7. Applications of Differential Calculus

EXERCISE – 7.1

1) A Point moves along a straight line in such a way that after t seconds its distance from the origin is $s = 2t^2 + 3t$ metres

(i) Find the average velocity of the points between t = 3 and t = 6 seconds

(ii) Find the instantaneous velocities at t = 3 and t = 6 seconds

Soln: Given: $s = 2t^2 + 3t$

(i) $s(t) = 2t^2 + 3t$

When t = 3,

$s(3) = 2(3)^2 + 3(3)$

$= 18 + 9$

$s(3) = 27$

When t = 6,

$s(6) = 2(6)^2 + 3(6)$

$= 72 + 18$

$s(6) = 90$

∴ Average velocity

$= \frac{s(6) - s(3)}{6 - 3}$

$= \frac{90 - 27}{3} = \frac{63}{3} = 21$ m/s

(ii) Given: $s(t) = 2t^2 + 3t$

Instantaneous velocity is $s'(t) = 4t + 3$

When t = 3, $s'(3) = 4(3) + 3$

$= 12 + 3$

$s'(3) = 15$ m/s

when t = 6, $s'(6) = 4(6) + 3$

$= 24 + 3$

$s'(6) = 27$ m/s

2) A Camera is accidentally knocked off an edge of a cliff 400 ft high. The Camera falls a distance of $s = 16t^2$ in t seconds

(i) How long does the Camera fall before it hits the ground?

(ii) What is the average velocity with which the Camera falls during the last 2 seconds?

(iii) What is the instantaneous velocity of the Camera when it hits the ground?

Soln: Given: $s = 16t^2$

(i) Given: The height of cliff is s = 400

∴ $\cancel{400}^{25} = \cancel{16}t^2$

$t^2 = 25$

$t = \pm 5$, $t = -5$ is not possible.

∴ t = 5 sec

(ii) Given: $s(t) = 16t^2$

when t = 3, $s(3) = 16(3)^2$

$s(3) = 16 \times 9 = 144$

When t = 5,

$s(5) = 16(5)^2 = 16 \times 25$

$s(5) = 400$

∴ Average velocity

$= \frac{s(5) - s(3)}{5 - 3}$

$= \frac{400 - 144}{2}$

$= \frac{\cancel{256}^{128}}{\cancel{2}} = 128$ ft/s

(iii) Given: $s(t) = 16t^2$

Instantaneous velocity is $s'(t) = 32t$

When t = 5,

$s'(5) = 32 \times 5 = 160$ ft/s

3) A Particle moves along a line according to the law $s(t) = 2t^3 - 9t^2 + 12t - 4$, where $t \geq 0$

(i) At what times the Particle changes direction?

(ii) Find the total distance travelled by the particle in the first 4 seconds.

(iii) Find the particle's acceleration each time the velocity is zero

Soln:

Given: $s(t)=2t^3-9t^2+12t-4$

$s'(t)=6t^2-18t+12$

(i) when $s'(t)=0$,

$6t^2-18t+12=0$

$\div 6,\ t^2-3t+2=0$ $\quad \overset{2}{\wedge}\ -1\ -2$

$(t-1)(t-2)=0$

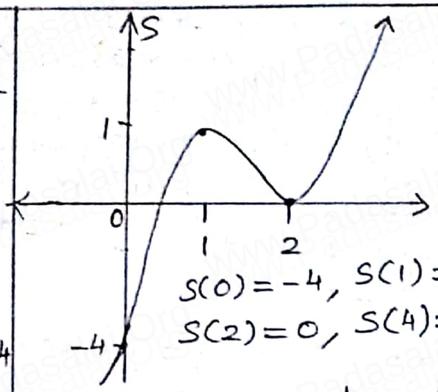
$t-1=0,\ t-2=0$

$t=1,\ t=2$

$\therefore t=1$ sec and $t=2$ sec

(ii) Given: $s(t)=2t^3-9t^2+12t-4$

t	0	1	2	4
s	-4	1	0	28



$s(0)=-4,\ s(1)=1$

$s(2)=0,\ s(4)=28$

Total distance travelled in first 4 seconds

$=|s(0)-s(1)|+|s(1)-s(2)|+|s(2)-s(4)|$

$=|-4-1|+|1-0|+|0-28|$

$=|-5|+|1|+|-28|$

$=5+1+28$

$=34\,m$

(iii) Given:

$s(t)=2t^3-9t^2+12t-4$

$s'(t)=6t^2-18t+12$

$s''(t)=12t-18$

when $s'(t)=0,\ 6t^2-18t+12=0$

$\div 6,\ t^2-3t+2=0$ $\quad \overset{2}{\wedge}\ -1\ -2$

$(t-1)(t-2)=0$

$t-1=0,\ t-2=0$

$t=1,\ t=2$

when $t=1$,

$s''(1)=12(1)-18=-6\,m/s^2$

when $t=2$,

$s''(2)=12(2)-18=24-18$

$=6\,m/s^2$

4) If the volume of a cube of side length x is $v=x^3$. Find the rate of change of the volume with respect to x when $x=5$ units

Soln: Given: $v=x^3$ and $x=5$

$\frac{dv}{dx}=3x^2$

When $x=5$, $\frac{dv}{dx}=3(5)^2$

$=3\times 25$

$=75$ units

5) If the mass $m(x)$ (in kilograms) of a thin rod of length x (in metres) is given by, $m(x)=\sqrt{3x}$ then what is the rate of change of mass with respect to the length when it is $x=3$ and $x=27$ metres

Soln: Given: $m(x)=\sqrt{3x}$

$\therefore m(x)=\sqrt{3}\sqrt{x}=\sqrt{3}\,x^{1/2}$

$m'(x)=\sqrt{3}\left[\frac{1}{2}x^{-1/2}\right]$

$=\sqrt{3}\left[\frac{1}{2x^{1/2}}\right]$

$m'(x)=\frac{\sqrt{3}}{2\sqrt{x}}$

when $x=3$, $m'(3)=\frac{\cancel{\sqrt{3}}}{2\cancel{\sqrt{3}}}$

$m'(3)=\frac{1}{2}$ kg/m

when $x=27$, $m'(27)=\frac{\sqrt{3}}{2\sqrt{27}}$

$=\frac{\cancel{\sqrt{3}}}{2\times 3\cancel{\sqrt{3}}}=\frac{1}{6}$

$m'(27)=\frac{1}{6}$ kg/m

6) A stone is dropped into a pond causing ripples in the form of concentric circles. The radius r of the outer ripple is increasing at a constant rate at 2 cm per second. When the radius is 5 cm find the rate of changing of the total area of the disturbed water?

Soln:

Let A be the Area of the circle at time t

$\therefore$ Area of the circle is

$$A = \pi r^2$$

$$\frac{dA}{dt} = 2\pi r \frac{dr}{dt}$$

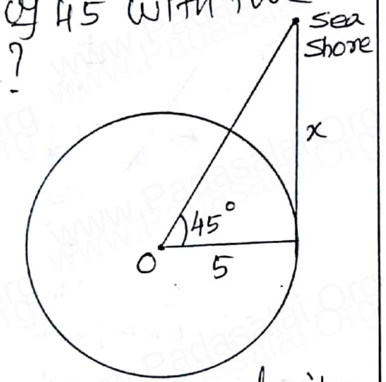
When $r = 5$, $\frac{dr}{dt} = 2$,

$$\frac{dA}{dt} = 2\pi(5)(2) = 20\pi$$

$\therefore$ The rate of changing of the total area is 20π sq cm/s

7) A beacon makes one revolution every 10 seconds. It is located on a ship which is anchored 5 km from a straight shore line. How fast is the beam moving along the shore line when it makes an angle of 45° with the shore?

Soln:



Let the angular velocity is $\frac{d\theta}{dt}$

$\therefore$ revolution in 10 sec $= 2\pi$

revolution in 1 sec $= \frac{\cancel{2}\pi}{\cancel{10}_5}$

$$\therefore \frac{d\theta}{dt} = \frac{\pi}{5}$$

$$\tan\theta = \frac{x}{5}$$

$$\Rightarrow x = 5\tan\theta$$

$$\frac{dx}{dt} = 5\sec^2\theta \frac{d\theta}{dt}$$

When $\theta = 45°$, $\frac{d\theta}{dt} = \frac{\pi}{5}$

$$\therefore \frac{dx}{dt} = 5\sec^2 45°\left(\frac{\pi}{5}\right)$$

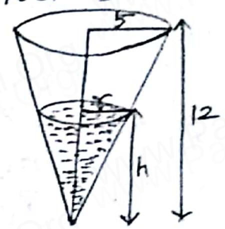
$$= \cancel{5}(\sqrt{2})^2\left(\frac{\pi}{\cancel{5}}\right)$$

$$\frac{dx}{dt} = 2\pi \text{ km/s}$$

$\therefore$ The beam moving along the shore line is 2π km/s

8) A conical water tank with vertex down of 12 metres height has a radius of 5 metres at the top. If water flows into the tank at a rate 10 cubic m/min, how fast is the depth of the water increases when the water is 8 metres deep?

Soln:



Let v, r and h be respectively the volume of the water, radius and height of the cone at time t

$$\frac{r}{h} = \frac{5}{12}$$

$$\boxed{r = \frac{5}{12}h}$$

Given: $\frac{dv}{dt} = 10\ m^3/min$ and $h = 8m$

Find: $\frac{dh}{dt}$

Volume of the cone

$$v = \frac{1}{3}\pi r^2 h$$

$$V = \frac{1}{3}\pi\left(\frac{5}{12}h\right)^2 h$$

$$= \frac{1}{3}\pi\left(\frac{25}{144}h^2\right)h$$

$$V = \frac{25}{3\times144}\pi h^3$$

$$\frac{dv}{dt} = \frac{25}{\cancel{3}\times144}\pi\, \cancel{3}h^2\frac{dh}{dt}$$

$$10 = \frac{25}{144}\pi(8)^2\frac{dh}{dt}$$

$$10 = \frac{25}{\cancel{144}_{9}}\pi(\cancel{64}^{4})\frac{dh}{dt}$$

$$9\cancel{0} = 10\cancel{0}\pi\frac{dh}{dt}$$

$$9 = 10\pi\frac{dh}{dt}$$

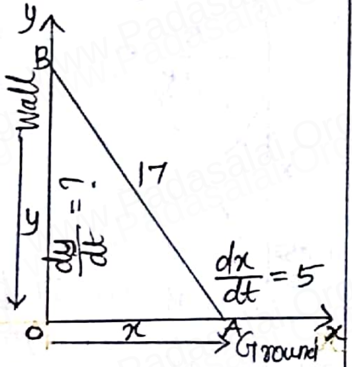
$$\Rightarrow \frac{dh}{dt} = \frac{9}{10\pi}\ m/min$$

9) A ladder 17 metre long is leaning against the wall. The base of the ladder is pulled away from the wall at a rate of 5 m/s. when the base of the ladder is 8 metres from the wall

(i) How fast is the top of the ladder moving down the wall?

(ii) At what rate, the area of the triangle formed by the ladder, wall, and the floor, is changing?

Soln:



Given: $\frac{dx}{dt} = 5\ m/s$,

$x = 8m$

By using Pythagoras thm,

$$x^2+y^2=289$$

$$(8)^2+y^2=289$$

$$64+y^2=289$$

$$y^2=289-64$$

$$y^2=225$$

$$\boxed{y=15}$$

(i) $x^2+y^2=289$

$$2x\frac{dx}{dt}+2y\frac{dy}{dt}=0$$

$\div 2$, $$x\frac{dx}{dt}+y\frac{dy}{dt}=0$$

$$y\frac{dy}{dt} = -x\frac{dx}{dt}$$

$$\frac{dy}{dt} = -\frac{x}{y}\frac{dx}{dt}$$

$$= -\frac{8}{\cancel{15}_{3}}(\cancel{5})$$

$$\frac{dy}{dt} = -\frac{8}{3}\ m/s$$

∴ The ladder is moving downward at the rate of $\frac{8}{3}$ m/s

(ii) Area of triangle ΔOAB,

$$\Delta = \frac{1}{2}xy$$

$$\frac{d\Delta}{dt} = \frac{1}{2}\left[x\frac{dy}{dt}+y\frac{dx}{dt}\right]$$

$$= \frac{1}{2}\left[8\left(-\frac{8}{3}\right)+15(5)\right]$$

$$= \frac{1}{2}\left[-\frac{64}{3}+75\right]$$

$$= \frac{1}{2}\left[\frac{-64+225}{3}\right]$$

$$= \frac{1}{2}\left[\frac{161}{3}\right]$$

$$= \frac{161}{6}$$

$$= 26.83$$

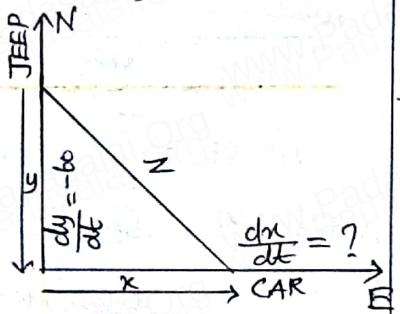
$$\therefore \frac{d\Delta}{dt} = 26.83$$

∴ The area of the triangle changing at the rate of 26.83 Sq m/s

10) A Police jeep, approaching an orthogonal intersection from the

northern direction, is chasing a speeding car that has turned and moving straight east. When the jeep is 0.6 km north of the intersection and the car is 0.8 km to the east. The police determine with a radar that the distance between them and the car is increasing at 20 km/hr. If the jeep is moving at 60 km/hr at the instant of measurement, what is the speed of the car?

Soln:



Let x, y be the distance and moving by car and jeep

Let z be the distance between car and jeep

Given: $\frac{dy}{dt} = -60$ km/hr, $\frac{dz}{dt} = 20$ km/hr, $x = 0.8$, $y = 0.6$, Find: $\frac{dx}{dt}$

By using Pythagoras thm,

$$x^2+y^2=z^2$$

$$(0.8)^2+(0.6)^2=z^2$$

$$0.64+0.36=z^2$$

$$1.00=z^2$$

$$\therefore z^2=1$$

$$\boxed{z=1}$$

$$x^2+y^2=z^2$$

$$2x\frac{dx}{dt}+2y\frac{dy}{dt}=2z\frac{dz}{dt}$$

$$\div 2,\quad x\frac{dx}{dt}+y\frac{dy}{dt}=z\frac{dz}{dt}$$

$$(0.8)\frac{dx}{dt}+(0.6)(-60)=(1)20$$

$$(0.8)\frac{dx}{dt}-36=20$$

$$0.8\frac{dx}{dt}=20+36$$

$$0.8\frac{dx}{dt}=56$$

$$\frac{dx}{dt}=\frac{56}{0.8}\times\frac{10}{10}$$

$$=\frac{\cancel{560}\ 70}{\cancel{8}}$$

$$\frac{dx}{dt}=70 \text{ km/hr}$$

$\therefore$ The speed of the car is increasing at the rate of 70 km/hr

EX 7.1

For the function $f(x)=x^2$, $x\in[0,2]$ compute the average rate of changes in the subintervals $[0,0.5]$, $[0.5,1]$, $[1,1.5]$, $[1.5,2]$ and the instantaneous rate of changes at the points $x=0.5, 1, 1.5, 2$

Soln:

Rate of Changes

a	b	x	Average rate is $\frac{f(b)-f(a)}{b-a}=b+a$	Instantaneous rate is $f'(x)=2x$
0	0.5	0.5	0.5	1
0.5	1	1	1.5	2
1	1.5	1.5	2.5	3
1.5	2	2	3.5	4

EX 7.2

The temperature in Celsius in a long rod of length 10 m, insulated at both ends, is a function of length x given by $T=x(10-x)$. Prove that the rate of change of the temperature at the midpoint of the rod is zero.

Soln: Given $T=x(10-x)$

$$\therefore T=10x-x^2$$

$$\frac{dT}{dx} = 10 - 2x$$

When $x = 5$, $\frac{dT}{dx} = 10 - 2(5)$

$= 10 - 10$

$\frac{dT}{dx} = 0$

EX 7.3 A person learnt 100 words for an English test. The number of words the person remembers in t days after learning is given by $W(t) = 100 \times (1 - 0.1t)^2$, $0 \le t \le 10$. What is the rate at which the person forgets the words 2 days after learning?

Soln: Given: $W(t) = 100 \times (1 - 0.1t)^2$, $0 \le t \le 10$

$\frac{d(W(t))}{dt} = 100 \times 2(1 - 0.1t)(-0.1)$

$\frac{d(W(t))}{dt} = -20(1 - 0.1t)$

When $t = 2$, $\frac{d(W(t))}{dt} = -20(1 - (0.1)(2))$

$= -20(1 - 0.2)$

$= -20(0.8)$

$= -16$

∴ The person forgets at the rate of 16 words after 2 days of studying

EX 7.4 A particle moves so that the distance moved is according to the law $S(t) = \frac{t^3}{3} - t^2 + 3$. At what time the velocity and acceleration are zero respectively?

Soln: Given: $S(t) = \frac{t^3}{3} - t^2 + 3$

Velocity $v = \frac{ds}{dt} = \frac{\cancel{3}t^2}{\cancel{3}} - 2t$

$v = \frac{ds}{dt} = t^2 - 2t$

Acceleration $A = \frac{dv}{dt} = 2t - 2$

Given: $v = 0$ and $A = 0$

$v = 0$

$t^2 - 2t = 0$

$t(t-2) = 0$

$\boxed{t = 0}$, $t - 2 = 0$

$\boxed{t = 2}$

∴ $t = 0$ and $t = 2$

Also $A = 0$

$2t - 2 = 0$

$\cancel{2}t = \cancel{2} \Rightarrow \boxed{t = 1}$

∴ $t = 1$

EX 7.5 A particle is fired straight up from the ground to reach a height of s feet in t seconds, where $s(t) = 128t - 16t^2$. (1) Compute the maximum height of the particle reached. (2) What is the velocity when the particle hits the ground?

Soln: Given:

(1) $S(t) = 128t - 16t^2$

$V(t) = \frac{ds}{dt} = 128 - 32t$

$V(t) = 0$, $128 - 32t = 0$

$\cancel{32}t = \cancel{128}\,4$

$\boxed{t = 4\text{sec}}$

When $t = 4$,

$S(4) = 128(4) - 16(4)^2$

$= 512 - 256$

$= 256$ ft

(2) When $s = 0$,

$128t - 16t^2 = 0$

$t(128 - 16t) = 0$

$t = 0$, $128 - 16t = 0$

$t = 0$ is not possible, $\cancel{16}t = \cancel{128}\,8$

$\boxed{t = 8 \text{ sec}}$

∴ $V(t) = 128 - 32t$

When $t = 8$,

$V(8) = 128 - 32(8)$

$= 128 - 256$

$V(8) = -128$ ft/s

EX 7.6 A particle moves along a horizontal line such that its position at any time $t \geq 0$ is given by $s(t) = t^3 - 6t^2 + 9t + 1$, where s is measured in metres and t in seconds?

(1) At what time the particle is at rest?

(2) At what time the particle changes direction?

(3) Find the total distance the particle travelled by the particle in the first 2 seconds.

Soln: Given:

$s(t) = t^3 - 6t^2 + 9t + 1$ and

$v(t) = 3t^2 - 12t + 9$

$A(t) = 6t - 12$

(1) when $v(t) = 0$,

$3t^2 - 12t + 9 = 0$

$\div 3$, $t^2 - 4t + 3 = 0$ $\quad$ 3, -1, -3

$(t-1)(t-3) = 0$

$t - 1 = 0,\ t - 3 = 0$

$\boxed{t=1}, \boxed{t=3}$

$\therefore t = 1$ and $t = 3$

(2) The particle changes direction when $v(t)$ changes its sign. Now

if $0 \leq t < 1$ then both $(t-1)$ and $(t-3) < 0$ and $v(t) > 0$

if $1 < t < 3$ then $(t-1) > 0$ and $(t-3) < 0$ and $v(t) < 0$

if $t > 3$ then both $(t-1)$ and $(t-3) > 0$ and $v(t) > 0$

$\therefore$ The particle changes direction when $t = 1$ and $t = 3$

(3) Given:

$s(t) = t^3 - 6t^2 + 9t + 1$

when $t = 0$, $s(0) = 1$

when $t = 1$, $s(1) = 1 - 6 + 9 + 1$

$s(1) = 5$

When $t = 2$,

$s(2) = (2)^3 - 6(2)^2 + 9(2) + 1$

$= 8 - 24 + 18 + 1$

$s(2) = 3$

The total distance travelled by the particle

$= |s(0) - s(1)| + |s(1) - s(2)|$

$= |1 - 5| + |5 - 3|$

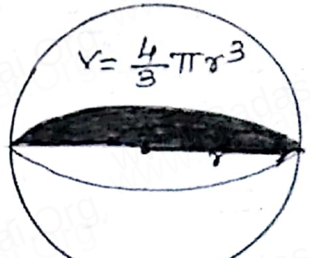
$= |-4| + |2|$

$= 4 + 2 = 6$ metres

Ex 7.7

If we blow air into a balloon of spherical shape at a rate of $1000\ cm^3$ per second. At what rate the radius of the baloon changes when the radius is 7cm? Also compute the rate at which the surface area changes.

Soln:



The volume of the balloon $v = \frac{4}{3}\pi r^3$

Given: $\frac{dv}{dt} = 1000\ cm^3$

and $r = 7$ cm

Find: $\frac{dr}{dt}$ and $\frac{ds}{dt}$

(1) $\frac{dv}{dt} = \frac{4}{\cancel{3}}\pi \cancel{3} r^2 \frac{dr}{dt}$

$\overset{250}{\cancel{1000}} = \cancel{4}\pi (7)^2 \frac{dr}{dt}$

$250 = 49\pi \frac{dr}{dt}$

$\therefore \frac{dr}{dt} = \frac{250}{49\pi}$ cm/sec

(2) The surface area of the balloon $S = 4\pi r^2$

$$\frac{ds}{dt} = 8\pi r \frac{dr}{dt}$$

When $r=7$, $\frac{dr}{dt} = \frac{250}{49\pi}$

$$\frac{ds}{dt} = 8\cancel{\pi}(\cancel{7})\left(\frac{250}{\cancel{49}_7\cancel{\pi}}\right)$$

$$\frac{ds}{dt} = \frac{2000}{7}\ cm^2/sec$$

∴ The rate of change of radius is $\frac{250}{49\pi}$ cm/sec and the rate of change of surface area is $\frac{2000}{7}\ cm^2/sec$

EX 7.8 The price of a product is related to the number of units available (supply) by the equation $Px+3P-16x=234$, where P is the price of the product per unit in Rupees (₹) and x is the number of units. Find the rate at which the price is changing with respect to time when 90 units are available and the supply is increasing at a rate of 15 units/week.

Soln: Given:

$$Px+3P-16x=234$$

$$Px+3P=234+16x$$

$$P(x+3)=234+16x$$

$$P=\frac{234+16x}{x+3}$$

$$\frac{dP}{dt} = \frac{(x+3)\left(16\frac{dx}{dt}\right)-(234+16x)\frac{dx}{dt}}{(x+3)^2}$$

$$=\frac{[(x+3)(16)-(234+16x)]\frac{dx}{dt}}{(x+3)^2}$$

$$=\frac{[\cancel{16x}+48-234-\cancel{16x}]\frac{dx}{dt}}{(x+3)^2}$$

$$\frac{dP}{dt} = \frac{-186}{(x+3)^2}\frac{dx}{dt}$$

When $x=90$, $\frac{dx}{dt}=15$,

$$\frac{dP}{dt} = \frac{-186}{(90+3)^2}\times 15$$

$$= \frac{-\cancel{186}^{2}}{(93)^{\cancel{2}}}\times 15$$

$$= \frac{-2}{\cancel{93}_{31}}\times \cancel{15}^{5}$$

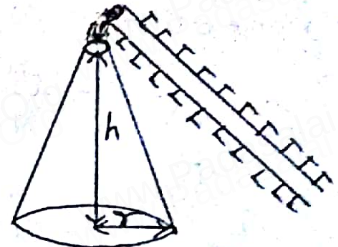
$$= \frac{-10}{31}$$

$$\approx -0.32$$

∴ The price is changing decreasing at the rate of ₹0.32 per unit

EX 7.9 Salt is poured from a conveyer belt at a rate of 30 cubic metre per minute forming a conical pile with a circular base whose height and diameter of base are always equal. How fast is the height of the pile increasing when the pile is 10 metre high?

Soln:



Let h and r be the height and the base radius of the salt cone. Let v be the volume of the salt cone.

Given: $\frac{dv}{dt} = 30\ m^3/min$

and $h=2r \Rightarrow \boxed{r=\frac{h}{2}}$

∴ The volume of the salt cone $V=\frac{1}{3}\pi r^2 h$

$$=\frac{1}{3}\pi\left(\frac{h}{2}\right)^2 h$$

$$=\frac{1}{3}\pi\frac{h^3}{4} = \frac{\pi}{12}h^3$$

$$\therefore V=\frac{\pi}{12}h^3$$

$$\frac{dv}{dt} = \frac{\pi}{\cancel{12}_{4}} \cancel{3}h^2 \frac{dh}{dt}$$

$$\frac{dv}{dt} = \frac{\pi}{4} h^2 \frac{dh}{dt}$$

When $h=10$, $\frac{dv}{dt} = 30$

$$30 = \frac{\pi}{4}(10)^2 \frac{dh}{dt}$$

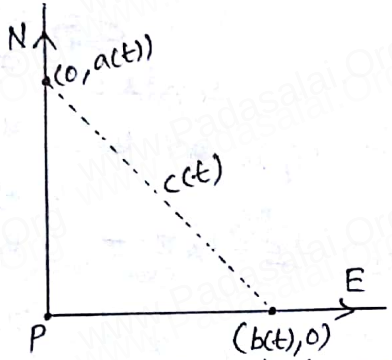
$$\cancel{120}^{6} = \cancel{100}^{5}\pi \frac{dh}{dt} \Rightarrow 6 = 5\pi \frac{dh}{dt}$$

$$\Rightarrow \frac{dh}{dt} = \frac{6}{5\pi} \text{ mtr/min}$$

Ex 7.10 (Two Variable related rate Problem)

A road running north to South Crosses a road going east to west at the Point P. Car A is driving north along the first road, and Car B is driving east along the second road. At a particular time Car A 10 kilometres to the north of P and traveling at 80 km/hr, while Car B is 15 Kilometres to the east of P and traveling at 100 km/hr. How fast is the distance between the two cars changing?

Soln:



Let $a(t)$ be the distance of Car A north of P at time t and $b(t)$ the distance of Car B east of P at time t and let $c(t)$ be the distance from Car A to Car B at time t.

