CHAPTER - 4

4. COMBINATORICS AND MATHEMATICAL INDUCTION

Exercise: 4.1

1) i) A Person went to a restaurant for dinner. In the menu card, the person saw 10 Indian and 7 chinese food items. In how many ways the person can select either an Indian or a chinese food?

Sol:

The Person can be selected,

10 Indian food in 10 ways

7 chinese food in 7 ways.

A Person selecting 10 Indian or chinese food is,

$= 10 + 7$

$= 17$ ways.

$\therefore$ In 17 ways the person can select either an Indian or a chinese food.

ii) There are 3 types of toy car and 2 types of toy train available in a shop. Find the number of ways a baby can buy a toy car and a toy train?

Sol:

Number of ways of buying a toy car from 3 types of car is 3.

Number of ways of buying a toy train from 2 types of train is 2.

$\therefore$ Number of ways of buying a toy and toy train,

$$= 3 \times 2$$

$$= 6.$$

$\therefore$ The Number of ways is 6.

---

iii) How many two-digit numbers can be formed using 1, 2, 3, 4, 5 without repition of digits?

Sol:

Tens	Ones
4	5

One's place can be filled in 5 ways.

Ten's place can be filled in 4 ways.

$\therefore$ Number of two digit number using digits 1, 2, 3, 4, 5 is

$= 4 \times 5$

$= 20.$

$\therefore$ 20 two digit numbers can be formed

---

iv) Three persons enter in to a conference hall in which there are 10 seats. In how many ways they can take their seats?

Sol

Number of ways of a seat getting, 1^{st} Person $= 10$

2^{nd} Person $= 9$

3^{rd} Person $= 8$

Number of ways getting seats for 3 persons are,

$$=10\times9\times8$$

$$=90\times8$$

$$=720.$$

$\therefore$ In 720 ways they can take their seats.

---

v) In how many ways 5 persons can be seated in a row?

Sol:

To arrange 5 person in a row We need 5 place.

Number of ways,

1^{st} Person = 5

2^{nd} Person = 4

3^{rd} Person = 3

4^{th} Person = 2

5^{th} Person = 1

Number of ways of 5 person can be selected in a row,

$= 5 \times 4 \times 3 \times 2 \times 1$

$= 20 \times 6 \times 1$

$= 120$

$\therefore$ In 120 ways 5 persons can be seated in a row.

---

2) i) A mobile Phone has a passcode of 6 distinct digits. What is the maximum number of attempts one makes to receive the passcode?

Sol:

Since the pass code has 6 distinct digits.

First digit can be tried in 10 ways using. 0, 1, 2, 3, 4, 5, 6, 7, 8, 9

Similarly

2^{nd}, 3^{rd}, 4^{th}, 5^{th}, 6^{th} digits are also.

9, 8, 7, 6, 5 respectively.

Maximum number of attempts made to retrieve the pass code,

$= 10 \times 9 \times 8 \times 7 \times 6 \times 5$

$= 90 \times 56 \times 30$

$= 2700 \times 56$

$= 151200$

$\therefore$ Maximum number of attempts one makes to retrieve the passcode is 151200.

---

ii) Given four flags of different colours, how many different signals can be generated if each signal requires the use of three flags, one below the other.

Sol:

The total number of signals in equal to the number of ways of filling 3 places in succession by 4 flags of different colour.

The upper place can be filled in 4 ways following.

Number of ways signals $= 4 \times 3 \times 2$

$= 4 \times 3 \times 2$

$= 12 \times 2$

$= 24$

$\therefore$ 24 different signals can be generated.

---

3) Four children are running a race.

i) In how many ways can the first two places be filled?

Sol. First Place can be given to any one of the 4 children.

Second Place can be given to any one of the 3 children.

$\therefore$ The first two places can be filled in,

$= 4 \times 3$

$= 12$ ways.

$\therefore$ In 12 ways can the first two places be filled.

ii) In how many diffe... they finish the race?

Sol: The race can be finished in,

$4! = 4 \times 3 \times 2 \times 1$

$= 12 \times 2 \times 1$

$= 24$

$\therefore$ 24 ways could finish the race.

Count the number of three-digit numbers which can be formed from the digits 2, 4, 6, 8 if

i) repititions of digits is allowed.

Sol:

Hundreds	Tens	Unit
4	4	4

The unit place can be filled in 4 ways.

Since, repitition is allowed, The tens place can also be filled in ways each

Total number of 3 digit numbers,

$= 4 \times 4 \times 4$

$= 16 \times 4$

$= 64.$

---

ii) Repititions of digits is not allowed.

Sol:- The unit Place can be filled in 4 ways.

Since, repitition of digits is not allowed the tens place can be filled in 3 ways.

