



Padalsalai's Telegram Groups!

(தலைப்பிற்கு கீழே உள்ள லிங்கை கிளிக் செய்து குழுவில் இணையவும்!)

- Padalsalai's NEWS - Group
https://t.me/joinchat/NIfCqVRBNj9hhV4wu6_NqA
- Padalsalai's Channel - Group
<https://t.me/padasalaichannel>
- Lesson Plan - Group
<https://t.me/joinchat/NIfCqVWwo5iL-21gpzrXLw>
- 12th Standard - Group
https://t.me/Padalsalai_12th
- 11th Standard - Group
https://t.me/Padalsalai_11th
- 10th Standard - Group
https://t.me/Padalsalai_10th
- 9th Standard - Group
https://t.me/Padalsalai_9th
- 6th to 8th Standard - Group
https://t.me/Padalsalai_6to8
- 1st to 5th Standard - Group
https://t.me/Padalsalai_1to5
- TET - Group
https://t.me/Padalsalai_TET
- PGTRB - Group
https://t.me/Padalsalai_PGTRB
- TNPSC - Group
https://t.me/Padalsalai_TNPSC

CHAPTER → 9

APPLICATIONS OF INTEGRATION

Formulas of integration :-

$$* \int 0 dx = c, \text{ where } c \text{ is a constant}$$

$$* \int k dx = kx + c, \text{ where } c \text{ is a constant}$$

$$* \int x^n dx = \frac{x^{n+1}}{n+1} + c, \quad n \neq -1 \text{ (power rule)}$$

$$* \int \frac{1}{x} dx = \log|x| + c$$

$$* \int \sin x dx = -\cos x + c$$

$$* \int \cos x dx = \sin x + c$$

$$* \int \sec^2 x dx = \tan x + c$$

$$* \int \csc^2 x dx = -\cot x + c$$

$$* \int \sec x \tan x dx = \sec x + c$$

$$* \int \csc x \cot x dx = -\csc x + c$$

$$* \int e^x dx = e^x + c$$

$$* \int a^x dx = \frac{a^x}{\log a} + c$$

$$* \int \frac{1}{\sqrt{1-x^2}} dx = \sin^{-1} x + c$$

$$* \int \frac{1}{1+x^2} dx = \tan^{-1} x + c$$

V. MURUGAN

PG ASST IN MATHS

7904171615

CHAPTER 4

APPLICATIONS OF INTEGRATION

$$* \int \frac{1}{x^2} dx = -\frac{1}{x} + c$$

- formula of integration

$$* \int \frac{1}{\sqrt{x}} dx = 2\sqrt{x}$$

$$* \int \sec x dx = \log |\sec x + \tan x| + c$$

$$* \int \tan x dx = \log |\sec x| + c \text{ (or) } -\log |\cos x| + c$$

$$* \int \cot x dx = \log |\sin x| + c$$

$$* \int a^x dx = \frac{a^x}{\log a} + c$$

$$* \int \operatorname{cosec} x dx = \log |\operatorname{cosec} x - \cot x| + c$$

$$* \int \frac{dx}{a^2 - x^2} = \frac{1}{2a} \log \left| \frac{a+x}{a-x} \right| + c$$

$$* \int \frac{dx}{x^2 - a^2} = \frac{1}{2a} \log \left| \frac{x-a}{x+a} \right| + c$$

$$* \int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1} \left(\frac{x}{a} \right) + c$$

$$* \int \frac{dx}{\sqrt{a^2 + x^2}} = \log |x|$$

$$* \int \frac{dx}{\sqrt{x^2 - a^2}} = \log |x|$$

$$* \int \sqrt{a^2 + x^2} dx = \frac{x}{2} \sqrt{a^2 + x^2} + \frac{a^2}{2} \log |x + \sqrt{a^2 + x^2}| + c$$

V. MURUGAN

PG ASST IN MATHS

7904171615

$$\int \sqrt{x^2 - a^2} dx = \frac{x}{2} \sqrt{x^2 - a^2} - \frac{a^2}{2} \log |x + \sqrt{x^2 - a^2}| + c$$

$$* \int \sqrt{a^2 - x^2} dx = \frac{x}{2} \sqrt{a^2 - x^2} + \frac{a^2}{2} \sin^{-1} \left(\frac{x}{a} \right) + c$$

$$* \int u dv = uv - u'v_1 + u''v_2 - u'''v_3 \dots \rightarrow \text{Bernoulli}$$

$$* \int e^{ax} \cos bx dx = \frac{e^{ax}}{a^2 + b^2} (a \cos bx + b \sin bx) + c$$

$$* \int e^{ax} \sin bx dx = \frac{e^{ax}}{a^2 + b^2} (a \sin bx - b \cos bx) + c$$

$$* \int \log x \cdot dx = x \log x - x + c$$

$$* \int e^{ax} \cdot dx = \frac{e^{ax}}{a} + c$$

Limits

Definite integral as the limits of a sum.

* Riemann Integral

* Left-end rule for evaluating the Riemann Integral.

$$\int_a^b f(x) dx = \lim_{\substack{n \rightarrow \infty \text{ and} \\ \max(x_i - x_{i-1}) \rightarrow 0}} \sum_{i=1}^n f(x_{i-1})(x_i - x_{i-1})$$

* Right-end rule for evaluating the Riemann Integral.

$$\int_a^b f(x) dx = \lim_{\substack{n \rightarrow \infty \text{ and} \\ \max(x_i - x_{i-1}) \rightarrow 0}} \sum_{i=1}^n f(x_i)(x_i - x_{i-1})$$

V. MURUGAN

PG ASST IN MATHS

7904171615

Riemann Integral.

$$\int_a^b f(x) dx = \lim_{\substack{n \rightarrow \infty \text{ and} \\ \max(x_i - x_{i-1}) \rightarrow 0}} \sum_{i=1}^n f\left(\frac{x_{i-1} + x_i}{2}\right) (x_i - x_{i-1})$$

Ex: 9.1

1. Find an approximate value of $\int_{1.1}^{1.5} x dx$ by applying the left-end rule with the partition $\{1.1, 1.2, 1.3, 1.4, 1.5\}$

Sol

$$\int_{1.1}^{1.5} x dx, \text{ interval} \Rightarrow 1.1, 1.2, 1.3, 1.4, 1.5$$

$$n = 5$$

$$\text{let } x_0 = 1.1, x_1 = 1.2, x_2 = 1.3, x_3 = 1.4, x_4 = 1.5$$

$$x_5 = 1.5$$

$$\Delta x = x_2 - x_1$$

$$= 1.2 - 1.1$$

$$\Delta x = 0.1$$

$$\text{let } f(x) = x$$

The left end rule for Riemann integral

$$S = [f(x_0) + f(x_1) + f(x_2) + \dots + f(x_{n-1})] \Delta x$$

$$S = [f(1.1) + f(1.2) + f(1.3) + f(1.4)] \Delta x$$

$$S = (1 + 1.1 + 1.2 + 1.3 + 1.4) 0.1$$

$$S = 6.0 \times 0.1$$

$$S = 0.6$$

2) find an approximate value of $\int_1^{1.5} x^2 dx$ by applying the right-end rule with the partition $\{1.1, 1.2, 1.3, 1.4, 1.5\}$

sol

$$\int_1^{1.5} x^2 dx, \text{ interval} = 1.1, 1.2, 1.3, 1.4, 1.5$$

$$n = 5$$

$$\text{let } x_1 = 1.1, x_2 = 1.2, x_3 = 1.3, x_4 = 1.4, x_5 = 1.5$$

$$\Delta x = x_2 - x_1 = 1.2 - 1.1$$

$$= 0.1$$

$$\Delta x = 0.1$$

$$\text{let } f(x) = x^2$$

By using right end rule

$$S = [f(x_1) + f(x_2) + f(x_3) + \dots + f(x_n)] \Delta x$$

$$S = [f(x_1) + f(x_2) + f(x_3) + f(x_4) + f(x_5)] \Delta x$$

$$S = [f(1.1) + f(1.2) + f(1.3) + f(1.4) + f(1.5)] \Delta x$$

$$S = [1.21 + 1.44 + 1.69 + 1.96 + 2.25] 0.1$$

$$S = (8.55) (0.1)$$

$$S = 0.855$$

3. Find an approximate value of $\int_1^{1.5} (2-x) dx$ by applying the mid-point rule with partition $\{1.1, 1.2, 1.3, 1.4, 1.5\}$

Sol

$$\int_1^{1.5} (2-x) dx, \text{ interval} = 1.1, 1.2, 1.3, 1.4, 1.5$$

$$n = 5$$

$$\text{let } x_0 = 1, x_1 = 1.1, x_2 = 1.2, x_3 = 1.3, x_4 = 1.4, x_5 = 1.5.$$

$$\Delta x = 0.1$$

$$\text{let } f(x) = 2-x$$

By using mid-point rule

$$S = \left[f\left(\frac{x_0+x_1}{2}\right) + f\left(\frac{x_1+x_2}{2}\right) + f\left(\frac{x_2+x_3}{2}\right) + f\left(\frac{x_3+x_4}{2}\right) + f\left(\frac{x_4+x_5}{2}\right) \right] \Delta x$$

$$S = \left[f\left(\frac{1+1.1}{2}\right) + f\left(\frac{1.1+1.2}{2}\right) + f\left(\frac{1.2+1.3}{2}\right) + f\left(\frac{1.3+1.4}{2}\right) + f\left(\frac{1.4+1.5}{2}\right) \right] \Delta x$$

$$S = \left[f(1.05) + f(1.15) + f(1.25) + f(1.35) + f(1.45) \right] (0.1)$$

$$S = (0.95 + 0.85 + 0.75 + 0.65 + 0.55)(0.1)$$

$$S = (3.75)(0.1)$$

$$S = 0.375$$

Limits of formula to evaluate $\int_a^b f(x) \cdot dx$.

$$\int_0^1 f(x) dx = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{r=1}^n f\left(\frac{r}{n}\right)$$

$$\int_a^b f(x) dx = \lim_{n \rightarrow \infty} \frac{b-a}{n} \sum_{r=1}^n f\left(a + (b-a) \frac{r}{n}\right)$$

$$\sum_{r=1}^n r \Rightarrow 1+2+3+\dots+n = \frac{n(n+1)}{2}$$

$$\sum_{r=1}^n r^2 \Rightarrow 1^2+2^2+3^2+\dots+n^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\sum_{r=1}^n r^3 \Rightarrow 1^3+2^3+3^3+\dots+n^3 = \left[\frac{n(n+1)}{2}\right]^2$$

Ex: 9.2

1. Evaluate the following integrals as the limits of sums.

i) $\int_0^1 (5x+4) dx$.

sol $\int_0^1 (5x+4) dx$

formula:

$$\int_0^1 f(x) dx = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{r=1}^n f\left(\frac{r}{n}\right)$$

$$f(x) = 5x+4$$

$$f\left(\frac{r}{n}\right) = \frac{5r}{n} + 4$$

$$\sum_{r=1}^n f\left(\frac{r}{n}\right) = \sum_{r=1}^n \frac{nr}{n} + 4$$

$$= \sum_{r=1}^n \frac{nr}{n} + \sum_{r=1}^n 4$$

$$= \frac{n}{n} \left(\sum_{r=1}^n r \right) + 4n$$

$$= \frac{n}{n} \times \frac{n(n+1)}{2} + 4n$$

$$= \frac{n}{2} (n+1) + 4n$$

$$\int_0^1 (5x+4) dx = \lim_{n \rightarrow \infty} \frac{1}{n} \left[\frac{n}{2} (n+1) + 4n \right]$$

$$= \lim_{n \rightarrow \infty} \frac{1}{n} \times n \left[\frac{n}{2} (1+4n) + 4 \right]$$

$$= \lim_{n \rightarrow \infty} \frac{n}{2} (1+4n) + 4$$

$$= \frac{n}{2} (1) + 4$$

$$= \frac{n}{2} + 4$$

$$= \frac{13}{2}$$

$$\therefore \int_0^1 (5x+4) dx = \frac{13}{2}$$

$$\text{ii) } \int_1^2 (4x^2-1) dx.$$

$$\text{sol} \int_1^2 (4x^2-1) dx$$

$$a=1, b=2, b-a=1$$

formula :

$$\int_a^b f(x) dx = \lim_{n \rightarrow \infty} \frac{b-a}{n} \sum_{r=1}^n f\left(a + (b-a) \frac{r}{n}\right)$$

$$f(x) = 4x^2 - 1$$

$$\sum_{r=1}^n f\left(a + (b-a) \frac{r}{n}\right) = \sum_{r=1}^n f\left(1 + \frac{r}{n}\right)$$

$$= \sum_{r=1}^n \left[4\left(1 + \frac{r}{n}\right)^2 - 1 \right]$$

$$= \sum_{r=1}^n \left[4 + \frac{4r^2}{n^2} + \frac{8r}{n} \right] - \sum_{r=1}^n 1$$

$$= \sum_{r=1}^n \left[4 + \frac{4r^2}{n^2} + \frac{8r}{n} \right] - n$$

$$= 4n + \frac{4}{n^2} \sum_{r=1}^n r^2 + \frac{8}{n} \sum_{r=1}^n r - n$$

$$= 4n + \frac{4}{n^2} \frac{n(n+1)(2n+1)}{6} + \frac{8}{n} \frac{n(n+1)}{2} - n$$

$$= 4n - n + \frac{4}{n^2} \frac{n^3(1+\frac{1}{n})(2+\frac{1}{n})}{6} + \frac{8}{n} \frac{n^2(1+\frac{1}{n})}{2}$$

$$= 3n + \frac{2n(1+\frac{1}{n})(2+\frac{1}{n})}{3} + 4(n+\frac{1}{n})$$

$$\int_1^2 (4x^2 - 1) dx = \lim_{n \rightarrow \infty} \frac{1}{n} \left[3n + \frac{2n(1+\frac{1}{n})(2+\frac{1}{n})}{3} + 4(n+\frac{1}{n}) \right]$$

$$= \lim_{n \rightarrow \infty} \frac{1}{n} \times n \left[3 + \frac{2(1+\frac{1}{n})(2+\frac{1}{n})}{3} + 4(1+\frac{1}{n}) \right]$$

$$= \lim_{n \rightarrow \infty} \left[3 + \frac{2(1+\frac{1}{n})(2+\frac{1}{n})}{3} + 4(1+\frac{1}{n}) \right]$$

$$= 3 + \frac{2(1)(2)}{3} + 4$$

$$= 3 + \frac{4}{3} + 4$$

$$= \frac{25}{3}$$

$$\int_1^2 (4x^2 - 1) dx = \frac{25}{3}$$

Fundamental theorems of Integral calculus and their applications:

Theorem: 9.1 (First fundamental theorem of Integral calculus)

If $f(x)$ be a continuous function defined on a closed interval $[a, b]$ and $\forall F(x) = \int_a^b f(u) \cdot du$, $a < x < b$ then,

$\frac{d}{dx} F(x) = f(x)$. In other words, $F(x)$ is an anti-derivative of $f(x)$.