By using Pythagoras thm,

$$c(t)^2 = a(t)^2 + b(t)^2$$
$$= (10)^2 + (15)^2$$
$$= 100 + 225$$
$$c(t)^2 = 325$$
$$c(t) = \sqrt{325} = \sqrt{25 \times 13} = 5\sqrt{13}$$
$$\boxed{c(t) = 5\sqrt{13}}$$

Also, $c(t)^2 = a(t)^2 + b(t)^2$

$$2c(t)c'(t) = 2a(t)a'(t) + 2b(t)b'(t)$$

$\div 2$, $c(t)c'(t) = a(t)a'(t) + b(t)b'(t)$

$$c'(t) = \frac{a(t)a'(t) + b(t)b'(t)}{c(t)}$$

$$c'(t) = \frac{(10\times 80) + (15 \times 100)}{5\sqrt{13}}$$
$$= \frac{800 + 1500}{5\sqrt{13}}$$
$$= \frac{\cancel{2300}\,460}{\cancel{5}\sqrt{13}}$$
$$c'(t) = \frac{460}{\sqrt{13}} = \frac{460}{3.605} \approx 127.6 \text{ km/hr}$$

$\therefore c'(t) \approx 127.6$ km/hr at the time of intersect

7.2.4 Equations of Tangent and Normal

EXERCISE - 7.2

1) Find the slope of the tangent to the curves at the respective given points

(i) $y = x^4 + 2x^2 - x$ at $x = 1$ Given:

Soln: $y = x^4 + 2x^2 - x$

$$\text{Slope } m = \left(\frac{dy}{dx}\right)_{x=1} = 4x^3 + 4x - 1$$

$$\text{Slope } m = 4(1)^3 + 4(1) - 1$$
$$= 4 + 4 - 1$$
$$\text{Slope } m = 7$$

$\therefore$ Slope is 7

(ii) $x = a\cos^3 t$, $y = b\sin^3 t$ at $t = \pi/2$

Soln: Given: $x = a\cos^3 t$ and $y = b\sin^3 t$

$\frac{dx}{dt} = 3a\cos^2 t(-\sin t) = -3a\cos^2 t\sin t$

$\frac{dy}{dt} = 3b\sin^2 t\cos t$

Slope $m = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\cancel{3}b\sin^2 t\cos t}{-\cancel{3}a\cos^2 t\sin t}$

$= \left(\frac{dy}{dx}\right) = -\frac{b}{a}\frac{\sin t}{\cos t}$

$= \left(\frac{dy}{dx}\right) = -\frac{b}{a}\tan t$

Slope $m = \left(\frac{dy}{dx}\right)_{t=\frac{\pi}{2}} = -\frac{b}{a}\tan\frac{\pi}{2}$

$= -\frac{b}{a}(\infty) = \infty$

$\therefore$ Slope is ∞

2) Find the point on the curve $y = x^2 - 5x + 4$ at which the tangent is parallel to the line $3x + y = 7$

Soln: Given: $y = x^2 - 5x + 4$

Let (x_1, y_1) be the point

The tangent is parallel to $3x + y = 7$

$\left(\frac{dy}{dx}\right)_{(x_1,y_1)} = -\frac{3}{1} = -3$ — ①

Now, $y = x^2 - 5x + 4$

$\frac{dy}{dx} = 2x - 5$

$\left(\frac{dy}{dx}\right)_{(x_1,y_1)} = 2x_1 - 5$ — ②

from ①&② we get,

$-3 = 2x_1 - 5$

$5 - 3 = 2x_1$

$\cancel{2} = \cancel{2}x_1$

$\therefore \boxed{x_1 = 1}$

(x_1, y_1) lies on $y = x^2 - 5x + 4$

$y_1 = x_1^2 - 5x_1 + 4$

$y_1 = (1)^2 - 5(1) + 4$

$y_1 = 1 - 5 + 4 = 0$

$\therefore \boxed{y_1 = 0}$

$\therefore$ The point is $(1, 0)$

3) Find the points on the curve $y = x^3 - 6x^2 + x + 3$ where the normal is parallel to the line $x + y = 1729$

Soln: Given: $y = x^3 - 6x^2 + x + 3$

Let (x_1, y_1) be the point

The normal is parallel to $x + y = 1729$

slope of normal $-\frac{1}{m} = -\frac{1}{1}$

Slope of normal $-\frac{1}{m} = -1$

$\therefore$ Slope of tangent $m = 1$

$\therefore \left(\frac{dy}{dx}\right)_{(x_1,y_1)} = 1$ — ①

Now, $y = x^3 - 6x^2 + x + 3$

$\frac{dy}{dx} = 3x^2 - 12x + 1$

$\left(\frac{dy}{dx}\right)_{(x_1,y_1)} = 3x_1^2 - 12x_1 + 1$ — ②

from ①&② we get,

$1 = 3x_1^2 - 12x_1 + 1$

$3x_1^2 - 12x_1 = 1 - 1$

$3x_1(x_1 - 4) = 0$

$3x_1 = 0,\quad x_1 - 4 = 0$

$\boxed{x_1 = 0},\quad \boxed{x_1 = 4}$

(x_1, y_1) lies on $y = x^3 - 6x^2 + x + 3$

$y_1 = x_1^3 - 6x_1^2 + x_1 + 3$

when $x_1 = 0$, $y_1 = 3$

when $x_1 = 4$,

$y_1 = (4)^3 - 6(4)^2 + 4 + 3$

$= 64 - 96 + 7$

$y_1 = -25$

$\therefore$ The points are $(0, 3)$ and $(4, -25)$

4) Find the points on the curve $y^2 - 4xy = x^2 + 5$ for which the tangent is horizontal

Soln: Given: $y^2-4xy=x^2+5$

Let (x_1,y_1) be the point

$2y\frac{dy}{dx}-4\left(x\frac{dy}{dx}+y(1)\right)=2x$

$2y\frac{dy}{dx}-4x\frac{dy}{dx}-4y=2x$

$2\frac{dy}{dx}(y-2x)=2x+4y$

$\div 2$, $\frac{dy}{dx}(y-2x)=x+2y$

$\left(\frac{dy}{dx}\right)_{(x_1,y_1)}=\frac{x_1+2y_1}{y_1-2x_1}$

Given the tangent is horizontal.

$\therefore \left(\frac{dy}{dx}\right)_{(x_1,y_1)}=0$

$\frac{x_1+2y_1}{y_1-2x_1}=0$

$x_1+2y_1=0$

$\boxed{x_1=-2y_1}$ — ①

(x_1,y_1) lies on $y^2-4xy=x^2+5$

$y_1^2-4x_1y_1=x_1^2+5$

$y_1^2-4(-2y_1)y_1=(-2y_1)^2+5$

$y_1^2+8y_1^2=4y_1^2+5$

$9y_1^2-4y_1^2=5$

$\cancel{5}y_1^2=\cancel{5}$

$y_1^2=1 \Rightarrow \boxed{y_1=\pm 1}$

① $\Rightarrow$ $x_1=-2y_1$

When $y_1=-1$, $x_1=-2(-1)=2$

When $y_1=1$, $x_1=-2(1)=-2$

$\therefore$ The points are $(2,-1)$ and $(-2,1)$

5) Find the tangent and normal to the following curves at the given points on the curve

(i) $y=x^2-x^4$ at $(1,0)$

Soln: Given: $y=x^2-x^4$

$\frac{dy}{dx}=2x-4x^3$

Slope $m=\left(\frac{dy}{dx}\right)_{(1,0)}=2(1)-4(1)^3=2-4$

$\therefore \boxed{m=-2}$ and

$(x_1,y_1)=(1,0)$

Equation of tangent is

$y-y_1=m(x-x_1)$

$y-0=-2(x-1)$

$y=-2x+2$

$2x+y=2$

Equation of normal is

$y-y_1=\frac{-1}{m}(x-x_1)$

$y-0=\frac{+1}{+2}(x-1)$

$y=\frac{1}{2}(x-1)$

$2y=x-1$

$1=x-2y$

$\therefore x-2y=1$

(ii) $y=x^4+2e^x$ at $(0,2)$

Soln: Given: $y=x^4+2e^x$

$\frac{dy}{dx}=4x^3+2e^x$

Slope $m=\left(\frac{dy}{dx}\right)_{(0,2)}=4(0)^3+2e^0$

Slope $m=2(1)=2$

$\therefore \boxed{m=2}$ and

$(x_1,y_1)=(0,2)$

Equation of tangent is

$y-y_1=m(x-x_1)$

$y-2=2(x-0)$

$y-2=2x$

$2x-y=-2$

Equation of normal is

$y-y_1=-\frac{1}{m}(x-x_1)$

$y-2=\frac{-1}{2}(x-0)$

$2y-4=-x$

$x+2y=4$

(iii) $y=x\sin x$ at $\left(\frac{\pi}{2},\frac{\pi}{2}\right)$

Soln: Given: $y=x\sin x$

$\frac{dy}{dx}=x\cos x+\sin x\cdot 1=x\cos x+\sin x$

Slope $m=\left(\frac{dy}{dx}\right)_{\left(\frac{\pi}{2},\frac{\pi}{2}\right)}=\frac{\pi}{2}\cos\frac{\pi}{2}+\sin\frac{\pi}{2}$

Slope $m = \frac{\pi}{\cancel{2}}(0)+1 = 1$

$\therefore \boxed{m=1}$ and $(x_1, y_1) = (\frac{\pi}{2}, \frac{\pi}{2})$

Equation of tangent is

$y - y_1 = m(x - x_1)$

$y - \frac{\pi}{2} = (1)(x - \frac{\pi}{2})$

$y - \frac{\pi}{2} = x - \frac{\pi}{2}$

$0 = x - y - \cancel{\frac{\pi}{2}} + \cancel{\frac{\pi}{2}}$

$\therefore x - y = 0$

Equation of normal is

$y - y_1 = -\frac{1}{m}(x - x_1)$

$y - \frac{\pi}{2} = \frac{-1}{1}(x - \pi/2)$

$y - \frac{\pi}{2} = -1(x - \pi/2)$

$y - \frac{\pi}{2} = -x + \frac{\pi}{2}$

$x + y = \frac{\pi}{2} + \frac{\pi}{2}$

$x + y = \frac{\cancel{2}\pi}{\cancel{2}}$

$x + y = \pi$

(iv) $x = \cos t$, $y = 2\sin^2 t$ at $t = \pi/3$

Soln: Given: $x = \cos t$ and $y = 2\sin^2 t$

$\frac{dx}{dt} = -\sin t$ and $\frac{dy}{dt} = 4\sin t\cos t$

Slope $m = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{4\cancel{\sin t}\cos t}{-\cancel{\sin t}}$

Slope $m = \left(\frac{dy}{dx}\right)_{t=\pi/3} = -4\cos t$

Slope $m = -4\cos\frac{\pi}{3} = -\cancel{4}^{2}\left(\frac{1}{\cancel{2}}\right)$

$\therefore \boxed{m=-2}$ and

$(x_1, y_1) = (\cos\frac{\pi}{3}, 2\sin^2\frac{\pi}{3})$

$= (\frac{1}{2}, 2(\frac{\sqrt{3}}{2})^2)$

$= (\frac{1}{2}, \cancel{2}(\frac{3}{\cancel{4}_2}))$

$\therefore (x_1, y_1) = (1/2, 3/2)$

Equation of tangent is

$y - y_1 = m(x - x_1)$

$y - \frac{3}{2} = -2(x - \frac{1}{2})$

$y - \frac{3}{2} = -2x + 1$

$2x + y = 1 + \frac{3}{2}$

$2x + y = \frac{5}{2}$

$4x + 2y = 5$

Equation of normal is

$y - y_1 = \frac{-1}{m}(x - x_1)$

$y - \frac{3}{2} = \frac{+1}{+2}(x - \frac{1}{2})$

$2y - 3 = x - \frac{1}{2}$

$\frac{1}{2} - 3 = x - 2y$

$\frac{1-6}{2} = x - 2y$

$-5 = 2x - 4y$

$\therefore 2x - 4y = -5$

6) Find the equations of the tangents to the curve $y = 1 + x^3$ for which the tangent is orthogonal with the line $x + 12y = 12$

Soln: Let (x_1, y_1) be the Point

Given: $y = 1 + x^3$

$\frac{dy}{dx} = 3x^2$

Slope $m_1 = \left(\frac{dy}{dx}\right)_{(x_1, y_1)} = 3x_1^2$

and Given: $x + 12y = 12$

Slope $m_2 = \frac{-1}{12}$

$\therefore m_1 m_2 = -1$

$(\cancel{3}x_1^2)\left(\frac{+1}{\cancel{12}_4}\right) = +1$

$x_1^2 = 4 \Rightarrow x_1 = \sqrt{4}$

$\boxed{x_1 = \pm 2}$

(x_1, y_1) lies on $y = 1 + x^3$

$y_1 = 1 + x_1^3$

(i) When $x_1 = 2$, $y_1 = 1+(2)^3$
$= 1+8$
$y_1 = 9$
$\therefore (x_1, y_1) = (2, 9)$ and
Slope $m = 3(2)^2 = 12$
$\therefore \boxed{m=12}$
Equation of tangent is
$y - y_1 = m(x - x_1)$
$y - 9 = 12(x-2)$
$y - 9 = 12x - 24$
$12x - y = 24 - 9$
$12x - y = 15$
(ii) when $x_1 = -2$, $y_1 = 1+(-2)^3$
$= 1-8$
$y_1 = -7$
$\therefore (x_1, y_1) = (-2, -7)$ and
Slope $m = 3(-2)^2 = 3(4) = 12$
$\therefore \boxed{m=12}$
Equation of tangent is
$y - y_1 = m(x - x_1)$
$y + 7 = 12(x+2)$
$y + 7 = 12x + 24$
$12x - y = 7 - 24$
$12x - y = -17$

7) Find the equations of the tangents to the curve $y = \frac{x+1}{x-1}$ which are parallel to the line $x + 2y = 6$

Soln: Let (x_1, y_1) be the Point

Given: $y = \frac{x+1}{x-1}$

$$\frac{dy}{dx} = \frac{(x-1)\cdot 1 - (x+1)\cdot 1}{(x-1)^2}$$
$$= \frac{\cancel{x} - 1 - \cancel{x} - 1}{(x-1)^2}$$
$$\frac{dy}{dx} = \frac{-2}{(x-1)^2}$$

Slope $m_1 = \left(\frac{dy}{dx}\right)_{(x_1, y_1)} = \frac{-2}{(x_1-1)^2}$

Also Given $x + 2y = 6$
slope $m_2 = -\frac{1}{2}$
$\therefore m_1 = m_2$
$$\frac{\cancel{-}2}{(x_1-1)^2} = \frac{\cancel{-}1}{2}$$
$(x_1 - 1)^2 = 4$
$x_1 - 1 = \sqrt{4}$
$x_1 - 1 = \pm 2$
$x_1 = 1 \pm 2$
$\therefore x_1 = 1+2 = 3$ and
$x_1 = 1 - 2 = -1$
$\therefore x_1 = 3$ and $x_1 = -1$
(x_1, y_1) lies on $y = \frac{x+1}{x-1}$
$y_1 = \frac{x_1+1}{x_1-1}$
When $x_1 = 3$, $y_1 = \frac{3+1}{3-1}$
$= \frac{\cancel{4}\,2}{\cancel{2}}$
$y_1 = 2$

When $x_1 = -1$,
$y_1 = \frac{-1+1}{-1-1} = \frac{0}{-2} = 0$
$y_1 = 0$
$\therefore$ The Points are $(3, 2)$ and $(-1, 0)$
(i) slope $m = -\frac{1}{2}$
and $(x_1, y_1) = (3, 2)$
Equation of tangent is
$y - y_1 = m(x - x_1)$
$y - 2 = -\frac{1}{2}(x-3)$
$2y - 4 = -x + 3$
$x + 2y = 4 + 3$
$x + 2y = 7$
(ii) Slope $m = -\frac{1}{2}$
and $(x_1, y_1) = (-1, 0)$
Equation of tangent is
$y - y_1 = m(x - x_1)$
$y - 0 = -\frac{1}{2}(x+1)$
$2y = -x - 1$
$x + 2y = -1$

8) Find the equation of tangent and normal to the curve given by $x = 7\cos t$ and $y = 2\sin t$, $t \in R$ at any Point on the curve

Soln: Given:

$x = 7\cos t$ and $y = 2\sin t$

$\frac{dx}{dt} = -7\sin t$ and $\frac{dy}{dt} = 2\cos t$

Slope $m = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2\cos t}{-7\sin t}$

$\therefore$ slope $m = -\frac{2\cos t}{7\sin t}$

Equation of tangent is

$y - y_1 = m(x - x_1)$

$y - 2\sin t = -\frac{2\cos t}{7\sin t}(x - 7\cos t)$

$(7\sin t)y - 14\sin^2 t = -(2\cos t)x + 14\cos^2 t$

$(2\cos t)x + (7\sin t)y = 14\sin^2 t + 14\cos^2 t$

$(2\cos t)x + (7\sin t)y = 14(\sin^2 t + \cos^2 t)$

$(2\cos t)x + (7\sin t)y = 14(1)$

$(2\cos t)x + (7\sin t)y = 14$

Equation of normal is

$y - y_1 = \frac{-1}{m}(x - x_1)$

$y - 2\sin t = \frac{+1}{+\frac{2\cos t}{7\sin t}}(x - 7\cos t)$

$y - 2\sin t = \frac{7\sin t}{2\cos t}(x - 7\cos t)$

$(2\cos t)y - 4\sin t\cos t = (7\sin t)x - 49\sin t\cos t$

$49\sin t\cos t - 4\sin t\cos t = (7\sin t)x - (2\cos t)y$

$\therefore (7\sin t)x - (2\cos t)y = 45\sin t\cos t$

9) Find the angle between the rectangular hyperbola $xy = 2$ and the parabola $x^2 + 4y = 0$

Soln: Given: $xy = 2$ and $x^2 + 4y = 0$

$xy = 2$ and $x^2 = -4y$

$y = \frac{2}{x}$ and $x^2 = -4\left(\frac{2}{x}\right)$

$x^2 = -\frac{8}{x}$

$x^3 = -8$

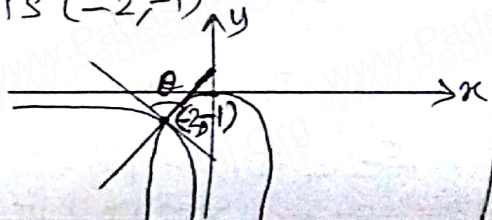
$x^3 = (-2)^3$

$\boxed{x = -2}$

$y = \frac{2}{x} = \frac{\cancel{2}}{-\cancel{2}} = -1$

$\therefore \boxed{y = -1}$

$\therefore$ The Point of intersection is $(-2, -1)$



$y = \frac{2}{x}$

$\frac{dy}{dx} = \frac{-2}{x^2}$

$m_1 = \left(\frac{dy}{dx}\right)_{(-2,-1)} = \frac{-2}{(-2)^2} = \frac{-\cancel{2}}{\cancel{4}_2}$

$\therefore \boxed{m_1 = -\frac{1}{2}}$

Also, $x^2 + 4y = 0$

$4y = -x^2$

$\cancel{4}^2 \frac{dy}{dx} = -\cancel{2}x$

$\frac{dy}{dx} = -\frac{x}{2}$

$m_2 = \left(\frac{dy}{dx}\right)_{(-2,-1)} = \frac{-(-\cancel{2})}{\cancel{2}}$

$\boxed{m_2 = 1}$

The angle between the curves is

$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$

$$= \left| \frac{-\frac{1}{2} - 1}{1 + \left(-\frac{1}{2}\right)(1)} \right|$$

$$= \left| \frac{-\frac{3}{\cancel{2}}}{\frac{1}{\cancel{2}}} \right|$$

$$= |-3|$$

$$\tan\theta = 3$$

$$\theta = \tan^{-1}(3)$$

10) Show that the two curves $x^2 - y^2 = r^2$ and $xy = c^2$ where c, r are constants cut orthogonally.

Soln: Let (x_1, y_1) be the point of intersection of the given curves

$\therefore\ x_1^2 - y_1^2 = r^2$ and

$$x_1 y_1 = c^2$$

Now, $x^2 - y^2 = r^2$

$$2x - 2y\frac{dy}{dx} = 0$$

$\div 2$, $x - y\dfrac{dy}{dx} = 0$

$$y\frac{dy}{dx} = x$$

$$\frac{dy}{dx} = \frac{x}{y}$$

$$\therefore\ m_1 = \left(\frac{dy}{dx}\right)_{(x_1, y_1)} = \frac{x_1}{y_1}$$

$$\therefore\ \boxed{m_1 = \frac{x_1}{y_1}}$$

and $xy = c^2$

$$y = \frac{c^2}{x}$$

$$\frac{dy}{dx} = -\frac{c^2}{x^2}$$

$$\therefore\ m_2 = \left(\frac{dy}{dx}\right)_{(x_1, y_1)} = \frac{-c^2}{x_1^2}$$

$$\therefore\ \boxed{m_2 = -\frac{c^2}{x_1^2}}$$

$$\therefore\ m_1 m_2 = \left(\frac{\cancel{x_1}}{y_1}\right)\left(\frac{-c^2}{x_1^{\cancel{2}}}\right)$$

$$= \frac{-c^2}{x_1 y_1}$$

$$= \frac{-\cancel{c^2}}{\cancel{c^2}} = -1$$

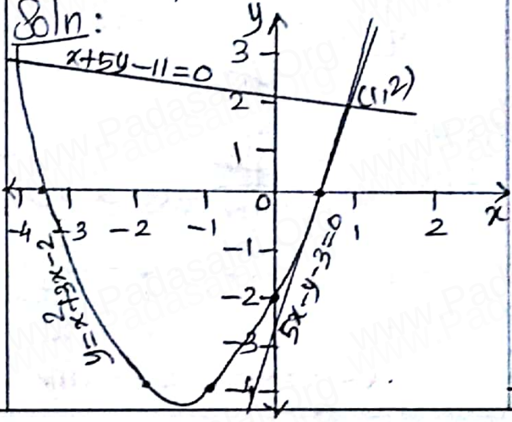
$\therefore\ m_1 m_2 = -1$

$\therefore$ The curves cut orthogonally

Ex 7.11

Find the equations of tangent and normal to the curve $y = x^2 + 3x - 2$ at the point $(1, 2)$

Soln:



Given: $y = x^2 + 3x - 2$ and $(x_1, y_1) = (1, 2)$

$$\frac{dy}{dx} = 2x + 3$$

$$\text{Slope } m = \left(\frac{dy}{dx}\right)_{(1,2)} = 2(1) + 3$$

$$\boxed{m = 5}$$

Equation of tangent is

$$y - y_1 = m(x - x_1)$$

$$y - 2 = 5(x - 1)$$

$$y - 2 = 5x - 5$$

$$5x - y = 5 - 2$$

$$5x - y = 3$$

$$5x - y - 3 = 0$$

Equation of normal is

$$y - y_1 = \frac{-1}{m}(x - x_1)$$

$$y - 2 = \frac{-1}{5}(x - 1)$$

$$5y - 10 = -x + 1$$

$$x + 5y - 10 - 1 = 0$$

$$x + 5y - 11 = 0$$

EX 7.12

For what value of x the tangent of the curve $y=x^3-3x^2+x-2$ is parallel to the line $y=x$

Soln: Given: $y=x$

Compare $y=mx$

$\therefore \boxed{m=1}$

and Given: $y=x^3-3x^2+x-2$

slope $m=\frac{dy}{dx}=3x^2-6x+1$

$\therefore 3x^2-6x+1=1$

$3x^2-6x=1-1$

$3x(x-2)=0$

$3x=0,\ x-2=0$

$\boxed{x=0},\ \boxed{x=2}$

When $x=0$, $y=-2$

When $x=2$, $y=(2)^3-3(2)^2+\cancel{2}-\cancel{2}$

$=8-12$

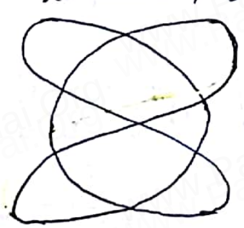
$y=-4$

$\therefore$ The points are $(0,-2)$ and $(2,-4)$

$\therefore$ The tangent is parallel to the line $y=x$

EX 7.13 Find the equation of the tangent and normal to the Lissajous curve given by $x=2\cos 3t$ and $y=3\sin 2t$, $t\in R$

Soln: $x=2\cos 3t$, $y=3\sin 2t$



Lissajous curve

Given: $x=2\cos 3t$ and $y=3\sin 2t$

$\frac{dx}{dt}=2(-\sin 3t(3))$

$\frac{dx}{dt}=-6\sin 3t$

$\frac{dy}{dt}=3\cos 2t(2)$

$\frac{dy}{dt}=6\cos 2t$

slope $m=\frac{\frac{dy}{dt}}{\frac{dx}{dt}}=\frac{\cancel{6}\cos 2t}{-\cancel{6}\sin 3t}$

$\boxed{m=-\frac{\cos 2t}{\sin 3t}}$

Eqn of tangent is

$y-y_1=m(x-x_1)$

$y-3\sin 2t=-\frac{\cos 2t}{\sin 3t}(x-2\cos 3t)$

$y\sin 3t-3\sin 2t\sin 3t=-x\cos 2t+2\cos 2t\cos 3t$

$x\cos 2t+y\sin 3t=3\sin 2t\sin 3t+2\cos 2t\cos 3t$

Eqn of normal is

$y-y_1=\frac{-1}{m}(x-x_1)$

$y-3\sin 2t=\frac{+1}{+\frac{\cos 2t}{\sin 3t}}(x-2\cos 3t)$

$y-3\sin 2t=\frac{\sin 3t}{\cos 2t}(x-2\cos 3t)$

$y\cos 2t-3\sin 2t\cos 2t=x\sin 3t-2\sin 3t\cos 3t$

$x\sin 3t-y\cos 2t=2\sin 3t\cos 3t-3\sin 2t\cos 2t$

$x\sin 3t-y\cos 2t=\sin 6t-\frac{3}{2}2\sin 2t\cos 2t$

$x\sin 3t-y\cos 2t=\sin 6t-\frac{3}{2}\sin 4t$

EX 7.14

Find the acute angle between $y=x^2$ and $y=(x-3)^2$

Soln: Given: $y=x^2$ and $y=(x-3)^2$

$\therefore \quad x^2 = (x-3)^2$

$x^2 = x^2 - 6x + 9$

$x^2 - x^2 = -6x + 9$

$0 = -6x + 9$

$\cancel{6}^{2}x = \cancel{9}^{3}$

$2x = 3 \Rightarrow x = \frac{3}{2}$

When $x = \frac{3}{2}$, $y = \frac{9}{4}$

$\therefore$ The point of intersection is $\left(\frac{3}{2}, \frac{9}{4}\right)$

Now, $y = x^2$

$\frac{dy}{dx} = 2x$

$m_1 = \left(\frac{dy}{dx}\right)_{\left(\frac{3}{2}, \frac{9}{4}\right)} = \cancel{2}\left(\frac{3}{\cancel{2}}\right) = 3$

$\therefore \boxed{m_1 = 3}$

and $y = (x-3)^2$

$\frac{dy}{dx} = 2(x-3)$

$m_2 = \left(\frac{dy}{dx}\right)_{\left(\frac{3}{2}, \frac{9}{4}\right)} = 2\left(\frac{3}{2} - 3\right)$

$= \cancel{2}\left(\frac{-3}{\cancel{2}}\right)$

$\boxed{m_2 = -3}$

The acute angle between two curves is $\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$

$\tan\theta = \left|\frac{3-(-3)}{1+(3)(-3)}\right|$

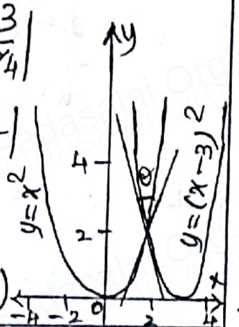
$= \left|\frac{3+3}{1-9}\right|$

$= \left|\frac{\cancel{6}3}{-\cancel{8}4}\right|$

$= \left|-\frac{3}{4}\right|$

$\tan\theta = \frac{3}{4}$

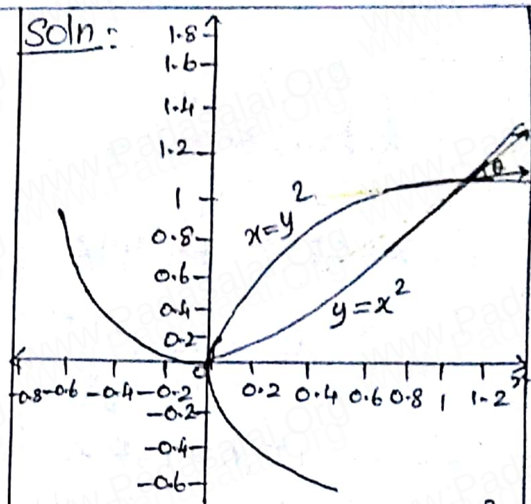
$\theta = \tan^{-1}\left(\frac{3}{4}\right)$