Hundred Place can be filled in 2 ways.

Total number of 3 digits number without repitition,

$= 4 \times 3 \times 2$

$= 12 \times 2$

$= 24.$

5) How many three digit numbers are there with 3 in the unit place?

i) With repitition.

Sol:

Hundreds	Tens	Units
9	10	1

The Given digits are 0, 1, 2, 3, 4, 5, 6, 7, 8, 9.

The unit place can be filled in only one way using 3.

Since repetition is allowed, the tens place can be filled in 10 ways using any one of the digits from 0, 1, 2, 3, 4, 5, 6, 7, 8, 9.

The hundreds place can be filled in 9 ways using the digits (1, 2, 3, 4, 5, 6, 7, 8, 9) excluding 0.

$\therefore$ By Fundamental Principle of multiplication, the total number of 3 digit numbers $\Rightarrow 9 \times 10$

$= 90.$

ii) without repetition.

Sol. Hundreds Tens Unit

8 8 1

The unit Place can be filled in only one way using the digits.

The hundreds place can be filled in 8 ways using the digits 1, 2, 3, 4, 5, 6, 7, 8, 9 (excluding 0 and 3).

Since repetition is not allowed and tens place can be filled in 8 ways, using the digits (0, 1, 2, 3, 4, 5, 6, 7, 8, 9) including 0.

∴ By, Fundamental principle of multiplication total number of 3 digits are

$\Rightarrow 8 \times 8 \times 1$

$= 64.$

Exercise: 4·4

1) By the principle of mathematical induction, prove that, for $n \geq 1$

$$1^3 + 2^3 + 3^3 + \dots + n^3 = \left(\frac{n(n+1)}{2}\right)^2.$$

Sol:

$$P(n) = 1^3 + 2^3 + 3^3 + \dots + n^3$$

$$= \left(\frac{n(n+1)}{2}\right)^2$$

Step: 1

Put $n = 1$

$$P(1) \Rightarrow 1^3 = \left(\frac{1(1+1)}{2}\right)^2$$

$$1 = \left[\frac{1(\cancel{2})}{\cancel{2}}\right]^2$$

$$1 = (1)^2$$

$$1 = \underline{1}$$

$n = 1$ is true.

Step 2:

Assume that $n = k$ is true.

$$P(k) \Rightarrow 1^3 + 2^3 + 3^3 + \dots + k^3 = \left(\frac{k(k+1)}{2}\right)^2$$

Step 3:

To Prove, $n = k+1$ is true.

$$P(k+1) = 1^3 + 2^3 + \ldots + k^3 + (k+1)^3$$

$$= \left[\frac{k(k+1)}{4}\right]^2 + (k+1)^3$$

$$= \frac{k^2(k+1)^2 + 4(k+1)^3}{4}$$

$$= \frac{(k+1)^2\,[k^2 + 4(k+1)]}{4}$$

$$= \frac{(k+1)^2\,[k^2 + 4k + 4]}{4}$$

$$= \frac{(k+1)^2\,(k+2)^2}{4}$$

$$= \left[\frac{(k+1)(k+2)}{2}\right]^2$$

$P(k+1)$ is true.

By the principle of mathematical induction $\forall n \geq 1$.

$$1^3 + 2^3 + 3^3 + \ldots + n^3 = \left[\frac{n(n+1)}{2}\right]^2.$$

2) By the principle of mathematical induction, prove that, for $n \geq 1$

$$1^2+3^2+5^2+\dots\dots+(2n-1)^2 = \frac{n(2n-1)(2n+1)}{3}$$

Sol:

$$P(n) \Rightarrow 1^2+3^2+5^2+\dots+(2n-1)^2 = \frac{n(2n-1)(2n+1)}{3}$$

Step : 1

Prove that, $P(1)$ is true.

$$P(1) \Rightarrow 1^2 = \frac{1(2(1)-1)(2(1)+1)}{3}$$

$$1 = \frac{1(2-1)(2+1)}{3}$$

$$1 = \frac{1(1)(3)}{3}$$

$$1 = \frac{3}{3}$$

$$1 = 1$$

$P(1)$ is a True.

Step : 2

Assume that $P(k)$ is true.

Put $n=k$

$$P(K) \Rightarrow 1^2 + 3^2 + 5^2 + \dots + (2K-1)^2 = \frac{K(2K-1)(2K+1)}{3}$$

P(K) is a true.