Theorem: 9.2 [Second fundamental theorem of Integral calculus]

If $f(x)$ be a continuous function defined on a closed interval $[a, b]$ and $F(x)$ is an anti-derivative of $f(x)$, then

$$\int_a^b f(x) dx = F(b) - F(a)$$

Properties :

Property : 1

$$\int_a^b f(x) dx = \int_a^b f(u) \cdot du, \quad a < b$$

Property : 2

$$\int_a^b f(x) dx = - \int_b^a f(x) dx.$$

Property : 3

$$\int_a^b f(x) dx = \int_a^c f(x) dx + \int_c^b f(x) dx, \quad a < c < b$$

Property : 4

$$\int_a^b [\alpha f(x) + \beta g(x)] dx = \alpha \int_a^b f(x) dx + \beta \int_a^b g(x) dx,$$

where α and β are constants

Property : 5

If $x = g(u)$, then $\int_a^b f(x) dx = \int_c^d f(g(u)) \frac{dg(u)}{du} du$

where $g(c) = a$ and $g(d) = b$.

Property : 6

$$\int_a^b f(x) dx = \int_a^b f(a+b-x) dx$$

Note:

$$\int_0^a f(x) dx = \int_0^a f(a-x) dx.$$

$$\int_0^{2a} f(x) dx = \int_0^a [f(x) + f(2a-x)] dx$$

Property: 8

If $f(x)$ is an even function, then

$$\int_{-a}^a f(x) dx = 2 \int_0^a f(x) dx$$

Property: 9

If $f(x)$ is an odd function, then

$$\int_{-a}^a f(x) dx = 0$$

Property: 10

$$\text{If } f(2a-x) = f(x), \text{ then } \int_0^{2a} f(x) dx = 2 \int_0^a f(x) dx$$

Property: 11

If $f(2a-x) = -f(x)$, then

$$\int_0^{2a} f(x) dx = 0$$

Property: 12

$$\int_0^a x f(x) dx = \frac{a}{2} \int_0^a f(x) dx \text{ if } f(a-x) = f(x)$$

Results:

$$* \int_0^{\pi} g(\sin x) dx = 2 \int_0^{\pi/2} g(\sin x) dx$$

$$* \int_0^{2\pi} g(\cos x) dx = 2 \int_0^{\pi} g(\cos x) dx$$

$$* \int_0^a \frac{f(x)}{f(x)+f(a-x)} dx = \frac{a}{2}$$

Ex: 9.3

1. Evaluate the following definite integrals

i) $\int_3^4 \frac{dx}{x^2-4}$

Sol.

$$\int_3^4 \frac{dx}{x^2-4}$$

formula :

$$\int \frac{dx}{x^2-a^2} = \frac{1}{2a} \log \left| \frac{x-a}{x+a} \right| + c$$

$$\int_3^4 \frac{dx}{x^2-2^2} = \frac{1}{2(2)} \left[\log \left| \frac{x-2}{x+2} \right| \right]_3^4$$

$$= \frac{1}{4} \left[\log \frac{2}{5} - \log \frac{1}{5} \right]$$

$$= \frac{1}{4} \left[\log \frac{1}{5} \div \frac{1}{5} \right]$$

$$\int_3^4 \frac{dx}{x^2-4} = \frac{1}{4} \log \frac{5}{3}$$

ii) $\int_{-1}^1 \frac{dx}{x^2+2x+5}$

Sol.

$$\int_{-1}^1 \frac{dx}{x^2+2x+5}$$

$$= \int_{-1}^1 \frac{dx}{x^2+2x+1+4}$$

$$= \int_{-1}^1 \frac{dx}{(x+1)^2+2^2}$$

Formula :

$$\int \frac{dx}{a^2+x^2} = \frac{1}{a} \tan^{-1}\left(\frac{x}{a}\right) + c$$

$$\begin{aligned} \int_{-1}^1 \frac{dx}{(x+1)^2+2^2} &= \frac{1}{2} \left[\tan^{-1}\left(\frac{x+1}{2}\right) \right]_{-1}^1 \\ &= \frac{1}{2} \left[\tan^{-1}(1) - \tan^{-1}(0) \right] \end{aligned}$$

$$\int_{-1}^1 \frac{dx}{x^2+2x+5} = \frac{\pi}{8}$$

iii) $\int_0^1 \frac{\sqrt{1-x}}{\sqrt{1+x}} dx$

Sol

$$\int_0^1 \frac{\sqrt{1-x}}{\sqrt{1+x}} dx$$

$$= \int_0^1 \frac{\sqrt{1-x} \cdot \sqrt{1-x}}{\sqrt{(1+x)(1-x)}} dx$$

$$\begin{aligned}
 &= \int_0^1 \frac{1-x}{\sqrt{1-x^2}} dx \\
 &= \int_0^1 \frac{dx}{\sqrt{1-x^2}} - \int_0^1 \frac{x}{\sqrt{1-x^2}} dx \\
 &= [\sin^{-1} x]_0^1 + \frac{1}{2} \int_0^1 (1-x^2)^{-1/2} \cdot -2x dx \\
 &= [\sin^{-1}(1) - \sin^{-1}(0)] + \frac{1}{2} \int_0^1 (1-x^2)^{-1/2} \cdot d(1-x^2) \\
 &= \frac{\pi}{2} + \frac{1}{2} \left[\frac{(1-x^2)^{+1/2}}{1/2} \right]_0^1 \\
 &= \frac{\pi}{2} - 1
 \end{aligned}$$

Alter:

$$= \int_0^1 \frac{1-x}{\sqrt{1-x^2}} dx$$

$$x = \sin \theta$$

$$dx = d(\sin \theta)$$

$$dx = \cos \theta d\theta$$

x	0	1
θ	0	$\pi/2$

$$= \int_0^{\pi/2} \frac{1 - \sin \theta}{\sqrt{1 - \sin^2 \theta}} \cos \theta d\theta$$

$$= \int_0^{\pi/2} \frac{1 - \sin \theta}{\sqrt{\cos^2 \theta}} \cos \theta d\theta$$

$$= \int_0^{\pi/2} \frac{1 - \sin \theta}{\cos \theta} \cos \theta d\theta$$

$$= \int_0^{\pi/2} 1 - \sin \theta \, d\theta$$

$$= \int_0^{\pi/2} d\theta - \int_0^{\pi/2} \sin \theta \, d\theta$$

$$= \left[\theta \right]_0^{\pi/2} - \left[-\cos \theta \right]_0^{\pi/2}$$

$$= \frac{\pi}{2} - (-\cos \pi/2 + \cos 0)$$

$$= \frac{\pi}{2} - (0 + 1)$$

$$= \frac{\pi}{2} - 1$$

iv) $\int_0^{\pi/2} e^x \left(\frac{1 + \sin x}{1 + \cos x} \right) dx.$

Sol

$$\int_0^{\pi/2} e^x \left(\frac{1 + \sin x}{1 + \cos x} \right) dx$$

$$= \int_0^{\pi/2} e^x \left(\frac{1 + 2 \sin x/2 \cdot \cos x/2}{2 \cos^2 x/2} \right) dx$$

$$= \int_0^{\pi/2} \frac{e^x}{2 \cos^2 x/2} dx + \int_0^{\pi/2} \frac{e^x \cancel{\sin x/2} \cos x/2}{\cancel{2} \cos^2 x/2} dx$$

$$= \frac{1}{2} \int_0^{\pi/2} e^x \cdot \sec^2 \frac{x}{2} dx + \int_0^{\pi/2} e^x \cdot \tan \frac{x}{2} dx$$

$$u = e^x$$

$$du = e^x \cdot dx$$

$$\int dv = \int \sec^2 \frac{x}{2} dx$$

$$v = \frac{\tan \frac{x}{2}}{\frac{1}{2}}$$

$$\int u dv = uv - \int v du$$

$$= \frac{1}{2} \left[e^x \cdot \frac{\tan \frac{x}{2}}{\frac{1}{2}} \right]_0^{\pi/2} - \frac{1}{2} \int_0^{\pi/2} \frac{\tan \frac{x}{2}}{\frac{1}{2}} \cdot e^x dx$$

$$+ \int_0^{\pi/2} \tan \frac{x}{2} e^x dx$$

$$= \left[e^x \tan \frac{x}{2} \right]_0^{\pi/2}$$

$$= e^{\pi/2} \tan \frac{\pi}{2}$$

$$= e^{\pi/2} (1)$$

$$= e^{\pi/2}$$

$$\int_0^{\pi/2} e^x \left(\frac{1+\sin x}{1+\cos x} \right) dx = e^{\pi/2}$$

(v)

$$\int_0^{\pi/2} \sqrt{\cos \theta} \sin^3 \theta \, d\theta$$

Sol

$$= \int_0^{\pi/2} \sqrt{\cos \theta} \cdot \sin^3 \theta \, d\theta$$

$$= \int_0^{\pi/2} \sqrt{\cos \theta} \cdot \sin^2 \theta \cdot \sin \theta \, d\theta$$

$$= \int_0^{\pi/2} \sqrt{\cos \theta} \cdot (1 - \cos^2 \theta) \cdot \sin \theta \, d\theta$$

let $t = \cos \theta$
 $dt = -\sin \theta \, d\theta$

0	0	$\pi/2$
t	1	0

$$= \int_1^0 t^{1/2} \cdot (1 - t^2) \cdot -dt$$

$$= - \int_1^0 t^{1/2} \cdot (1 - t^2) \, dt$$

changing the limit

$$= \int_0^1 (t^{1/2} - t^{5/2}) \, dt$$

$$= \left[\frac{t^{3/2}}{3/2} - \frac{t^{7/2}}{7/2} \right]_0^1$$

$$= \frac{2}{3} - \frac{2}{7}$$

$$= \frac{14 - 6}{21}$$

$$= \frac{8}{21}$$

$$(vi) \int_0^1 \frac{1-x^2}{(1+x^2)^2} dx.$$

Sol

$$\int_0^1 \frac{1-x^2}{(1+x^2)^2} dx.$$

$$x = \tan \theta$$

$$dx = \sec^2 \theta d\theta$$

x	0	1
θ	0	$\pi/4$

$$= \int_0^{\pi/4} \frac{1 - \tan^2 \theta}{(1 + \tan^2 \theta)^2} \cdot \sec^2 \theta d\theta$$

$$= \int_0^{\pi/4} \frac{1 - \frac{\sin^2 \theta}{\cos^2 \theta}}{(\sec^2 \theta)^2} \sec^2 \theta d\theta$$

$$= \int_0^{\pi/4} \frac{\cos^2 \theta - \sin^2 \theta}{\cos^2 \theta} \cdot \frac{1}{\cos^2 \theta} d\theta$$

$$= \int_0^{\pi/4} \cos^2 \theta - \sin^2 \theta d\theta$$

$$= \int_0^{\pi/4} \cos 2\theta d\theta$$

$$= \left[\frac{\sin 2\theta}{2} \right]_0^{\pi/4}$$

$$= \frac{\sin 2(\pi/4)}{2} - 0 = \frac{\sin \pi/2}{2}$$

$$= \frac{1}{2}$$

0	$\pi/2$
1	0

Limit

2. Evaluate the following using properties of integration :

i) $\int_{-5}^5 x \cos \left(\frac{e^x - 1}{e^x + 1} \right) dx.$

Sol

$$\int_{-5}^5 x \cos \left(\frac{e^x - 1}{e^x + 1} \right) dx$$

$$\text{Let } f(x) = x \cos \left(\frac{e^x - 1}{e^x + 1} \right)$$

$$f(-x) = (-x) \cos \left(\frac{e^{-x} - 1}{e^{-x} + 1} \right)$$

$$f(-x) = -x \cdot \cos \left(\frac{\frac{1}{e^x} - 1}{\frac{1}{e^x} + 1} \right)$$

$$f(-x) = -x \cdot \cos \left(\frac{\frac{1 - e^x}{e^x}}{\frac{1 + e^x}{e^x}} \right)$$

$$f(-x) = -x \cdot \cos \left(- \left(\frac{e^x - 1}{e^x + 1} \right) \right)$$

$$f(-x) = -f(x)$$

$\therefore f(x)$ is an odd.