EX 7.15

Find the acute angle between the curves $y = x^2$ and $x = y^2$ at their points of intersection $(0,0)$, $(1,1)$

Soln:



Given: $y = x^2$ and $x = y^2$

$\frac{dy}{dx} = 2x$ and $1 = 2y\frac{dy}{dx}$

$\frac{dy}{dx} = \frac{1}{2y}$

$\therefore m_1 = \frac{dy}{dx} = 2x$ and

$m_2 = \frac{dy}{dx} = \frac{1}{2y}$

Let θ_1 and θ_2 be the acute angles at $(0,0)$ and $(1,1)$ respectively

At $(0,0)$,

$\tan\theta_1 = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$

$= \left|\frac{2x - \frac{1}{2y}}{1 + (2x)\left(\frac{1}{2y}\right)}\right|$

$\tan\theta_1 = \lim_{(x,y)\to(0,0)} \left|\frac{2x - \frac{1}{2y}}{1 + (2x)\left(\frac{1}{2y}\right)}\right|$

$= \lim_{(x,y)\to(0,0)} \left|\frac{\frac{4xy-1}{2y}}{\frac{2y+2x}{2y}}\right|$

$= \lim_{(x,y)\to(0,0)} \left|\frac{4xy-1}{2(x+y)}\right|$

$= \left|\frac{4(0)-1}{2(0+0)}\right|$

$= \left|\frac{-1}{0}\right|$

$\tan\theta_1 = \frac{1}{0} = \infty$

$\theta_1 = \tan^{-1}(\infty)$

$\boxed{\theta_1 = \frac{\pi}{2}}$

At $(1,1)$,

$$\tan\theta_2 = \left|\frac{2x - \frac{1}{2y}}{1+(2x)\left(\frac{1}{2y}\right)}\right|$$

$$= \left|\frac{2(1) - \frac{1}{2(1)}}{1+(2(1))\left(\frac{1}{2(1)}\right)}\right|$$

$$= \left|\frac{2 - \frac{1}{2}}{1+(2)\left(\frac{1}{2}\right)}\right|$$

$$= \left|\frac{\frac{3}{\cancel{2}}}{\frac{4}{\cancel{2}}}\right|$$

$$= \left|\frac{3}{4}\right|$$

$$\tan\theta_2 = \frac{3}{4}$$

$$\theta_2 = \tan^{-1}\left(\frac{3}{4}\right)$$

Ex 7.16

Find the angle of intersection of the curve $y = \sin x$ with the positive x-axis

Soln:

The curve $y = \sin x$ intersects the positive x-axis

When $y = 0$, $\sin x = 0$

$\Rightarrow x = n\pi$, $n = 1, 2, 3 \ldots$

Given: $y = \sin x$

$$m = \left(\frac{dy}{dx}\right)_{x=n\pi} = \cos x = \cos n\pi$$

$$m = (-1)^n$$

$\therefore m = \tan\theta = (-1)^n$

$$\theta = \tan^{-1}((-1)^n)$$

$\therefore$ The required angle of intersection is

$$\tan^{-1}((-1)^n) = \begin{cases} \frac{\pi}{4}, & \text{when } n \text{ is even} \\ \frac{3\pi}{4}, & \text{when } n \text{ is odd} \end{cases}$$

Ex 7.17

If the curves $ax^2 + by^2 = 1$ and $cx^2 + dy^2 = 1$ intersect each other orthogonally then $\frac{1}{a} - \frac{1}{b} = \frac{1}{c} - \frac{1}{d}$

Soln:

The two curves intersect at a point (x_0, y_0)

$$ax_0^2 + by_0^2 = 1$$
$$cx_0^2 + dy_0^2 = 1$$
$$(a-c)x_0^2 + (b-d)y_0^2 = 0$$

Given: $ax^2 + by^2 = 1$ and $cx^2 + dy^2 = 1$

$$ax^2 + by^2 = 1$$

$$2ax + 2by\frac{dy}{dx} = 0$$

$$\cancel{2}by\frac{dy}{dx} = -\cancel{2}ax$$

$$\frac{dy}{dx} = -\frac{ax}{by}$$

$$m_1 = \left(\frac{dy}{dx}\right)_{(x_0,y_0)} = -\frac{ax_0}{by_0}$$

$$\boxed{m_1 = -\frac{ax_0}{by_0}}$$

and $cx^2 + dy^2 = 1$

$$2cx + 2dy\frac{dy}{dx} = 0$$

$$\cancel{2}dy\frac{dy}{dx} = -\cancel{2}cx$$

$$\frac{dy}{dx} = -\frac{cx}{dy}$$

$$m_2 = \left(\frac{dy}{dx}\right)_{(x_0,y_0)} = -\frac{cx_0}{dy_0}$$

$$\boxed{m_2 = -\frac{cx_0}{dy_0}}$$

$\therefore m_1 m_2 = -1$

$$\left(\frac{-ax_0}{by_0}\right)\left(\frac{-cx_0}{dy_0}\right) = -1$$

$$\frac{ac x_0^2}{bd y_0^2} = -1$$

$$ac x_0^2 = -bd y_0^2$$

$ac x_0^2 + bd y_0^2 = 0$, together with $(a-c)x_0^2 + (b-d)y_0^2 = 0$

$$\begin{vmatrix} ac & bd \\ a-c & b-d \end{vmatrix} = 0 \Rightarrow ac(b-d) - (a-c)bd = 0$$

$$ac(b-d) = (a-c)bd$$

$$\Rightarrow \frac{a-c}{ac} = \frac{b-d}{bd}$$

$$\frac{\cancel{a}}{\cancel{a}c} - \frac{\cancel{c}}{a\cancel{c}} = \frac{\cancel{b}}{\cancel{b}d} - \frac{\cancel{d}}{b\cancel{d}}$$

$$\frac{1}{c} - \frac{1}{a} = \frac{1}{d} - \frac{1}{b}$$

$$\frac{1}{a} - \frac{1}{b} = \frac{1}{c} - \frac{1}{d}$$

EX 7.18 Prove that the ellipse $x^2+4y^2=8$ and the hyperbola $x^2-2y^2=4$ intersect orthogonally

Soln: Let the point of intersection of the two curves be (a,b)

Given: $x^2+4y^2=8$ and $x^2-2y^2=4$

(a,b) lies on $x^2+4y^2=8$ and $x^2-2y^2=4$

$\therefore a^2+4b^2=8$ and $a^2-2b^2=4$ —— ①

Given: $x^2+4y^2=8$

$$2x+8y\frac{dy}{dx}=0$$

$\div 2$, $x+4y\frac{dy}{dx}=0$

$$4y\frac{dy}{dx}=-x$$

$$m_1=\left(\frac{dy}{dx}\right)_{(a,b)}=\frac{-x}{4y}=\frac{-a}{4b}$$

$$\therefore \boxed{m_1=\frac{-a}{4b}}$$

and $x^2-2y^2=4$

$$2x-4y\frac{dy}{dx}=0$$

$\div 2$, $x-2y\frac{dy}{dx}=0$

$$2y\frac{dy}{dx}=x$$

$$m_2=\left(\frac{dy}{dx}\right)_{(a,b)}=\frac{x}{2y}=\frac{a}{2b}$$

$$\therefore \boxed{m_2=\frac{a}{2b}}$$

$$\therefore m_1m_2=\left(\frac{-a}{4b}\right)\left(\frac{a}{2b}\right)$$

$$m_1m_2=\frac{-a^2}{8b^2} \quad \text{—— ②}$$

from ①, Cross multiplication we get

	a^2	b^2	1
4	-8	1	4
-2	-4	1	-2

$$\frac{a^2}{-16-16}=\frac{b^2}{-8+4}=\frac{1}{-2-4}$$

$$\frac{a^2}{-32}=\frac{b^2}{-4}=\frac{1}{-6}$$

$$\Rightarrow \frac{a^2}{-32}=\frac{b^2}{-4}$$

$$\frac{a^2}{b^2}=\frac{+\cancel{32}\,8}{+\cancel{4}}$$

$$\boxed{\frac{a^2}{b^2}=8}$$

$$② \Rightarrow m_1m_2=\frac{-a^2}{8b^2}$$

$$=-\frac{1}{\cancel{8}}(\cancel{8})$$

$$m_1m_2=-1$$

$\therefore$ The curves cut orthogonally

7.3 Mean Value Theorem

EXERCISE 7.3

1) Explain why Rolle's theorem is not applicable to the following functions in the respective intervals

(i) $f(x) = \left|\frac{1}{x}\right|, x \in [-1,1]$

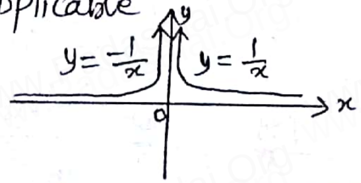
Soln:

Given: $f(x) = \left|\frac{1}{x}\right|, x \in [-1,1]$

$f(x)$ is not continuous on $[-1,1]$

$\therefore f(x)$ is not continuous at $x=0$, $f(x)$ tends to $\pm\infty$ at $x=0$

$\therefore$ Rolle's theorem is not applicable



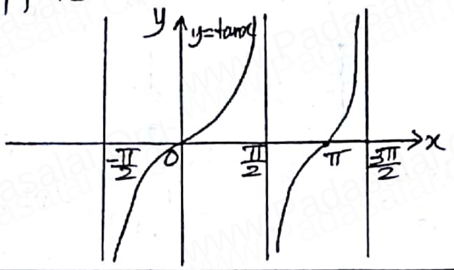
(ii) $f(x) = \tan x, \; x \in [0,\pi]$

Soln: Given: $f(x) = \tan x$, $x \in [0,\pi]$

$f(x)$ is not continuous on $[0,\pi]$, As $\tan x$ tends to $+\infty$ at $x = \frac{\pi}{2}$

$\therefore f(x)$ is not continuous at $x = \pi/2$

$\therefore$ Rolle's theorem is not applicable.



(iii) $f(x) = x - 2\log x, \; x \in [2,7]$

Soln: Given: $f(x) = x - 2\log x$, $x \in [2,7]$

$f(x)$ is continuous on $[2,7]$ and differentiable on $(2,7)$

$$f(2) = 2 - 2\log 2 = 2 - 2(0.301) = 2 - 0.602$$

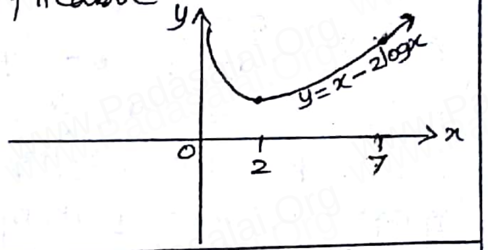
$$f(2) = 1.398$$

$$f(7) = 7 - 2\log 7 = 7 - 2(0.845) = 7 - 1.690$$

$$f(7) = 5.310$$

$\therefore f(2) \neq f(7)$

$\therefore$ Rolle's theorem is not applicable



2) Using the Rolle's theorem, determine the values of x at which the tangent is parallel to the x-axis for the following functions:

(i) $f(x) = x^2 - x, \; x \in [0,1]$

Soln: $f(x)$ is continuous on $[0,1]$ and differentiable on $(0,1)$

$\therefore f(0) = 0, \; f(1) = 0$

$\therefore f(0) = f(1) = 0$

Since $f(x)$ satisfies all the three conditions of Rolle's theorem.

$$f(x) = x^2 - x$$

$$f'(x) = 2x - 1$$

Put $x = c$, $f'(c) = 2c - 1$

$f'(c) = 0$, $2c - 1 = 0$

$$c = \frac{1}{2} \in (0,1)$$

When $x = \frac{1}{2}$, $f(\frac{1}{2}) = -\frac{1}{4}$

$\therefore$ The tangent to the curve $f(x) = x^2 - x$ is parallel to the x-axis at $\left(\frac{1}{2}, -\frac{1}{4}\right)$

(ii) $f(x) = \frac{x^2 - 2x}{x+2}$, $x \in [-1,6]$

Soln:

$f(x)$ is continuous on $[-1,6]$ and differentiable on $(-1,6)$

$$\therefore f(-1) = \frac{(-1)^2 - 2(-1)}{-1+2}$$

$= \frac{1+2}{1} = \frac{3}{1} = 3$

$\boxed{f(-1) = 3}$ and

$f(6) = \frac{(6)^2 - 2(6)}{6+2} = \frac{36-12}{8}$

$= \frac{24^3}{8} = 3$

$\boxed{f(6) = 3}$

$\therefore f(-1) = f(6) = 3$

Since $f(x)$ satisfies all the three conditions of Rolle's theorem

$f(x) = \frac{x^2 - 2x}{x+2}$

$f'(x) = \frac{(x+2)(2x-2) - (x^2-2x) \cdot 1}{(x+2)^2}$

$= \frac{2x^2 - \cancel{2x} + 4x - 4 - x^2 + \cancel{2x}}{(x+2)^2}$

$f'(x) = \frac{x^2 + 4x - 4}{(x+2)^2}$

Put $x = c$, $f'(c) = \frac{c^2+4c-4}{(c+2)^2}$

$f'(c) = 0$, $\frac{c^2+4c-4}{(c+2)^2} = 0$

$c^2 + 4c - 4 = 0$

$a = 1, b = 4, c = -4$

$c = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$= \frac{-4 \pm \sqrt{(4)^2 - 4(1)(-4)}}{2(1)}$

$= \frac{-4 \pm \sqrt{16+16}}{2}$

$= \frac{-4 \pm \sqrt{32}}{2}$

$= \frac{-4 \pm \sqrt{16 \times 2}}{2} = \frac{-4 \pm 4\sqrt{2}}{2}$

$= \frac{\cancel{2}(-2 \pm 2\sqrt{2})}{\cancel{2}}$

$c = -2 \pm 2\sqrt{2}$

$\therefore c = -2 + 2\sqrt{2} \in (-1, 6)$

and $c = -2 - 2\sqrt{2} \notin (-1, 6)$

$f(x) = \frac{x^2 - 2x}{x+2}$

When $x = -2 + 2\sqrt{2}$,

$f(-2+2\sqrt{2}) = \frac{(-2+2\sqrt{2})^2 - 2(-2+2\sqrt{2})}{-\cancel{2} + 2\sqrt{2} + \cancel{2}}$

$= \frac{4 - 8\sqrt{2} + 8 + 4 - 4\sqrt{2}}{2\sqrt{2}}$

$= \frac{16 - 12\sqrt{2}}{2\sqrt{2}} = \frac{2\sqrt{2}(4\sqrt{2} - 6)}{2\sqrt{2}}$

$f(-2+2\sqrt{2}) = 4\sqrt{2} - 6 = -0.344$

$\therefore$ The tangent to the curve $f(x) = \frac{x^2-2x}{x+2}$ is parallel to the x-axis at $(-2+2\sqrt{2}, -0.344)$

(iii) $f(x) = \sqrt{x} - \frac{x}{3}$, $x \in [0, 9]$

Soln: $f(x)$ is continuous on $[0, 9]$ and differentiable on $(0, 9)$

$\therefore f(0) = 0$ and $f(9) = 0$

$\therefore f(0) = f(9) = 0$

Since $f(x)$ satisfies all the three conditions of Rolle's theorem

$f(x) = \sqrt{x} - \frac{x}{3} = x^{1/2} - \frac{x}{3}$

$f'(x) = \frac{1}{2} x^{-1/2} - \frac{1}{3}$

$f'(x) = \frac{1}{2x^{1/2}} - \frac{1}{3}$

$f'(x) = \frac{1}{2\sqrt{x}} - \frac{1}{3}$

Put $x = c$,

$f'(c) = \frac{1}{2\sqrt{c}} - \frac{1}{3}$

$f'(c) = 0$, $\frac{1}{2\sqrt{c}} - \frac{1}{3} = 0$

$\frac{1}{2\sqrt{c}} = \frac{1}{3}$

$3 = 2\sqrt{c}$

On squaring, $9 = 4c$

$\Rightarrow c = \frac{9}{4} \in (0, 9)$

When $x = \frac{9}{4}$,

$f\left(\frac{9}{4}\right) = \sqrt{\frac{9}{4}} - \frac{9/4}{3}$

$= \frac{3}{2} - \frac{9^3}{4} \times \frac{1}{3}$

$= \frac{3}{2} - \frac{3}{4}$

$= \frac{6-3}{4} = \frac{3}{4}$

$f\left(\frac{9}{4}\right) = \frac{3}{4}$

$\therefore$ The tangent to the curve $f(x) = \sqrt{x} - \frac{x}{3}$ is parallel to the x-axis at $\left(\frac{9}{4}, \frac{3}{4}\right)$

3) Explain why Lagrange's mean value theorem is not applicable to the following functions in the respective intervals:

(i) $f(x) = \frac{x+1}{x}, x \in [-1, 2]$

Soln:

Given: $f(x) = \frac{x+1}{x}$, $x \in [-1, 2]$

f(x) is not continuous in $[-1, 2]$

when $x = 0$, $f(0) = \frac{0+1}{0}$

$f(0) = \frac{1}{0} = \infty$

$\therefore$ f(x) is not defined at $x = 0$

$\therefore$ f(x) is not continuous at $x = 0$

$\therefore$ Lagrange's Mean value theorem is not applicable

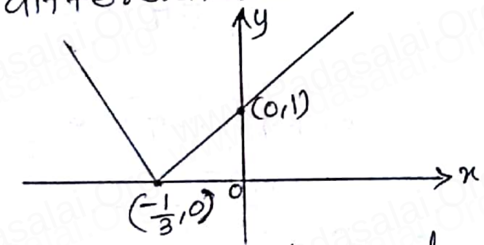
(ii) $f(x) = |3x+1|, x \in [-1, 3]$

Soln:

Given: $f(x) = |3x+1|$, $x \in [-1, 3]$

f(x) is continuous on $[-1, 3]$ but f(x) is not differentiable on $(-1, 3)$

Since f(x) is not differentiable at $x = -1/3$



$\therefore$ Lagrange's Mean value theorem is not applicable

4) Using the Lagrange's mean value theorem determine the value of x at which the tangent is parallel to the secant line at the end points of the given interval:

(i) $f(x) = x^3 - 3x + 2$, $x \in [-2, 2]$

Soln:

f(x) is continuous on $[-2, 2]$ and differentiable on $(-2, 2)$

$\therefore$ By Mean Value Theorem, there exists atleast one point $c \in (-2, 2)$ such that

$$f'(c) = \frac{f(2) - f(-2)}{2+2} \quad \text{—} ①$$

$f'(x) = 3x^2 - 3$

Put $x = c$, $f'(c) = 3c^2 - 3$

$f(2) = (2)^3 - 3(2) + 2 = 4$

$f(-2) = (-2)^3 - 3(-2) + 2 = 0$

$① \Rightarrow 3c^2 - 3 = \frac{4-0}{2+2}$

$3c^2 - 3 = \frac{4}{4}$

$3c^2 - 3 = 1$

$3c^2 = 4 \Rightarrow c^2 = \frac{4}{3}$

$c = \sqrt{\frac{4}{3}} \Rightarrow c = \pm \frac{2}{\sqrt{3}}$

$\therefore c = \pm \frac{2}{\sqrt{3}} \in (-2, 2)$

At $x = \pm \frac{2}{\sqrt{3}}$ is the value of tangent line is parallel to the secant line

(ii) $f(x) = (x-2)(x-7)$, $x \in [3, 11]$

Soln: Given: $f(x) = (x-2)(x-7)$, $x \in [3, 11]$

$f(x) = x^2 - 9x + 14$

f(x) is continuous on $[3, 11]$

and differentiable on $(3,11)$

$\therefore$ By Mean Value Theorem, there exists atleast one Point $c \in (3,11)$ Such that

$$f'(c) = \frac{f(11)-f(3)}{11-3} \quad \text{--- ①}$$

$f'(x) = 2x-9$

Put $x=c$, $f'(c)=2c-9$

$f(11)=(11)^2-9(11)+14$

$f(11)=121-99+14=36$

$f(3)=(3)^2-9(3)+14$

$f(3)=9-27+14=-4$

① $\Rightarrow 2c-9 = \dfrac{36+4}{8} = \cancel{\dfrac{40}{8}}\,5$

$2c-9=5$

$2c=9+5=14$

$\cancel{2}c=\cancel{14}\,7$

$c=7 \in (3,11)$

At $x=7$ is the value of tangent line is parallel to the Secant line

5) Show that the value in the Conclusion of the mean value theorem for

(i) $f(x)=\frac{1}{x}$ on a closed interval of positive numbers $[a,b]$ is $\sqrt{ab}$

Soln: Given: $f(x)=\frac{1}{x}$, $x\in[a,b]$, where a and b are positive numbers

$f(x)$ is Continuous on $[a,b]$ and differentiable on (a,b)

$\therefore$ By Mean Value Theorem, there exists atleast one Point $c\in(a,b)$ Such that

$$f'(c)=\frac{f(b)-f(a)}{b-a} \quad \text{--- ①}$$

$f'(x)=\frac{-1}{x^2}$

Put $x=c$, $f'(c)=\frac{-1}{c^2}$

$f(b)=\frac{1}{b}$, $f(a)=\frac{1}{a}$

① $\Rightarrow \dfrac{-1}{c^2} = \dfrac{\frac{1}{b}-\frac{1}{a}}{b-a}$

$\dfrac{-1}{c^2} = \dfrac{a-b}{ab}\times\dfrac{1}{b-a}$

$= \dfrac{\cancel{(a-b)}}{ab}\times\dfrac{1}{-\cancel{(a-b)}}$

$\cancel{-}\dfrac{1}{c^2} = \cancel{-}\dfrac{1}{ab}$

$\Rightarrow c^2=ab \Rightarrow c=\sqrt{ab}$

$\therefore c=\sqrt{ab}\in(a,b)$

$\therefore$ The value of the given function is $\sqrt{ab}$

(ii) $f(x)=Ax^2+Bx+C$ on any interval $[a,b]$ is $\frac{a+b}{2}$

Soln: Given $f(x)=Ax^2+Bx+C$, $x\in[a,b]$

$f(x)$ is Continuous on $[a,b]$ and differentiable on (a,b)

$\therefore$ By Mean Value Theorem there exists atleast one point $c\in(a,b)$ Such that

$$f'(c)=\frac{f(b)-f(a)}{b-a} \quad \text{--- ①}$$

$f'(x)=2Ax+B$

Put $x=c$, $f'(c)=2AC+B$

$f(b)=Ab^2+Bb+C$

$f(a)=Aa^2+Ba+C$

① $\Rightarrow 2AC+B = \dfrac{Ab^2+Bb+\cancel{C}-Aa^2-Ba-\cancel{C}}{b-a}$

$2AC+B=\dfrac{Ab^2-Aa^2+Bb-Ba}{b-a}$

$2AC+B=\dfrac{A(b^2-a^2)+B(b-a)}{b-a}$

$=\dfrac{A(b-a)(b+a)+B(b-a)}{b-a}$

$=\dfrac{\cancel{(b-a)}[A(b+a)+B]}{\cancel{(b-a)}}$

$2AC+\cancel{B}=A(b+a)+\cancel{B}$

$2\cancel{A}C=\cancel{A}(b+a)$

$2c = a+b$

$c = \frac{a+b}{2} \in (a,b)$

∴ The value of the given function is $\frac{a+b}{2}$

6) A race car driver is racing at 20th km. If his speed never exceeds 150 km/hr, what is the maximum distance he can cover in the next two hours

Soln:

Let $s(t)$ be the function of distance at time t

Given: $s(t_1)=20$, $s(t_2)=?$

$t_1=0$, $t_2=2$

$s(t)$ is continuous on $[0,2]$ and differentiable on $(0,2)$

∴ By Mean Value Theorem, there exists atleast one point $c \in (0,2)$ such that $s'(t) = \frac{s(t_2)-s(t_1)}{t_2-t_1}$ —(1)

$\frac{s(t_2)-20}{2-0} = s'(t)$

$\frac{s(t_2)-20}{2} \leq 150$

$s(t_2)-20 \leq 300$

$s(t_2) \leq 300+20$

$s(t_2) \leq 320$

∴ The maximum distance he can cover in the next two hours is 320 km

7) Suppose that for a function $f(x)$, $f'(x) \leq 1$ for all $1 \leq x \leq 4$. Show that $f(4)-f(1) \leq 3$

Soln: Given: $f'(x) \leq 1$

$f(x)$ is continuous on $[1,4]$ and differentiable on $(1,4)$

∴ By Mean Value Theorem, there exists atleast one point $c \in (1,4)$ such that

$f'(x) = \frac{f(4)-f(1)}{4-1}$

$\frac{f(4)-f(1)}{3} = f'(x)$

$\frac{f(4)-f(1)}{3} \leq 1$

$f(4)-f(1) \leq 3$

8) Does there exist a differentiable function $f(x)$ such that $f(0)=-1$, $f(2)=4$ and $f'(x) \leq 2$ for all x. Justify your answer.