Step : 3

To Prove P(K+1) is true.

$$P(K+1) = \frac{(K+1)[2(K+1)-1][2(K+1)+1]}{3}$$

$$= \frac{(K+1)(2K+1)(2K+3)}{3}$$

$$P(K+1) = \frac{(K+1)(2K+1)(2K+3)}{3}$$

Step 4 : To Proof :

$$P(K+1) = 1^2 + 3^2 + 5^2 + \dots (2K-1)^2 + (2(K+1)-1)^2$$

$$= \frac{K(2K-1)(2K+1)}{3} + (2K+2-1)^2$$

$$= \frac{K(2K-1)(2K+1)}{3} + (2K+1)^2$$

$$= \frac{K(2K-1)(2K+1) + 3(2K+1)^2}{3}$$

$$= \frac{(2K+1)[K(2K-1)+3(2K+1)]}{3}$$

$$= \frac{(2K+1)\left[K(2K-1)+3(2K+1)\right]}{3}$$

$$= \frac{(2K+1)(2K^2-K+6K+3)}{3}$$

$$= \frac{(2K+1)(2K^2+5K+3)}{3}$$

$$= \frac{(2K+1)(2K^2+2K+3K+3)}{3}$$

$$= \frac{(2K+1)(2K^2+2K+3K+3)}{3}$$

$$= \frac{(2K+1)\left[2K(K+1)+3(K+1)\right]}{3}$$

$$= \frac{(2K+1)(2K+3)(K+1)}{3}$$

= R.H.S

$\therefore$ P(K+1) is true.

By Principle of mathematical induction.

It is true for $n \geq 1$.

$$1^2+3^2+5^2+\ldots\ldots+(2n-1)^2 = \frac{n(2n-1)(2n+1)}{3}$$

3) Prove that the sum of the first n non-zero even numbers is n^2+n.

Sol:

Given that,

$P(n) = 2+4+6+\ldots\ldots\ldots+2n = n^2+n$

Step 1:

$n=1$

$2(1) = (1)^2+1$

$2 = 1+1$

$2 = 2$

$\therefore P(1)$ is true.

Step 2:

Assume that $n=k$ is true.

$P(k) = 2+4+6+\ldots\ldots+2k = k^2+k$.

Step 3:

To Prove:

$n=k+1$ is true.

$P(k+1) = 2+4+6+\dots+2k+2(k+1)$

$= k^2+k+2(k+1)$

$= k^2+k+2k+2$

$= k^2+3k+2$

$= k^2+k+2k+2$

$= (k^2+2k+1)+(k+1)$

$= (k+1)^2+(k+1)$

$\therefore P(k+1)$ is true.

By Principle of mathematical induction,

$2+4+6+\dots+2n = n^2+n.$

---

4) By the principle of Mathematical induction, prove that, for $n \geq 1$.

$$1\cdot 2+2\cdot 3+3\cdot 4+\dots+n\cdot(n+1) = \frac{n(n+1)(n+2)}{3}$$

Sol:

Step 1:

Put $n = 1$

$P(n) = 1\cdot 2+2\cdot 3+3\cdot 4+\dots n(n+1)$

$p(n) \Rightarrow 1\cdot 2 + 2\cdot 3 + 3\cdot 4 + \dots + n(n+1)$

$$= \frac{n(n+1)(n+2)}{3}$$

$$P(1) \Rightarrow 1(1+1) = \frac{1(1+1)(1+2)}{3}$$

$$1(2) = \frac{1(2)(3)}{3}$$

$$2 = \frac{2\times 3}{3}$$

$$2 = \frac{\cancel{6}}{\cancel{3}}$$

$$2 = 2$$

$\therefore P(1)$ is true.

Step 2:

Assume that $P(k)$ is true.

$$1\cdot 2 + 2\cdot 3 + 3\cdot 4 + \dots + k(k+1) = \frac{k(k+1)(k+2)}{3}$$

Step 3:

To Prove that,

$$P(k+1) = \frac{(k+1)(k+2)(k+3)}{3}$$

Proof:

$$P(K+1) = 1\cdot 2 + 2\cdot 3 + 3\cdot 4 + \dots + K(K+1) + (K+1)(K+2)$$

$$= \frac{K(K+1)(K+2)}{3} + (K+1)(K+2)$$

$$= \frac{K(K+1)(K+2) + 3(K+1)(K+2)}{3}$$

$$= \frac{(K+1)(K+2)(K+3)}{3}$$

$= R.H.S$

By Principal of mathematical induction,

$$1\cdot 2 + 2\cdot 3 + 3\cdot 4 + \dots + n(n+1) = \frac{n(n+1)(n+2)}{3}$$

$\therefore$ Hence it is Proved.

Using the Mathematical induction, show that for any natural number $n \geq 2$.

$$\left(1 - \frac{1}{2^2}\right)\left(1 - \frac{1}{3^2}\right)\left(1 - \frac{1}{4^2}\right)\dots\left(1 - \frac{1}{n^2}\right) = \frac{n+1}{2n}$$

30.