By the property of integration

$$\int_{-a}^a f(x) \cdot dx = 0$$

$$\int_{-5}^5 x \cos \left(\frac{e^x - 1}{e^x + 1} \right) dx = 0$$

$$\int_{-\pi/2}^{\pi/2} (x^5 + x \cos x + \tan^3 x + 1) dx.$$

Sol

$$\int_{-\pi/2}^{\pi/2} (x^5 + x \cos x + \tan^3 x + 1) dx.$$

Formula:

$$\int_a^b f(x) dx = \int_a^b f(a+b-x) dx.$$

$$I = \int_{-\pi/2}^{\pi/2} (x^5 + x \cos x + \tan^3 x + 1) dx.$$

$$\therefore x = \frac{\pi}{2} - \frac{\pi}{2} - x$$

$$x = -x.$$

$$I = \int_{-\pi/2}^{\pi/2} (-x^5 - x \cos x - \tan^3 x + 1) dx.$$

Adding ① & ②

$$2I = \int_{-\pi/2}^{\pi/2} (\cancel{x^5} - \cancel{x^5} + \cancel{x \cos x} - \cancel{x \cos x} + \cancel{\tan^3 x} - \cancel{\tan^3 x} + 1 + 1) dx$$

$$2I = \int_{-\pi/2}^{\pi/2} 2 dx.$$

$$2I = 2 \int_{-\pi/2}^{\pi/2} dx$$

$$I = \left[x \right]_{-\pi/2}^{\pi/2}$$

$$I = \frac{\pi}{2} + \frac{\pi}{2} = \frac{2\pi}{2}$$

$$I = \pi$$

iii)

$$\int_{-\pi/4}^{\pi/4} \sin^2 x \, dx.$$

Sol.

$$f(x) = \sin^2 x$$

$$f(x) = \sin x \cdot \sin x.$$

$$f(-x) = \sin(-x) \cdot \sin(-x)$$

$$f(-x) = -\sin x \cdot -\sin x.$$

$$f(-x) = \sin^2 x$$

$\therefore f(x)$ is even

By the property of Integration

$$\int_{-a}^a f(x) \, dx = 2 \int_0^a f(x) \, dx$$

$$\int_{-\pi/4}^{\pi/4} \sin^2 x \, dx = 2 \int_0^{\pi/4} \sin^2 x \, dx.$$

$$= 2 \int_0^{\pi/4} \frac{1 - \cos 2x}{2} \, dx$$

$$= \int_0^{\pi/4} 1 - \cos 2x \, dx.$$

$$= \left[x - \frac{\sin 2x}{2} \right]_0^{\pi/4}$$

$$= \frac{\pi}{4} - \frac{1}{2}$$

$$= \frac{\pi - 2}{4}$$

(iv)

$$\int_0^{\pi} x \log \left(\frac{3 + \cos x}{3 - \cos x} \right) dx.$$

sol

$$f(x) = \log \left(\frac{3 + \cos x}{3 - \cos x} \right)$$

$$\int_0^a x f(x) dx = \frac{a}{2} \int_0^a f(x) dx$$

$$\text{if } f(x) = f(a-x)$$

$$x \rightarrow 2\pi - x$$

$$f(2\pi - x) = \log \left(\frac{3 + \cos(2\pi - x)}{3 - \cos(2\pi - x)} \right) dx.$$

$$f(2\pi - x) = \log \left(\frac{3 + \cos x}{3 - \cos x} \right) dx = f(x)$$

By property of integral

$$\int_0^{2\pi} g(\cos x) dx = 2 \int_0^{\pi} g(\cos x) dx.$$

$$\int_0^a x f(x) dx = \frac{a}{2} \int_0^a f(x) dx \quad \text{if } f(x) = f(a-x)$$

$$\int_0^{2\pi} x \log \left(\frac{3 + \cos x}{3 - \cos x} \right) dx = \frac{2\pi}{2} \int_0^{2\pi} \log \left(\frac{3 + \cos x}{3 - \cos x} \right) dx$$

$$= \pi \int_0^{2\pi} \log \left(\frac{3 + \cos x}{3 - \cos x} \right) dx$$

By property of integral

$$\int_0^{2\pi} g(\cos x) dx = 2 \int_0^{\pi} g(\cos x) dx$$

$$= 2\pi \int_0^{\pi} \log \left(\frac{3 + \cos x}{3 - \cos x} \right) dx$$

$$= 2\pi \left[\int_0^{\pi} \log(3 + \cos x) dx - \int_0^{\pi} \log(3 - \cos x) dx \right]$$

$$\int_0^a f(x) dx = \int_0^a f(a-x) dx$$

$$= 2\pi \left[\int_0^{\pi} \log(3 + \cos(\pi - x)) dx - \int_0^{\pi} \log(3 - \cos x) dx \right]$$

$$= 2\pi \int_0^{\pi} \log(3 - \cos x) dx - \int_0^{\pi} \log(3 + \cos x) dx$$

$$= \left(\frac{x \cos x + \sin x}{x^2} \right) \bigg|_0^{\pi} = (x) \bigg|_0^{\pi}$$

$$= 0.$$

V) $\int_0^{2\pi} \sin^4 x \cos^3 x dx$

Sol

$$= \int_0^{2\pi} (\sin^2 x)^2 \cos^3 x dx$$

$$= \int_0^{2\pi} (1 - \cos^2 x)^2 \cos^3 x dx$$

$$= \int_0^{2\pi} (1 + \cos^4 x - 2\cos^2 x) \cos^3 x dx$$

$$= \int_0^{2\pi} (\cos^3 x + \cos^7 x - 2\cos^5 x) dx$$

Formula.

$$\int_0^{2\pi} g(\cos x) dx = 2 \int_0^{\pi} g(\cos x) dx$$

$$\text{let } I = 2 \int_0^{\pi} (\cos^3 x + \cos^7 x - 2\cos^5 x) dx \rightarrow 0$$

Formula.

$$\int_0^a f(x) dx = \int_0^a f(a-x) dx$$

$\sin x dx$

$$I = 2 \int_0^{\pi} \cos^3(\pi-x) + \cos^7(\pi-x) - 2\cos^5(\pi-x) dx$$

$$I = 2 \int_0^{\pi} -\cos^3 x - \cos^7 x + 2\cos^5 x dx \rightarrow \textcircled{2}$$

By adding ① & ②

$$2I = \int_0^{\pi} 0 \cdot dx.$$

$$2I = 0$$

$$I = 0$$

$$\int_0^{2\pi} \sin^4 x \cos^3 x dx = 0.$$

vii) $\int_0^1 |\pi x - 3| dx.$

sol $\int_0^1 |\pi x - 3| dx$

$$f(x) = |\pi x - 3|$$

$$\pi x - 3 = 0$$

$$\pi x = 3$$

$$x = 3/\pi$$

$$f(x) = |\pi x - 3| = \begin{cases} \pi x - 3 & \text{if } \frac{3}{\pi} < x < 1 \\ -(\pi x - 3) & \text{if } 0 < x < \frac{3}{\pi} \end{cases}$$

By property of integration

$$\int_a^b f(x) dx = \int_a^c f(x) dx + \int_c^b f(x) dx, a < c < b$$

$$\begin{aligned}
 \int_0^1 (5x-3) dx &= \int_0^{3/5} -(5x-3) dx + \int_{3/5}^1 (5x-3) dx \\
 &= \int_0^{3/5} -5x+3 \cdot dx + \int_{3/5}^1 5x-3 dx \\
 &= \int_0^{3/5} 3 dx - \int_0^{3/5} 5x dx + \int_{3/5}^1 5x dx - \int_{3/5}^1 3 dx \\
 &= \left[3x \right]_0^{3/5} - \left[\frac{5x^2}{2} \right]_0^{3/5} + \left[\frac{5x^2}{2} \right]_{3/5}^1 - \left[3x \right]_{3/5}^1 \\
 &= \frac{9}{5} - \frac{5(3/5)^2}{2} + \frac{5}{2} - \frac{5(3/5)^2}{2} - 3 + 3(3/5) \\
 &= \frac{9}{5} - \frac{9(5/5)}{2} + \frac{5}{2} - \frac{9(5/5)}{2} - 3 + \frac{9}{5} \\
 &= \frac{9}{5} - \frac{9}{10} + \frac{5}{2} - \frac{9}{10} - 3 + \frac{9}{5} \\
 &= \frac{18}{10} - \frac{9}{10} + \frac{25}{10} - \frac{9}{10} - \frac{30}{10} + \frac{18}{10} \\
 &= \frac{13}{10}
 \end{aligned}$$

(vii) $\int_0^{\sin^2 x} \sin^{-1} \sqrt{t} dt + \int_0^{\cos^2 x} \cos^{-1} \sqrt{t} dt$

$$\int u dv = uv - \int v du$$

Sol. $\int_0^{\sin^2 x} \sin^{-1} \sqrt{t} dt + \int_0^{\cos^2 x} \cos^{-1} \sqrt{t} dt$

$$\begin{aligned}
 &\left[t \sin^{-1}(\sqrt{t}) \right]_0^{\sin^2 x} - \int_0^{\sin^2 x} t \cdot \frac{1}{\sqrt{1-t}} \cdot \frac{1}{2\sqrt{t}} dt + \left[t \cos^{-1} \sqrt{t} \right]_0^{\cos^2 x} \\
 &\quad + \int_0^{\cos^2 x} t \cdot \frac{1}{\sqrt{1-t}} \cdot \frac{1}{2\sqrt{t}} dt
 \end{aligned}$$

$$x \sin^2 x + x \cos^2 x + \int \frac{\sqrt{t}}{2\sqrt{1-t}} dt + \int \frac{\sqrt{t}}{2\sqrt{1-t}} dt$$

$$x \sin^2 x + \cos^2 x + \frac{1}{2} \int \frac{\sqrt{t}}{2\sqrt{1-t}} dt$$

$$x(1) + \frac{1}{2} \int \frac{\sqrt{t}}{2\sqrt{1-t}} dt$$

$$x + \frac{1}{2} \int \frac{\sqrt{t}}{2\sqrt{1-t}} dt \rightarrow \textcircled{1}$$

$$t = \sin^2 \theta$$

$$dt = 2 \sin \theta \cdot \cos \theta d\theta$$

$$\sin^2 x = \sin^2 \theta$$

$$x = \theta$$

$$\cos^2 x = \sin^2 \theta$$

$$\sin^2(\pi/2 - x) = \sin^2 \theta$$

$$\theta = \pi/2 - x$$

$$= \frac{1}{2} \int_x^{\pi/2 - x} \frac{\sin \theta}{\sqrt{\cos^2 \theta}} 2 \sin \theta \cos \theta d\theta$$

$$= \int_x^{\pi/2 - x} \sin^2 \theta d\theta$$

$$= \int_x^{\pi/2 - x} \frac{1 - \cos 2\theta}{2} d\theta$$

$$= \frac{1}{2} \left(\theta - \frac{\sin 2\theta}{2} \right)_x^{\pi/2 - x}$$

$$= \frac{1}{2} \left(\frac{\pi}{2} - x - \frac{\sin(\pi - 2x)}{2} - x + \frac{\sin 2x}{2} \right)$$

$$= \frac{1}{2} \left(\frac{\pi}{2} - x - \frac{\sin 2x}{2} - x + \frac{\sin 2x}{2} \right)$$

$$= \frac{1}{2} (\pi/2 - 2x)$$

$$= \frac{\pi}{4} - x.$$

sub in eqn ①

$$x + \frac{\pi}{4} - x$$

$$= \pi/4$$

viii) $\int_0^1 \frac{\log(1+x)}{1+x^2} dx.$

$$= \int_0^1 \frac{\log(1+x)}{1+x^2} dx.$$

let $x = \tan \theta$

$$dx = \sec^2 \theta \cdot d\theta$$

x	0	1
θ	0	$\pi/4$

$$= \int_0^{\pi/4} \frac{\log(1+\tan \theta)}{1+\tan^2 \theta} \sec^2 \theta d\theta$$

$$= \int_0^{\pi/4} \frac{\log(1+\tan \theta)}{\cancel{\sec^2 \theta}} \cancel{\sec^2 \theta} d\theta$$

$$I = \int_0^{\pi/4} \log(1+\tan \theta) d\theta \rightarrow ①$$

By property

$$\int_0^a f(x) dx = \int_0^a f(a-x) dx$$

$$= \int_0^{\pi/4} \log (1 + \tan (\frac{\pi}{4} - \theta)) d\theta$$

$$= \int_0^{\pi/4} \log \left(1 + \frac{\tan \frac{\pi}{4} - \tan \theta}{1 + \tan \frac{\pi}{4} \tan \theta} \right) d\theta$$

$$= \int_0^{\pi/4} \log \left(1 + \frac{1 - \tan \theta}{1 + \tan \theta} \right) d\theta$$

$$= \int_0^{\pi/4} \log \left(\frac{1 + \tan \theta + 1 - \tan \theta}{1 + \tan \theta} \right) d\theta$$

$$I = \int_0^{\pi/4} \log \left(\frac{2}{1 + \tan \theta} \right) d\theta \quad \rightarrow \textcircled{2}$$