Soln: Given: $f'(x) \leq 2$ —(1)

$f(x)$ is continuous on $[0,2]$ and differentiable on $(0,2)$

∴ By Mean Value Theorem, there exists atleast one point $c \in (0,2)$ such that

$f'(x) = \frac{f(2)-f(0)}{2-0}$

$= \frac{4+1}{2}$

$= \frac{5}{2}$

$f'(x) = 2.5 \notin (0,2)$ —(2)

From (1) & (2) we get,

No. Since $f'(x)$ cannot be 2.5 at any point in $(0,2)$

9) Show that there lies a point on the curve $f(x)=x(x+3)e^{-\pi/2}$, $-3 \leq x \leq 0$

where tangent drawn is parallel to the x-axis

Soln: Given: $f(x)=x(x+3)e^{-\pi/2}$, $x\in[-3,0]$

$f(x)$ is continuous on $[-3,0]$ and differentiable on $(-3,0)$

$f(-3)=0$ and $f(0)=0$

$\therefore f(-3)=f(0)=0$

By Rolle's theorem, there exists atleast one point $c\in(-3,0)$ such that $f'(c)=0$

$$f(x)=(x^2+3x)e^{-\pi/2}$$

$$f'(x)=(2x+3)e^{-\pi/2}$$

Put $x=c$, $f'(c)=(2c+3)e^{-\pi/2}$

$f'(c)=0$, $(2c+3)e^{-\pi/2}=0$

$$2c+3=0$$

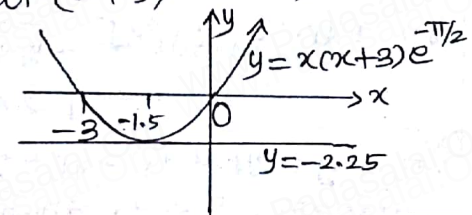
$$2c=-3 \Rightarrow c=-\frac{3}{2}$$

$$c=-1.5\in(-3,0)$$

when $x=-1.5$,

$$f(-1.5)=((-1.5)^2+3(-1.5))e^{-\pi/2}$$
$$=(2.25-4.50)(1)$$
$$f(-1.5)=-2.25$$

$\therefore$ The tangent to the curve $f(x)=x(x+3)e^{-\pi/2}$ is parallel to the x-axis at $(-1.5,-2.25)$



10) Using mean value theorem prove that for, $a>0, b>0$, $|e^{-a}-e^{-b}|<|a-b|$

Soln: $f(x)=e^{-x}$

$f(x)$ is continuous on $[a,b]$ and differentiable on (a,b)

By Mean Value Theorem, there exists atleast one point $c\in(a,b)$ such that

$$f'(c)=\frac{f(b)-f(a)}{b-a} \quad \to (1)$$

$$f(x)=e^{-x}$$

$$f'(x)=-e^{-x}$$

Put $x=c$, $f'(c)=-e^{-c}$

$f(a)=e^{-a}$ and $f(b)=e^{-b}$

$$(1)\Rightarrow \frac{e^{-b}-e^{-a}}{b-a}=-e^{-c}$$

$$\frac{-(e^{-a}-e^{-b})}{-(a-b)}=-e^{-c}$$

$$\frac{e^{-a}-e^{-b}}{a-b}=-e^{-c}$$

Taking modulus on both sides, we get,

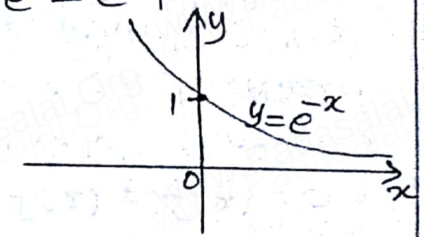
$$\left|\frac{e^{-a}-e^{-b}}{a-b}\right|=|-e^{-c}|$$

$$\frac{|e^{-a}-e^{-b}|}{|a-b|}=|e^{-c}|\le 1$$

[since $e^{-x}\le 1$]

$$\frac{|e^{-a}-e^{-b}|}{|a-b|}\le 1$$

$$|e^{-a}-e^{-b}|\le|a-b|$$



EX 7.19 Compute the value of 'c' satisfied by the Rolle's theorem for the function $f(x)=x^2(1-x)^2$, $x\in[0,1]$

Soln: $f(x)$ is continuous on $[0,1]$ and differenti

-able on $(0,1)$

$f(0)=0$ and $f(1)=0$

$\therefore f(0)=f(1)=0$

By Rolle's theorem, there exists atleast one point $c\in(0,1)$ such that $f'(c)=0$

$f(x)=x^2(1-x)^2$

$f'(x)=x^2\,2(1-x)(-1)+(1-x)^2(2x)$

$=2x(1-x)[-x+1-x]$

$=2x(1-x)(1-2x)$

$f'(x)=2x(1-x)(1-2x)$

Put $x=c$,

$f'(c)=2c(1-c)(1-2c)$

$f'(c)=0,\ 2c(1-c)(1-2c)=0$

$2c=0,\ 1-c=0,\ 1-2c=0$

$c=0,\ c=1,\ c=\frac{1}{2}$

$\Rightarrow c=\frac{1}{2}\in(0,1)$

EX 7.20

Find the values in the interval $(\frac{1}{2},2)$ satisfied by the Rolle's theorem for the function $f(x)=x+\frac{1}{x}$, $x\in[\frac{1}{2},2]$

Soln: $f(x)$ is continuous on $[\frac{1}{2},2]$ and differen-tiable on $(\frac{1}{2},2)$

$f(\frac{1}{2})=\frac{1}{2}+\frac{1}{1/2}=\frac{1}{2}+2=\frac{5}{2}$

$f(2)=2+\frac{1}{2}=\frac{5}{2}$

$\therefore f(\frac{1}{2})=f(2)=\frac{5}{2}$

By Rolle's theorem, there exists atleast one point $c\in(\frac{1}{2},2)$ such that $f'(c)=0$

$f(x)=x+\frac{1}{x}$

$f'(x)=1-\frac{1}{x^2}$

Put $x=c$, $f'(c)=1-\frac{1}{c^2}$

$f'(c)=0,\ 1-\frac{1}{c^2}=0$

$\Rightarrow \frac{1}{c^2}=1$

$\Rightarrow c^2=1$

$\Rightarrow c=\pm1$

$\Rightarrow c=1\in(\frac{1}{2},2)$ and

$c=-1\notin(\frac{1}{2},2)$

$\therefore c=1$

EX 7.21

Compute the value of 'c' satisfied by Rolle's theorem for the function $f(x)=\log\left(\frac{x^2+6}{5x}\right)$ in the interval $[2,3]$

Soln: $f(x)$ is continuous on $[2,3]$ and differenti-able on $(2,3)$

$f(2)=\log\left(\frac{4+6}{10}\right)=\log\left(\frac{10}{10}\right)=\log 1=0$

$f(3)=\log\left(\frac{9+6}{15}\right)=\log\left(\frac{15}{15}\right)=\log 1=0$

$\therefore f(2)=f(3)=0$

By Rolle's theorem, there exists atleast one point $c\in(2,3)$ such that $f'(c)=0$

$f(x)=\log\left(\frac{x^2+6}{5x}\right)$

$f'(x)=\frac{1}{\frac{x^2+6}{5x}}\left(\frac{(5x)(2x)-(x^2+6)5}{(5x)^2}\right)$

$=\frac{5x}{x^2+6}\left(\frac{10x^2-5x^2-30}{(5x)^{\cancel{2}}}\right)$

$=\frac{1}{x^2+6}\left(\frac{5x^2-30}{5x}\right)$

$= \frac{1}{x^2+6}\left(\frac{\cancel{2}(x^2-6)}{\cancel{2}x}\right)$

$f'(x) = \frac{x^2-6}{x(x^2+6)}$

Put $x=c$, $f'(c) = \frac{c^2-6}{c(c^2+6)}$

$f'(c)=0$, $\frac{c^2-6}{c(c^2+6)} = 0$

$c^2-6=0$

$c^2=6$

$c=\pm\sqrt{6}$

$\therefore c=\sqrt{6}\in(2,3)$ and

$c=-\sqrt{6}\notin(2,3)$

$\therefore c=\sqrt{6}$ Satisfies the Rolle's theorem

Ex 7.22 without actually solving show that the equation $x^4+2x^3-2=0$ has only one real root in the interval $(0,1)$

Soln:

Let $f(x) = x^4+2x^3-2$

$f(x)$ is continuous on $[0,1]$ and differentiable on $(0,1)$

$f'(x) = 4x^3+6x^2$

$f'(x)=0$, $4x^3+6x^2=0$

$2x^2(2x+3)=0$

$2x^2=0$, $2x+3=0$

$x^2=0$, $2x=-3$

$x=0$ (twice), $x=-\frac{3}{2}$

$\therefore x=0$ and $x=-\frac{3}{2}$

$\therefore x=0, -\frac{3}{2}\notin(0,1)$

Thus $f'(x)>0$, $\forall x\in(0,1)$

The Rolle's theorem there do not exist $a,b\in(0,1)$ such that $f(a)=0=f(b)$

$\therefore$ The eqn $f(x)=0$ can not have two roots in the interval $(0,1)$. But $f(0)=-2<0$ and $f(1)=1>0$

$\therefore$ The curve $y=f(x)$ crosses the x-axis between 0 and 1 only once by the intermediate value theorem.

$\therefore x^4+2x^3-2=0$ has only one real root in the interval $(0,1)$

Ex 7.23 Prove using the Rolle's theorem that between any two distinct real zeros of the polynomial $a_nx^n+a_{n-1}x^{n-1}+\cdots+a_1x+a_0$ there is a zero of the polynomial $na_nx^{n-1}+(n-1)a_{n-1}x^{n-2}+\cdots+a_1$

Soln:

Let $p(x)=a_nx^n+a_{n-1}x^{n-1}+\cdots+a_1x+a_0$

Let $\alpha<\beta$ be two real zeros of $p(x)$.

$\therefore p(x)$ is continuous on $[\alpha,\beta]$ and differentiable on (α,β)

$\therefore p(\alpha)=p(\beta)=0$

By Rolle's theorem, there exists atleast one point $\gamma\in(\alpha,\beta)$ such that $p'(\gamma)=0$

$p'(x)=na_nx^{n-1}+(n-1)a_{n-1}x^{n-2}+\cdots+a_1$

which completes the proof

EX 7.24

Prove that there is a zero of the polynomial, $2x^3-9x^2-11x+12$ in the interval $(2,7)$ given that 2 and 7 are the zeros of the polynomial $x^4-6x^3-11x^2+24x+28$

Soln: Let $\alpha=2$, $\beta=7$

Let $P(x)=x^4-6x^3-11x^2+24x+28$

Let $2<7$ be two real zeros of $P(x)$.

$P(x)$ is continuous on $[2,7]$ and differentiable on $(2,7)$

$P(2)=(2)^4-6(2)^3-11(2)^2+24(2)+28=16-48-44+48+28=0$

$P(7)=(7)^4-6(7)^3-11(7)^2+24(7)+28$
$=2401-2058-539+168+28$

$P(7)=0$

$\therefore P(2)=P(7)=0$

By Rolle's theorem, there exists atleast one point $\gamma\in(2,7)$ such that

$P'(\gamma)=0$

$P'(x)=4x^3-18x^2-22x+24$

$P'(x)=2(2x^3-9x^2-11x+12)$

$\frac{P'(x)}{2}=2x^3-9x^2-11x+12=Q(x)$

$\Rightarrow$ There is a zero of the polynomial $Q(x)$ in the interval $(2,7)$

EX 7.25 Find the values in the interval $(1,2)$ of the mean value theorem satisfied by the function $f(x)=x-x^2$ for $1\le x\le 2$

Soln: $f(x)$ is continuous on $[1,2]$ and differentiable on $(1,2)$

By Mean Value Theorem, there exists atleast one point $c\in(1,2)$ such that

$f'(c)=\frac{f(2)-f(1)}{2-1}$ —①

$f'(x)=1-2x$

Put $x=c$, $f'(c)=1-2c$

$f(2)=2-4=-2$

$f(1)=1-1=0$

①$\Rightarrow$ $1-2c=\frac{-2-0}{1}$

$1-2c=-2$

$2c=1+2$

$2c=3$

$c=\frac{3}{2}\in(1,2)$

$\therefore$ The value of c is $\frac{3}{2}$

EX 7.26

A truck travels on a toll road with a speed limit of 80 km/hr. The truck completes a 164 km journey in 2 hours. At the end of the toll road the trucker is issued with a speed violation ticket. Justify this using the Mean Value Theorem

Soln:

Let $f(t)$ be the distance travelled by the trucker in 't' hours.

$f(t)$ is continuous on $[0,2]$ and differentiable on $(0,2)$

$\therefore$ By Mean value theorem, there exists atleast one point $c\in(0,2)$ such that $f'(c)=\frac{f(2)-f(0)}{2-0}$

$=\frac{164-0}{2}=\frac{\cancel{164}^{82}}{\cancel{2}}=82>80$

$\therefore$ The during the travel in 2 hours the trucker must have travelled with a speed more than 80 km which justifies the issuance of a Speed violation ticket

EX 7.27

Suppose $f(x)$ is a differentiable function for all x with $f'(x) \le 29$ and $f(2)=17$. what is the maximum value of $f(7)$?

Soln:

$f(x)$ is continuous on $[2,7]$ and differentiable on $(2,7)$

$\therefore$ By Mean Value Theorem, there exists atleast one point $c \in (2,7)$ such that

$$f'(c) = \frac{f(7)-f(2)}{7-2} \quad \text{—①}$$

Given: $f(2)=17$ and $f'(x) \le 29$

$$① \Rightarrow \frac{f(7)-f(2)}{7-2} = f'(c)$$

$$\frac{f(7)-17}{5} \le 29$$

$$f(7)-17 \le 29 \times 5$$

$$f(7)-17 \le 145$$

$$f(7) \le 145+17$$

$$f(7) \le 162$$

$\therefore$ The maximum value of $f(7)$ is 162

EX 7.28

Prove, using mean value theorem, that $|\sin\alpha - \sin\beta| \le |\alpha-\beta|$, $\alpha, \beta \in R$

Soln: Let $f(x) = \sin x$

$f(x)$ is continuous on $[\alpha,\beta]$ and differentiable on (α,β)

$\therefore$ By Mean Value Theorem, there exists atleast one point $c \in (\alpha,\beta)$ such that

$$f'(c) = \frac{f(\beta)-f(\alpha)}{\beta-\alpha} \quad \text{—①}$$

$$f(x) = \sin x$$

$$\Rightarrow f'(x) = \cos x$$

Put $x=c$, $f'(c) = \cos c$

$f(\beta) = \sin\beta$, $f(\alpha) = \sin\alpha$

$$① \Rightarrow \frac{f(\beta)-f(\alpha)}{\beta-\alpha} = f'(c)$$

$$\frac{\sin\beta - \sin\alpha}{\beta-\alpha} = \cos c$$

$$\left|\frac{\sin\beta-\sin\alpha}{\beta-\alpha}\right| = |\cos c|$$

$$\left|\frac{+(\sin\alpha-\sin\beta)}{+(\alpha-\beta)}\right| = |\cos c|$$

$$\frac{|\sin\alpha-\sin\beta|}{|\alpha-\beta|} = |\cos c| \le 1$$

$$|\sin\alpha - \sin\beta| \le |\alpha-\beta|, \quad \alpha,\beta \in R$$

EX 7.29

A thermometer was taken from a freezer and placed in a boiling water. It took 22 seconds for the thermometer to raise from $-10°c$ to $100°c$. Show that the rate of change of temperature at some time t is $5°c$ per second

Soln:

Let $f(t)$ be the temperature at time t,

By Mean value Theorem, we have $f'(c) = \frac{f(b)-f(a)}{b-a}$

$$= \frac{100-(-10)}{22}$$

$$= \frac{100+10}{22} = \frac{\cancel{110}^{5}}{\cancel{22}}$$

$$= 5°c \text{ per second}$$

$\therefore$ The instantaneous rate of change of temperature at some time t should be $5°c$ per second

7.4 Series Expansions

EXERCISE 7.4

1) Write the Maclaurin Series expansion of the following functions:

(i) e^x

Functions and its derivatives	e^x and its derivatives	Value at $x=0$
$f(x)$	e^x	1
$f'(x)$	e^x	1
$f''(x)$	e^x	1
⋮	⋮	⋮

$f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \cdots$

$e^x = 1 + \frac{1}{1!}x + \frac{1}{2!}x^2 + \cdots$

$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots$ for all x

(ii) $\sin x$

Functions and its derivatives	$\sin x$ and its derivatives	Value at $x=0$
$f(x)$	$\sin x$	0
$f'(x)$	$\cos x$	1
$f''(x)$	$-\sin x$	0
$f'''(x)$	$-\cos x$	-1
$f^{IV}(x)$	$\sin x$	0
$f^{V}(x)$	$\cos x$	1
$f^{VI}(x)$	$-\sin x$	0
$f^{VII}(x)$	$-\cos x$	-1
⋮	⋮	⋮

$f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{IV}(0)}{4!}x^4 + \frac{f^{V}(0)}{5!}x^5 + \frac{f^{VI}(0)}{6!}x^6 + \frac{f^{VII}(0)}{7!}x^7 + \cdots$

$\sin x = 0 + \frac{1}{1!}x + \frac{0}{2!}x^2 - \frac{1}{3!}x^3 + \frac{0}{4!}x^4 + \frac{1}{5!}x^5 + \frac{0}{6!}x^6 - \frac{1}{7!}x^7 + \cdots$

$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$

(iii) $\cos x$

Functions and its derivatives	$\cos x$ and its derivatives	Value at $x=0$
$f(x)$	$\cos x$	1
$f'(x)$	$-\sin x$	0
$f''(x)$	$-\cos x$	-1
$f'''(x)$	$\sin x$	0
$f^{IV}(x)$	$\cos x$	1
$f^{V}(x)$	$-\sin x$	0
$f^{VI}(x)$	$-\cos x$	-1
⋮	⋮	⋮

$f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{IV}(0)}{4!}x^4 + \frac{f^{V}(0)}{5!}x^5 + \frac{f^{VI}(0)}{6!}x^6 + \cdots$

$\cos x = 1 + \frac{0}{1!}x - \frac{1}{2!}x^2 + \frac{0}{3!}x^3 + \frac{1}{4!}x^4 + \frac{0}{5!}x^5 - \frac{1}{6!}x^6 + \cdots$

$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$

(iv) $\log(1-x)$; $-1 \le x < 1$

Functions and its derivatives	$\log(1-x)$ and its derivatives	Value at $x=0$
$f(x)$	$\log(1-x)$	0
$f'(x)$	$\frac{-1}{1-x}$	-1
$f''(x)$	$\frac{-1}{(1-x)^2}$	-1
$f'''(x)$	$\frac{-2}{(1-x)^3}$	-2
$f^{IV}(x)$	$\frac{-6}{(1-x)^4}$	-6

⋮	⋮	⋮

$f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{IV}(0)}{4!}x^4 + \cdots$

$\log(1-x) = 0 - \frac{1}{1!}x - \frac{1}{2!}x^2 - \frac{2}{3!}x^3 - \frac{6}{4!}x^4 - \cdots$

$\log(1-x) = -x - \frac{x^2}{2!} - \frac{\cancel{2}x^3}{\cancel{6}3} - \frac{\cancel{6}x^4}{\cancel{24}4} - \cdots$

$\log(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \cdots$

$\log(1-x) = -\left(x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \cdots\right)$

$-1 \le x < 1$

(v) $\tan^{-1}(x)$; $-1 \le x \le 1$

Functions and its derivatives	$\tan^{-1}(x)$ and its derivatives	Value at $x=0$
$f(x)$	$\tan^{-1}(x)$	0
$f'(x)$	$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \cdots$	1
$f''(x)$	$-2x + 4x^3 - 6x^5 \cdots$	0
$f'''(x)$	$-2 + 12x^2 - 30x^4 + \cdots$	-2
$f^{IV}(x)$	$24x - 120x^3 + \cdots$	0
$f^{V}(x)$	$24 - 360x^2 + \cdots$	24
$f^{VI}(x)$	$-720x + \cdots$	0
$f^{VII}(x)$	$-720 + \cdots$	-720

$f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{IV}(0)}{4!}x^4 + \frac{f^{V}(0)}{5!}x^5 + \frac{f^{VI}(0)}{6!}x^6 + \frac{f^{VII}(0)}{7!}x^7 + \cdots$

$\tan^{-1}x = 0 + \frac{1}{1!}x + \frac{0}{2!}x^2 - \frac{2}{3!}x^3 + \frac{0}{4!}x^4 + \frac{24}{5!}x^5 + \frac{0}{6!}x^6 - \frac{720}{7!}x^7 + \cdots$

$\tan^{-1}x = x - \frac{\cancel{2}}{\cancel{6}3}x^3 + \frac{\cancel{24}}{\cancel{120}5}x^5 - \frac{\cancel{720}}{7\times\cancel{720}}x^7 + \cdots$

$\tan^{-1}x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$

(vi) $\cos^2 x$

Functions and its derivatives	$\cos^2 x$ and its derivatives	Value at $x=0$
$f(x)$	$\cos^2 x$	1
$f'(x)$	$2\cos x(-\sin x) = -\sin 2x$	0
$f''(x)$	$-2\cos 2x$	-2
$f'''(x)$	$4\sin 2x$	0
$f^{IV}(x)$	$8\cos 2x$	8
$f^{V}(x)$	$-16\sin 2x$	0
$f^{VI}(x)$	$-32\cos 2x$	-32
⋮	⋮	

$f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{IV}(0)}{4!}x^4 + \frac{f^{V}(0)}{5!}x^5 + \frac{f^{VI}(0)}{6!}x^6 + \cdots$

$\cos^2 x = 1 + \frac{0}{1!}x - \frac{2}{2!}x^2 + \frac{0}{3!}x^3 + \frac{8}{4!}x^4 + \frac{0}{5!}x^5 - \frac{32}{6!}x^6 + \cdots$

$\cos^2 x = 1 - \frac{2}{2!}x^2 + \frac{2^3}{4!}x^4 - \frac{2^5}{6!}x^6 + \cdots$

$\cos^2 x = 1 - \frac{2x^2}{2!} + \frac{2^3x^4}{4!} - \frac{2^5x^6}{6!} + \cdots$

2) Write down the Taylor Series expansion, of the function $\log x$ about $x=1$ upto three non-zero terms for $x>0$

Soln:

Functions and its derivatives	$\log x$ and its derivatives	value at $x=1$
$f(x)$	$\log x$	0
$f'(x)$	$\frac{1}{x}$	1
$f''(x)$	$\frac{-1}{x^2}$	-1
$f'''(x)$	$\frac{2}{x^3}$	2
$f^{IV}(x)$	$\frac{-6}{x^4}$	-6
⋮	⋮	⋮

$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \frac{f^{IV}(a)}{4!}(x-a)^4 + \cdots$

$\log x = f(1) + \frac{f'(1)}{1!}(x-1) + \frac{f''(1)}{2!}(x-1)^2 + \frac{f'''(1)}{3!}(x-1)^3 + \frac{f^{IV}(1)}{4!}(x-1)^4 + \cdots$

$$\log x = 0 + \frac{1}{1!}(x-1) - \frac{1}{2!}(x-1)^2 + \frac{2}{3!}(x-1)^3 - \frac{6}{4!}(x-1)^4 + \cdots$$

$$\log x = (x-1) - \frac{1}{2}(x-1)^2 + \frac{\cancel{2}^{1}}{\cancel{6}_3}(x-1)^3 - \frac{\cancel{6}^{1}}{\cancel{24}_4}(x-1)^4 + \cdots$$

$$\log x = (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4 + \cdots$$

3) Expand $\sin x$ in ascending powers $x - \frac{\pi}{4}$ upto three non-zero terms

Soln:

Functions and its derivatives	$\sin x$ and its derivatives	Value at $x = \pi/4$
$f(x)$	$\sin x$	$\frac{1}{\sqrt{2}}$
$f'(x)$	$\cos x$	$\frac{1}{\sqrt{2}}$
$f''(x)$	$-\sin x$	$-\frac{1}{\sqrt{2}}$
$f'''(x)$	$-\cos x$	$-\frac{1}{\sqrt{2}}$
$\vdots$	$\vdots$	$\vdots$

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots$$

$$\sin x = f\left(\frac{\pi}{4}\right) + \frac{f'\left(\frac{\pi}{4}\right)}{1!}\left(x - \pi/4\right) + \frac{f''(\pi/4)}{2!}(x-\pi/4)^2 + \frac{f'''(\pi/4)}{3!}(x-\pi/4)^3 + \cdots$$

$$\sin x = \frac{1}{\sqrt{2}} + \frac{\frac{1}{\sqrt{2}}}{1!}\left(x - \frac{\pi}{4}\right) + \frac{(-1/\sqrt{2})}{2!}\left(x-\frac{\pi}{4}\right)^2 + \frac{(-1/\sqrt{2})}{3!}\left(x-\frac{\pi}{4}\right)^3 + \cdots$$

$$\sin x = \frac{1}{\sqrt{2}} + \frac{1}{1!\sqrt{2}}\left(x - \frac{\pi}{4}\right) - \frac{1}{2!\sqrt{2}}\left(x-\frac{\pi}{4}\right)^2 - \frac{1}{3!\sqrt{2}}\left(x-\frac{\pi}{4}\right)^3 + \cdots$$

$$\sin x = \frac{1}{\sqrt{2}}\left[1 + \frac{1}{1!}\left(x-\frac{\pi}{4}\right) - \frac{1}{2!}\left(x-\frac{\pi}{4}\right)^2 - \frac{1}{3!}\left(x-\frac{\pi}{4}\right)^3 + \cdots\right]$$

$$= \frac{1}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}}\left[1 + \frac{1}{1!}\left(x-\frac{\pi}{4}\right) - \frac{1}{2!}\left(x-\frac{\pi}{4}\right)^2 - \frac{1}{3!}\left(x-\frac{\pi}{4}\right)^3 + \cdots\right]$$

$$\sin x = \frac{\sqrt{2}}{2}\left[1 + \frac{1}{1!}\left(x-\frac{\pi}{4}\right) - \frac{1}{2!}\left(x-\frac{\pi}{4}\right)^2 - \frac{1}{3!}\left(x-\frac{\pi}{4}\right)^3 + \cdots\right]$$

4) Expand the polynomial $f(x) = x^2 - 3x + 2$ in powers of $x-1$

Soln:

Functions and its derivatives	x^2-3x+2 and its derivatives	Value at $x=1$
$f(x)$	x^2-3x+2	0
$f'(x)$	$2x-3$	-1
$f''(x)$	2	2
$f'''(x)$	0	0

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots$$

$$x^2 - 3x + 2 = f(1) + \frac{f'(1)}{1!}(x-1) + \frac{f''(1)}{2!}(x-1)^2 + \frac{f'''(1)}{3!}(x-1)^3 + \cdots$$

$$x^2 - 3x + 2 = 0 + \frac{(-1)}{1!}(x-1) + \frac{2}{2!}(x-1)^2 + \frac{0}{3!}(x-1)^3 + \cdots$$

$$x^2 - 3x + 2 = -(x-1) + \frac{\cancel{2}}{\cancel{2}}(x-1)^2 + 0$$

$$x^2 - 3x + 2 = -(x-1) + (x-1)^2$$

EX 7.30

Expand $\log(1+x)$ as a Maclaurin's series upto 4 non-zero terms for $-1 < x \le 1$

Soln:

Function and its derivatives	$\log(1+x)$ and its derivatives	Value at $x=0$
$f(x)$	$\log(1+x)$	0

$f'(x)$	$\frac{1}{1+x}$	1
$f''(x)$	$\frac{-1}{(1+x)^2}$	-1
$f'''(x)$	$\frac{2}{(1+x)^3}$	2
$f^{IV}(x)$	$\frac{-6}{(1+x)^4}$	-6

$$f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{IV}(0)}{4!}x^4 + \cdots$$

$$\log(1+x) = 0 + \frac{1}{1!}x - \frac{1}{2!}x^2 + \frac{2}{3!}x^3 - \frac{6}{4!}x^4 + \cdots$$

$$\log(1+x) = x - \frac{x^2}{2} + \frac{\cancel{2}}{\cancel{6}3}x^3 - \frac{\cancel{6}}{\cancel{24}4}x^4 + \cdots$$

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots;$$

$$-1 < x \le 1$$

Ex 7.31

Expand $\tan x$ in ascending Powers of x upto 5th Power for $-\frac{\pi}{2} < x < \frac{\pi}{2}$

Soln:

Function and its derivatives	$\tan x$ and its derivatives	Value at $x=0$
$f(x)$	$\tan x$	0
$f'(x)$	$\sec^2 x = 1+\tan^2 x = 1+f^2(x)$	1
$f''(x)$	$2f(x)f'(x)$	0
$f'''(x)$	$2[f(x)f''(x)+f'(x)f'(x)]$ $=2[f(x)f''(x)+(f'(x))^2]$	2
$f^{IV}(x)$	$2[f(x)f'''(x)+f'(x)f''(x) + 2f'(x)f''(x)]$	0
$f^{V}(x)$	$2[f(x)f^{IV}(x)+f'(x)f'''(x) + f''(x)f''(x)+f'(x)f'''(x) + 2(f'(x)f'''(x)+f''(x)f''(x))]$	16
⋮	⋮	⋮

$$f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{IV}(0)}{4!}x^4 + \frac{f^{V}(0)}{5!}x^5 + \cdots$$

$$\tan x = 0 + \frac{1}{1!}x + \frac{0}{2!}x^2 + \frac{2}{3!}x^3 + \frac{0}{4!}x^4 + \frac{16}{5!}x^5 + \cdots$$

$$\tan x = x + \frac{\cancel{2}1}{\cancel{6}3}x^3 + \frac{\cancel{16}2}{\cancel{120}15}x^5 + \cdots$$

$$\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \cdots$$

$$\tan x = x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \cdots;$$

$$-\frac{\pi}{2} < x < \frac{\pi}{2}$$

Ex 7.32

Write the Taylor Series expansion of $\frac{1}{x}$ about $x=2$ by finding the first three non-zero terms