Step 1:

Put, $n = 2$

$$P(2) \Rightarrow \left[1 - \frac{1}{2^2}\right] = \frac{2+1}{2(2)}$$

$$1 - \frac{1}{4} = \frac{3}{4}$$

$$\frac{4-1}{4} = \frac{3}{4}$$

$$\frac{3}{4} = \frac{3}{4}$$

$\therefore P(2)$ is true.

Step 2:

Assume that $P(k)$ is true.

$$P(K) \Rightarrow \left[1 - \frac{1}{2^2}\right]\left[1 - \frac{1}{3^2}\right]\left[1 - \frac{1}{4^2}\right] \cdots \left[1 - \frac{1}{K^2}\right] = \frac{K+1}{2K} \rightarrow (1)$$

Step 3:

$$P(K+1) = \frac{(K+1)+1}{2(K+1)} = \frac{K+2}{2(K+1)}$$

$$P(K+1) = \left[1 - \frac{1}{2^2}\right]\left[1 - \frac{1}{3^2}\right]\left[1 - \frac{1}{4^2}\right]\cdots$$

$$\left[1 - \frac{1}{K^2}\right]\left[1 - \frac{1}{(K+1)^2}\right]$$

$$P(K+1) = P(K)\left[1 - \frac{1}{(K+1)^2}\right]$$

$$= \frac{K+1}{2K}\left[1 - \frac{1}{(K+1)^2}\right]$$

$$= \frac{K+1}{2K}\left[\frac{(K+1)^2 - 1}{(K+1)^2}\right]$$

$$= \frac{K+1}{2K}\left(\frac{(K+1)^2 - 1}{(K+1)^2}\right)$$

$$= \frac{K+1}{2K}\left[\frac{K^2 + \cancel{1} + 2K - \cancel{1}}{(K+1)^2}\right]$$

$$= \frac{K+1}{2K}\left[\frac{K^2 + 2K}{(K+1)^2}\right]$$

$$= \frac{(K+1)}{2\cancel{K}} \, \frac{\cancel{K}(K+2)}{(K+1)^{\cancel{2}}}$$

$$= \frac{K+2}{2(K+1)} \rightarrow (2)$$

By Principle of mathematical induction,

$$\left[1-\frac{1}{2^2}\right]\left[1-\frac{1}{3^2}\right]\left[1-\frac{1}{4^2}\right]\ldots\ldots\left(1-\frac{1}{n^2}\right) = \frac{n+1}{2n}.$$

---

6) Using the Mathematical induction, Show that for any natural number $n \geq 2$,

$$\frac{1}{1+2}+\frac{1}{1+2+3}+\frac{1}{1+2+3+4}+\ldots\frac{1}{1+2+3+\ldots+n} = \frac{n-1}{n+1}.$$

Sol:

$$P(n) = \frac{1}{1+2}+\frac{1}{1+2+3}+\frac{1}{1+2+3+4}+\ldots\ldots+\frac{1}{1+2+3+\ldots+n} = \frac{n-1}{n+1}.$$

Step 1:

Put $n=2$

$$P(n) = \frac{n-1}{n+1}$$

$$P(2) = \frac{2-1}{2+1}$$

$$= \frac{2-1}{2+1}$$

$$= \frac{1}{3}$$

$$P(2) = \frac{1}{3}$$

L.H.S = R.H.S

$\therefore$ P(2) is a true.

Step 2:

Assume that p(k) is true.

$$P(k) = \frac{1}{1+2} + \frac{1}{1+2+3} + \dots + \frac{1}{1+2+3+\dots+k}$$

$$= \frac{k-1}{k+1}$$

Step 3:

To Prove that.

$$P(k+1) = \frac{(k+1)-1}{(k+1)+1}$$

$$= \frac{k+\cancel{1}-\cancel{1}}{k+2}$$

$$= \frac{k}{k+2}$$

$$P(k+1) = \frac{k}{k+2} \rightarrow 1$$

$$P(k+1) = \frac{1}{1+2} + \frac{1}{1+2+3} + \dots \frac{1}{1+2+3+\dots k} + \frac{1}{1+2+\dots k(k+1)}$$

$$= P(k) + \frac{1}{1+2+3+\dots+(k+1)}$$

$$= P(k) + \frac{1}{\frac{(k+1)[(k+1)+1]}{2}}$$

$$= P(k) + \frac{2}{(k+1)(k+2)}$$

$$= P(k) + \frac{2}{(k+1)(k+2)}$$

$$= \frac{(k-1)}{k+1} + \frac{2}{(k+1)(k+2)}$$

$$= \frac{(k-1)(k+2)+2}{(k+1)(k+2)}$$

$$= \frac{k^2+2k-k-\cancel{2}+\cancel{2}}{(k+1)(k+2)}$$

$$= \frac{k^2+k}{(k+1)(k+2)}$$

$$= \frac{k\cancel{(k+1)}}{\cancel{(k+1)}(k+2)}$$

$$= \frac{k}{(k+2)}$$

$P(k+1)$ is true.