① + ②

$$2I = \int_0^{\pi/4} \log (1 + \tan \theta) \left(\frac{2}{1 + \tan \theta} \right) d\theta$$

$$2I = \int_0^{\pi/4} \log 2 \cdot d\theta$$

$$2I = \log_2 \left(\frac{\pi}{4} \right)$$

$$2I = \log_2 \left(\frac{\pi}{4} \right)$$

$$I = \frac{\pi}{8} \log_2$$

$$\therefore \int_0^1 \frac{\log(1+x)}{1+x^2} dx = \frac{\pi}{8} \log_2$$

ix)

$$\int_0^{\pi} \frac{x \sin x}{1 + \sin x} dx$$

sol

$$f(x) = \frac{\sin x}{1 + \sin x}$$

$$f(\pi - x) = \frac{\sin(\pi - x)}{1 + \sin(\pi - x)}$$

$$f(\pi - x) = \frac{\sin x}{1 + \sin x} = f(x)$$

$$\int_0^a x f(x) dx = \frac{a}{2} \int_0^a f(x) dx$$

if $f(a-x) = f(x)$

$$\int_0^{\pi} x \frac{\sin x}{1 + \sin x} dx = \frac{\pi}{2} \int_0^{\pi} \frac{\sin x}{1 + \sin x} dx$$

$$= \pi \int_0^{\pi/2} \frac{\sin x}{1 + \sin x} dx$$

$$= \pi \int_0^{\pi/2} \frac{\sin(\pi/2 - x)}{1 + \sin(\pi/2 - x)} dx$$

$$\int_0^{\pi} g(\sin x) dx = 2 \int_0^{\pi/2} g(\sin x) dx$$

$$\int_0^a f(x) dx = \int_0^a f(a-x) dx$$

$$= \pi \int_0^{\pi/2} \frac{\cos x}{1 + \cos x} dx \cdot \frac{x^{1-\cos x}}{1-\cos x}$$

$$= \pi \int_0^{\pi/2} \frac{\cos x - \cos^2 x}{1 - \cos^2 x} dx$$

$$= \pi \int_0^{\pi/2} \frac{\cos x}{\sin^2 x} - \frac{\cos^2 x}{\sin^2 x} dx$$

$$= \pi \int_0^{\pi/2} (\cot x \operatorname{cosec} x - \cot^2 x) dx$$

$$= \pi \int_0^{\pi/2} (\cot x \operatorname{cosec} x + 1 - \operatorname{cosec}^2 x) dx$$

$$= \pi \left[-\operatorname{cosec} x + x + \cot x \right]_0^{\pi/2}$$

$$= \pi \left[-1 + \frac{\pi}{2} + 0 - 0 \right]$$

$$= \pi \left(\frac{\pi-2}{2} \right)$$

$$= \frac{\pi}{2} (\pi-2)$$

$$\int_0^{\pi} \frac{x \sin x}{1 + \sin x} dx = \frac{\pi}{2} (\pi-2)$$

$$\int_0^a f(x) dx$$

$$f(x) = f(x)$$

$$dx =$$

$$\int_0^a f(\sin x) dx$$

$$\int_0^a f(a-x) dx$$

7)

$$\int_{\pi/8}^{3\pi/8} \frac{1}{1 + \sqrt{\tan x}} dx$$

$$I = \int_{\pi/8}^{3\pi/8} \frac{1}{1 + \sqrt{\tan x}} dx \rightarrow \textcircled{1}$$

$$\int_a^b f(x) dx = \int_a^b f(a+b-x) dx$$

$$I = \int_{\pi/8}^{3\pi/8} \frac{1}{1 + \sqrt{\tan(\frac{\pi}{2} - x)}} dx$$

$$x \Rightarrow \frac{\pi}{8} + \frac{3\pi}{8} - x$$

$$= \frac{4\pi}{8} - x$$

$$x = \frac{\pi}{2} - x$$

$$= \int_{\pi/8}^{3\pi/8} \frac{1}{1 + \sqrt{\cot x}} dx$$

$$= \int_{\pi/8}^{3\pi/8} \frac{1}{1 + \frac{1}{\sqrt{\tan x}}} dx$$

$$I = \int_{\pi/8}^{3\pi/8} \frac{\sqrt{\tan x}}{1 + \sqrt{\tan x}} dx \rightarrow \textcircled{2}$$

① + ②

$$2I = \int_{\pi/8}^{3\pi/8} \frac{1 + \sqrt{\tan x}}{1 + \sqrt{\tan x}} dx$$

$$2I = \int_{\pi/8}^{3\pi/8} dx$$

$$2I = \left[x \right]_{\pi/8}^{3\pi/8} = I$$

$$2I = \frac{3\pi}{8} - \frac{\pi}{8}$$

$$2I = \frac{2\pi}{8}$$

$$I = \pi/8$$

$$\int_{\pi/8}^{3\pi/8} \frac{1}{1+\sqrt{\tan x}} dx = \pi/8$$

$$x_i) \int_0^{\pi} x [\sin^2(\sin x) + \cos^2(\cos x)] dx.$$

$$\text{Sol} \int_0^{\pi} x [\sin^2(\sin x) + \cos^2(\cos x)] dx$$

$$= \frac{\pi}{2} \int_0^{\pi} [\sin^2(\sin x) + \cos^2(\cos x)] dx$$

$$f(x) = \sin^2(\sin x) + \cos^2(\cos x)$$

$$f(\pi-x) = \sin^2(\sin(\pi-x)) + \cos^2(\cos(\pi-x))$$

$$f(\pi-x) = \sin^2(\sin x) + \cos^2(\cos x)$$

$$f(\pi-x) = f(x)$$

$$= \frac{\pi}{2} \cdot 2 \int_0^{\pi/2} [\sin^2(\sin x) + \cos^2(\cos x)] dx$$

$$I = \pi \int_0^{\pi/2} [\sin^2(\sin x) + \cos^2(\cos x)] dx \rightarrow \textcircled{1}$$

$$I = \pi \int_0^{\pi/2} [\sin^2(\sin(\frac{\pi}{2}-x)) + \cos^2(\cos(\frac{\pi}{2}-x))] dx$$

$$I = \pi \int_0^{\pi/2} [\sin^2(\cos x) + \cos^2(\sin x)] dx \rightarrow (2)$$

(1) + (2)

$$2I = \pi \int_0^{\pi/2} 1 dx$$

$$I = \pi [x]_0^{\pi/2}$$

$$\int_0^{\pi/2} x [\sin^2(\sin x) + \cos^2(\cos x)] dx = \frac{\pi^2}{2}$$

Bernoulli's formula.

$$\int u dv = uv - u'v_1 - u''v_2 - u'''v_3 + u^{(4)}v_4 \dots$$

$u', u'', u''', \dots$ successive differentiation

$v_1, v_2, v_3, \dots$ successive integration

① ←

Evaluate the following:

$$i) \int_0^1 x^3 e^{-2x} dx.$$

Sol.

$$u = x^3$$

$$u' = 3x^2$$

$$u'' = 6x$$

$$u''' = 6$$

$$u^{iv} = 0$$

$$dv = e^{-2x} dx$$

$$v = -\frac{e^{-2x}}{2}$$

$$v_1 = \frac{e^{-2x}}{4}$$

$$v_2 = -\frac{e^{-2x}}{8}$$

$$v_3 = \frac{e^{-2x}}{16}$$

$$\int u dv = uv - u'v_1 + u''v_2 - u'''v_3 + \dots$$

$$\int_0^1 x^3 \cdot e^{-2x} dx = \left[\frac{-x^3 e^{-2x}}{2} - \frac{3x^2 e^{-2x}}{4} - \frac{6x e^{-2x}}{8} - \frac{6e^{-2x}}{16} \right]_0^1$$

$$= \left[-\frac{e^{-2x}}{2} \left(x^3 + \frac{3x^2}{2} + \frac{3x}{2} + \frac{3}{4} \right) \right]_0^1$$

$$= -\frac{e^{-2}}{2} \left[1 + \frac{3}{2} + \frac{3}{2} + \frac{3}{4} \right] + \left(\frac{1}{2} \times \frac{3}{4} \right)$$

$$= -\frac{e^{-2}}{2} \left[\frac{19}{4} \right] + \frac{3}{8}$$

$$\int_0^1 x^3 \cdot e^{-2x} dx = \frac{3}{8} - \frac{19}{8} e^{-2}$$

8)

$$\int_0^1 \frac{\sin(3 \tan^{-1} x) \tan^{-1} x}{1+x^2} dx.$$

Sol.

$$t = \tan^{-1} x$$

$$dt = \frac{1}{1+x^2} dx$$

x	0	1
t	0	$\pi/4$

$$\int_0^{\pi/4} t \sin 3t \cdot dt$$

$$u = t$$

$$\int dv = \int \sin 3t \cdot dt$$

$$u' = 1$$

$$v = -\frac{\cos 3t}{3}$$

$$u'' = 0$$

$$v_1 = -\frac{\sin 3t}{9}$$

$$\int u dv = uv - u'v_1 + u''v_2 \dots \dots \dots$$

$$= \left[-t \frac{\cos 3t}{3} + \frac{\sin 3t}{9} \right]_0^{\pi/4}$$

$$= \left[-\frac{\pi}{12} \cos\left(\frac{3\pi}{4}\right) + \frac{\sin\left(\frac{3\pi}{4}\right)}{9} \right]$$

$$= \left[-\frac{\pi}{12} \cos\left(\pi - \frac{\pi}{4}\right) + \frac{1}{9} \sin\left(\pi - \frac{\pi}{4}\right) \right]$$

$$= -\frac{\pi}{12} (-\cos \frac{\pi}{4}) + \frac{1}{9} \sin \frac{\pi}{4}$$

$$= \frac{\pi}{12} \cos \frac{\pi}{4} + \frac{1}{9} \sin \frac{\pi}{4}$$

$$= \frac{\pi}{12} \left(\frac{1}{\sqrt{2}} \right) + \frac{1}{9} \left(\frac{1}{\sqrt{2}} \right)$$

$$= \frac{1}{\sqrt{2}} \left(\frac{\pi}{12} + \frac{1}{9} \right)$$

$$\int_0^1 \frac{\sin(3 \tan^{-1} x) \tan^{-1} x}{1+x^2} dx = \frac{1}{\sqrt{2}} \left(\frac{\pi}{12} + \frac{1}{9} \right)$$

$$3) \int_0^{1/\sqrt{2}} \frac{e^{\sin^{-1} x} \sin^{-1} x}{\sqrt{1-x^2}} dx$$

sol $t = \sin^{-1} x$

x	0	$1/\sqrt{2}$
t	0	$\pi/4$

$$dt = \frac{1}{\sqrt{1-x^2}} dx$$

$$= \int_0^{\pi/4} e^t \cdot t \cdot dt$$

$$\begin{array}{l|l} u = t & \int dv = \int e^t \cdot dt \\ u' = 1 & v = e^t \\ u'' = 0 & v_1 = e^t \end{array}$$

$$\int u dv = uv - u'v_1 + u''v_2 \dots \dots$$

$$= \left[t \cdot e^t - e^t \right]_0^{\pi/4}$$

$$= \frac{\pi}{4} e^{\pi/4} - e^{\pi/4} + 1$$

$$= 1 + e^{\pi/4} \left(\frac{\pi}{4} - 1 \right)$$

$$(iv) \int_0^{\pi/2} x^2 \cos 2x \, dx.$$

Sol

$$\int_0^{\pi/2} x^2 \cos 2x \, dx.$$

$$u = x^2$$

$$u' = 2x$$

$$u'' = 2$$

$$u''' = 0$$

$$\int dv = \int \cos 2x \, dx$$

$$v = \frac{\sin 2x}{2}$$

$$v_1 = -\frac{\cos 2x}{4}$$

$$v_2 = -\frac{\sin 2x}{8}$$

$$\int u \, dv = uv - u'v_1 + u''v_2 - u'''v_3 \dots$$

$$= \left[x^2 \frac{\sin 2x}{2} + \frac{2x \cdot \cos 2x}{4} - \frac{\sin 2x}{8} \right]_0^{\pi/2}$$

$$= \left[0 - \frac{\pi}{4} + 0 \right]$$

$$= -\frac{\pi}{4}$$

$$\int_0^{\pi/2} x^2 \cos 2x \, dx = -\frac{\pi}{4}$$

Improper Integrals

$$* \int_0^{\infty} \frac{1}{a^2+x^2} dx = \frac{\pi}{2a}$$

$$* \int_a^{\infty} \frac{1}{a^2+x^2} dx = \frac{\pi}{4a}$$

$$* \int_{-\infty}^{\infty} \frac{1}{a^2+x^2} dx = \frac{\pi}{a} \quad \text{since } \frac{1}{a^2+x^2} \text{ is even function.}$$