Soln:

Functions and its derivatives	$\frac{1}{x}$ and its derivatives	Value at $x=2$
$f(x)$	$\frac{1}{x}$	$\frac{1}{2}$
$f'(x)$	$\frac{-1}{x^2}$	$-\frac{1}{4}$
$f''(x)$	$\frac{2}{x^3}$	$\frac{1}{4}$
$f'''(x)$	$\frac{-6}{x^4}$	$-\frac{3}{8}$

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots$$

$$\frac{1}{x} = f(2) + \frac{f'(2)}{1!}(x-2) + \frac{f''(2)}{2!}(x-2)^2 + \frac{f'''(2)}{3!}(x-2)^3 + \cdots$$

$$\frac{1}{x} = \frac{1}{2} - \frac{1/4}{1!}(x-2) + \frac{\frac{1}{4}}{2!}(x-2)^2 - \frac{3/8}{3!}(x-2)^3 + \cdots$$

$$\frac{1}{x} = \frac{1}{2} - \frac{1}{4}(x-2) + \frac{1}{8}(x-2)^2 - \frac{\cancel{3}}{8}\times\frac{1}{\cancel{6}2}(x-2)^3 + \cdots$$

$$\frac{1}{x} = \frac{1}{2} - \frac{(x-2)}{4} + \frac{(x-2)^2}{8} - \frac{(x-2)^3}{16} + \cdots$$

7.5 Indeterminate Forms

EXERCISE 7.5

Evaluate the following limits, if necessary use l'hôpital Rule:

1) $\lim_{x\to 0} \frac{1-\cos x}{x^2}$

Soln:

$\lim_{x\to 0} \frac{1-\cos x}{x^2} = \frac{1-\cos 0}{0}$

$= \frac{1-1}{0} = \frac{0}{0}$

This is an indeterminate of the Form

Applying l'hôpital's Rule,

$\lim_{x\to 0} \frac{0-(-\sin x)}{2x} = \lim_{x\to 0} \frac{\sin x}{2x}$

$= \frac{\sin 0}{2(0)} = \left(\frac{0}{0}\text{ form}\right)$

Applying l'hôpital's Rule,

$\lim_{x\to 0} \frac{\sin x}{2x} = \lim_{x\to 0} \frac{\cos x}{2}$

$= \frac{\cos 0}{2} = \frac{1}{2}$

2) $\lim_{x\to\infty} \frac{2x^2-3}{x^2-5x+3}$

Soln:

$\lim_{x\to\infty} \frac{2x^2-3}{x^2-5x+3} = \frac{2(\infty)^2-3}{(\infty)^2-5(\infty)+3}$

$= \frac{\infty}{\infty}$

This is an indeterminate of the form

Applying l'hôpital's Rule,

$\lim_{x\to\infty} \frac{2x^2-3}{x^2-5x+3} = \lim_{x\to\infty} \frac{4x}{2x-5}$

$= \frac{4(\infty)}{2(\infty)-5} = \left(\frac{\infty}{\infty}\text{ form}\right)$

Applying l'hôpital's Rule,

$\lim_{x\to\infty} \frac{4x}{2x-5} = \lim_{x\to\infty} \frac{4}{2}$

$= \frac{\cancel{4}\,2}{\cancel{2}} = 2$

3) $\lim_{x\to\infty} \frac{x}{\log x}$

Soln:

$\lim_{x\to\infty} \frac{x}{\log x} = \frac{\infty}{\log\infty} = \frac{\infty}{\infty}$

This is an indeterminate of the form

Applying l'hôpital's Rule,

$\lim_{x\to\infty} \frac{x}{\log x} = \lim_{x\to\infty} \frac{1}{1/x}$

$= \lim_{x\to\infty} x = \infty$

4) $\lim_{x\to\pi/2^-} \frac{\sec x}{\tan x}$

Soln:

$\lim_{x\to\frac{\pi}{2}^-} \frac{\sec x}{\tan x} = \frac{\sec\pi/2}{\tan\pi/2}$

$= \frac{\infty}{\infty}$

This is an indeterminate of the form

Applying l'hôpital's Rule,

$\lim_{x\to\pi/2^-} \frac{\sec x}{\tan x} = \lim_{x\to\pi/2^-} \frac{\sec x\tan x}{\sec^2 x}$

$= \lim_{x\to\frac{\pi}{2}^-} \frac{\tan x}{\sec x}$

$= \lim_{x\to\frac{\pi}{2}^-} \frac{\frac{\sin x}{\cancel{\cos x}}}{\frac{1}{\cancel{\cos x}}}$

$= \lim_{x\to\frac{\pi}{2}^-} \sin x = \sin\frac{\pi}{2}$

$= 1$

5) $\lim_{x\to\infty} e^{-x}\sqrt{x}$

Soln:

$\lim_{x\to\infty} e^{-x}\sqrt{x} = \lim_{x\to\infty} \frac{\sqrt{x}}{e^x}$

$= \frac{\sqrt{\infty}}{e^\infty} = \frac{\infty}{\infty}$

This is an indeterminate of the form

Applying l'hôpital's Rule,

$\lim_{x\to\infty} \frac{\sqrt{x}}{e^x} = \lim_{x\to\infty} \frac{x^{1/2}}{e^x}$

$= \lim_{x\to\infty} \dfrac{\frac{1}{2}x^{-1/2}}{e^x}$

$= \lim_{x\to\infty} \dfrac{1}{2x^{1/2}} \times \dfrac{1}{e^x}$

$= \lim_{x\to\infty} \dfrac{1}{2\sqrt{x}\, e^x}$

$= \dfrac{1}{2\sqrt{\infty}\, e^{\infty}} = \dfrac{1}{2(\infty)(\infty)}$

$= \dfrac{1}{2(\frac{1}{0})(\frac{1}{0})} = \dfrac{1}{\frac{2}{0}}$

$= \dfrac{1}{\infty} = 0$

6) $\lim_{x\to 0}\left(\dfrac{1}{\sin x} - \dfrac{1}{x}\right)$

Soln:

$\lim_{x\to 0}\left(\dfrac{1}{\sin x} - \dfrac{1}{x}\right)$

$= \left(\dfrac{1}{\sin 0} - \dfrac{1}{0}\right) = \left(\dfrac{1}{0} - \dfrac{1}{0}\right)$

$= (\infty - \infty)$

This is an indeterminate of the form

$\lim_{x\to 0}\left(\dfrac{1}{\sin x} - \dfrac{1}{x}\right)$

$= \lim_{x\to 0}\left(\dfrac{x - \sin x}{x \sin x}\right)$

$= \dfrac{0 - \sin 0}{0 \sin 0} = \dfrac{0-0}{0} = \left(\dfrac{0}{0}\text{ form}\right)$

Applying l'hôpital's Rule,

$\lim_{x\to 0}\left(\dfrac{x - \sin x}{x \sin x}\right)$

$= \lim_{x\to 0}\left(\dfrac{1 - \cos x}{x\cos x + \sin x \cdot 1}\right)$

$= \lim_{x\to 0}\left(\dfrac{1 - \cos x}{x\cos x + \sin x}\right)$

$= \dfrac{1 - \cos 0}{0\cos 0 + \sin 0} = \dfrac{1-1}{0+0}$

$= \left(\dfrac{0}{0}\text{ form}\right)$

Applying l'hôpital's Rule,

$\lim_{x\to 0}\left(\dfrac{1 - \cos x}{x\cos x + \sin x}\right)$

$= \lim_{x\to 0}\left(\dfrac{0 - (-\sin x)}{x(-\sin x) + \cos x \cdot 1 + \cos x}\right)$

$= \lim_{x\to 0}\left(\dfrac{\sin x}{-x\sin x + \cos x + \cos x}\right)$

$= \dfrac{\sin 0}{-0\sin 0 + \cos 0 + \cos 0}$

$= \dfrac{0}{0+1+1} = \dfrac{0}{2}$

$= 0$

7) $\lim_{x\to 1^+}\left(\dfrac{2}{x^2-1} - \dfrac{x}{x-1}\right)$

Soln:

$\lim_{x\to 1^+}\left(\dfrac{2}{x^2-1} - \dfrac{x}{x-1}\right)$

$= \left(\dfrac{2}{1-1} - \dfrac{1}{1-1}\right)$

$= \left(\dfrac{2}{0} - \dfrac{1}{0}\right) = (\infty - \infty)$

This is an indeterminate of the form

$= \lim_{x\to 1^+}\left(\dfrac{2(x-1) - x(x^2-1)}{(x^2-1)(x-1)}\right)$

$= \lim_{x\to 1^+}\left(\dfrac{2x - 2 - x^3 + x}{x^3 - x^2 - x + 1}\right)$

$= \lim_{x\to 1^+}\left(\dfrac{3x - x^3 - 2}{x^3 - x^2 - x + 1}\right)$

$= \dfrac{3(1) - 1 - 2}{1 - 1 - 1 + 1}$

$= \dfrac{3-3}{2-2} = \left(\dfrac{0}{0}\text{ form}\right)$

Applying l'hôpital's Rule,

$\lim_{x\to 1^+}\left(\dfrac{3x - x^3 - 2}{x^3 - x^2 - x + 1}\right)$

$= \lim_{x\to 1^+}\left(\dfrac{3 - 3x^2}{3x^2 - 2x - 1}\right)$

$= \dfrac{3 - 3(1)}{3(1) - 2(1) - 1} = \dfrac{3-3}{3-3}$

$= \left(\dfrac{0}{0}\text{ form}\right)$

Applying l'hôpital's Rule,

$\lim_{x\to 1^+}\left(\dfrac{3 - 3x^2}{3x^2 - 2x - 1}\right) = \lim_{x\to 1^+}\left(\dfrac{-6x}{6x-2}\right)$

$= \dfrac{-6}{6-2} = \dfrac{-\cancel{6}3}{\cancel{4}2} = -3/2$

8) $\lim_{x \to 0^+} x^x$

Soln:

$\lim_{x \to 0^+} x^x = (0^0 \text{ form})$

This is an indeterminate of the form

Let $y = x^x$

Taking log on bothsides we get,

$\log y = \log x^x$

$\log y = x \log x$

$\log y = \frac{\log x}{1/x}$

$\lim_{x \to 0^+} \log y = \lim_{x \to 0^+} \frac{\log x}{1/x}$

$= \frac{\log 0}{1/0} = \left(\frac{-\infty}{\infty} \text{ form}\right)$

Applying l'hôpital's Rule,

$\lim_{x \to 0^+} \log y = \lim_{x \to 0^+} \frac{1/x}{-1/x^2}$

$= \lim_{x \to 0^+} \frac{1}{\cancel{x}} \times \frac{x^{\cancel{2}}}{-1}$

$= \lim_{x \to 0^+} \frac{x}{-1} = \frac{0}{-1} = 0$

$\lim_{x \to 0^+} \log y = 0$

By Composite function theorem,

$\log\left(\lim_{x \to 0^+} y\right) = 0$

Taking exponential on both sides we get,

$\log_e\left(\lim_{x \to 0^+} y\right) = e^0$

$\lim_{x \to 0^+} y = 1$

$\lim_{x \to 0^+} x^x = 1$

9) $\lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^x$

Soln:

$\lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^x = \left(1 + \frac{1}{\infty}\right)^\infty$

$= (1+0)^\infty = (1^\infty \text{ form})$

This is an indeterminate of the form

Let $y = \left(1 + \frac{1}{x}\right)^x$

Taking log on bothsides we get,

$\log y = \log\left(1 + \frac{1}{x}\right)^x$

$\log y = x \log\left(1 + \frac{1}{x}\right)$

$\log y = \frac{\log\left(1 + \frac{1}{x}\right)}{1/x}$

$\lim_{x \to \infty} \log y = \lim_{x \to \infty} \frac{\log\left(1 + \frac{1}{x}\right)}{\frac{1}{x}}$

$= \frac{\log\left(1 + \frac{1}{\infty}\right)}{\frac{1}{\infty}} = \frac{\log(1+0)}{0}$

$= \frac{\log 1}{0} = \left(\frac{0}{0} \text{ form}\right)$

Applying l'hôpital's Rule,

$\lim_{x \to \infty} \frac{\log\left(1 + \frac{1}{x}\right)}{\frac{1}{x}}$

$= \lim_{x \to \infty} \left[\frac{\frac{1}{1 + \frac{1}{x}} \cancel{\left(\frac{-1}{x^2}\right)}}{\cancel{\left(\frac{-1}{x^2}\right)}}\right]$

$= \lim_{x \to \infty} \frac{1}{1 + \frac{1}{x}}$

$= \frac{1}{1 + \frac{1}{\infty}} = \frac{1}{1+0}$

$= \frac{1}{1} = 1$

$\lim_{x \to \infty} \log y = 1$

By Composite function theorem,

$\log\left(\lim_{x\to\infty} y\right) = 1$

Taking exponential on both sides we get,

$\log_e\left(\lim_{x\to\infty} y\right) = e^1$

$\lim_{x\to\infty} y = e$

$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = e$

10) $\lim_{x\to\frac{\pi}{2}} (\sin x)^{\tan x}$

Soln:

$\lim_{x\to\frac{\pi}{2}} (\sin x)^{\tan x} = \left(\sin\frac{\pi}{2}\right)^{\tan\frac{\pi}{2}}$

$= (1^\infty \text{ form})$

This is an indeterminate form

Let $y = (\sin x)^{\tan x}$

Taking log on both sides, we get

$\log y = \log (\sin x)^{\tan x}$

$\log y = \tan x \log(\sin x)$

$\log y = \frac{\log(\sin x)}{\frac{1}{\tan x}}$

$\log y = \frac{\log(\sin x)}{\cot x}$

$\lim_{x\to\frac{\pi}{2}} \log y = \lim_{x\to\frac{\pi}{2}} \frac{\log(\sin x)}{\cot x}$

$= \frac{\log\left(\sin\frac{\pi}{2}\right)}{\cot\frac{\pi}{2}}$

$= \frac{\log 1}{0} = \left(\frac{0}{0} \text{ form}\right)$

Applying l'hôpital's Rule,

$\lim_{x\to\frac{\pi}{2}} \log y = \lim_{x\to\frac{\pi}{2}} \frac{\frac{1}{\sin x}(\cos x)}{-\operatorname{cosec}^2 x}$

$= \lim_{x\to\frac{\pi}{2}} \frac{\frac{\cos x}{\sin x}}{-\frac{1}{\sin^2 x}}$

$= \lim_{x\to\frac{\pi}{2}} \frac{\cos x}{\cancel{\sin x}} \times \frac{\sin^2 x}{-1}$

$= \lim_{x\to\frac{\pi}{2}} \frac{\sin x \cos x}{-1}$

$= \frac{\sin\frac{\pi}{2}\cos\frac{\pi}{2}}{-1}$

$= \frac{(1)(0)}{-1} = \frac{0}{-1} = 0$

$\lim_{x\to\frac{\pi}{2}} \log y = 0$

By Composite function theorem,

$\log\left(\lim_{x\to\frac{\pi}{2}} y\right) = 0$

Taking exponential on both sides we get,

$\log_e\left(\lim_{x\to\frac{\pi}{2}} y\right) = e^0$

$\lim_{x\to\pi/2} y = 1$

$\lim_{x\to\frac{\pi}{2}} (\sin x)^{\tan x} = 1$

11) $\lim_{x\to 0^+} (\cos x)^{\frac{1}{x^2}}$

Soln:

$\lim_{x\to 0^+} (\cos x)^{\frac{1}{x^2}} = (\cos 0)^{\frac{1}{0}}$

$= (1^\infty \text{ form})$

This is an indeterminate form

Let $y = (\cos x)^{\frac{1}{x^2}}$

Taking log on both sides we get,

$\log y = \log(\cos x)^{1/x^2}$

$\log y = \frac{1}{x^2}\log(\cos x)$

$\log y = \frac{\log(\cos x)}{x^2}$

$\lim_{x\to 0^+} \log y = \lim_{x\to 0^+} \frac{\log(\cos x)}{x^2}$

$= \frac{\log(\cos 0)}{0} = \frac{\log 1}{0} = \left(\frac{0}{0} \text{ form}\right)$

Applying l'hôpital's Rule,

$$\lim_{x\to 0^+} \frac{\log(\cos x)}{x^2} = \lim_{x\to 0^+} \frac{\frac{1}{\cos x}(-\sin x)}{2x}$$

$$= \lim_{x\to 0^+} \left(-\frac{\sin x}{\cos x} \times \frac{1}{2x}\right)$$

$$= \lim_{x\to 0^+} \left(\frac{-\sin x}{2x\cos x}\right)$$

$$= \lim_{x\to 0^+} \left(\frac{-\cos x}{2[x(-\sin x)+\cos x \cdot 1]}\right)$$

$$= \lim_{x\to 0^+} \left(\frac{-\cos x}{2[-x\sin x+\cos x]}\right)$$

$$= \frac{-\cos 0}{2[-0\sin 0+\cos 0]}$$

$$= \frac{-1}{2[0+1]} = \frac{-1}{2(1)} = \frac{-1}{2}$$

$$\lim_{x\to 0^+} \log y = \frac{-1}{2}$$

By Composite Function theorem,

$$\log\left(\lim_{x\to 0^+} y\right) = -\frac{1}{2}$$

Taking exponential on both sides, we get

$$\log_e\left(\lim_{x\to 0^+} y\right) = e^{-\frac{1}{2}}$$

$$\lim_{x\to 0^+} y = \frac{1}{e^{1/2}} = \frac{1}{\sqrt{e}}$$

$$\lim_{x\to 0^+} (\cos x)^{\frac{1}{x^2}} = \frac{1}{\sqrt{e}}$$

12) If an initial amount A_0 of money is invested at an interest rate r compounded n times a year, the value of the investment after t years is $A = A_0\left(1+\frac{r}{n}\right)^{nt}$. If the interest is compounded continuously, (that is as $n\to\infty$), show that the amount after t years is $A = A_0 e^{rt}$

Soln:

$$\lim_{n\to\infty} A = \lim_{n\to\infty} A_0\left(1+\frac{r}{n}\right)^{nt}$$

$$= A_0\left(1+\frac{r}{\infty}\right)^{(\infty)t}$$

$$= A_0(1+0)^{\infty} = A_0(1)^{\infty}$$

$$= A_0(1^{\infty})$$

This is an indeterminate form

Let $y = \left(1+\frac{r}{n}\right)^{nt}$

Taking log on bothsides, we get,

$$\log y = \log\left(1+\frac{r}{n}\right)^{nt}$$

$$= nt\log\left(1+\frac{r}{n}\right)$$

$$= t\,\frac{\log\left(1+\frac{r}{n}\right)}{1/n}$$

$$\lim_{n\to\infty} \log y = \lim_{n\to\infty} t\,\frac{\log\left(1+\frac{r}{n}\right)}{\frac{1}{n}}$$

$$= t\lim_{n\to\infty} \frac{\log\left(1+\frac{r}{n}\right)}{\frac{1}{n}}$$

$$= t\left[\frac{\log\left(1+\frac{r}{\infty}\right)}{1/\infty}\right]$$

$$= t\left[\frac{\log(1+0)}{0}\right]$$

$$= t\left[\frac{\log 1}{0}\right]$$

$$= t\left[\frac{0}{0}\right] = \left(\frac{0}{0}\text{ form}\right)$$

Applying l'hôpital's Rule,

$$= \lim_{n\to\infty} t\,\frac{\log\left(1+\frac{r}{n}\right)}{1/n}$$

$$= t\lim_{n\to\infty} \frac{\log\left(1+\frac{r}{n}\right)}{1/n}$$

$$= t \lim_{n\to\infty} \frac{\frac{1}{1+r/n}\left(\frac{-r}{n^2}\right)}{\left(\frac{-1}{n^2}\right)}$$

$$= rt \lim_{n\to\infty} \frac{\frac{1}{1+\frac{r}{n}}\left(\cancel{\frac{-1}{n^2}}\right)}{\cancel{\left(\frac{-1}{n^2}\right)}}$$

$$= rt \lim_{n\to\infty} \frac{1}{1+\frac{r}{n}}$$

$$= rt\left[\frac{1}{1+\frac{r}{\infty}}\right]$$

$$= rt\left[\frac{1}{1+0}\right]$$

$$= rt(1) = rt$$

$$\lim_{n\to\infty} \log y = rt$$

By Composite function theorem, we get

$$\log\left(\lim_{n\to\infty} y\right) = rt$$

Taking exponential on both sides we get,

$$\log_e\left(\lim_{n\to\infty} y\right) = e^{rt}$$

$$\lim_{n\to\infty} y = e^{rt}$$

$$\lim_{n\to\infty}\left(1+\frac{r}{n}\right)^{nt} = e^{rt}$$

Multiple by A_0 on both sides, we get

$$\lim_{n\to\infty} A_0\left(1+\frac{r}{n}\right)^{nt} = A_0 e^{rt}$$

$$\lim_{n\to\infty} A = A_0 e^{rt}$$

[OR]

$$\lim_{n\to\infty} A = \lim_{n\to\infty} A_0\left(1+\frac{r}{n}\right)^{nt}$$

$$= A_0 \lim_{n\to\infty}\left[\left(1+\frac{r}{n}\right)^n\right]^t$$

$$= A_0\left(e^r\right)^t$$

$\left[\text{Since} \lim_{n\to\infty}\left(1+\frac{k}{n}\right)^n = e^k\right]$

$$\lim_{n\to\infty} A = A_0 e^{rt}$$

EX 7.33

Evaluate: $\lim_{x\to 1}\left(\frac{x^2-3x+2}{x^2-4x+3}\right)$

Soln:

$$\lim_{x\to 1}\left(\frac{x^2-3x+2}{x^2-4x+3}\right) = \frac{1-3+2}{1-4+3} = \frac{0}{0}$$

This is an indeterminate form

Applying l'hôpital's Rule,

$$\lim_{x\to 1}\left(\frac{x^2-3x+2}{x^2-4x+3}\right) = \lim_{x\to 1}\frac{2x-3}{2x-4}$$

$$= \frac{2(1)-3}{2(1)-4} = \frac{-1}{-2} = \frac{1}{2}$$

EX 7.34

Compute the limit $\lim_{x\to a}\left(\frac{x^n-a^n}{x-a}\right)$

Soln:

$$\lim_{x\to a}\left(\frac{x^n-a^n}{x-a}\right) = \frac{a^n-a^n}{a-a} = \frac{0}{0}$$

This is an indeterminate form

Applying l'hôpital's Rule,

$$\lim_{x\to a}\left(\frac{x^n-a^n}{x-a}\right) = \lim_{x\to a}\frac{nx^{n-1}}{1}$$

$$= \lim_{x\to a} nx^{n-1} = na^{n-1}$$

EX 7.35

Evaluate the limit $\lim_{x\to 0}\left(\frac{\sin mx}{x}\right)$

Soln:

$$\lim_{x\to 0}\left(\frac{\sin mx}{x}\right) = \frac{\sin 0}{0} = \frac{0}{0}$$

Applying l'hôpital's Rule,

$$\lim_{x\to 0}\left(\frac{\sin mx}{x}\right) = \lim_{x\to 0}\frac{(\cos mx)(m)}{1}$$

$$= \lim_{x\to 0}(\cos mx)(m)$$

$$= (\cos 0)(m) = (1)(m)$$

$$= m$$

EX 7.36

Evaluate the limit $\lim_{x\to 0}\left(\frac{\sin x}{x^2}\right)$

Soln:

$\lim_{x\to 0}\left(\frac{\sin x}{x^2}\right) = \frac{\sin 0}{0} = \frac{0}{0}$

This is an indeterminate form

Applying l'hôpital's Rule,

$\lim_{x\to 0^+}\left(\frac{\sin x}{x^2}\right) = \lim_{x\to 0^+}\frac{(\cos x)}{2x}$

$= \frac{\cos 0}{2(0)} = \frac{1}{0} = \infty$

Also, $\lim_{x\to 0^-}\left(\frac{\sin x}{x^2}\right) = \lim_{x\to 0^-}\left(\frac{\cos x}{2x}\right)$

$= \frac{\cos 0}{2(0)} = \frac{1}{0} = -\infty$

$\therefore$ The left limit and the right limit are not the same

$\therefore$ The limit does not exist

EX 7.37

If $\lim_{\theta\to 0}\left(\frac{1-\cos m\theta}{1-\cos n\theta}\right) = 1$, then Prove that $m = \pm n$

Soln:

$\lim_{\theta\to 0}\left(\frac{1-\cos m\theta}{1-\cos n\theta}\right)$

$= \frac{1-\cos 0}{1-\cos 0} = \frac{1-1}{1-1} = \frac{0}{0}$

This is an indeterminate form

Applying l'hôpital's Rule,

$\lim_{\theta\to 0}\left(\frac{1-\cos m\theta}{1-\cos n\theta}\right)$

$= \lim_{\theta\to 0}\left(\frac{-(-\sin m\theta)m}{-(-\sin n\theta)n}\right)$

$= \lim_{\theta\to 0}\left(\frac{m\sin m\theta}{n\sin n\theta}\right)$

$= \lim_{\theta\to 0}\frac{m}{n}\times\left(\frac{\frac{\sin m\theta}{\theta}}{\frac{\sin n\theta}{\theta}}\right)$

$= \lim_{\theta\to 0}\frac{m}{n}\times\left(\frac{\cos m\theta(m)\frac{1}{}}{\frac{\cos n\theta(n)}{1}}\right)$

$= \lim_{\theta\to 0}\frac{m^2}{n^2}\left(\frac{\cos m\theta}{\cos n\theta}\right)$

$= \frac{m^2}{n^2}\left(\frac{\cos 0}{\cos 0}\right)$

$= \frac{m^2}{n^2}\left(\frac{1}{1}\right) = \frac{m^2}{n^2}$

Given: $\lim_{\theta\to 0}\left(\frac{1-\cos m\theta}{1-\cos n\theta}\right) = 1$

$\frac{m^2}{n^2} = 1$

$m^2 = n^2$

$m = \pm n$

EX 7.38

Evaluate: $\lim_{x\to 1}\left(\frac{\log(1-x)}{\cot(\pi x)}\right)$

Soln:

$\lim_{x\to 1}\left(\frac{\log(1-x)}{\cot(\pi x)}\right) = \frac{\log(1-1)}{\cot\pi(1)}$

$= \frac{\log 0}{\cot\pi} = \frac{-(-\infty)}{\infty} = \frac{\infty}{\infty}$

This is an indeterminate form

Applying l'hôpital's Rule,

$\lim_{x\to 1}\left(\frac{\log(1-x)}{\cot(\pi x)}\right)$

$= \lim_{x\to 1}\left(\frac{\frac{1}{1-x}(-1)}{-\operatorname{cosec}^2(\pi x)(\pi)}\right)$

$= \lim_{x\to 1}\left(\frac{\frac{\cancel{-}1}{1-x}}{\cancel{-}\pi\operatorname{cosec}^2(\pi x)}\right)$