By using mathematical induction,

$$\frac{1}{1+2} + \frac{1}{1+2+3} + \frac{1}{1+2+3+4} + \dots \frac{1}{1+2+3+\dots+n} = \frac{n-1}{n+1}.$$

7) Using the Mathematical induction, show that for any natural number n,

$$\frac{1}{1\cdot 2\cdot 3}+\frac{1}{2\cdot 3\cdot 4}+\frac{1}{3\cdot 4\cdot 5}+\frac{1}{n\cdot(n+1)\cdot(n+2)}=\frac{n(n+3)}{4(n+1)(n+2)}$$

Sol:

Step 1:

$$P(1)=\frac{1}{1\cdot(1+1)(1+2)}$$

$$=\frac{1}{1\cdot 2(3)}$$

$$=\frac{1}{6}$$

$$P(1)=\frac{1}{6}$$

$$P(1)=\frac{1(1+3)}{4(1+1)(1+2)}$$

$$=\frac{1(4)}{4(2)(3)}$$

$$=\frac{4}{4\cdot 6}$$

$$= \frac{4}{4 \cdot 6}$$

$$= \frac{4}{24}$$

$$= \frac{1}{6}$$

$$P(1) = \frac{1}{6}$$

P(1) is True.

Step 2:

Assume that $n = k$ is true.

$$P(k) = \frac{1}{1 \cdot 2 \cdot 3} + \frac{1}{2 \cdot 3 \cdot 4} + \dots + \frac{1}{k(k+1)}$$

$$= \frac{k(k+3)}{4(k+1)(k+2)}$$

$$P(k) = \frac{k(k+3)}{4(k+1)(k+2)}$$

Step 3:

To Proof:

$$n = k+1$$

To Prove:

$$P(K+1) = \frac{(K+1)(K+1+3)}{4(K+1+1)(K+1+2)}$$

$$= \frac{(K+1)(K+4)}{4(K+2)(K+3)}$$

$$P(K+1) = \frac{(K+1)(K+4)}{4(K+2)(K+3)}$$

$$P(K+1) = \frac{1}{1\cdot 2\cdot 3} + \frac{1}{2\cdot 3\cdot 4} + \frac{1}{3\cdot 4\cdot 5} + \dots\dots$$

$$\frac{1}{K(K+1)(K+2)} + \frac{1}{(K+1)(K+2)(K+3)}$$

$$= P(K) + \frac{1}{(K+1)(K+2)(K+3)}$$

$$= \frac{K(K+3)}{4(K+1)(K+2)} + \frac{1}{(K+1)(K+2)(K+3)}$$

$$= \frac{K(K+3)(K+3)+4}{4(K+1)(K+2)(K+3)}$$

$$= \frac{K(K+3)^2+4}{4(K+1)(K+2)(K+3)}$$

$$= \frac{k(k+3)^2+4}{4(k+1)(k+2)(k+3)}$$

$$= \frac{k(k^2+9+6k)+4}{4(k+1)(k+2)(k+3)}$$

$$= \frac{k^3+9k+6k^2+4}{4(k+1)(k+2)(k+3)}$$

$$\begin{array}{r|rrrr}
-1 & 1 & 6 & 9 & 4 \\
 & 0 & -1 & -5 & -4 \\
\hline
-1 & 1 & 5 & 4 & \boxed{0} \\
 & 0 & -1 & -4 & \\
\hline
-4 & 1 & 4 & \boxed{0} & \\
 & 0 & -4 & & \\
\hline
 & 1 & \boxed{0} & &
\end{array}$$

$$= \frac{(k+1)^2(k+4)}{4(k+1)(k+2)(k+3)}$$

$k = -1, -1, -4$

$(k-1)(k+1)(k+4)$

$= (k+1)^2(k+4)$

$$= \frac{\cancel{(k+1)}^2(k+4)}{4\cancel{(k+1)}(k+2)(k+3)}$$

$$= \frac{(k+1)(k+4)}{4(k+2)(k+3)}.$$

$P(k+1)$ is true.