Ex: 9.5

1. Evaluate the following:

$$i) \int_0^{\pi/2} \frac{dx}{1+5\cos^2 x}$$

Sol

$$= \int_0^{\pi/2} \frac{dx}{1+5\cos^2 x}$$

$$= \int_0^{\pi/2} \frac{1}{\cos^2 x \left(\frac{1}{\cos^2 x} + 5 \right)} dx$$

$$= \int_0^{\pi/2} \frac{\sec^2 x}{\sec^2 x + 5} dx$$

V. MURUGAN

PG ASST IN MATH

7904171615

Improper Integrals

$$* \int_0^{\infty} \frac{1}{a^2+x^2} dx = \frac{\pi}{2a}$$

$$* \int_a^{\infty} \frac{1}{a^2+x^2} dx = \frac{\pi}{4a}$$

$$* \int_{-\infty}^{\infty} \frac{1}{a^2+x^2} dx = \frac{\pi}{a} \quad \text{since } \frac{1}{a^2+x^2} \text{ is even function.}$$

Ex: 9.5

1. Evaluate the following:

$$i) \int_0^{\pi/2} \frac{dx}{1+5\cos^2 x}$$

Sol

$$= \int_0^{\pi/2} \frac{dx}{1+5\cos^2 x}$$

$$= \int_0^{\pi/2} \frac{1}{\cos^2 x \left(\frac{1}{\cos^2 x} + 5 \right)} dx$$

$$= \int_0^{\pi/2} \frac{\sec^2 x}{\sec^2 x + 5} dx$$

Y. MURUGAN

PG ASST IN MATHS

7904171615

$$= \int_0^{\pi/2} \frac{\sec^2 x}{\tan^2 x + 6} dx$$

$$t = \tan x$$

x	0	$\pi/2$
t	0	∞

$$dt = \sec^2 x dx$$

$$= \int_0^{\infty} \frac{dt}{t^2 + (\sqrt{6})^2}$$

$$\therefore \int_0^{\infty} \frac{1}{a^2 + x^2} dx = \frac{\pi}{2a}$$

$$= \frac{\pi}{2\sqrt{6}}$$

$$\therefore \int_0^{\pi/2} \frac{dx}{1 + 5 \cos^2 x} = \frac{\pi}{2\sqrt{6}}$$

$$\text{ii)} \int_0^{\pi/2} \frac{dx}{5 + 4 \sin^2 x}$$

sol

$$= \int_0^{\pi/2} \frac{dx}{\sin^2 x \left(\frac{5}{\sin^2 x} + 4 \right)}$$

$$= \int_0^{\pi/2} \frac{\operatorname{cosec}^2 x dx}{5 \operatorname{cosec}^2 x + 4}$$

$$= \int_0^{\pi/2} \frac{\operatorname{cosec}^2 x \cdot dx}{5(\cot^2 x + 9/5)}$$

Y. MURUGAN

PG ASST IN MATHS

7904171615

$$dt = -\operatorname{cosec}^2 x \cdot dx$$

$$= \frac{1}{5} \int_8^0 \frac{-dt}{t^2 + \left(\frac{3}{\sqrt{5}}\right)^2}$$

x	0	$\pi/2$
t	∞	0

$$= \frac{1}{5} \int_0^{\infty} \frac{dt}{t^2 + \left(\frac{3}{\sqrt{5}}\right)^2}$$

$$= \frac{1}{5} \cdot \frac{\pi}{2\left(\frac{3}{\sqrt{5}}\right)}$$

$$= \frac{\pi}{5\left(\frac{6}{\sqrt{5}}\right)}$$

$$= \frac{\pi}{6\sqrt{5}}$$

$$\therefore \int_0^{\pi/2} \frac{dx}{5 + 4\sin^2 x} = \frac{\pi}{6\sqrt{5}}$$

V. MURUGAN

PG ASST IN MATHS

7904171615

R.F-I $\Rightarrow$ If $I_n = \int_0^{\pi/2} \sin^n x dx$, then $I_n = \frac{(n-1)}{n} I_{n-2}$, $n \geq 2$

R.F-II : If $I_n = \int_0^{\pi/2} \cos^n x dx$, then $I_n = \frac{(n-1)}{n} I_{n-2}$, $n \geq 2$

R.F-III : If $I_{m,n} = \int_0^{\pi/2} \sin^m x \cos^n x dx$,
then $I_{m,n} = \frac{(n-1)}{m+n} I_{m,n-2}$, $n \geq 2$

R.F-IV : If $I_{m,n} = \int_0^1 x^m (1-x)^n dx$, then
 $I_{m,n} = \frac{n}{m+n+1} I_{m,n-1}$, $n \geq 1$

By using the reduction formula I and II

$$\int_0^{\pi/2} \sin^n x dx = \int_0^{\pi/2} \cos^n x dx = \begin{cases} \frac{(n-1)}{n} \times \frac{(n-3)}{(n-2)} \times \dots \times \frac{1}{2} \times \frac{\pi}{2}, & \text{if } n=2,4 \\ \frac{(n-1)}{n} \times \frac{(n-3)}{(n-2)} \times \dots \times \frac{2}{3}, & \text{if } n=3,5 \end{cases}$$

By applying the reduction formula III iteratively, we get the following result.

$$\int_0^{\pi/2} \sin^m x \cos^n x dx = \frac{(n-1)}{(m+n)} \frac{(n-3)}{(m+n-2)} \frac{(n-5)}{(m+n-4)} \dots$$

$$\frac{1}{(m+2)} \frac{(m-1)}{m} \frac{(m-3)}{(m-2)} \frac{(m-5)}{(m-4)} \dots \frac{1}{2} \frac{\pi}{2}$$

(ii) If n is odd and m is any positive integer

(even or odd) then.

$$\int_0^{\pi/2} \sin^m x \cos^n x dx = \frac{(n-1)}{(m+n)} \frac{(m-3)}{(m+n-2)} \frac{(n-5)}{(m+n-4)} \dots \frac{2}{(m+3)} \frac{1}{(m+1)}$$

Note:

If m is odd and n is even.

$$\int_0^{\pi/2} \sin^m x \cos^n x dx = \int_0^{\pi/2} \sin^n x \cos^m x dx = \frac{(m-1)}{(n+m)} \frac{(m-3)}{(n+m-2)} \frac{(m-5)}{(n+m-4)} \dots \frac{2}{(n+3)} \frac{1}{(n+1)}$$

Ex: 9.6

Evaluate the following

i) $\int_0^{\pi/2} \sin^{10} x dx$

Sol $\int_0^{\pi/2} \sin^{10} x dx$ is positive (even)

Formula :

$$\int_0^{\pi/2} \sin^n x dx = \left\{ \frac{n-1}{n} \times \frac{n-3}{n-2} \times \dots \times \frac{1}{2} \times \frac{\pi}{2} \right\} \text{ if } n=2$$

$n=10$

$$= \frac{10-1}{10} \times \frac{10-3}{10-2} \times \dots \times \frac{1}{2} \times \frac{\pi}{2}$$

$$= \frac{9}{10} \times \frac{7}{8} \times \frac{5}{6} \times \frac{3}{4} \times \frac{1}{2} \times \frac{\pi}{2}$$

$$= \frac{63\pi}{512}$$

$$\int_0^{\pi/2} \cos^7 x \, dx.$$

Sol

$$\int_0^{\pi/2} \cos^7 x \, dx \quad (\text{it is odd})$$

Formula

$$\int_0^{\pi/2} \cos^n x \, dx = \frac{n-1}{n} \times \frac{n-3}{n-2} \times \frac{n-5}{n-4} \times \dots \times \frac{2}{3}$$

$n=7$

$$= \frac{7-1}{7} \times \frac{7-3}{7-2} \times \frac{7-5}{7-4} \times \frac{2}{3}$$

$$= \frac{6}{7} \times \frac{4}{5} \times \frac{2}{3}$$

$$= \frac{16}{35}$$

iii)

$$\int_0^{\pi/4} \sin^6 x \, dx.$$

Sol.

$$2x = t$$

$$dt = 2dx$$

x	0	$\pi/4$
t	0	$\pi/2$

$$= \int_0^{\pi/2} \sin^6 t \cdot \frac{dt}{2}$$

$$= \frac{1}{2} \int_0^{\pi/2} \sin^6 t \cdot dt$$

even formula = $\frac{n-1}{n} \times \frac{n-3}{n-2} \times \dots \times \frac{1}{2} \times \frac{\pi}{2}$

$$= \frac{1}{2} \times \frac{6-1}{6} \times \frac{6-3}{6-2} \times \frac{6-5}{6-4} \times \frac{\pi}{2}$$

$$= \frac{1}{2} \times \frac{5}{6} \times \frac{3}{4} \times \frac{1}{2} \times \frac{\pi}{2}$$

$$= \frac{5\pi}{64}$$

iv)

$$\int_0^{\pi/6} \sin^5 3x \cdot dx$$

sol

$$\int_0^{\pi/6} \sin^5 3x \cdot dx$$

$$3x = t$$

$$3dx = dt$$

$$dx = \frac{dt}{3}$$

x	0	$\pi/6$
t	0	$\pi/2$

$$= \int_0^{\pi/2} \sin^5 t \cdot \frac{dt}{3}$$

$$= \frac{1}{3} \int_0^{\pi/2} \sin^5 t \cdot dt$$

odd formula = $\int_0^{\pi/2} \sin^n x \cdot dx = \frac{n-1}{n} \times \frac{n-3}{n-2} \times \dots \times \frac{2}{3}$

$$= \frac{1}{3} \left(\frac{5-1}{5} \times \frac{5-3}{5-2} \right)$$

$$\frac{\pi}{2} \times \frac{1}{3} \times \dots = \frac{1}{3} \times \frac{4}{5} \times \frac{2}{3}$$

$$\frac{\pi}{8} \times \frac{2-1}{2-0} = \frac{1}{3} \times \frac{8}{15} \times \frac{1}{2}$$

$$\frac{\pi}{2} \times \frac{1}{3} \times \frac{8}{15} \times \frac{1}{2}$$

v) $\int_0^{\pi/2} \sin^2 x \cos^4 x dx$

Sol

$$\int_0^{\pi/2} \sin^2 x \cos^4 x dx$$

Formula for even

$$\int_0^{\pi/2} \sin^m x \cos^n x dx = \frac{n-1}{m+n} \cdot \frac{n-3}{(m+n-2)} \cdot \frac{n-5}{(m+n-4)} \dots \frac{1}{(m+2)}$$

$$\frac{(m-1)}{m} \cdot \frac{(m-3)}{(m-2)} \cdot \frac{(m-5)}{(m-4)} \cdot \frac{1}{2} \cdot \frac{\pi}{2}$$

$$m=2, n=4$$

$$= \frac{4-1}{2+4} \cdot \frac{4-3}{2+4-2} \dots \frac{4-5}{2+4-4} \cdot \frac{1}{2+2} \cdot \frac{2-1}{2} \cdot \frac{2-3}{2-4} \cdot \frac{1}{2} \cdot \frac{\pi}{2}$$

$$= \frac{3}{6} \cdot \frac{1}{4} \cdot \frac{1}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2}$$

$$= \frac{\pi}{128}$$

$$\int_0^{2\pi} \sin^7 \frac{x}{4} dx$$

Sol.