$= \frac{\frac{1}{1-1}}{\pi\operatorname{cosec}^2(\pi(1))}$

$= \frac{\frac{1}{0}}{\pi\operatorname{cosec}^2\pi}$

$= \frac{\frac{1}{0}}{\pi\left(\frac{1}{0}\right)}$

$= \left(\frac{\infty}{\infty}\text{ form}\right)$

Applying l'hôpital's Rule,

$$\lim_{x\to 1^-}\left(\frac{\frac{1}{1-x}}{\pi\,\mathrm{cosec}^2(\pi x)}\right)$$

$$=\lim_{x\to 1^-}\left(\frac{\frac{1}{1-x}}{\pi\left(\frac{1}{\sin^2(\pi x)}\right)}\right)$$

$$=\lim_{x\to 1^-}\left(\frac{1}{1-x}\times\frac{\sin^2(\pi x)}{\pi}\right)$$

$$=\lim_{x\to 1^-}\left(\frac{\sin^2(\pi x)}{\pi(1-x)}\right)$$

$$=\frac{\sin^2(\pi(1))}{\pi(1-1)}=\left(\frac{0}{0}\text{ form}\right)$$

Applying l'hôpital's Rule,

$$=\lim_{x\to 1^-}\left(\frac{2\sin(\pi x)\cos(\pi x)(\pi)}{-\pi}\right)$$

$$=\lim_{x\to 1^-}\left(-2\sin(\pi x)\cos(\pi x)\right)$$

$$=-2\sin(\pi)\cos(\pi)$$

$$=-2(0)(-1)=2(0)=0$$

EX 7.39 Evaluate:

$$\lim_{x\to 0^+}\left(\frac{1}{x}-\frac{1}{e^x-1}\right)$$

Soln:

$$\lim_{x\to 0^+}\left(\frac{1}{x}-\frac{1}{e^x-1}\right)$$

$$=\frac{1}{0}-\frac{1}{e^0-1}$$

$$=\frac{1}{0}-\frac{1}{1-1}$$

$$=\frac{1}{0}-\frac{1}{0}=(\infty-\infty)$$

This is an indeterminate form

$$\lim_{x\to 0^+}\left(\frac{1}{x}-\frac{1}{e^x-1}\right)$$

$$=\lim_{x\to 0^+}\left(\frac{e^x-1-x}{x(e^x-1)}\right)$$

$$=\frac{e^0-1-0}{0(e^0-1)}=\frac{1-1}{0(1-1)}$$

$$=\left(\frac{0}{0}\text{ form}\right)$$

Applying l'hôpital's Rule,

$$\lim_{x\to 0^+}\left(\frac{e^x-1-x}{x(e^x-1)}\right)$$

$$=\lim_{x\to 0^+}\left(\frac{e^x-1}{x(e^x)+(e^x-1)\cdot 1}\right)$$

$$=\lim_{x\to 0^+}\left(\frac{e^x-1}{xe^x+e^x-1}\right)$$

$$=\frac{e^0-1}{(0)e^0+e^0-1}=\frac{1-1}{(0)(1)+(1)-1}$$

$$=\frac{1-1}{1-1}=\left(\frac{0}{0}\text{ form}\right)$$

Applying l'hôpital's Rule,

$$=\lim_{x\to 0^+}\left(\frac{e^x}{xe^x+e^x+e^x}\right)$$

$$=\lim_{x\to 0^+}\left(\frac{e^x}{xe^x+2e^x}\right)$$

$$=\frac{e^0}{(0)e^0+2e^0}=\frac{1}{(0)(1)+2(1)}$$

$$=\frac{1}{2}$$

EX 7.40

Evaluate: $\lim_{x\to 0^+} x\log x$

Soln:

$$\lim_{x\to 0^+} x\log x=(0)\log 0$$

$$=(0)(-\infty)$$

$$=(0\times-\infty)$$

This is an indeterminate form.

$$\lim_{x\to 0^+} x\log x=\lim_{x\to 0^+}\frac{\log x}{\frac{1}{x}}$$

$$=\frac{\log 0}{\frac{1}{0}}=\left(\frac{-\infty}{\infty}\text{ form}\right)$$

Applying l'hôpital's Rule,

$$\lim_{x\to 0^+}\frac{\log x}{\frac{1}{x}}=\lim_{x\to 0^+}\frac{\frac{1}{x}}{\frac{-1}{x^2}}$$

$$=\lim_{x\to 0^+}\frac{1}{\cancel{x}}\times\frac{x^2}{-1}$$

$$=\lim_{x\to 0^+}\frac{x}{-1}=\frac{0}{-1}=0$$

EX 7.41 Evaluate:

$$\lim_{x\to\infty}\left(\frac{x^2+17x+29}{x^4}\right)$$

Soln:

$$\lim_{x\to\infty}\left(\frac{x^2+17x+29}{x^4}\right) = \frac{\infty}{\infty}$$

This is an indeterminate form

Applying l'hôpital's Rule,

$$\lim_{x\to\infty}\left(\frac{x^2+17x+29}{x^4}\right) = \lim_{x\to\infty}\left(\frac{2x+17}{4x^3}\right)$$

$= \left(\frac{\infty}{\infty}\text{ form}\right)$

Applying l'hôpital's Rule,

$$= \lim_{x\to\infty}\left(\frac{2}{12x^2}\right) = \frac{2}{12(\infty)^2} = \frac{2}{\infty} = 0$$

EX 7.42

Evaluate: $\lim_{x\to\infty}\left(\frac{e^x}{x^m}\right)$, $m \in \mathbb{N}$

Soln:

$$\lim_{x\to\infty}\left(\frac{e^x}{x^m}\right) = \left(\frac{e^\infty}{(\infty)^m}\right) = \left(\frac{\infty}{\infty}\right)$$

This is an indeterminate of the Form

Applying l'hôpital's Rule,

$$\lim_{x\to\infty}\left(\frac{e^x}{x^m}\right) = \lim_{x\to\infty}\frac{e^x}{m!}$$

$$= \frac{e^\infty}{m!} = \frac{\infty}{m!} = \infty$$

EX 7.43

Using the l'hôpital's Rule prove that,

$$\lim_{x\to 0^+}(1+x)^{\frac{1}{x}} = e$$

Soln:

$$\lim_{x\to 0^+}(1+x)^{\frac{1}{x}} = (1+0)^{\frac{1}{0}} = 1^\infty$$

This is an indeterminate of the Form

Let $g(x) = (1+x)^{\frac{1}{x}}$

Taking log on both sides we get,

$$\log g(x) = \log(1+x)^{\frac{1}{x}}$$

$$\log g(x) = \frac{\log(1+x)}{x}$$

$$\lim_{x\to 0^+}\log g(x) = \lim_{x\to 0^+}\frac{\log(1+x)}{x}$$

$$= \frac{\log(1+0)}{0} = \frac{\log 1}{0} = \left(\frac{0}{0}\text{ form}\right)$$

Applying l'hôpital's Rule,

$$= \lim_{x\to 0^+}\frac{\frac{1}{1+x}}{1}$$

$$= \lim_{x\to 0^+}\frac{1}{1+x} = \frac{1}{1+0} = \frac{1}{1} = 1$$

$$\lim_{x\to 0^+}\log g(x) = 1$$

By Composite function theorem,

$$\log\left(\lim_{x\to 0^+} g(x)\right) = 1$$

Taking exponential on both sides we get,

$$\log_e\left(\lim_{x\to 0^+} g(x)\right) = e^1$$

$$\lim_{x\to 0^+} g(x) = e$$

$$\lim_{x\to 0^+}(1+x)^{\frac{1}{x}} = e$$

EX 7.44

Evaluate: $\lim_{x\to\infty}(1+2x)^{\frac{1}{2\log x}}$

Soln:

$$\lim_{x\to\infty}(1+2x)^{\frac{1}{2\log x}}$$

$$= (1+2(\infty))^{\frac{1}{2\log\infty}}$$

$$= (\infty)^{\frac{1}{2(\infty)}} = (\infty)^{\frac{1}{\infty}}$$

$= (\infty^0 \text{ form})$

This is an indeterminate form

Let $g(x) = (1+2x)^{\frac{1}{2\log x}}$

Taking log on both sides,

$\log g(x) = \log(1+2x)^{\frac{1}{2\log x}}$

$\log g(x) = \frac{\log(1+2x)}{2\log x}$

$\lim_{x\to\infty} \log g(x) = \lim_{x\to\infty} \frac{\log(1+2x)}{2\log x}$

$= \frac{\log(1+2(\infty))}{2\log\infty}$

$= \frac{\log\infty}{2(\infty)} = (\frac{\infty}{\infty} \text{ form})$

Applying l'hôpital's Rule,

$= \lim_{x\to\infty} \frac{\frac{1}{1+2x}(2)}{2(\frac{1}{x})}$

$= \lim_{x\to\infty} \frac{\cancel{2}}{1+2x} \times \frac{x}{\cancel{2}}$

$= \lim_{x\to\infty} \frac{x}{1+2x}$

$= \frac{\infty}{1+2(\infty)} = (\frac{\infty}{\infty} \text{ form})$

Applying l'hôpital's Rule,

$= \lim_{x\to\infty} \frac{1}{2}$

$= \frac{1}{2}$

$\lim_{x\to\infty} \log g(x) = \frac{1}{2}$

By Composite function theorem,

$\log(\lim_{x\to\infty} g(x)) = \frac{1}{2}$

Taking exponential on both sides we get,

$\log_e(\lim_{x\to\infty} g(x)) = e^{\frac{1}{2}}$

$\lim_{x\to\infty} g(x) = \sqrt{e}$

$\lim_{x\to\infty} (1+2x)^{\frac{1}{2\log x}} = \sqrt{e}$

Ex 7.45

Evaluate: $\lim_{x\to 1} x^{\frac{1}{1-x}}$

Soln:

$\lim_{x\to 1} x^{\frac{1}{1-x}} = 1^{\frac{1}{1-1}} = 1^{\frac{1}{0}}$

$= 1^{\infty}$

This is an indeterminate form

Let $g(x) = x^{\frac{1}{1-x}}$

Taking log on both sides,

$\log g(x) = \log x^{\frac{1}{1-x}}$

$\log g(x) = \frac{\log x}{1-x}$

$\lim_{x\to 1} \log g(x) = \lim_{x\to 1} \frac{\log x}{1-x}$

$= \frac{\log 1}{1-1} = (\frac{0}{0} \text{ form})$

Applying l'hôpital's Rule,

$= \lim_{x\to 1} \frac{\frac{1}{x}}{-1}$

$= \lim_{x\to 1} -\frac{1}{x} = \frac{-1}{1} = -1$

$\lim_{x\to 1} \log g(x) = -1$

By Composite function theorem,

$\log(\lim_{x\to 1} g(x)) = -1$

Taking exponential on both sides,

$\log_e(\lim_{x\to 1} g(x)) = e^{-1}$

$\lim_{x\to 1} g(x) = e^{-1}$

$\lim_{x\to 1} x^{\frac{1}{1-x}} = \frac{1}{e}$

7.6 Applications of First Derivative

EXERCISE 7.6

1) Find the absolute extrema of the following functions on the given closed interval.

(i) $f(x) = x^2 - 12x + 10; [1, 2]$

Soln:

$f(x) = x^2 - 12x + 10$

$f(x)$ is continuous on $[1,2]$

$f'(x) = 2x - 12$

Let $f'(x) = 0$, $2x - 12 = 0$

$\cancel{2}x = \cancel{12}\,6$

$\boxed{x = 6}$

$x = 6 \in [1,2]$

$\therefore x = 6$ is the only critical point

The End points are $x = 1$ and $x = 2$

when $x = 6$, $f(6) = (6)^2 - 12(6) + 10$

$f(6) = 36 - 72 + 10 = -26$

when $x = 1$, $f(1) = (1)^2 - 12(1) + 10$

$f(1) = 1 - 12 + 10 = -1$

when $x = 2$, $f(2) = (2)^2 - 12(2) + 10$

$f(2) = 4 - 24 + 10 = -10$

$\therefore$ The absolute maximum is -1 and absolute minimum is -26

(ii) $f(x) = 3x^4 - 4x^3$; $[-1,2]$

Soln: $f(x) = 3x^4 - 4x^3$

$f(x)$ is continuous on $[-1,2]$

$f'(x) = 12x^3 - 12x^2$

Let $f'(x) = 0$, $12x^3 - 12x^2 = 0$

$12x^2(x-1) = 0$

$12x^2 = 0$, $x - 1 = 0$

$x^2 = 0$, $x = 1$

$x = 0$ (twice), $x = 1$

$\therefore x = 0, 1 \in [-1,2]$

$\therefore$ The critical points are $x = 0$ and $x = 1$

The End Points are $x = -1$ and $x = 2$

when $x = 0$, $f(0) = 3(0)^4 - 4(0)^3 = 0$

when $x = 1$, $f(1) = 3(1)^4 - 4(1)^3$

$f(1) = 3 - 4 = -1$

When $x = -1$, $f(-1) = 3(-1)^4 - 4(-1)^3$

$f(-1) = 3 + 4 = 7$

When $x = 2$, $f(2) = 3(2)^4 - 4(2)^3$

$f(2) = 48 - 32 = 16$

$\therefore$ The absolute Maximum is 16 and absolute minimum is -1

(iii) $f(x) = 6x^{4/3} - 3x^{1/3}$; $[-1,1]$

Soln:

$f(x) = 6x^{4/3} - 3x^{1/3}$

$f(x)$ is continuous on $[-1,1]$

$f'(x) = \frac{\cancel{24}\,8}{\cancel{3}} x^{1/3} - \frac{\cancel{3}}{\cancel{3}} x^{1/3 - 1}$

$= 8x^{1/3} - x^{1/3 - 1}$

$= x^{1/3}(8 - x^{-1})$

$= x^{1/3}\left(8 - \frac{1}{x}\right)$

$= x^{1/3}\left(\frac{8x-1}{x}\right)$

$= x^{1/3} x^{-1}(8x-1)$

$= x^{1/3 - 1}(8x-1)$

$f'(x) = x^{-2/3}(8x-1)$

$f'(x) = \frac{1}{x^{2/3}}(8x-1)$

Let $f'(x) = 0$, $8x - 1 = 0$

$8x = 1$

$x = 1/8$ and

$f'(x)$ does not exist for $x = 0$

$\therefore$ The critical points are $x = 0$ and $x = 1/8$

The End points are $x = -1$ and $x = 1$

when $x = 0$, $f(0) = 6(0)^{4/3} - 3(0)^{1/3}$

$f(0) = 0$

when $x = \frac{1}{8}$, $f\left(\frac{1}{8}\right) = 6\left(\frac{1}{8}\right)^{4/3} - 3\left(\frac{1}{8}\right)^{1/3}$

$= 6\left(\left(\frac{1}{2}\right)^{\cancel{3}}\right)^{4/\cancel{3}} - 3\left(\left(\frac{1}{2}\right)^{\cancel{3}}\right)^{1/\cancel{3}}$

$= 6\left(\frac{1}{2}\right)^4 - 3\left(\frac{1}{2}\right)$

$= \cancel{6}^{\,3}\left(\frac{1}{\cancel{16}_{\,8}}\right) - 3\left(\frac{1}{2}\right)$

$= \frac{3}{8} - \frac{3}{2} = \frac{3-12}{8} = \frac{-9}{8}$

when $x=-1$, $f(-1)=6(-1)^{4/3}-3(-1)^{1/3}$

$=6((-1)^4)^{1/3}-3(-1)^{1/3}$

$=6(1)^{1/3}-3(-1)^{1/3}$

$=6((1)^{1/3})^3-3((-1)^{1/3})^3$

$=6(1)-3(-1) = 6+3=9$

when $x=1$, $f(1)=6(1)^{4/3}-3(1)^{1/3}$

$=6((1)^4)^{1/3}-3(1)^{1/3}$

$=6(1)^{1/3}-3(1)^{1/3}$

$=6((1)^{1/3})^3-3((1)^{1/3})^3$

$=6(1)-3(1)$

$=6-3=3$

$\therefore$ The absolute maximum is 9 and absolute minimum is $-\frac{9}{8}$

(iv) $f(x)=2\cos x+\sin 2x$; $[0, \pi/2]$

Soln: $f(x)=2\cos x+\sin 2x$

$f(x)$ is continuous on $[0,\pi/2]$

$f'(x)=-2\sin x+2\cos 2x$

$f'(x)=2(\cos 2x-\sin x)$

Let $f'(x)=0$,

$2(\cos 2x-\sin x)=0$

$\cos 2x-\sin x=0$

$1-2\sin^2 x-\sin x=0$

$2\sin^2 x+\sin x-1=0$

$2\sin^2 x+2\sin x-\sin x-1=0$

$2\sin x(\sin x+1)-1(\sin x+1)=0$

$(2\sin x-1)(\sin x+1)=0$

$2\sin x-1=0$	$\sin x+1=0$
$\sin x=\frac{1}{2}$	$\sin x=-1$
$\sin x=\sin\frac{\pi}{6}$	$\sin x=-\sin\frac{\pi}{2}$
$\boxed{x=\frac{\pi}{6}}$	$\sin x=\sin(\pi+\frac{\pi}{2})$
	$\sin x=\sin\frac{3\pi}{2}$
	$\boxed{x=\frac{3\pi}{2}}$

$\therefore x=\frac{\pi}{6}\in[0,\pi/2]$ and

$x=\frac{3\pi}{2}\notin[0,\pi/2]$

$\therefore$ The Critical Point is $x=\frac{\pi}{6}$

The End points are $x=0$ and $x=\pi/2$

when $x=\frac{\pi}{6}$, $f(\frac{\pi}{6})=2\cos\frac{\pi}{6}+\sin\cancel{2}(\frac{\pi}{\cancel{6}_3})$

$f(\frac{\pi}{6})=2\cos\frac{\pi}{6}+\sin\frac{\pi}{3}$

$=2\left(\frac{\sqrt{3}}{2}\right)+\frac{\sqrt{3}}{2}$

$=\frac{2\sqrt{3}+\sqrt{3}}{2}$

$f(\frac{\pi}{6})=\frac{3\sqrt{3}}{2}$

when $x=0$, $f(0)=2\cos 0+\sin 0$

$=2(1)+0$

$f(0)=2$

when $x=\frac{\pi}{2}$, $f(\frac{\pi}{2})=2\cos\frac{\pi}{2}+\sin\cancel{2}(\frac{\pi}{\cancel{2}})$

$f(\frac{\pi}{2})=2\cos\frac{\pi}{2}+\sin\pi$

$f(\frac{\pi}{2})=2(0)+0=0$

$\therefore$ The absolute maximum is $\frac{3\sqrt{3}}{2}$ and absolute minimum is 0

2) Find the intervals of monotonicities and hence find the local extremum for the following functions:

(i) $f(x)=2x^3+3x^2-12x$

Soln:

$f(x)=2x^3+3x^2-12x$

$f'(x)=6x^2+6x-12$

Let $f'(x)=0$,

$6x^2+6x-12=0$

$\div 6$, $x^2+x-2=0$ $\quad$ -2 ; $2 \wedge -1$

$(x+2)(x-1)=0$

$x+2=0$, $x-1=0$

$x=-2$, $x=1$

$\therefore$ The Stationary Points are $x=-2$ and $x=1$

The intervals are $(-\infty,-2)$, $(-2,1)$ and $(1,\infty)$

Interval	Sign of $f'(x)$	Monotonicity
$(-\infty,-2)$	+	strictly increasing
$(-2,1)$	−	strictly decreasing
$(1,\infty)$	+	strictly increasing

$f(x)$ is strictly increasing on $(-\infty,-2)$ and $(1,\infty)$ and strictly decreasing on $(-2,1)$

Since $f'(x)$ changes from positive to negative when passing through $x=-2$, $f(x)$ has local maximum at $x=-2$ and the local maximum value is $f(-2)$

$=2(-2)^3+3(-2)^2-12(-2)$

$=-16+12+24=20$

Also, since $f'(x)$ changes from negative to positive when passing through $x=1$, $f(x)$ has local minimum at $x=1$ and the local minimum value is $f(1)=2(1)^3+3(1)^2-12(1)=2+3-12=-7$

$\therefore$ local maximum $=20$

local minimum $=-7$

(ii) $f(x)=\frac{x}{x-5}$

Soln:

$$f(x)=\frac{x}{x-5}$$

$$f'(x)=\frac{(x-5).1-x(1)}{(x-5)^2}$$

$$=\frac{\cancel{x}-5-\cancel{x}}{(x-5)^2}$$

$$f'(x)=\frac{-5}{(x-5)^2}\neq 0,\ \forall x\in R$$

$f'(x)$ does not exist at $x=5$

$\therefore$ There are no Stationary Points but there is a Critical Point at $x=5$

$-\infty \qquad 5 \qquad \infty$

The intervals are $(-\infty,5)$ and $(5,\infty)$

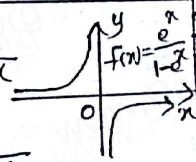
Interval	Sign of $f'(x)$	Monotonicity
$(-\infty,5)$	−	strictly decreasing
$(5,\infty)$	−	strictly decreasing

$f(x)$ is strictly decreasing on $(-\infty,5)$ and $(5,\infty)$

Since there is no sign change in $f'(x)$ when passing through $x=5$

$\therefore$ There is no local extremum

(iii) $f(x)=\frac{e^x}{1-e^x}$



Soln:

$$f(x)=\frac{e^x}{1-e^x}$$

$$f'(x)=\frac{(1-e^x)e^x-e^x(-e^x)}{(1-e^x)^2}$$

$$=\frac{e^x-\cancel{e^{2x}}+\cancel{e^{2x}}}{(1-e^x)^2}$$

$$f'(x)=\frac{e^x}{(1-e^x)^2}\neq 0,\ \forall x\in R$$

and $f'(x)$ does not exist at $x=0$

$\therefore$ There is no Stationary Points but there is a Critical Point at $x=0$

$-\infty \qquad 0 \qquad \infty$

The intervals are $(-\infty,0)$ and $(0,\infty)$

Interval	Sign of $f'(x)$	Monotonicity
$(-\infty,0)$	+	strictly increasing
$(0,\infty)$	+	strictly increasing

$f(x)$ is strictly increasing on $(-\infty,\infty)$

Since there is no Sign change in $f'(x)$ when passing through $x=0$

$\therefore$ There is no local extremum

(iv) $f(x) = \dfrac{x^3}{3} - \log x$

Soln:

$f(x) = \dfrac{x^3}{3} - \log x$

$f'(x) = \dfrac{\cancel{3}x^2}{\cancel{3}} - \dfrac{1}{x}$

$f'(x) = x^2 - \dfrac{1}{x}$

$f'(x) = \dfrac{x^3-1}{x}$

Let $f'(x) = 0$, $\dfrac{x^3-1}{x} = 0$

$x^3 - 1 = 0 \Rightarrow x^3 = 1$

$x^3 = 1^3$

$x^3 - 1^3 = 0$

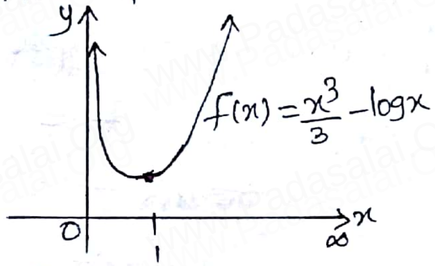
$(x-1)(x^2+x+1) = 0$

$x-1=0$, $x^2+x+1=0$ is not possible

$\boxed{x=1}$

$f'(x)$ does not exist at $x=0$

$\therefore$ There is no Stationary Points but there is a Critical point at $x=1$



$\therefore$ $f(x)$ lies only on $(0,\infty)$ but $f(x)$ is not defined on $(-\infty, 0)$

0 1 ∞

The intervals are $(0,1)$ and $(1,\infty)$

Interval	Sign of $f'(x)$	Monotonicity
$(0,1)$	$-$	strictly decreasing
$(1,\infty)$	$+$	strictly increasing

$f(x)$ is strictly decreasing on $(0,1)$ and strictly increasing on $(1,\infty)$

Since $f'(x)$ changes from negative to positive when passing through $x=1$

$f(x)$ has local minimum at $x=1$ and the local minimum value is $f(1) = \dfrac{1}{3} - \log 1$

$= \dfrac{1}{3} - 0 = \dfrac{1}{3}$

$\therefore$ local minimum $= \dfrac{1}{3}$

(v) $f(x) = \sin x \cos x + 5$, $x \in (0, 2\pi)$

Soln: $f(x) = \sin x \cos x + 5$

$f(x) = \dfrac{1}{2} 2\sin x \cos x + 5$

$f(x) = \dfrac{1}{2}\sin 2x + 5$

$f'(x) = \dfrac{1}{\cancel{2}}\cos 2x (\cancel{2})$

$f'(x) = \cos 2x$

Let $f'(x) = 0$, $\cos 2x = 0$,

$2x = \dfrac{\pi}{2}, \dfrac{3\pi}{2}, \dfrac{5\pi}{2}, \dfrac{7\pi}{2} \ldots$

$x = \dfrac{\pi}{4}, \dfrac{3\pi}{4}, \dfrac{5\pi}{4}, \dfrac{7\pi}{4} \ldots$

$\therefore$ The Stationary Points are $x = \dfrac{\pi}{4}, \dfrac{3\pi}{4}, \dfrac{5\pi}{4}, \dfrac{7\pi}{4}$

0 $\pi/4$ $3\pi/4$ $5\pi/4$ $7\pi/4$ 2π

The intervals are $(0,\pi/4)$, $(\pi/4, 3\pi/4)$, $(3\pi/4, 5\pi/4)$, $(5\pi/4, 7\pi/4)$ and $(7\pi/4, 2\pi)$

Interval	Sign of $f'(x)$	Monotonicity
$(0,\pi/4)$	$+$	Strictly increasing
$(\pi/4, 3\pi/4)$	$-$	Strictly decreasing
$(3\pi/4, 5\pi/4)$	$+$	Strictly increasing
$(5\pi/4, 7\pi/4)$	$-$	Strictly decreasing
$(7\pi/4, 2\pi)$	$+$	Strictly increasing

$f(x)$ is strictly increasing on $(0,\pi/4)$ $(3\pi/4, 5\pi/4)$ and $(7\pi/4, 2\pi)$ and strictly decreasing on $(\pi/4, 3\pi/4)$ and $(5\pi/4, 7\pi/4)$

Since $f'(x)$ changes from positive to negative when passing through $x=\frac{\pi}{4}, \frac{5\pi}{4}$

$f(x)$ has local maximum at $x=\frac{\pi}{4}, x=\frac{5\pi}{4}$ and the local maximum value is

$f(\frac{\pi}{4}) = \frac{1}{2}\sin\cancel{2}(\frac{\pi}{\cancel{4}_2})+5$

$= \frac{1}{2}(1)+5 = \frac{1}{2}+5 = \frac{11}{2}$

Also, $f(\frac{5\pi}{4}) = \frac{1}{2}\sin\cancel{2}(\frac{5\pi}{\cancel{4}_2})+5$

$= \frac{1}{2}\sin(2\pi+\frac{\pi}{2})+5$

$= \frac{1}{2}\sin\frac{\pi}{2}+5$

$= \frac{1}{2}(1)+5 = \frac{1}{2}+5 = \frac{11}{2}$

Also, since $f'(x)$ changes from negative to positive when passing through $x=\frac{3\pi}{4}, \frac{7\pi}{4}$

$f(x)$ has local minimum at $x=\frac{3\pi}{4}$, $x=\frac{7\pi}{4}$ and the local minimum value is $f(\frac{3\pi}{4}) = \frac{1}{2}\sin\cancel{2}(\frac{3\pi}{\cancel{4}_2})+5$