By Principle mathematical induction

$$\frac{1}{1\cdot2\cdot3} + \frac{1}{2\cdot3\cdot4} + \frac{1}{3\cdot4\cdot5} + \dots + \frac{1}{(3n-1)(3n+2)} = \frac{n}{6n+4}.$$

8) Using the Mathematical induction, show that for any natural number n,

$$\frac{1}{2\cdot5}+\frac{1}{5\cdot8}+\frac{1}{8\cdot11}+\dots+\frac{1}{(3n-1)(3n+2)} = \frac{n}{6n+4}.$$

Sol:

$$\frac{1}{2\cdot5}+\frac{1}{5\cdot8}+\frac{1}{8\cdot11}+\dots+\frac{1}{(3n-1)(3n+2)} = \frac{n}{6n+4}$$

Step 1:

Put $n=1$

L.H.S =

$$\frac{1}{(3n-1)(3n+2)} = \frac{1}{3(1)-1)(3(1)+2)}$$

$$= \frac{1}{2\cdot5}$$

$$= \frac{1}{10}$$

R.H.S =

$$\frac{n}{6n+4} = \frac{1}{6(1)+4}$$

$$= \frac{1}{6(1)+4}$$

$$= \frac{1}{10}$$

L.H.S = R.H.S

P(1) is true.

Step 2:

Assume that P(K) is true.

$$P(K) = \frac{1}{2\cdot5} + \frac{1}{5\cdot8} + \frac{1}{8\cdot11} + \dots + \frac{1}{3(K-1)(3K+2)}$$

$$= \frac{K}{6K+4}.$$

where,

$$P(K) = \frac{K}{6K+4}$$

Step 3:

To Prove that,

$n = K+1.$

$$P(K+1) = \frac{K+1}{6(K+1)4}$$

$$= \frac{k+1}{6(k+1)4}$$

$$= \frac{k+1}{6k+6+4}$$

$$= \frac{k+1}{6k+10} \rightarrow (1)$$

L.H.S

$$= \frac{1}{2\cdot 5} + \frac{1}{5\cdot 8} + \frac{1}{(3k-1)(3k+2)} + \frac{1}{(3(k+1))-1]\ (3(k+1)+2)}$$

$$= P(k) + \frac{1}{(3k+3-1)(3k+3+2)}$$

$$= \frac{k}{6k+4} + \frac{1}{(3k+2)(3k+5)}$$

$$= \frac{k}{2(3k+2)} + \frac{1}{(3k+2)(3k+5)}$$

$$= \frac{1}{(3k+2)}\left[\frac{k}{2} + \frac{1}{3k+5}\right]$$

$$= \frac{1}{3k+2}\left[\frac{k(3k+5)+2}{2(3k+5)}\right]$$

$$= \frac{3k^2+5k+2}{2(3k+2)(3k+5)}$$

$$= \frac{(k+1)\cancel{(3k+2)}}{2\cancel{(3k+2)}(3k+5)}$$

$$= \frac{(k+1)}{2(3k+5)}$$

$$= \frac{k+1}{6k+10}$$

$$= \frac{k+1}{6k+10}$$

$= R \cdot H \cdot S$

$\therefore P(k+1)$ is True.

9) Prove by Mathematical Inducti

that,

$1! + (2 \times 2!) + (3 \times 3!) + \ldots + (n \times$

$= (n+1)! - 1$

Sol:

Let,

P(n) be the statement.

$1! + (2\times2)! + (3\times3)! + \ldots + (n\times n)! = (n+1)! - 1$

Step 1:

Putting $n = 1$,

we get

$P(1) \Rightarrow 1! = (1+1)! - 1$

$P(1) = 2! - 1$

$P(1) = 1$

L.H.S = R.H.S

P(1) is true.

Step 2:

Let us assume that P(K) is true.

$1! + (2\times2)! + (3\times3)! + \ldots (K\times K)! = (K+1)! - 1$

Step 3:

To Prove that P(K+1) is true.

Prove that

$1! + (2\times 2)! + (3\times 3)! + (K\times K\ !) + (K+1)$

$(K+1)! = (K+1)! - 1$

$= 1! \times (2\times 2!) + (3\times 3!) + \dots + (K\times K) + (K\times 1)!$

$= P(K) + (K+1)((K+1))$

$= (K+1)! - 1 + (K+1)(K+1)!$

$= (K+1)! + (K+1)(K+1)! - 1$

$= (K+1)! [1+K+1] - 1$

$= (K+1)! [2+K] - 1$

$= (K+2)! - 1$

L.H.S = R.H.S

P(K+1) is true.

Hence by the principle of mathematical induction P(n) for all values of n.

Using the Mathematical induction, show that for any natural number n, $x^{2n} - y^{2n}$ is divisible by $x+y$.