$$\int_0^{2\pi} \sin^7 \frac{x}{4} dx$$

$$t = \frac{x}{4}$$

$$4t = x$$

$$dt = dx$$

x	0	2π
t	0	π/2

odd formula

$$\int_0^{\pi/2} \sin^n x dx = \frac{(n-1)}{n} \times \frac{(n-3)}{(n-2)} \times \dots \times \frac{2}{3} \text{ if } n=3,5$$

$$= 4 \left(\frac{7-1}{7} \times \frac{7-3}{7-2} \times \frac{7-5}{7-4} \right)$$

$$= 4 \left(\frac{6}{7} \times \frac{4}{5} \times \frac{2}{3} \right)$$

$$= \frac{64}{35}$$

$$\int_0^{\pi/2} \sin^3 \theta \cos^5 \theta d\theta$$

Sol

$$\int_0^{\pi/2} \sin^3 \theta \cos^5 \theta d\theta$$

Formula for odd or even in n power

It is odd

$$\int_0^{\pi/2} \sin^m x \cos^n x dx = \frac{n-1}{(m+n)} \cdot \frac{n-3}{(m+n-2)} \cdots \frac{n-5}{(m+n-4)} \cdots \frac{2}{(m+3)} \cdot \frac{1}{(m+1)}$$

$$n=5, m=3$$

$$= \frac{5-1}{5+3} \cdots \frac{5-3}{5+3-2} \cdots \frac{5-5}{(3+5-4)} \cdots \frac{2}{3+3} \cdots \frac{1}{3+1}$$

$$= \left(\frac{4}{8} \cdot \frac{2}{6} \cdot \frac{2}{6} \cdot \frac{1}{4} \right)$$

$$= \frac{8}{48} \cdot \frac{2}{6} \cdot \frac{1}{4}$$

$$= \frac{1}{6} \cdot \frac{1}{3} \cdot \frac{1}{4}$$

$$= \frac{1}{72}$$

$$\int_0^1 x^2 (1-x)^3 \cdot dx$$

sol

$$\int_0^1 x^2 (1-x)^3 \cdot dx$$

formula $\int_0^1 x^m (1-x)^n dx = \frac{m! \times n!}{(m+n+1)!}$

$$= \frac{2! \times 3!}{(2+3+1)!}$$

$$= \frac{1 \times 2 \times 1 \times 2 \times 3}{1 \times 2 \times 3 \times 4 \times 5 \times 6}$$

$$= \frac{1}{6 \times 10}$$

$$= \frac{1}{60}$$

Gamma Integral

$$\int_0^{\infty} e^{-ax} \cdot x^n dx, n > 0 = \frac{n!}{a^{n+1}} \rightarrow \text{General form.}$$

$$n = \int_0^{\infty} e^{-x} x^{n-1} dx$$

$$\int_0^{\infty} \frac{x^n}{n^x} dx = \frac{n!}{(\log n)^{n+1}} \text{ where } n \geq 2.$$

Evaluate the following

i) $\int_0^{\infty} x^5 e^{-3x} dx$

Sol

formula :

$$\int_0^{\infty} e^{-ax} x^n dx = \frac{n!}{a^{n+1}}$$

$$\int_0^{\infty} x^5 e^{-3x} dx = \frac{5!}{3^{5+1}} = \frac{5!}{3^6}$$

ii) $\int_0^{\pi/2} \frac{e^{-\tan x}}{\cos^6 x} dx$

Sol

$$\int_0^{\pi/2} \frac{e^{-\tan x}}{\cos^6 x} dx = \int_0^{\pi/2} \sec^6 x \cdot e^{-\tan x} dx$$

$$= \int_0^{\pi/2} \sec^4 x \cdot e^{-\tan x} \cdot \sec^2 x dx$$

$$= \int_0^{\pi/2} (1 + \tan^2 x)^2 \cdot e^{-\tan x} \cdot \sec^2 x dx$$

$$t = \tan x$$

$$dt = \sec^2 x \, dx$$

x	0	$\pi/2$
t	0	∞

$$= \int_0^{\infty} (1+t^2)^2 \cdot e^{-t} \cdot dt$$

$$= \int_0^{\infty} (1+t^4+2t^2) e^{-t} \cdot dt$$

$$= \int_0^{\infty} e^{-t} dt + \int_0^{\infty} t^4 e^{-t} dt + \int_0^{\infty} 2t^2 e^{-t} dt$$

$$= \left[\frac{e^{-t}}{-1} \right]_0^{\infty} + \frac{4!}{1+1} + (2) \frac{2!}{1^2}$$

$$= e^0 + 24 + 4$$

$$= 1 + 28$$

$$= 29.$$

$$\int_0^{\pi/2} \frac{e^{-\tan x}}{\cos^6 x} dx = 29.$$

2) If $\int_0^{\infty} e^{-\alpha x^2} x^3 dx = 32$, $\alpha > 0$, find α

Sol.

$$\int_0^{\infty} e^{-\alpha x^2} x^3 dx = 32$$

Compare with general form.

$$\int_0^{\infty} e^{-\alpha x} x^n dx = \frac{n!}{\alpha^{n+1}}$$

$$t = x^2$$

$$dt = 2x dx$$

$$\frac{dt}{2} = x dx$$

x	0	∞
t	0	∞

$$\frac{1}{2} \int_0^{\infty} e^{-\alpha t} \cdot t dt = 32$$

$$\int_0^{\infty} e^{-\alpha t} \cdot t dt = 64$$

$$\frac{1!}{\alpha^{1+1}} = 64$$

$$\frac{1}{\alpha^2} = 64$$

$$\alpha^2 = \frac{1}{64}$$

$$\alpha = \frac{1}{8}$$

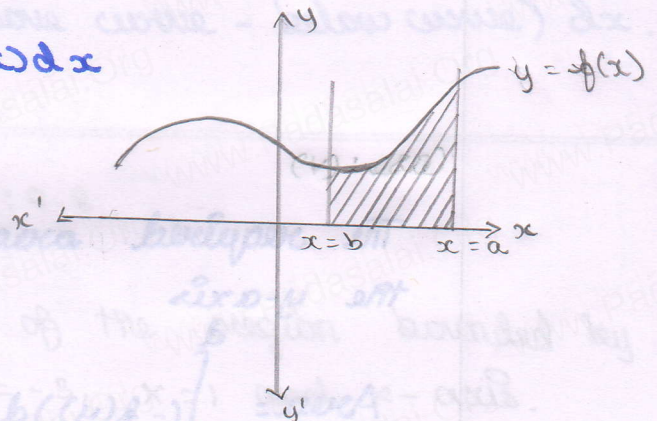
Evaluation of a Bounded plane area by integration.

Area of the region bounded by a curve, x -axis and the lines $x=a$ and $x=b$.

Case : (i)

Let $y = f(x)$, $a \leq x \leq b$ be the equation of the portion of the continuous curve that lies above the x -axis

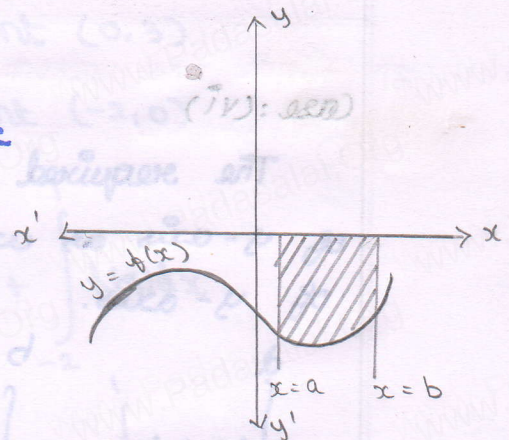
$$\text{Area} = \int_a^b f(x) dx$$



Case : (ii)

The required area lies below the x -axis

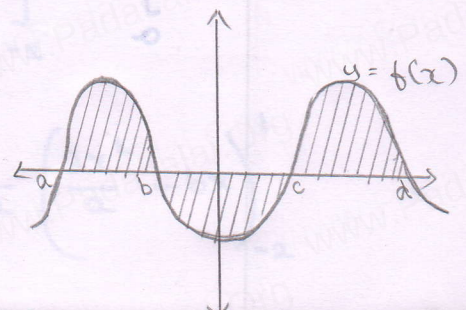
$$\text{Area} = \int_a^b (-f(x)) dx$$



Case : (iii)

The required areas lies above as well as below the x -axis

$$\int_a^d f(x) dx = \int_a^b f(x) dx - \int_b^c f(x) dx + \int_c^d f(x) dx$$

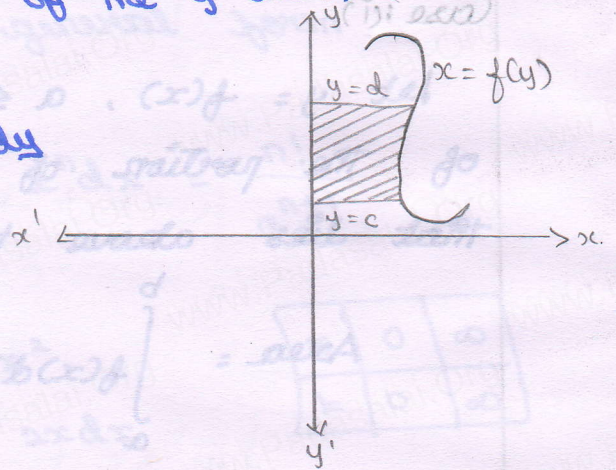


Area of the region bounded by a curve, y-axis and lines $y=c$ & $y=d$

Case: (iv)

Let $x = f(y)$, $c \leq y \leq d$ be the equation of the portion of the continuous curve that lies to the right side of the y-axis.

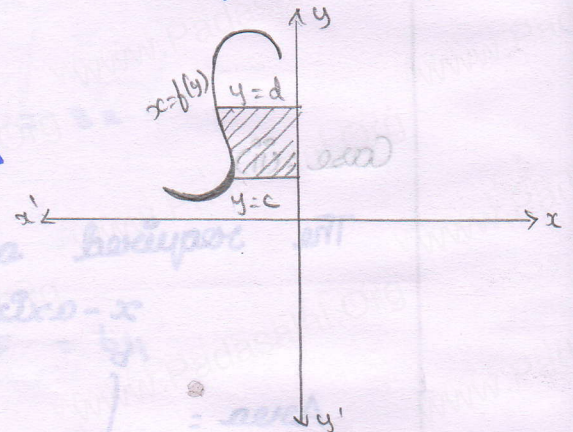
$$\text{Area} = \int_c^d f(y) dy$$



Case: (v)

The required area lies left side of the y-axis

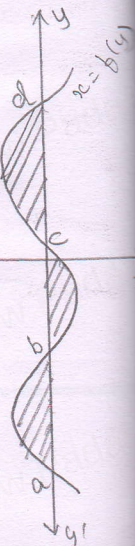
$$\text{Area} = \int_c^d (-f(y)) dy$$



Case: (vi)

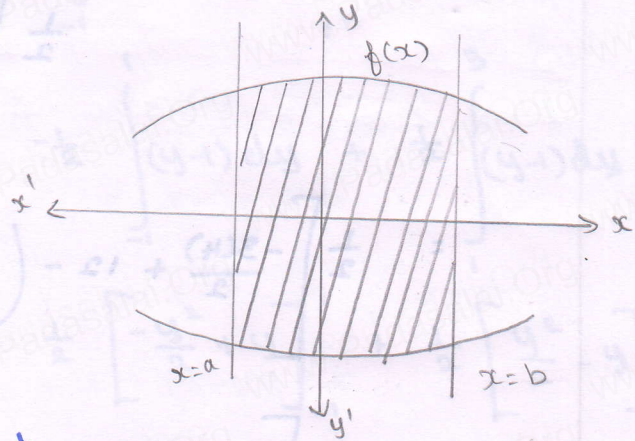
The required area lies to the left side of y-axis as well as to the right side of y-axis.

$$\int_a^d f(x) dx = \int_a^b f(x) dx + \int_b^c f(x) dx + \int_c^d f(x) dx$$



Area of the region bounded between two curves.

Case: 1



$$\text{Area} = \int_a^b (\text{above curve} - \text{below curve}) dx.$$

Ex: 9.8.

1. Find the area of the region bounded by $3x - 2y + 6 = 0$, $x = -3$, $x = 1$ and x -axis.

sol

$$3x - 2y + 6 = 0, \quad x = -3, \quad x = 1$$

$$y = \frac{1}{2}(3x + 6)$$

$$x = 0, \quad y = 3 \quad \text{The point } (0, 3)$$

$$y = 0, \quad x = -2 \quad \text{The point } (-2, 0)$$

$$\int_{-3}^1 f(x) dx = - \int_{-3}^{-2} f(x) dx + \int_{-2}^1 f(x) dx$$

$$= -\frac{1}{2} \int_{-3}^{-2} (3x + 6) dx + \frac{1}{2} \int_{-2}^1 (3x + 6) dx$$

$$= \frac{1}{2} \left(-\frac{3x^2}{2} - 6x \right)_{-3}^{-2} + \frac{1}{2} \left(\frac{3x^2}{2} + 6x \right)_{-2}^1$$

$$= \frac{1}{2} \left[\frac{-3(-2)^2}{2} - 6(-2) - \left(\frac{-3(-3)^2}{2} - 6(-3) \right) \right] + \frac{1}{2} \left[\frac{3(1)^2}{2} + 6(1) - \left(\frac{3(-2)^2}{2} + 6(-2) \right) \right]$$

$$= \frac{1}{2} \left[\frac{-3(4)}{2} + 12 - \left(\frac{-27}{2} + 18 \right) + \frac{3}{2} + 6 - \left(\frac{12}{2} - 12 \right) \right]$$

$$= \frac{1}{2} \left[-6 + 12 + \frac{27}{2} - 18 - 6 + 12 + \frac{3}{2} + 6 \right]$$

$$= \frac{1}{2} \left[\frac{27+3}{2} \right]$$

$$= \frac{1}{2} \left[\frac{30}{2} \right]$$

$$= 7.5 \text{ sq. units}$$

∴ The area of region is 7.5 sq units.

- 8) Find the area of the region bounded by $2x - y + 1 = 0$, $y = -1$, $y = 3$ and y -axis.