$= \frac{1}{2}(-1)+5 = \frac{-1}{2}+5 = \frac{9}{2}$

Also,

$f(\frac{7\pi}{4}) = \frac{1}{2}\sin\cancel{2}(\frac{7\pi}{\cancel{4}_2})+5$

$= \frac{1}{2}\sin(3\pi+\frac{\pi}{2})+5$

$= \frac{1}{2}\sin(2\pi+(\pi+\frac{\pi}{2}))+5$

$= \frac{1}{2}\sin(\pi+\frac{\pi}{2})+5$

$= \frac{1}{2}\sin\frac{3\pi}{2}+5$

$= \frac{1}{2}(-1)+5 = \frac{-1}{2}+5 = \frac{9}{2}$

$\therefore$ local maximum $= \frac{11}{2}$

local minimum $= \frac{9}{2}$

EX 7.46 Prove that the function $f(x) = x^2+2$ is strictly increasing in the interval $(2,7)$ and strictly decreasing in the interval $(-2,0)$

Soln:

$f(x) = x^2+2$

$f'(x) = 2x > 0 \ \forall x \in (2,7)$

and $f'(x) = 2x < 0 \ \forall x \in (-2,0)$

$\therefore f(x)$ is strictly increasing on $(2,7)$ and strictly decreasing on $(-2,0)$

EX 7.47

Prove that the function $f(x) = x^2-2x-3$ is strictly increasing in $(2,\infty)$

Soln:

$f(x) = x^2-2x-3$

$f'(x) = 2x-2 > 0$, $\forall x \in (2,\infty)$

$\therefore f(x)$ is strictly increasing on $(2,\infty)$

EX 7.48

Find the absolute maximum and absolute minimum values of the function $f(x) = 2x^3+3x^2-12x$ on $[-3,2]$

Soln: $f(x)$ is continuous on $[-3,2]$

$f(x) = 2x^3+3x^2-12x$

$f'(x) = 6x^2+6x-12$

Let $f'(x) = 0$,

$6x^2+6x-12 = 0$

$\div 6$, $x^2+x-2 = 0$

$(x+2)(x-1) = 0$

$x+2=0$, $x-1=0$

$x=-2$, $x=1$

$\therefore x = -2, 1 \in [-3,2]$

$\therefore$ The critical points are $x=-2$ and $x=1$

The end points are $x=-3$ and $x=2$

when $x=-2$, $f(-2)=2(-2)^3+3(-2)^2-12(-2)$

$f(-2)=-16+12+24=20$

when $x=1$, $f(1)=2(1)^3+3(1)^2-12(1)$

$f(1)=2+3-12=-7$

when $x=-3$, $f(-3)=2(-3)^3+3(-3)^2-12(-3)$

$f(-3)=-54+27+36=9$

when $x=2$, $f(2)=2(2)^3+3(2)^2-12(2)$

$f(2)=16+12-24=4$

$\therefore$ The absolute maximum is 20 and the absolute minimum is -7

EX 7.49

Find the absolute extrema of the function $f(x)=3\cos x$ on the closed interval $[0,2\pi]$

Soln: $f(x)=3\cos x$, $f(x)$ continuous on $[0,2\pi]$

$f'(x)=-3\sin x$

Let $f'(x)=0$, $-3\sin x=0$

$\sin x=0$,

$x=\pi\in[0,2\pi]$

$\therefore$ The critical point is $x=\pi$

The End points are $x=0$, $x=2\pi$

when $x=\pi$, $f(\pi)=3\cos\pi$

$f(\pi)=3(-1)=-3$

when $x=0$, $f(0)=3\cos 0$

$f(0)=3(1)=3$

when $x=2\pi$, $f(2\pi)=3\cos 2\pi$

$f(2\pi)=3(1)=3$

$\therefore$ The absolute maximum is 3 and the absolute minimum is -3

EX 7.50

Find the intervals of monotonicity and hence find the local extrema for the function $f(x)=x^2-4x+4$

Soln: $f(x)=x^2-4x+4$

$f'(x)=2x-4$

Let $f'(x)=0$, $2x-4=0$

$\cancel{2}x=\cancel{4}2$

$x=2$

$\therefore$ The stationary point is $x=2$

$-\infty$ 2 ∞

The intervals are $(-\infty,2)$ and $(2,\infty)$

Interval	Sign of $f'(x)$	Monotonicity
$(-\infty,2)$	$-$	Strictly decreasing
$(2,\infty)$	$+$	Strictly increasing

$f(x)$ is strictly decreasing on $(-\infty,2)$ and strictly increasing on $(2,\infty)$

Since $f'(x)$ changes from negative to positive when passing through $x=2$, $f(x)$ has local minimum at $x=2$ and the local minimum value is $f(2)=(2)^2-4(2)+4=4-8+4=0$

$\therefore$ local minimum $=0$

EX 7.51

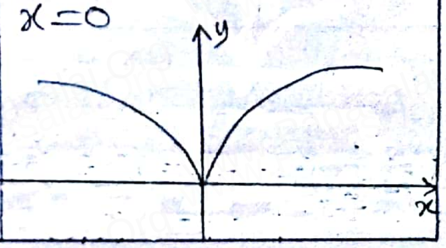
Find the intervals of monotonicity and hence find the local extrema for the function $f(x)=x^{2/3}$

Soln: $f(x)=x^{2/3}$

$f'(x)=\frac{2}{3}x^{-1/3}=\frac{2}{3x^{1/3}}$

$f'(x)\neq 0\ \forall\ x\in R$ and $f'(x)$ does not exist at $x=0$

$\therefore$ There are no stationary points but there is a critical point at $x=0$



$-\infty \quad 0 \quad \infty$

The intervals are $(-\infty,0)$ and $(0,\infty)$

Interval	Sign of $f'(x)$	Monotonicity
$(-\infty,0)$	$-$	strictly decreasing
$(0,\infty)$	$+$	strictly increasing

$f(x)$ is strictly decreasing on $(-\infty,0)$ and strictly increasing on $(0,\infty)$

Since $f'(x)$ changes from negative to positive when passing through $x=0$

$f(x)$ has local minimum at $x=0$ and the local minimum value is $f(0)=0$

$\therefore$ local minimum $=0$

$\therefore$ The local minimum occurs at $x=0$ which is not a stationary point

EX 7.52 Prove that the function $f(x)=x-\sin x$ is increasing on the real line. Also discuss for the existence of local extrema

Soln:

$f(x)=x-\sin x$

$f'(x)=1-\cos x \geq 0$

Let $f'(x)=0$, $1-\cos x=0$

$\cos x=1$,

$x=2n\pi, n\in Z$

$\therefore f(x)$ is increasing on the real line.

Since there is no sign change in $f'(x)$ when passing through $x=2n\pi$, $n\in z$, by the first derivative test, there is no local extrema

EX 7.53

Discuss the monotonicity and local extrema of the function $f(x)=\log(1+x)-\frac{x}{1+x}$, $x>-1$ and hence find the domain where $\log(1+x)>\frac{x}{1+x}$

Soln:

$f(x)=\log(1+x)-\frac{x}{1+x}$

$f'(x)=\frac{1}{1+x}-\left[\frac{(1+x)\cdot 1-x\cdot 1}{(1+x)^2}\right]$

$=\frac{1}{1+x}-\left[\frac{1+\cancel{x}-\cancel{x}}{(1+x)^2}\right]$

$=\frac{1}{1+x}-\frac{1}{(1+x)^2}$

$=\frac{\cancel{1}+x-\cancel{1}}{(1+x)^2}$

$f'(x)=\frac{x}{(1+x)^2}$

Let $f'(x)=0$, $\frac{x}{(1+x)^2}=0$

$\Rightarrow x=0$

$\therefore$ The stationary point is $x=0$

$-\infty \quad 0 \quad \infty$

The intervals are $(-\infty,0)$ and $(0,\infty)$

Interval	Sign of $f'(x)$	Monotonicity
$(-1,0)$	$-$	strictly decreasing
$(0,\infty)$	$+$	strictly increasing

But $x=0$, $f'(x)=0$

$f(x)$ is strictly decreasing for $x<0$ and strictly increasing for $x>0$

Since $f'(x)$ changes from negative to positive when passing through $x=0$

$f(x)$ has local minimum at $x=0$ and the local minimum value is $f(0) = \log(1+0) - \frac{0}{1+0}$

$= \log 1 - 0 = \log 1$

$f(0) = 0$

$\therefore$ local minimum $= 0$

Further, for $x > 0$

$f(x) > f(0)$

$\log(1+x) - \frac{x}{1+x} > 0$

$\log(1+x) > \frac{x}{1+x}$ on $(0, \infty)$

$\therefore$ The domain is $(0, \infty)$

EX 7.54 Find the intervals of monotonicity and local extrema of the function $f(x) = x\log x + 3x$

Soln:

Given: $f(x) = x\log x + 3x$, $x \in (0, \infty)$

$f'(x) = x\left(\frac{1}{x}\right) + \log x \cdot 1 + 3$

$= 1 + \log x + 3$

$f'(x) = 4 + \log x$

Let $f'(x) = 0$, $4 + \log x = 0$

$\log x = -4$

$x = e^{-4}$

$\therefore$ The stationary point is $x = e^{-4}$

$0 \quad e^{-4} \quad \infty$

The intervals are $(0, e^{-4})$ and (e^{-4}, ∞)

Interval	Sign of $f'(x)$	Monotonicity
$(0, e^{-4})$	$-$	strictly decreasing
(e^{-4}, ∞)	$+$	strictly increasing

$f(x)$ is strictly decreasing on $(0, e^{-4})$ and strictly increasing on (e^{-4}, ∞)

Since $f'(x)$ changes from negative to positive when passing through $x = e^{-4}$.

$f(x)$ has local minimum at $x = e^{-4}$ and the local minimum value is

$f(e^{-4}) = e^{-4}\log e^{-4} + 3e^{-4}$

$= e^{-4}(-4) + 3e^{-4}$

$= -4e^{-4} + 3e^{-4}$

$f(e^{-4}) = -e^{-4}$

$\therefore$ local minimum $= -e^{-4}$

EX 7.55

Find the intervals of monotonicity and local extrema of the function $f(x) = \frac{1}{1+x^2}$

Soln: Given: $f(x) = \frac{1}{1+x^2}$

$f'(x) = \frac{\cancel{(1+x^2)(0)} - 1(2x)}{(1+x^2)^2}$

$f'(x) = \frac{-2x}{(1+x^2)^2}$

Let $f'(x) = 0$,

$\frac{-2x}{(1+x^2)^2} = 0$

$-2x = 0$

$x = 0$

$\therefore$ The stationary point is $x = 0$

$-\infty \quad 0 \quad \infty$

The intervals are $(-\infty, 0)$ and $(0, \infty)$

Interval	Sign of $f'(x)$	Monotonicity
$(-\infty, 0)$	$+$	strictly increasing
$(0, \infty)$	$-$	strictly decreasing

$f(x)$ is strictly increasing on $(-\infty, 0)$ and strictly decreasing on $(0, \infty)$

Since $f'(x)$ changes from positive to negative when passing through $x=0$.

$f(x)$ has local maximum at $x=0$ and the local maximum value is

$f(0) = \frac{1}{1+(0)^2} = \frac{1}{1} = 1$

local maximum $= 1$

Ex 7.56

Find the intervals of monotonicity and local extrema of the function $f(x) = \frac{x}{1+x^2}$

Soln: Given: $f(x) = \frac{x}{1+x^2}$

$f'(x) = \frac{(1+x^2)\cdot 1 - x(2x)}{(1+x^2)^2}$

$= \frac{1+x^2-2x^2}{(1+x^2)^2}$

$f'(x) = \frac{1-x^2}{(1+x^2)^2}$

Let $f'(x) = 0$, $\frac{1-x^2}{(1+x^2)^2} = 0$

$1-x^2 = 0$

$x^2 = 1$

$x = \pm 1$

$\therefore$ The stationary points are $x = -1$ and $x = 1$

$-\infty \quad -1 \quad 1 \quad \infty$

The intervals are $(-\infty,-1)$, $(-1,1)$ and $(1,\infty)$

Interval	sign of $f'(x)$	Monotonicity
$(-\infty,-1)$	$-$	Strictly decreasing
$(-1,1)$	$+$	Strictly increasing
$(1,\infty)$	$-$	Strictly decreasing

$\therefore f(x)$ is strictly decreasing on $(-\infty,-1)$ and $(1,\infty)$ and strictly increasing on $(-1,1)$

Since $f'(x)$ changes from negative to positive when passing through $x=-1$

$f(x)$ has local minimum at $x=-1$ and the local minimum value is $f(-1)$

$= \frac{-1}{1+(-1)^2} = \frac{-1}{1+1} = \frac{-1}{2}$

Also, since $f'(x)$ changes from positive to negative when passing through $x=1$

$f(x)$ has local maximum at $x=1$ and the local maximum value is $f(1) = \frac{1}{1+(1)^2} = \frac{1}{1+1} = \frac{1}{2}$

$\therefore$ local minimum $= -\frac{1}{2}$

local maximum $= \frac{1}{2}$

7.7. Applications of Second Derivative

EXERCISE 7.7

1) Find intervals of concavity and points of inflexion for the following functions:

(i) $f(x) = x(x-4)^3$

Soln: Given: $f(x) = x(x-4)^3$

$f'(x) = x\,3(x-4)^2 + (x-4)^3\cdot 1$

$= (x-4)^2(3x+x-4)$

$= (x-4)^2(4x-4)$

$= (x-4)^2\, 4(x-1)$

$f(x) = 4(x-4)^2(x-1)$

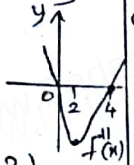
$f''(x) = 4[(x-4)^2 \cdot 1 + (x-1)2(x-4)]$

$= 4(x-4)[x-4+2(x-1)]$

$= 4(x-4)[x-4+2x-2]$

$= 4(x-4)(3x-6)$

$= 4(x-4)3(x-2)$



$f''(x) = 12(x-4)(x-2)$

Let $f''(x) = 0$,

$12(x-4)(x-2) = 0$

$12(x-4) = 0$, $x-2=0$

$x-4=0$

$x = 4$ and $x = 2$

$-\infty \quad 2 \quad 4 \quad \infty$

The intervals are $(-\infty, 2)$, $(2,4)$ and $(4,\infty)$

Interval	sign of $f''(x)$	Concavity
$(-\infty,2)$	+	Concave upward
$(2,4)$	−	Concave downward
$(4,\infty)$	+	Concave upward

The curve is concave upwards on $(-\infty, 2)$ and $(4,\infty)$.

The curve is concave downwards on $(2,4)$

$f''(x)$ changes its sign when it passes through $x=2$ and $x=4$

when $x=2$, $f(2) = 2(2-4)^3$

$= 2(-2)^3$

$f(2) = -16$

when $x=4$, $f(4) = 4(4-4)^3$

$f(4) = 0$

$\therefore$ The point of inflection are $(2,-16)$ and $(4,0)$

(ii) $f(x) = \sin x + \cos x$, $0 < x < 2\pi$

Soln: Given: $f(x) = \sin x + \cos x$

$x \in (0, 2\pi)$

$f'(x) = \cos x - \sin x$

$f''(x) = -\sin x - \cos x$

Let $f''(x) = 0$, $-\sin x - \cos x = 0$

$\sin x = -\cos x$

$\frac{\sin x}{\cos x} = -1$

$\tan x = -1$, $x = \frac{3\pi}{4}, \frac{7\pi}{4} \in (0, 2\pi)$

$0 \quad \frac{3\pi}{4} \quad \frac{7\pi}{4} \quad 2\pi$

The intervals are $(0, \frac{3\pi}{4})$, $(\frac{3\pi}{4}, \frac{7\pi}{4})$ and $(\frac{7\pi}{4}, 2\pi)$

Interval	Sign of $f''(x)$	Concavity
$(0,\frac{3\pi}{4})$	−	Concave downward
$(\frac{3\pi}{4},\frac{7\pi}{4})$	+	Concave upward
$(\frac{7\pi}{4},2\pi)$	−	Concave downward

The curve is concave upwards on $(\frac{3\pi}{4}, \frac{7\pi}{4})$

The curve is concave downwards on $(0, \frac{3\pi}{4})$ and $(\frac{7\pi}{4}, 2\pi)$

$f''(x)$ changes its sign when it passes through $x = \frac{3\pi}{4}$ and $x = \frac{7\pi}{4}$

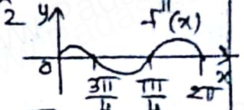
when $x = \frac{3\pi}{4}$,

$f(\frac{3\pi}{4}) = \sin\frac{3\pi}{4} + \cos\frac{3\pi}{4}$

$= \sin(\pi - \frac{\pi}{4}) + \cos(\pi - \frac{\pi}{4})$

$= \sin\frac{\pi}{4} - \cos\frac{\pi}{4}$

$= \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} = 0$



$f(\frac{3\pi}{4}) = 0$

when $x = \frac{7\pi}{4}$,

$f(\frac{7\pi}{4}) = \sin\frac{7\pi}{4} + \cos\frac{7\pi}{4}$

$= \sin(2\pi - \frac{\pi}{4}) + \cos(2\pi - \frac{\pi}{4})$

$= -\sin\frac{\pi}{4} + \cos\frac{\pi}{4}$

$= -\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = 0$

$f(\frac{7\pi}{4}) = 0$

$\therefore$ The point of inflection are $(\frac{3\pi}{4}, 0)$ and $(\frac{7\pi}{4}, 0)$

(iii) $f(x) = \frac{1}{2}(e^x - e^{-x})$

Soln: Given: $f(x) = \frac{1}{2}(e^x - e^{-x})$

$f'(x) = \frac{1}{2}(e^x + e^{-x})$

$f''(x) = \frac{1}{2}(e^{x} - e^{-x})$

Let $f''(x)=0$, $\frac{1}{2}(e^{x}-e^{-x})=0$

$e^{x} - e^{-x} = 0$

$e^{x} = e^{-x}$

$e^{x} = \frac{1}{e^{x}}$

$e^{x}\cdot e^{x} = 1$

$e^{2x} = 1$, $x = 0 \in (-\infty,\infty)$

$-\infty \quad 0 \quad \infty$

The intervals are $(-\infty,0)$ and $(0,\infty)$

Interval	Sign of $f''(x)$	Concavity
$(-\infty,0)$	$-$	Concave downward
$(0,\infty)$	$+$	Concave upward

The curve is concave upwards on $(0,\infty)$

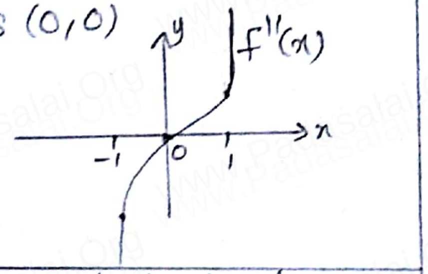
The curve is concave downwards on $(-\infty,0)$

$f''(x)$ changes its sign when it passes through $x=0$

when $x=0$, $f(0)=\frac{1}{2}(e^{0}-e^{0})$

$= \frac{1}{2}(1-1) = \frac{1}{2}(0) = 0$

$\therefore$ The point of inflection is $(0,0)$



2) Find the local extrema for the following functions using second derivative test:

(i) $f(x) = -3x^{5}+5x^{3}$

Soln: Given:

$f(x) = -3x^{5}+5x^{3}$

$f'(x) = -15x^{4}+15x^{2}$

Let $f'(x)=0$,

$-15x^{4}+15x^{2}=0$

$-15x^{2}(x^{2}-1)=0$

$-15x^{2}=0$, $x^{2}-1=0$

$x^{2}=0$, $x^{2}=1$

$x=0$ (twice), $x=\pm 1$

The critical points are $x = 0, 1, -1$

$f''(x) = -60x^{3}+30x$

At $x=-1$, $f''(x)>0$,

$f(x)$ has local minimum at $x=-1$ and the local minimum value is $f(-1) = -3(-1)^{5}+5(-1)^{3} = 3-5 = -2$

At $x=1$, $f''(x)<0$,

$f(x)$ has local maximum at $x=1$ and the local maximum value is $f(1) = -3(1)^{5}+5(1)^{3} = -3+5 = 2$

At $x=0$, $f''(0)=0$

The second derivative test does not give any information about local extrema at $x=0$

$\therefore$ local minimum $= -2$

local maximum $= 2$

(ii) $f(x) = x\log x$

Soln: Given:

$f(x) = x\log x$

$f'(x) = \cancel{x}\left(\frac{1}{\cancel{x}}\right) + \log x \cdot 1$

$f'(x) = 1+\log x$

Let $f'(x)=0$,

$1+\log x = 0$

$\log x = -1$

$x = e^{-1}$

$\boxed{x = \frac{1}{e}}$

The critical point is $x = \frac{1}{e}$

$f''(x) = 0 + \frac{1}{x}$

$f''(x) = \frac{1}{x}$

At $x = \frac{1}{e}$, $f''\left(\frac{1}{e}\right) = \frac{1}{1/e}$

$f''\left(\frac{1}{e}\right) = e > 0$,

$f(x)$ has local minimum at $x = \frac{1}{e}$ and the local minimum value is $f\left(\frac{1}{e}\right)$

$= \frac{1}{e}\log\left(\frac{1}{e}\right)$

$= \frac{1}{e}[\log 1 - \log e]$

$= \frac{1}{e}[0-1] = \frac{1}{e}(-1) = \frac{-1}{e}$

$\therefore$ local minimum $= \frac{-1}{e}$

(iii) $f(x) = x^2 e^{-2x}$

Soln: Given: $f(x) = x^2 e^{-2x}$

$f'(x) = x^2(e^{-2x}(-2)) + e^{-2x}(2x)$

$f'(x) = 2e^{-2x}(-x^2+x)$

Let $f'(x) = 0$,

$2e^{-2x}(-x^2+x) = 0$

$2e^{-2x} \neq 0$, $-x^2+x=0$

$e^{-2x} \neq 0$, $x(-x+1) = 0$

$\boxed{x=0}$, $-x+1=0$

$\boxed{x=1}$

$\therefore$ The critical points are $x=0$ and $x=1$

$f''(x) = 2e^{-2x}[-2x+1]$

$+(-x^2+x)2e^{-2x}(-2)$

$f''(x) = 2e^{-2x}[-2x+1+2x^2-2x]$

$f''(x) = 2e^{-2x}[2x^2-4x+1]$

At $x=0$, $f''(x) > 0$,

$f(x)$ has local minimum at $x=0$ and the local minimum value is $f(0) = 0$

At $x=1$, $f''(x) < 0$,

$f(x)$ has local maximum at $x=1$ and the local maximum value is $f(1) = e^{-2}$

$= \frac{1}{e^2}$

$\therefore$ local minimum $= 0$

local maximum $= \frac{1}{e^2}$

3) For the function $f(x) = 4x^3+3x^2-6x+1$ find the intervals of monotonicity, local extrema, intervals of concavity and points of inflection.

Soln: Given:

$f(x) = 4x^3+3x^2-6x+1$

$f'(x) = 12x^2+6x-6$

Let $f'(x) = 0$, $12x^2+6x-6=0$

$\div 6$, $2x^2+x-1=0$ $\quad \overset{-2}{\underset{2\ \ -1}{\wedge}}$

$2x^2+2x-x-1=0$

$2x(x+1)-1(x+1)=0$

$(2x-1)(x+1)=0$

$2x-1=0$, $x+1=0$

$x=\frac{1}{2}$, $x=-1$

$\therefore$ The stationary points are $x=-1$ and $x=\frac{1}{2}$

$-\infty \quad -1 \quad 1/2 \quad \infty$

The intervals are $(-\infty,-1)$, $(-1,1/2)$ and $(1/2,\infty)$

Interval	Sign of $f'(x)$	Monotonicity
$(-\infty,-1)$	$+$	Strictly increasing
$(-1,1/2)$	$-$	Strictly decreasing
$(1/2,\infty)$	$+$	Strictly increasing

$f(x)$ is strictly increasing on $(-\infty,-1)$ and $(1/2,\infty)$ and strictly decreasing on $(-1,1/2)$

Since $f'(x)$ changes from positive to negative when passing through $x=-1$

f(x) has local maximum at $x=-1$ and the local maximum value is $f(-1)$

$= 4(-1)^3+3(-1)^2-6(-1)+1$

$= -4+3+6+1$

$= 6$

Also, since $f'(x)$ changes from negative to positive when passing through $x=\frac{1}{2}$

f(x) has local minimum at $x=\frac{1}{2}$ and the local minimum value is $f(\frac{1}{2})$

$= 4(\frac{1}{2})^3+3(\frac{1}{2})^2-\cancel{6}(\frac{1}{\cancel{2}})+1$

$= \cancel{4}(\frac{1}{\cancel{8}_2})+3(\frac{1}{4})-3+1$

$= \frac{1}{2}+\frac{3}{4}-2$

$= \frac{2+3-8}{4} = \frac{-3}{4}$

$\therefore$ local maximum $= 6$

local minimum $= \frac{-3}{4}$

Also,

Given: $f(x)=4x^3+3x^2-6x+1$

$f'(x) = 12x^2+6x-6$

$f''(x) = 24x+6$

Let $f''(x)=0$, $24x+6=0$

$\cancel{24}4x = -\cancel{6}$

$4x = -1$

$x = -\frac{1}{4}$

$-\infty \quad -\frac{1}{4} \quad \infty$

The intervals are $(-\infty,-\frac{1}{4})$ and $(-\frac{1}{4},\infty)$

Interval	Sign of $f''(x)$	Concavity
$(-\infty,-\frac{1}{4})$	$-$	Concave downward
$(-\frac{1}{4},\infty)$	$+$	Concave upward

The curve is concave upwards on $(-\frac{1}{4},\infty)$

The curve is concave downwards on $(-\infty,-\frac{1}{4})$

$f''(x)$ changes its sign when it passes through $x=-\frac{1}{4}$

When $x=-\frac{1}{4}$,

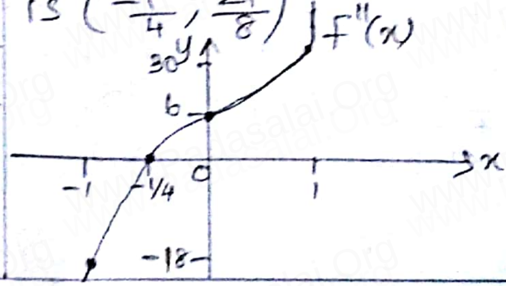
$f(-\frac{1}{4}) = 4(-\frac{1}{4})^3+3(-\frac{1}{4})^2-6(-\frac{1}{4})+1$

$= \cancel{4}(\frac{-1}{\cancel{64}_{16}})+3(\frac{1}{16})+\frac{6}{4}+1$

$= \frac{-1+3+24+16}{16}$

$= \frac{\cancel{42}21}{\cancel{16}8} = \frac{21}{8}$

$\therefore$ The point of inflection is $(-\frac{1}{4}, \frac{21}{8})$



Ex 7.57

Determine the intervals of concavity of the curve $f(x)=(x-1)^3.(x-5)$, $x\in R$ and, points of inflection if any.