Sol:

$$P(n) = x^{2n} - y^{2n}$$

where $x^{2n} - y^{2n}$ is divisible by $x+y$

Step 1:

Put, $n = 1$

$$P(1) = x^{2n} - y^{2n}$$

$$= x^{2(1)} - y^{2(1)}$$

$$= x^2 - y^2$$

$x^2 - y^2 = x^2 - y^2$ is divisible by $x+y$.

$$\div \text{ by } x+y = \frac{x^2 - y^2}{x+y}$$

$$= \frac{(x-y)\cancel{(x+y)}}{\cancel{(x+y)}}$$

$$= (x-y)$$

$P(1)$ is true.

Step 2:

$n = k$

$P(k) = x^{2k} - y^{2k}$ is divisible by $(x+y)$.

$$\frac{x^{2k}-y^{2k}}{x+y} = c$$

$$x^{2k}-y^{2k} = c(x+y)$$

$$x^{2k} = y^{2k} + [c(x+y)] \rightarrow (1)$$

Step 3:

To Prove

$$n = k+1$$

$P(k+1) = x^{2(k+1)} - y^{2(k+1)}$ is divisible by $x+y$

$$P(k+1) = x^{2(k+1)} - y^{2(k+1)}$$

$$= x^{2k+2} - y^{2k+2}$$

$$= x^{2k}x^{2} - y^{2k}y^{2}$$

$$= \{y^{2k} + c(x+y)\}x^{2} - y^{2k}y^{2}$$

$$= y^{2k}(x^{2}-y^{2}) + c(x+y)x^{2}$$

$$= y^{2k}(x-y)(x+y) + c(x+y)x^{2}$$

$$= (x+y)(y^{2k}(x-y) + cx^{2})$$

It is divisible by $x+y$

P(k+1) is true.

By principle of mathematical induction.

P(n) is true $n \in N$.

$\therefore x^{2n} - y^{2n}$ is divisible by $x+y$.

By the principle of mathematical induction, prove that for $n \geq 1$.

$$1^2 + 2^2 + 3^2 + \dots + n^2 > \frac{n^3}{3}$$

Sol:

If, $n=1$

$1 > \frac{1}{3}$ is true.

P(1) is true

$\therefore$ P(k) is true.

$$P(k) = 1^2 + 2^2 + 3^2 + \dots + k^2 > \frac{k^3}{3}$$

To Prove:

$$P(K+1) = 1^2 + 2^2 + \dots + K^2 (K+1)^2 > \frac{(K+1)}{3}$$

$$= \frac{K^3}{3} + (K+1)^2$$

$$= \frac{K^3 + 3(K+1)^2}{3}$$

$$= \frac{K^3 + 3(K^2 + 2K + 1)}{3}$$

$$= \frac{K^3 + 3K^2 + 6K + 3}{3}$$

$$= \frac{(K^3 + 3K^2 + 3K + 1) + (3K + 2)}{3}$$

$$= \frac{K^3 + 3K^2 + 3K + 1}{3}$$

$$= \frac{(K+1)^3}{3}$$

$\therefore P(K+1)$ is true.

By the principle of mathematical induction $P(n)$ is true $\forall n \in N$ is true.

Use induction to prove that n^3-7n+3, is divisible by 3, for all natural numbers n.

Sol:

$$P(n) = n^3-7n+3$$

Step 1:

Put,

$n=1$

$$P(1) = 1^3-7(1)+3$$

$$=1-7+3$$

$$=-6+3$$

$$=-3$$

$$P(1)=-3$$

$P(1)$ is divisible by 3

$\therefore P(1)$ is true.

Step 2:

Assume that, $n=k$

$$P(k) = k^3-7(k)+3$$

$$=k^3-7k+3$$

$P(k)$ is divisible by 3.

Step 3:

To Prove:

$$k^3 - 7k + 3 = 3m$$

$$n = k+1$$

$$P(k+1) = (k+1)^3 - 7(k+1) + 3$$

$$= k^3 + 3k^3 + 3k + 1 - (7(k+1)) + 3$$

$$= k^3 + 3k^3 + 3k + 1 - 7k - 7 + 3$$

$$= (k^3 - 7k + 3) + (3k^3 + 3k + 1 - 7)$$

$$= 3m + (3k^2 + 3k - 6)$$

$$= 3m + 3(k^2 + k - 2)$$

$$= 3(m + k^2 + k - 2)$$

$$= 3c$$

$\therefore$ $c = (m + k^2 + k - 2) \in N$

$P(k+1)$ is true

$P(k+1)$ is divisible by 3.

By the principle of mathematical induction $n^3 - 7n + 3$ is divisible by 3, for all natural numbers n.

13) Use induction to prove that $5^{n+1}+4\times 6^n$ when divided by 20 leaves a remainder 9, for all natural numbers n.