Sol

$$2x - y + 1 = 0, \quad y = -1, \quad y = 3, \quad y\text{-axis}$$

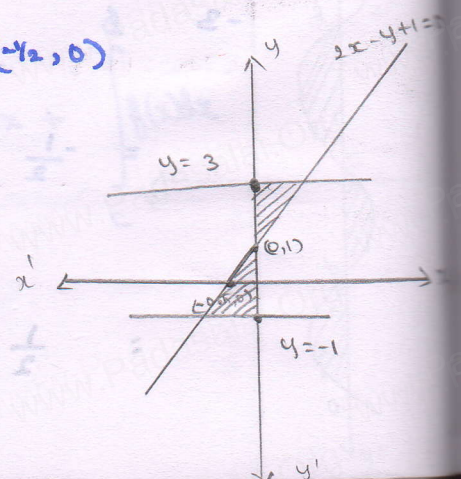
$$x = 0, \quad y = 1 \quad \therefore \text{The point } (0, 1)$$

$$y = 0, \quad x = -\frac{1}{2} \quad \therefore \text{The point } \left(-\frac{1}{2}, 0\right)$$

$$x = f(y)$$

$$2x = y - 1$$

$$x = \frac{1}{2}(y - 1)$$



$$\int_{-1}^3 f(y) dy = - \int_{-1}^1 f(y) dy + \int_1^3 f(y) dy$$

$$= -\frac{1}{2} \int_{-1}^1 (y-1) dy + \frac{1}{2} \int_1^3 (y-1) dy$$

$$= \frac{1}{2} \left[-\frac{y^2}{2} + y \right]_{-1}^1 + \frac{1}{2} \left[\frac{y^2}{2} - y \right]_1^3$$

$$= \frac{1}{2} \left[\left[-\frac{1}{2} + 1 + \frac{1}{2} + 1 \right] + \left[\frac{9}{2} - 3 - \frac{1}{2} + 1 \right] \right]$$

$$= \frac{1}{2} \left[-\frac{1}{2} + 1 + \frac{1}{2} + 1 + \frac{9}{2} - 3 - \frac{1}{2} + 1 \right]$$

$$= \frac{1}{2} \left[1 + 1 + 1 + \frac{9}{2} - 3 - \frac{1}{2} \right]$$

$$= \frac{1}{2} \left[\frac{8}{2} \right]$$

$$= \frac{4^2}{2}$$

$$= 2 \text{ sq. units}$$

∴ The area of the region is 2 sq. units

→ x

3) Find the area of the region bounded by the curve $2+x-x^2+y=0$, x -axis, $x=-3$ and $x=3$.

sol

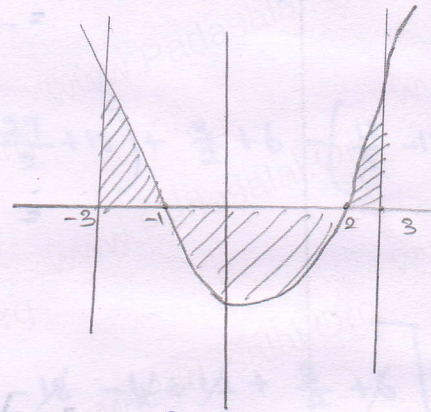
$$2+x-x^2+y=0, \quad x\text{-axis}, \quad x=-3, \quad x=3$$

$$y = x^2 - x - 2$$

$$y=0, \quad x^2 - x - 2 = 0$$

$$x = 2, -1$$

$$(2, 0) \quad (-1, 0)$$



$$\int_{-3}^3 f(x) dx = \int_{-3}^{-1} f(x) dx - \int_{-1}^2 f(x) dx + \int_2^3 f(x) dx$$

$$= \int_{-3}^{-1} (x^2 - x - 2) dx - \int_{-1}^2 (x^2 - x - 2) dx +$$

$$\int_2^3 (x^2 - x - 2) dx$$

$$= \left[\frac{x^3}{3} - \frac{x^2}{2} - 2x \right]_{-3}^{-1} - \left[\frac{x^3}{3} - \frac{x^2}{2} - 2x \right]_{-1}^2$$

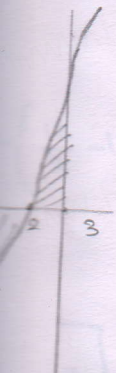
$$+ \left[\frac{x^3}{3} - \frac{x^2}{2} - 2x \right]_2^3$$

$$= \left[-\frac{1}{3} - \frac{1}{2} + 2 + \frac{27}{3} + \frac{9}{2} - 6 \right] - \left[\frac{8}{3} - \frac{4}{2} - 4 + \frac{1}{3} + \frac{1}{2} - 2 \right]$$

$$+ \left[\frac{27}{3} - \frac{9}{2} - 6 - \frac{8}{3} + \frac{4}{2} + 4 \right]$$

ded by
= -3 and

$$x = 3$$



x

dx +

dx

$$\left[-2x \right]_{-1}^2$$

$$\left[3 \right]_2$$

$$\left[\frac{1}{3} + \frac{1}{2} - 2 \right]$$

$$\left[\frac{1}{2} + 4 \right]$$

$$= -\frac{1}{3} - \frac{1}{2} + 2 + \frac{27}{3} + \frac{9}{2} - 6 - \frac{8}{3} + \frac{4}{2} + 4 - \frac{1}{3} - \frac{1}{2}$$

$$= 2 + \frac{27}{3} - \frac{9}{2} - 6 - \frac{8}{3} + \frac{4}{2} + 4$$

$$= -\frac{2}{3} - \frac{2}{2} + 8 + 8 + 18 - 12 - \frac{16}{3}$$

$$= -\frac{18}{3} - 1 + 16 + 18 - 12$$

$$= 16 + 16 + 18 - 13$$

$$= 34 - 19$$

$$= 15$$

$\therefore$ The area of the region is
15 sq. units.

4) Find the area of the region bounded by the line $y = 2x + 5$ and the parabola $y = x^2 - 2x$.

Sol.

$$y = 2x + 5, \text{ parabola } y = x^2 - 2x$$

$$2x + 5 = x^2 - 2x$$

$$x^2 - 2x - 2x - 5 = 0$$

$$x^2 - 4x - 5 = 0$$

$$(x+1)(x-5) \quad x = -1, 5$$

Sub x value in line $y = 2x + 5$

$$\boxed{x = -1}$$

$$y = -2 + 5$$

$$\boxed{y = 3}$$

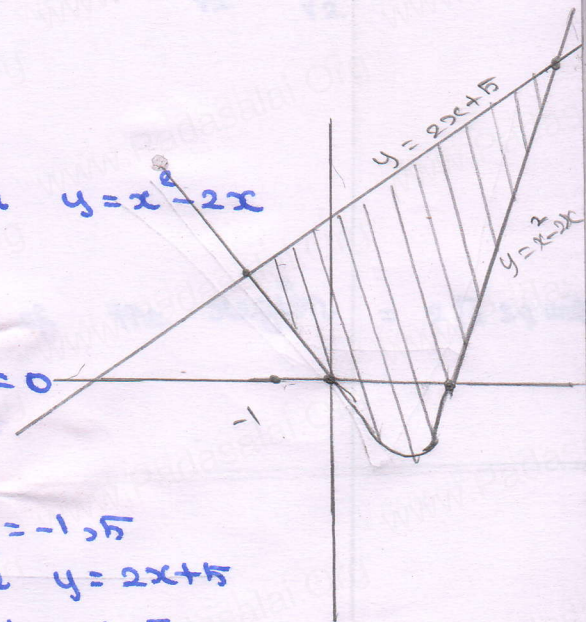
$\therefore$ The point $(-1, 3)$

$$x = 5$$

$$y = 10 + 5$$

$$y = 15$$

$\therefore$ The point $(5, 15)$



$$\int_{-1}^5 (\text{above curve} - \text{below curve}) dx =$$

$$\int_{-1}^5 [(2x+5) - (x^2-2x)] dx$$

$$\int_{-1}^5 (-x^2 + 4x + 5) dx$$

$$= \left[-\frac{x^3}{3} + \frac{4x^2}{2} + 5x \right]_{-1}^5$$

$$= \left[-\frac{x^3}{3} + 2x^2 + 5x \right]_{-1}^5$$

$$= -\frac{(5)^3}{3} + 2(5)^2 + 5(5) - \left[-\frac{(-1)^3}{3} + 2(-1)^2 + 5(-1) \right]$$

$$= -\frac{125}{3} + 50 + 25 - \left[-\frac{1}{3} - 2 + 5 \right]$$

$$= -\frac{126}{3} + 75 - 2 + 5$$

$$= -\frac{126}{3} + 80 - 2$$

$$= \frac{-126 + 240 - 6}{3}$$

$$= \frac{-126 - 6 + 240}{3}$$

$$= \frac{240 - 132}{3}$$

$$= \frac{108}{3}$$

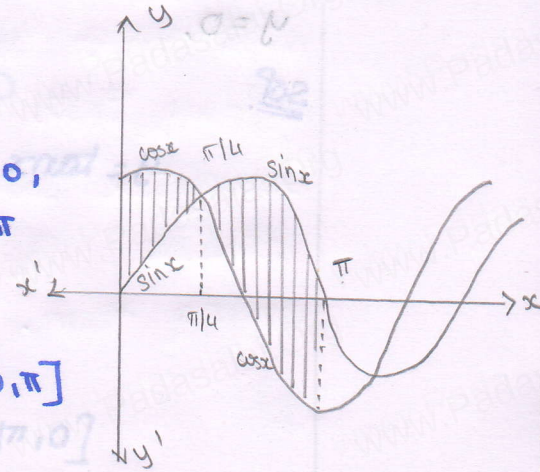
$$= 36 \text{ sq. units.}$$

the area of the region bounded between the curves $y = \sin x$ and $y = \cos x$ and the lines $x = 0$ and $x = \pi$.

Sol.

Given

$$y = \sin x, y = \cos x, x = 0, x = \pi$$



$$\sin x = \cos x$$

$$x = \pi/4 \quad \frac{\pi}{4} \in [0, \pi]$$

$$0 \text{ to } \pi/4 \rightarrow \cos x > \sin x$$

$$\pi/4 \text{ to } \pi \rightarrow \sin x > \cos x$$

$$\text{Required area} = \int_0^{\pi/4} (\cos x - \sin x) dx + \int_{\pi/4}^{\pi} (\sin x - \cos x) dx$$

$$= [\sin x + \cos x]_0^{\pi/4} + [-\cos x - \sin x]_{\pi/4}^{\pi}$$

$$= \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} - 1 + 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}$$

$$= \frac{4}{\sqrt{2}}$$

$$= 2\sqrt{2}$$

$\therefore$ The area of the region = $2\sqrt{2}$ sq units

6) Find the area of the region bounded by $y = \tan x$, $y = \cot x$ and the lines $x = 0$, $x = \frac{\pi}{2}$, $y = 0$.

Sol.

$$y = \tan x, \quad y = \cot x, \quad x = 0, x = \frac{\pi}{2}, y = 0$$

$$\tan x = \cot x$$

$$x = \frac{\pi}{4} \quad \frac{\pi}{4} \in [0, \frac{\pi}{2}]$$

$$[0, \frac{\pi}{4}] \rightarrow \cot x > \tan x$$

$$[\frac{\pi}{4}, \frac{\pi}{2}] \rightarrow \tan x > \cot x$$

$$\text{Area} = \int_0^{\frac{\pi}{4}} (\cot x - \tan x) dx + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} (\tan x - \cot x) dx$$

$$= (\log \sin x - \log \sec x) \Big|_0^{\frac{\pi}{4}} + (\log \sec x - \log \sin x) \Big|_{\frac{\pi}{4}}^{\frac{\pi}{2}}$$

$$= (\log \frac{1}{\sqrt{2}} - \log \sqrt{2} - \phi) + (\phi - \log \sqrt{2} + \log \frac{1}{\sqrt{2}})$$

$$= 2 \log \frac{1}{\sqrt{2}} - 2 \log \sqrt{2}$$

$$= 2 \log \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}}$$

$$= 2 \log \frac{1}{2}$$

$$= \log \left(\frac{1}{2}\right)^2$$

$$= \log \frac{1}{4}$$

the area of the region bounded by the parabola $y^2 = x$ and the line $y = x - 2$.