Soln: Given:

$f(x) = (x-1)^3.(x-5)$

$f(x) = (x^3-3x^2+3x-1)(x-5)$

$= x^4-3x^3+3x^2-x$

$\quad -5x^3+15x^2-15x+5$

$f(x) = x^4-8x^3+18x^2-16x+5$

$f'(x) = 4x^3-24x^2+36x-16$

$f''(x) = 12x^2-48x+36$

Let $f''(x)=0$,

$12x^2-48x+36=0$

$\div 3$, $4x^2-16x+12=0$

$4x^2 - 4x - 12x + 12 = 0$ 48, -4, -12

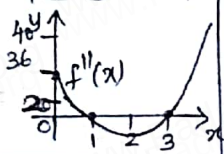
$4x(x-1) - 12(x-1) = 0$

$(4x-12)(x-1) = 0$

$4x - 12 = 0,\ x - 1 = 0$

$\cancel{4}x = \cancel{12}\,3$

$x = 3, x = 1$



$-\infty$ 1 3 ∞

The intervals are $(-\infty, 1)$, $(1, 3)$ and $(3, \infty)$

Interval	Sign of $f''(x)$	Concavity
$(-\infty, 1)$	+	Concave upward
$(1, 3)$	−	Concave downward
$(3, \infty)$	+	Concave upward

The curve is concave upwards on $(-\infty, 1)$ and $(3, \infty)$

The curve is concave downwards on $(1, 3)$

$f''(x)$ changes its sign when it passes through $x = 1$ and $x = 3$

when $x = 1$,

$f(1) = (1-1)^3 \cdot (1-5)$

$= (0) \cdot (-4) = 0$

$f(1) = 0$

when $x = 3$,

$f(3) = (3-1)^3 \cdot (3-5)$

$= (2)^3 \cdot (-2)$

$= (8) \cdot (-2)$

$f(3) = -16$

$\therefore$ The point of inflection are $(1, 0)$ and $(3, -16)$

EX 7.58

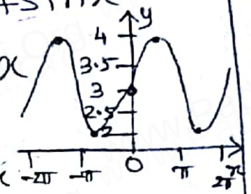
Determine the intervals of concavity of the curve $y = 3 + \sin x$

Soln: The given function is a periodic function with period 2π

Given: $y = 3 + \sin x$

$\frac{dy}{dx} = \cos x$

$\frac{d^2y}{dx^2} = -\sin x$



Let $\frac{d^2y}{dx^2} = 0 \Rightarrow -\sin x = 0$

$\sin x = 0,\ x = n\pi$

Now, Consider an interval $(-\pi, \pi)$

$-\pi$ 0 π

The intervals are $(-\pi, 0)$ and $(0, \pi)$

Interval	Sign of d^2y/dx^2	Concavity
$(-\pi, 0)$	+	Concave upward
$(0, \pi)$	−	Concave downward

The curve is concave upwards on $(-\pi, 0)$

The curve is concave downwards on $(0, \pi)$

$\frac{d^2y}{dx^2}$ changes its sign when it passes through $x = 0$

when $x = 0$, $y = 3 + \sin 0$

$y = 3 + 0$

$y = 3$

$\therefore$ The point of inflection is $(0, 3)$

The general intervals are $(n\pi, (n+1)\pi), n \in Z$

$\therefore$ The point of inflection are $(n\pi, 3)$

EX 7.59

Find the local extremum of the function $f(x) = x^4 + 32x$

Soln: Given: $f(x) = x^4 + 32x$

$f'(x) = 4x^3 + 32$

Let $f'(x) = 0$, $4x^3 + 32 = 0$

$4x^3 = -32$

$x^3 = (-2)^3$

$x = -2$

The critical point is $x = -2$

$f''(x) = 12x^2$

At $x = -2$, $f''(x) > 0$,

$f(x)$ has local minimum at $x = -2$ and the local minimum value is $f(-2) = (-2)^4 + 32(-2) = 16 - 64$

$= -48$

$\therefore$ local minimum $= -48$

$\therefore$ The extreme point is $(-2, -48)$

Ex 7.60

Find the local extrema of the function $f(x) = 4x^6 - 6x^4$

Soln:

Given: $f(x) = 4x^6 - 6x^4$

$f'(x) = 24x^5 - 24x^3$

Let $f'(x) = 0$,

$24x^5 - 24x^3 = 0$

$24x^3(x^2 - 1) = 0$

$24x^3 = 0$, $x^2 - 1 = 0$

$x^3 = 0$, $x^2 = 1$

$x = 0$ (thrice), $x = \pm 1$

$\therefore$ The critical points are $x = -1, 0, 1$

Now, $f'(x) = 24x^5 - 24x^3$

$f''(x) = 120x^4 - 72x^2$

At $x = -1$, $f''(x) > 0$

$f(x)$ has local minimum at $x = -1$

At $x = 1$, $f''(x) > 0$

$f(x)$ has local minimum at $x = 1$

At $x = 0$, $f''(x) = 0$

The second derivative test does not give any information about local extrema at $x = 0$

$-\infty \quad -1 \quad 0 \quad 1 \quad \infty$

The intervals are $(-\infty, -1)$, $(-1, 0)$, $(0, 1)$ and $(1, \infty)$

Interval	sign of $f'(x)$	Monotonicity
$(-\infty, -1)$	$-$	Strictly decreasing
$(-1, 0)$	$+$	Strictly increasing
$(0, 1)$	$-$	Strictly decreasing
$(1, \infty)$	$+$	Strictly increasing

$\therefore f(x)$ is strictly increasing on $(-1, 0)$ and $(1, \infty)$ and strictly decreasing on $(-\infty, -1)$ and $(0, 1)$

Since $f'(x)$ changes from negative to positive when passing through $x = -1$

$f(x)$ has local minimum at $x = -1$ and the local minimum value is $f(-1) = 4(-1)^6 - 6(-1)^4$

$f(-1) = 4 - 6 = -2$

Also, since $f'(x)$ changes from positive to negative when passing through $x = 0$

$f(x)$ has local maximum at $x = 0$ and the local maximum value is $f(0) = 0$

Since $f'(x)$ changes from negative to positive when passing through $x=1$

$f(x)$ has local minimum at $x=1$ and the local minimum value is

$f(1)=4(1)^6-6(1)^4=4-6=-2$

$\therefore$ local minimum : -2

local maximum : 0

Ex 7.61

Find the local maximum and minimum of the function x^2y^2 on the line $x+y=10$

Soln: Given: $f(x)=x^2y^2$

and $x+y=10$ —①

$\boxed{y=10-x}$

① ⇒ $f(x)= x^2(10-x)^2$

$f(x)=x^2(100-20x+x^2)$

$f(x)=x^4-20x^3+100x^2$

$f'(x)=4x^3-60x^2+200x$

Let $f'(x)=0$,

$4x^3-60x^2+200x=0$

$\div 4$, $x^3-15x^2+50x=0$

$x(x^2-15x+50)=0$

$\boxed{x=0}$, $x^2-15x+50=0$

$(x-10)(x-5)=0$

$x-10=0, x-5=0$

$\boxed{x=10}$, $\boxed{x=5}$

The critical points are $x=0,5,10$

$f'(x)=4x^3-60x^2+200x$

$f''(x)=12x^2-120x+200$

At $x=0$, $f''(x)>0$,

$f(x)$ has local minimum at $x=0$ and the local minimum value is $f(0)=0$

At $x=5$, $f''(x)<0$,

$f(x)$ has local maximum at $x=5$ and the local maximum value is $f(5)=(5)^2(10-5)^2=(25)(25)=625$

At $x=10$, $f''(x)>0$,

$f(x)$ has local minimum at $x=10$ and the local minimum value is $f(10)=0$

$\therefore$ local minimum $=0$

local maximum $=625$

7.8 Applications in optimization

EXERCISE 7.8

1) Find two positive numbers whose sum is 12 and their product is maximum

Soln:

Let x and y be the two numbers

$\therefore x+y=12 \Rightarrow y=12-x$ —①

Let P denote their product

$P=xy$

$P=x(12-x)$

$P=12x-x^2$

$P'=12-2x$

$P''=-2$

Let $P'=0$, $12-2x=0$

$\cancel{2}x=\cancel{12}\,6$

$\boxed{x=6}$

When $x=6$, $P''=-2<0$

The product is maximum

① ⇒ $y=12-6=6$ $\therefore \boxed{y=6}$

$\therefore$ The two numbers are 6, 6

2) Find two positive numbers whose product is 20 and their sum is minimum

Soln:

Let x and y be the two Positive numbers

$$\therefore xy = 20$$

$$y = \frac{20}{x} \quad \text{—— ①}$$

Let S denote their sum

$$\therefore S = x + y$$

$$S = x + \frac{20}{x}$$

$$S' = 1 - \frac{20}{x^2}$$

$$S'' = \frac{40}{x^3}$$

Let $S' = 0$, $1 - \frac{20}{x^2} = 0$

$$\frac{20}{x^2} = 1 \Rightarrow x^2 = 20$$

$$x = \pm\sqrt{20}$$

$$x = \pm\sqrt{4\times5}$$

$$x = \pm 2\sqrt{5}$$

But x is positive,

$$\therefore \boxed{x = 2\sqrt{5}}$$

When $x = 2\sqrt{5}$, $S'' = \frac{40}{(2\sqrt{5})^3} > 0$

$\therefore$ S is minimum.

When $x = 2\sqrt{5}$,

$$y = \frac{\cancel{20}\,10}{\cancel{2}\sqrt{5}} = \frac{2\times\sqrt{5}\times\cancel{\sqrt{5}}}{\cancel{\sqrt{5}}}$$

$$\boxed{y = 2\sqrt{5}}$$

$\therefore$ The two positive numbers are $2\sqrt{5}, 2\sqrt{5}$

3) Find the Smallest Possible value of x^2+y^2 given that $x+y=10$

Soln: Given: $x+y=10$

$$y = 10 - x$$

Let $f(x) = x^2+y^2$

$$f(x) = x^2 + (10-x)^2$$

$$f'(x) = 2x + 2(10-x)(-1)$$

$$= 2x - 2(10-x)$$

$$= 2x - 20 + 2x$$

$$f'(x) = 4x - 20$$

Let $f'(x) = 0$,

$$4x - 20 = 0$$

$$\cancel{4}x = \cancel{20}\,5$$

$$\boxed{x = 5}$$

$$f''(x) = 4$$

When $x = 5$, $f''(x) = 4 > 0$

When $x = 5$, function is minimum

When $x = 5$, $f(5) = (5)^2 + (10-5)^2$

$$= (5)^2 + (5)^2$$

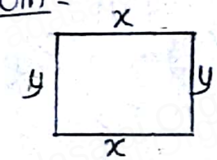
$$= 25 + 25$$

$$= 50$$

$\therefore$ The Smallest Possible value is 50

4) A garden is to be laid out in a rectangular area and protected by wire fence. What is the largest possible area of the fenced garden with 40 metres of wire

Soln:



Let x and y be the length and breadth of the rectangular garden

The total length of the fencing is 40 m

$$\therefore 2x + 2y = 40$$

$\div 2$, $x + y = 20$

$$\boxed{y = 20 - x} \quad \text{—— ①}$$

$$\therefore \text{Area} = xy$$

$$A(x) = x(20-x)$$

$$A(x) = 20x - x^2$$

$A'(x) = 20 - 2x$

$A''(x) = -2$

Let $A'(x)=0$, $20-2x=0$

$\cancel{2}x = \cancel{20}^{10}$

$\boxed{x=10}$

when $x=10$, $A''(x)=-2<0$

when $x=10$, Area is maximum

when $x=10$, $y=20-10$

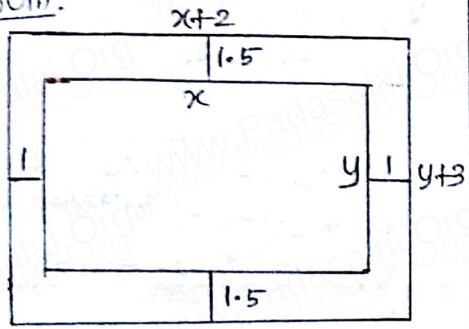
$y=10$

$\therefore x=10, y=10$

The largest possible Area $= xy = (10)(10) = 100 m^2$

5) A rectangular page is to contain $24 cm^2$ of print. The margins at the top and bottom of the page are 1.5 cm and the margins at other sides of the page is 1 cm. What should be the dimensions of the page so that the area of the paper used in minimum

Soln:



Let x and y be the length and breadth of the printed Area, then the area $xy=24$

$y = \frac{24}{x}$ —①

Dimensions of the page are $x+2$ and $y+3$

paper Area $A = (x+2)(y+3)$

$= xy + 3x + 2y + 6$

$= 24 + 3x + 2\left(\frac{24}{x}\right) + 6$

$A(x) = 3x + \frac{48}{x} + 30$

$A'(x) = 3 - \frac{48}{x^2}$

$A''(x) = \frac{96}{x^3}$

$A'(x) = 0$, $3 - \frac{48}{x^2} = 0$

$\frac{\cancel{48}^{16}}{x^2} = \cancel{3}$

$x^2 = 16 \Rightarrow x^2 = 4^2$

$\boxed{x=\pm 4}$

But $x>0$, $\therefore \boxed{x=4}$

when $x=4$, $A''(x) = \frac{96}{(4)^3}$

$= \frac{\cancel{96}^{3}}{\cancel{64}_{2}} = \frac{3}{2} > 0$

when $x=4$, The Area of the page is minimum

when $x=4$, $y = \frac{\cancel{24}^{6}}{\cancel{4}} = 6$

$\boxed{y=6}$

length of the page

$= x+2 = 4+2 = 6cm$

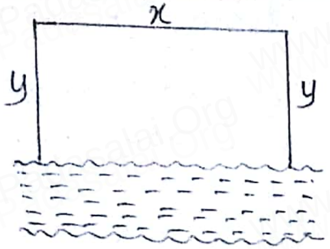
breadth of the page

$= y+3 = 6+3 = 9cm$

$\therefore$ The dimensions of the page are 6cm and 9cm

6) A Farmer plans to fence a rectangular pasture adjacent to a river. The pasture must contain 1,80,000 sq. mtrs. in order to provide enough grass for herds. No fencing is needed along the river. What is the length of the minimum needed fencing material?

Soln:



Let x and y be the length and breadth of the pasture

Total Area $= 1,80,000$

$\therefore xy = 1,80,000$

$y = \dfrac{1,80,000}{x}$ —(1)

Length of the Fencing $= x+2y$

$L(x) = x+2\left(\dfrac{1,80,000}{x}\right)$

$L(x) = x+\dfrac{3,60,000}{x}$

$L'(x) = 1-\dfrac{3,60,000}{x^2}$

$L''(x) = \dfrac{7,20,000}{x^3}$

$L'(x)=0, \; 1-\dfrac{3,60,000}{x^2} = 0$

$\dfrac{3,60,000}{x^2} = 1$

$x^2 = 3,60,000$

$x^2 = (600)^2$

$x = 600$ m

when $x=600$,

$L''(x) = \dfrac{7,20,000}{(600)^3} > 0$

when $x=600$, Length of fencing is minimum

When $x=600$,

(1) $\Rightarrow y = \dfrac{\cancel{1,80,000}^{\,300}}{\cancel{600}}$

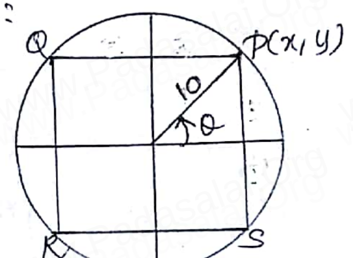
$\therefore y = 300$ m

$\therefore$ Minimum length of the fencing $= x+2y = 600+2(300)$

$= 600+600 = 1200$ m

7) Find the dimensions of the rectangle with maximum area that can be inscribed in a circle of radius 10 cm

Soln :



Let us take the circle to be a circle with centre $(0,0)$ and radius r and PQRS be the rectangle inscribed in it. Let $P(x,y)$ be the vertex of the rectangle that lies on the first quadrant. Let θ be the angle made by OP with the x-axis

Then $x = 10\cos\theta$,

$y = 10\sin\theta$

Now the dimension of the rectangle are

$2x = 20\cos\theta$,

$2y = 20\sin\theta, \; 0 \le \theta \le \frac{\pi}{2}$

Area of the rectangle $A = (2x)(2y)$

$\therefore$ Area of the rectangle

$A(\theta) = (20\cos\theta)(20\sin\theta)$

$= 400\sin\theta\cos\theta$

$= 200 \times 2\sin\theta\cos\theta$

$A(\theta) = 200\sin 2\theta$

$A'(\theta) = 200\cos 2\theta (2)$

$A'(\theta) = 400\cos 2\theta$

$A''(\theta) = 400(-\sin 2\theta)(2)$

$A''(\theta) = -800\sin 2\theta$

Let $A'(\theta) = 0$

$400\cos 2\theta = 0$

$\cos 2\theta = 0, \; 2\theta = \frac{\pi}{2}$

$\Rightarrow \theta = \frac{\pi}{4}$

$\therefore \boxed{\theta = \frac{\pi}{4}}$

when $\theta = \frac{\pi}{4}$, $A''(\theta)$

$= -800\sin\cancel{2}\left(\frac{\pi}{\cancel{4}_2}\right)$

$= -800(1) = -800 < 0$

when $\theta = \frac{\pi}{4}$, Area is maximum

when $\theta = \frac{\pi}{4}$,

$2x = 20\cos\frac{\pi}{4}$

$= 20\left(\frac{1}{\sqrt{2}}\right) = (\cancel{\sqrt{2}})(\sqrt{2})10\left(\frac{1}{\cancel{\sqrt{2}}}\right)$

$= 10\sqrt{2}$

$2y = 20\sin\frac{\pi}{4} = 20\left(\frac{1}{\sqrt{2}}\right)$

$= (\cancel{\sqrt{2}})(\sqrt{2})10\left(\frac{1}{\cancel{\sqrt{2}}}\right) = 10\sqrt{2}$

$\therefore$ The dimensions of the rectangle are $10\sqrt{2}$, $10\sqrt{2}$

8) Prove that among all the rectangles of the given perimeter, the square has the maximum area.

Soln: Let x and y be the length and breadth of the rectangle. Let L be the perimeter

$\therefore L = 2x + 2y$ —①

$L - 2x = 2y$

$y = \frac{L-2x}{2}$

Let A be the Area

$A = xy$

$A = x\left[\frac{L-2x}{2}\right]$

$A(x) = \frac{1}{2}[Lx - 2x^2]$

$A'(x) = \frac{1}{2}[L - 4x]$

$A''(x) = \frac{1}{\cancel{2}}[-\cancel{4}^{\,2}] = -2$

$A''(x) = -2$

Let $A'(x) = 0$, $\frac{1}{2}[L-4x] = 0$

$L - 4x = 0$

$L = 4x$

when $L = 4x$, $A''(x) = -2 < 0$

when $L = 4x$, The Area is maximum

when $L = 4x$,

① $\Rightarrow 4x = 2x + 2y$

$4x - 2x = 2y$

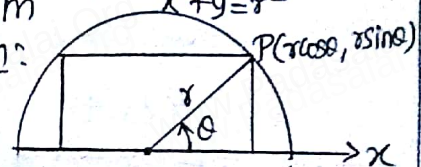
$\cancel{2}x = \cancel{2}y$

$x = y$

$\therefore$ The rectangle is a square when the area is maximum

9) Find the dimensions of the largest rectangle that can be inscribed in a semi circle of radius r cm

Soln:



Let θ be the angle made by OP with the positive direction of x-axis.

Then the area of the rectangle $A = (2x)(y)$

$A(\theta) = (2r\cos\theta)(r\sin\theta)$

$= r^2 2\sin\theta\cos\theta$

$A(\theta) = r^2\sin 2\theta$

$A'(\theta) = r^2\cos 2\theta\,(2)$

$A'(\theta) = 2r^2\cos 2\theta$

$A''(\theta) = 2r^2(-\sin 2\theta)(2)$

$A''(\theta) = -4r^2\sin 2\theta$

Let $A'(\theta)=0$,

$2r^2\cos 2\theta = 0$

$\cos 2\theta = 0$, $2\theta = \frac{\pi}{2}$

$\theta = \frac{\pi}{4}$

When $\theta=\frac{\pi}{4}$,

$A''(\theta) = -4r^2\sin 2\theta$

$A''(\frac{\pi}{4}) = -4r^2 \sin \cancel{2}(\frac{\pi}{\cancel{4}_{2}})$

$= -4r^2(1)$

$A''(\frac{\pi}{4}) = -4r^2 < 0$

∴ Area is Maximum when $\theta = \frac{\pi}{4}$,

$2x = 2r\cos\theta$

$= 2r\cos\frac{\pi}{4}$

$= \overset{\sqrt{2}}{\cancel{2}}r(\frac{1}{\cancel{\sqrt{2}}})$

$2x = \sqrt{2}\, r$

$y = r\sin\theta = r\sin\frac{\pi}{4}$

$y = r(\frac{1}{\sqrt{2}}) = \frac{r}{\sqrt{2}}$

The dimensions of the largest rectangle that can be inscribed in a semicircle are $\sqrt{2}r, \frac{r}{\sqrt{2}}$

Also, when $\theta = \frac{\pi}{4}$,

$A(\theta) = r^2\sin\cancel{2}(\frac{\pi}{\cancel{4}_{2}})$

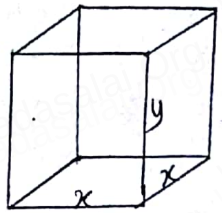
$= r^2\sin\frac{\pi}{2}$

$= r^2(1)$

$A(\theta) = r^2$ ∴ Area is r^2

10) A manufacturer wants to design an open box having a square base and a surface area of 108 sq. cm. Determine the dimensions of the box for the maximum volume

Soln:



Let x, x and y be the length, breadth and height of the box

Area of base $= x^2$

Area of sides $= 4xy$

Given: Surface Area $=108$

$\therefore x^2+4xy=108$

$4xy = 108 - x^2$

$y = \frac{108-x^2}{4x}$ —— (1)

Volume of the box $= lbh$

$V = (x)(x)y$

$V = x^2 y$

$V(x) = \cancel{x^2}\left[\frac{108-x^2}{4\cancel{x}}\right]$

$V(x) = \frac{1}{4}(108x - x^3)$

$V'(x) = \frac{1}{4}(108-3x^2)$

$V''(x) = \frac{1}{\cancel{4}_{2}}(-\overset{3}{\cancel{6}}x)$

$V''(x) = -\frac{3}{2}x$

Let $V'(x)=0$,

$\frac{1}{4}(108-3x^2)=0$

$108-3x^2=0$

$\cancel{3}x^2 = \cancel{108}\,36$

$x^2=36$

$x=\pm 6$

But $x>0$, $x=-6$ is not possible.

$\therefore x=6$ cm

When $x=6$,

$V''(x) = -\frac{3}{\cancel{2}}(\overset{3}{\cancel{6}}) = -9<0$

$v''(x) = -9 < 0$, when $x=6$,

∴ volume of box is maximum when $x=6$,

① ⇒ $y = \frac{108-(6)^2}{4(6)}$

$= \frac{108-36}{24} = \frac{\cancel{72}}{\cancel{24}} = 3$

∴ $y = 3$ cm

∴ dimension of box are $x=6$cm, $x=6$cm and $y=3$cm

11) The volume of a cylinder is given by the formula $V = \pi r^2 h$. Find the greatest and least values of V if $r+h=6$

Soln: Given: $r+h=6$

$\Rightarrow h = 6-r$ —— ①

Also Given: $V = \pi r^2 h$

$V(r) = \pi r^2(6-r)$

$V(r) = \pi(6r^2 - r^3)$

$V'(r) = \pi(12r - 3r^2)$

$V''(r) = \pi(12 - 6r)$

Let $V'(r) = 0$,

$\pi(12r - 3r^2) = 0$

$12r - 3r^2 = 0$

$r(12 - 3r) = 0$

$r = 0$, $12 - 3r = 0$

$\cancel{3}r = \cancel{12}\,4$

$\boxed{r=0}$, $\boxed{r=4}$

when $r=0$, $V''(r) = 12\pi > 0$

when $r=0$, volume is minimum

when $r=4$, $V''(r) = -12\pi < 0$

when $r=4$, volume is maximum

∴ Given: $h+r=6$, $0 \le h, r \le 6$

$V(r) = \pi(6r^2 - r^3)$

$V(0) = \pi(6(0)^2 - (0)^3)$

$= \pi(0)$

$V(0) = 0$

$V(4) = \pi(6(4)^2 - (4)^3)$

$= \pi(96 - 64)$

$V(4) = 32\pi$

$V(0) = \pi(0) = 0$

$V(6) = \pi(6(6)^2 - (6)^3)$

$= \pi(216 - 216)$

$= \pi(0)$

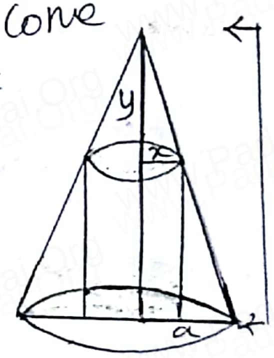
$V(6) = 0$

∴ greatest value $= 32\pi$

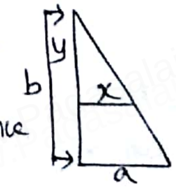
least value $= 0$

12) A hollow Cone with base radius a cm and height b cm is placed on a table. Show that the volume of the largest cylinder that can be hidden underneath is $\frac{4}{9}$ times volume of the cone

Soln:



Let x be the radius of the cylinder and y be the distance from the top of cone to the inscribed cylinder.



Given: height of cone $= b$

height of cylinder $= b - y$

Volume of cylinder $V = \pi r^2 h$

$V = \pi x^2 (b-y)$ — ①

$\frac{y}{x} = \frac{b}{a}$

$y = \frac{bx}{a}$

① $\Rightarrow V(x) = \pi x^2 \left(b - \frac{bx}{a}\right)$ — ②

$V(x) = \pi b \left(x^2 - \frac{x^3}{a}\right)$

$V'(x) = \pi b \left(2x - \frac{3x^2}{a}\right)$

$V''(x) = \pi b \left(2 - \frac{6x}{a}\right)$

Let $V'(x) = 0$,

$\pi b \left(2x - \frac{3x^2}{a}\right) = 0$

$2x - \frac{3x^2}{a} = 0$

$x\left(2 - \frac{3x}{a}\right) = 0$

$\boxed{x=0}$, $2 - \frac{3x}{a} = 0$

$\frac{3x}{a} = 2$

$3x = 2a$

$\boxed{x = \frac{2a}{3}}$

$\therefore x = 0$ (or) $x = \frac{2a}{3}$

$x = 0$ is not possible

when $x = \frac{2a}{3}$,

$V''(x) = \pi b \left(2 - \frac{\cancel{6}^{2}\left(\frac{2a}{\cancel{3}}\right)}{\cancel{a}}\right)$

$= \pi b (2 - 4)$

$V''(x) = -2\pi b < 0$

when $x = \frac{2a}{3}$, $V(x)$ is maximum

$\therefore$ when $x = \frac{2a}{3}$, volume of cylinder is maximum when $x = \frac{2a}{3}$,

② $\Rightarrow V(x) = \pi x^2 \left(b - \frac{bx}{a}\right)$

$= \pi \left(\frac{2a}{3}\right)^2 \left(b - \frac{b\left(\frac{2a}{3}\right)}{a}\right)$

$= \pi \left(\frac{4a^2}{9}\right)\left(b - \frac{2\cancel{a}b}{3} \times \frac{1}{\cancel{a}}\right)$

$= \pi \left(\frac{4a^2}{9}\right)\left(b - \frac{2b}{3}\right)$

$= \pi \left(\frac{4a^2}{9}\right)\left(\frac{b}{3}\right)$

$= \frac{4}{9}\left(\frac{1}{3}\pi a^2 b\right)$

$= \frac{4}{9}$ (volume of cone)

$\therefore$ Volume of the largest Cylinder $= \frac{4}{9}$ (volume of Cone)

Ex 7.62

We have a 12 square unit piece of thin material and want to make an open box by cutting small squares from the corners of our material and folding the sides up. The question is, which cut produces the box of maximum volume?

Soln: