^2)}$ $= \dfrac{1+0}{2(3+0+0)}$ $= 1/6$.

$= \dfrac{1+0}{2(3+0+0)} = \dfrac{1}{6} = R.H.S.$

$$\boxed{\lim_{n\to\infty} \frac{1+2+3+\cdots+n}{3n^2+7n+2} = 1/6}$$

ii) $\lim\limits_{n\to\infty} \dfrac{1^2+2^2+\cdots+(3n)^2}{(1+2+\cdots+5n)(2n+3)} = \dfrac{9}{25}$

Soln:

$L.H.S = \lim\limits_{n\to\infty} \dfrac{1^2+2^2+\cdots+(3n)^2}{(1+2+\cdots+5n)(2n+3)}$

$1^2+2^2+\cdots+n^2 = \dfrac{n(n+1)(2n+1)}{6}.$

$= \lim\limits_{n\to\infty} \dfrac{\frac{3n(3n+1)(6n+1)}{6}}{\frac{5n(5n+1)}{2}\times(2n+3)}$

$= \lim\limits_{n\to\infty} \dfrac{\cancel{3n}(3n+1)(6n+1)}{\cancel{6}\,3} \times \dfrac{\cancel{2}\,1}{5\cancel{n}(5n+1)(2n+3)}$

$= \lim\limits_{n\to\infty} \dfrac{\cancel{n^2}(3+\frac{1}{n})(6+1/n)}{\cancel{n^2}(5+1/n)(2+3/n)}$

$= \dfrac{(3+0)(6+0)}{5(5+0)(2+0)} = \dfrac{18}{50} = \dfrac{9}{25} = R.H.S.$

$$\boxed{\lim_{n\to\infty} \frac{1^2+2^2+\cdots+(3n)^2}{(1+2+\cdots+5n)(2n+3)} = \frac{9}{25}}$$

iii) $\lim_{n\to\infty} \frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\cdots+\frac{1}{n(n+1)}=1$

Soln:

$$L.H.S = \lim_{n\to\infty}\left[\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\cdots+\frac{1}{n(n+1)}\right]$$

Rough work:

$\frac{1}{1\cdot 2}=\left[\frac{1}{1}-\frac{1}{2}\right]$

$\frac{1}{2\cdot 3}=\frac{1}{2}-\frac{1}{3}$

$\frac{1}{3\cdot 4}=\frac{1}{3}-\frac{1}{4}$.

$$=\lim_{n\to\infty}\left\{\left[1-\cancel{\frac{1}{2}}\right]+\left[\cancel{\frac{1}{2}}-\cancel{\frac{1}{3}}\right]+\left[\cancel{\frac{1}{3}}-\cancel{\frac{1}{4}}\right]+\cdots+\left[\cancel{\frac{1}{n}}-\frac{1}{n+1}\right]\right\}$$

$$=\lim_{n\to\infty}\left[1-\frac{1}{n+1}\right]$$

$$=1-0=1=R.H.S.$$

$$\therefore \boxed{\lim_{n\to\infty}\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\cdots+\frac{1}{n(n+1)}=1}$$

9. An important problem in fishery science is to estimate the number of fish presently spawing in streams and use this information to predict the number of mature fish or "recruits" that will return to the rivers during the reproductive period. If S is the number of spawners and R the number of recruits, "Beverton – Holt spawner recruit function" is $R(S)=\frac{S}{(\alpha S+\beta)}$ where α and β are the positive constants. Show that this function predicts approximately constant recruitmen when the number of spawners is sufficiently large.

Soln:

Given that, $R(S)=\frac{S}{\alpha S+\beta}$.

To prove:

$S\to\infty$ as $R(S)\longrightarrow$ constant.

$$\lim_{S\to\infty}R(S)=\lim_{S\to\infty}\frac{S}{\alpha S+\beta}=\lim_{S\to\infty}\frac{S}{S(\alpha+\beta/S)}$$

$$=\lim_{S\to\infty}\frac{1}{\alpha+\beta/S}=\frac{1}{\alpha+0}=1/\alpha \text{ (constant)}.$$

$\therefore S\to\infty$ as $R(S)\to 1/\alpha$ (constant).

10. A tank contains 500 litres of pure water. Brine (very salty water) that contains 30 grams of salt per litre of water is pumped into the tank at a rate of 25 litres per minute. The concentration of salt water after t minutes (in grams per litre) is $c(t) = \frac{30t}{200+t}$. What happens to the concentration as $t \to \infty$?

Soln:

$$\lim_{t\to\infty} c(t) = \lim_{t\to\infty} \frac{30t}{200+t}$$

$$= \lim_{t\to\infty} \frac{30\cancel{t}}{\cancel{t}\left(\frac{200}{t}+1\right)}$$

$$= \lim_{t\to\infty} \frac{30}{\left(\frac{200}{t}+1\right)}$$

$$= \frac{30}{0+1} = 30$$

$$\therefore \boxed{\lim_{t\to\infty} c(t) = 30}$$