sol

$$y^2 = x, \quad y = x - 2 \rightarrow \textcircled{1}$$

$$(x - 2)^2 = x$$

$$x^2 + 4 - 4x - x = 0$$

$$x^2 - 5x + 4 = 0$$

$$(x - 1)(x - 4) = 0$$

$$x = 1, 4$$

Sub in $\textcircled{1}$

$$y = 1 - 2, \quad y = 4 - 2$$

$$\boxed{y = -1}$$

$$\boxed{y = 2}$$

$\therefore$ The point $(1, -1), (4, 2)$

$$\int_{-1}^2 (\text{above curve} - \text{below curve}) dx = \int_{-1}^2 (y + 2 - y^2) dy$$

$$= \int_{-1}^2 (-y^2 + y + 2) dy$$

$$= \left[-\frac{y^3}{3} + \frac{y^2}{2} + 2y \right]_{-1}^2$$

$$= -\frac{(2)^3}{3} + \frac{(2)^2}{2} + 2(2) - \left(-\frac{(-1)^3}{3} + \frac{(-1)^2}{2} - 2 \right)$$

$$= -\frac{8}{3} + 2 + 4 - \frac{1}{3} - \frac{1}{2} + 2$$

$$= -\frac{9}{3} - \frac{1}{2} + 8$$

$$= \frac{-18 - 3 + 48}{6}$$

$$= \frac{27}{6}$$

$$= \frac{9}{2}$$

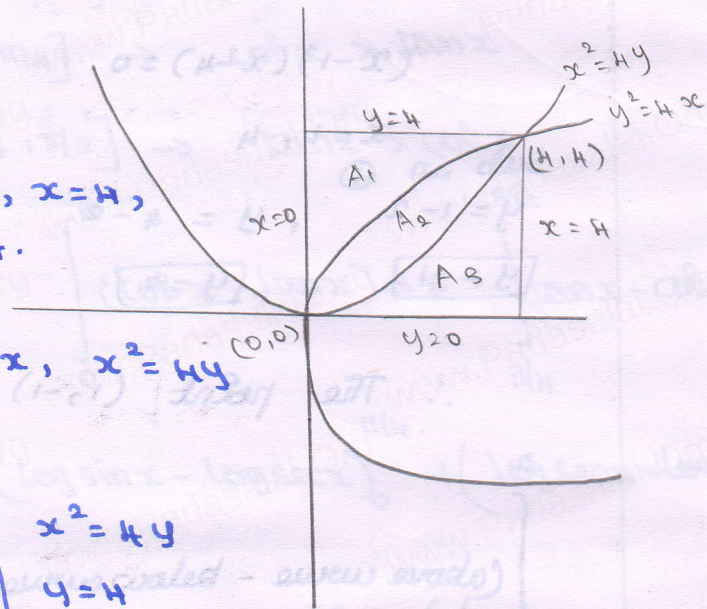
8) Father of a family wishes to divide his square field bounded by $x=0$, $x=h$, $y=h$ and $y=0$ along the curve $y^2=4x$ and $x^2=4y$ into three equal parts for his wife, daughter and son. Is it possible to divide? If so, find the area to be divided among them.

sol

Given

$$x=0, y=0, x=h, y=h.$$

Curve $y^2=4x, x^2=4y$



$$y^2=4x$$

$$x=h$$

$$y=\pm h$$

Point (h, h)

$$x^2=4y$$

$$y=h$$

$$x=\pm h$$

Point (h, h)

$$\begin{aligned} \text{Area of square} &= h \times h \\ &= 16 \text{ sq. units} \end{aligned}$$

$$\text{Area of } A_1 = \int_0^h (h - 2\sqrt{x}) dx$$

$$= \left[hx - \frac{2x^{3/2}}{3/2} \right]_0^h$$

$$= \left(16 - \frac{4(h)^{3/2}}{3} \right)$$

$$= \left(16 - \frac{32}{3} \right)$$

$$\text{Area of } A_1 = \frac{16}{3} \text{ sq. units} \rightarrow \textcircled{1}$$

$$\text{Area of } A_2 = \int_0^4 \left(2\sqrt{x} - \frac{x^2}{4} \right) dx$$

$$= \left[\frac{2x^{3/2}}{3/2} - \frac{x^3}{12} \right]_0^4$$

$$= \left[\frac{4 \times 8}{3} - \frac{64}{12} \right]$$

$$= \left[\frac{32}{3} - \frac{16}{3} \right]$$

$$\text{Area of } A_2 = \frac{16}{3} \rightarrow \textcircled{2}$$

$$\text{Area of } A_3 = \int_0^4 \left(\frac{x^2}{4} - 0 \right) dx$$

$$= \left[\frac{x^3}{12} \right]_0^4$$

$$= \frac{64}{12}$$

$$\text{Area of } A_3 = \frac{16}{3} \rightarrow \textcircled{3}$$

From $\textcircled{1}, \textcircled{2} \text{ \& } \textcircled{3}$

The area of square is equally divided by the curve $y^2 = 4x$ & $x^2 = 4y$

Each part is divided by $\frac{16}{3}$

10)

Find the area of the region common to the circle $x^2 + y^2 = 16$ and the parabola $y^2 = 6x$.

sol

$$x^2 + y^2 = 16, \text{ parabola } y^2 = 6x$$

$$x^2 + 6x = 16$$

$$x^2 + 6x - 16 = 0$$

$$(x+8)(x-2) = 0$$

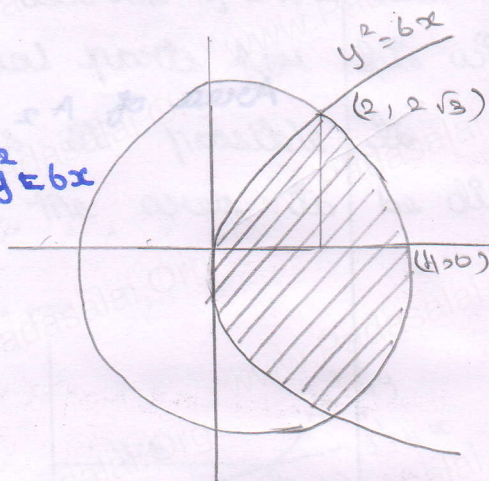
$$x = -8, 2$$

sub [2] in eqn parabola

$$y^2 = 6(2)$$

$$y = 2\sqrt{3}$$

The point $(2, 2\sqrt{3})$



$$\text{Required area} = 2 \left(\int_0^2 \sqrt{6x} dx + \int_2^4 \sqrt{4^2 - x^2} dx \right)$$

$$= 2 \left[\sqrt{6} \left[\frac{x^{3/2}}{3/2} \right]_0^2 + \left[\frac{x}{2} \sqrt{4^2 - x^2} + \frac{16}{2} \sin^{-1} \frac{x}{4} \right]_2^4 \right]$$

$$= 2 \left[\sqrt{6} \left(\frac{2}{3} \right) 2\sqrt{2} + 8 \sin^{-1}(1) - 2\sqrt{3} - 8 \sin^{-1} \frac{1}{2} \right]$$

$$= \frac{8}{3} \sqrt{12} + 8\pi - 4\sqrt{3} - \frac{16\pi}{6}$$

$$= \frac{4\sqrt{3}}{3} + \frac{16\pi}{3}$$

$$\text{Required area} = \frac{4}{3} (4\pi + \sqrt{3}) \text{ sq. units.}$$

Volume of a solid obtained by revolving area about an x -axis.

The curve $y = f(x)$ and the line $x = a$ and $x = b$. Then the volume of solid of revolution about x -axis is

$$V = \pi \int_a^b y^2 \cdot dx.$$

Volume of a solid obtained by revolving area about an y -axis.

The curve $x = f(y)$ and the line $y = c$ and $y = d$. Then the volume of solid of revolution about y -axis is

$$V = \pi \int_c^d x^2 \cdot dy$$

Ex: 9.9.

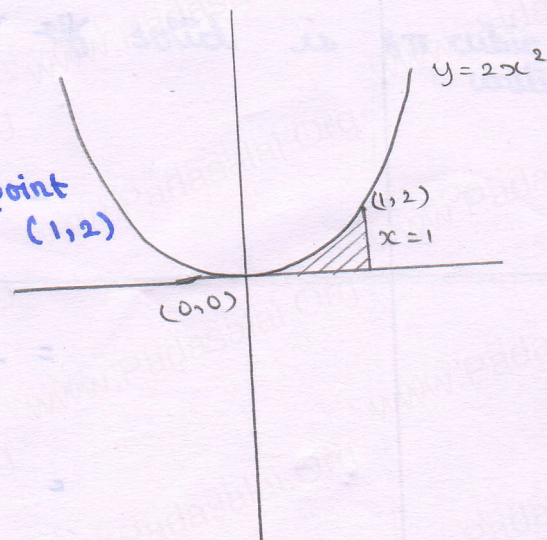
- ① Find, by integration, the volume of the solid generated by revolving about the x -axis, the region enclosed by $y = 2x^2$, $y = 0$ and $x = 1$

sol

$$y = 2x^2, y = 0, x = 1$$

$$x = 1, y = 2 \therefore \text{The point } (1, 2)$$

$$\begin{aligned} V &= \pi \int_a^b y^2 \cdot dx \\ &= \pi \int_0^1 4x^4 \cdot dx \end{aligned}$$



$$= 4\pi \left[\frac{x^5}{5} \right]_0^1$$

$$= 4\pi \left[\frac{1}{5} \right]$$

$$= \frac{4\pi}{5}$$

$\therefore$ The Volume of the solid is

$\frac{4\pi}{5}$ cubic units.

- a) Find, by integration, the Volume of the solid generated by revolving about the x-axis, the region enclosed by $y = e^{-2x}$, $y = 0$, $x = 0$ and $x = 1$.

Sol.

Given,

$$y = e^{-2x}, y = 0, x = 0, x = 1$$

about x-axis

$$\text{Volume, } V = \pi \int_a^b y^2 \cdot dx \quad (1)$$

$$V = \pi \int_0^1 e^{-4x} dx$$

$$= \pi \left[\frac{-e^{-4x}}{4} \right]_0^1$$

$$= -\frac{\pi}{4} [e^{-4} - e^0]$$

$$= -\frac{\pi}{4} [e^{-4} - 1]$$

$$V = \frac{\pi}{4} [1 - e^{-4}] \text{ cubic units}$$

Find the volume of the solid generated by revolving about the y-axis, the region enclosed by $x^2 = 1+y$ and $y = 3$.

Sol

$$x^2 = 1+y, \quad y = 3, \quad x = 0, \quad y = -1$$

$$V = \pi \int_{-1}^3 x^2 \cdot dy$$

$$= \pi \int_{-1}^3 (y+1) dy$$

$$= \pi \left[\frac{y^2}{2} + y \right]_{-1}^3$$

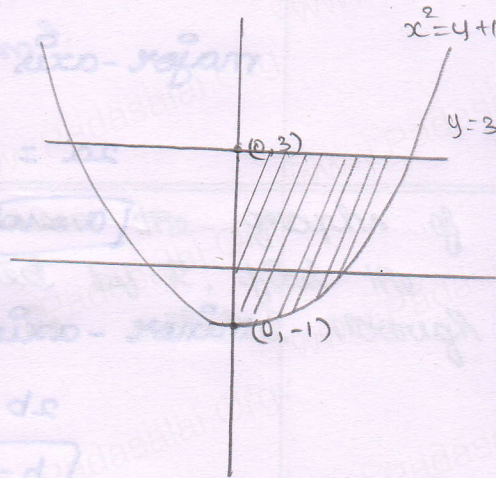
$$= \pi \left[\frac{9}{2} + 3 - \frac{1}{2} + 1 \right]$$

$$= \pi \left[\frac{9}{2} + 4 \right]$$

$$= \frac{16\pi}{2}$$

$$= 8\pi$$

$\therefore$ The Volume of solid is 8π cubic units.



6) A watermelon has an ellipsoid shape which can be obtained by revolving an ellipse with major-axis 20cm and minor-axis 10cm about its major axis. Find its Volume using Integration.

Sol.

$$\text{major-axis} = 20$$

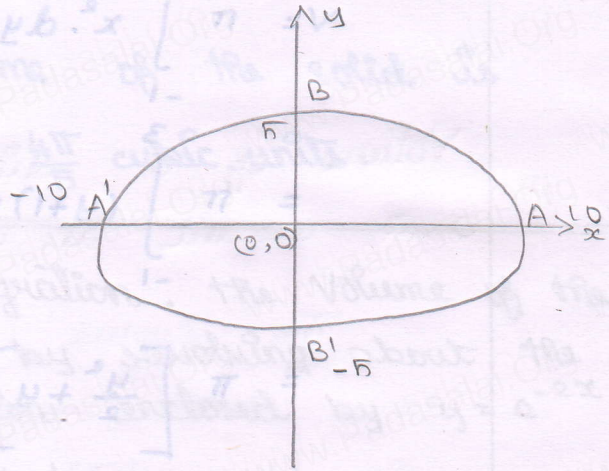
$$2a = 20$$

$$\boxed{a = 10}$$

$$\text{minor-axis} = 10$$

$$2b = 10$$

$$\boxed{b = 5}$$



Equation of ellipse

$$\frac{x^2}{100} + \frac{y^2}{25} = 1$$

$$y^2 = 25 \left(\frac{100 - x^2}{100} \right)$$

$$y^2 = \frac{1}{4} (100 - x^2)$$

$$\text{Volume, } V = \pi \int_{-10}^{10} \frac{1}{4} (100 - x^2) dx$$

$$= \frac{\pi}{4} \cdot 2 \int_0^{10} (100 - x^2) dx$$

$$= \frac{\pi}{2} \left[100x - \frac{x^3}{3} \right]_0^{10}$$

$$= \frac{\pi}{2} \left[1000 - \frac{1000}{3} \right]$$

$$= \frac{\pi}{2} \left[\frac{2000}{3} \right]$$

$$= \frac{1000\pi}{3} \text{ cm}^3.$$

$\therefore$ The Volume of the solid is

$$\frac{1000\pi}{3} \text{ cm}^3.$$

4. The region enclosed between the graphs of $y=x$ and $y=x^2$ is denoted by R , find the Volume generated when R is rotated through 360° about x -axis.

Y. MURUGAN

PG ASST IN MATHS

7904171615