Sol:

$5^{n+1}+4\times 6^n$ is divided by 20 and the remainder is 9.

If the remainder is 9 means $5^{n+1}+4\times 6^n-9$ is divisible by 20.

To Prove:

$5^{n+1}+4\times 6^n-9$ is divisible by 20

Put, $n=1$

$P(n) = 5^{1+1}+4\times 6^1-9$

$P(1) = 5^2+4\times 6-9$

$= 25+24-9$

$= 49-9$

$= 40.$

40 is divisible by 20

$\therefore$ P(1) is true.

$P(K)$ is true.

$\therefore 5^{k+1}+4.6^{k}-9$, 20 is divisible by 20.

To Prove

$P(K+1)$ is true.

$= 5^{k+1}+46 = 20C+9$

$= 5^{k+1+1}+4\times6^{k+1}-9$

$= 5^{k+1}.5+4.6^{k}.6-9$

$= 5\times5^{k+1}+46^{k}(5+1)-9$

$= 5\times5^{k+1}+4\times6^{k}\times5+46^{k}-9$

$= 5(20C+9)+4.6^{k}-9$

$= 100C+45+4.6^{k}-9$

$= 100C+4.6^{k}+36$

$4.6^{k}+36$ is the multiplier of 20

Put, $K=1$

$= 4.6^{k}+36$

$$= 4 \cdot 6^1 + 36$$

$$= 24 + 36$$

$$= 60$$

$$= 3(20)$$

3(20) is divisible by 20.

Let us assume,

$$k = m$$

$$4 \cdot 6^m + 36 = 20\lambda$$

$$k = m+1$$

$$4 \cdot 6^{m+1} + 36 = 4 \cdot 6^m \cdot 6 + 36$$

$$= (20\lambda - 36)6 + 36$$

$$= 20\lambda - 216 + 36$$

$$= 20\lambda - 180$$

$$= 20(\lambda - 9)$$

$20(\lambda - 9)$ is the multiplier of 20.

$\therefore$ $k = m+1$ is true.

$\forall k \in N$

$$5^{k+1+1}+4\times 6^{k+1}-9=100c+20a$$

$$=20(5c+a)$$

$5^{k+1+1}+4\times 6^{k+1}-9$ is divisible by 20.

$\therefore P(k+1)$ is true.

$P(n)$ is true for $n \in N$.

Use induction to prove that $10^n+3\times 4^{n+2}+5$, divisible by 9, for all natural numbers n.

Sol:

$$P(n)=10^n+3\times 4^{n+2}+5$$

Put, $n=1$

$$P(1)=10^1+3\times 4^{1+2}+5$$

$$=10+3\times 4^3+5$$

$$=10+3\times 64+5$$

$$=10+192+5$$

$$=207$$

$$P(1)=207$$

P(1) is divisible by 9

P(1) is true.

P(K) is true.

$$10^{K} + 3 \times 4^{K+2} + 5 = 9C \in N \quad \dots (1)$$

P(K+1) is true,

$$10^{K} = 9C - 3 \times 4^{K+2} - 5$$

$$P(K+1) = 10^{K+1} + 3 \times 4^{(K+1)+2} + 5$$

$$= 10^{K} \cdot 10 + 3 \times 4^{K+3} + 5$$

$$= (9C - 3 \times 4^{K+2} - 5)10 + 3 \times 4^{K+3} + 5$$

$$= [90C - 30 \times 4^{K+2} - 50] + 3 \cdot 4 \cdot 4^{K+2} + 5$$

$$= 90C - 30 \times 4^{K+2} + 12 4^{K+2} - 45$$

$$= 90C - 18 4^{K+2} - 45$$

$$= 9(10C - 2 \cdot 4^{K+2} - 5)$$

$$= 9m \qquad \therefore m = 10C - 2 \cdot 4^{K+2} - 5$$

Prove that using the Mathematic[al] induction.

$$\sin(\alpha)+\sin\left(\alpha+\frac{\pi}{6}\right)+\sin\left(\alpha+\frac{2\pi}{6}\right)+\ldots+\sin\left(\alpha+\frac{(n-1)\pi}{6}\right)=\frac{\sin\left(\alpha+\frac{(n-1)\pi}{12}\right)\times\sin\ldots}{\sin\left(\frac{\pi}{12}\right)}.$$

Sol:

$$P(n)=\sin\alpha+\sin\left(\alpha+\frac{\pi}{6}\right)+\sin\left(\alpha+\frac{2\ldots}{\ldots}\right)+\ldots\sin\left(\alpha+\frac{(n-1)}{6}\pi\right)$$

$$=\frac{\sin\left[\alpha+\frac{(n-1)\pi}{12}\right]\times\sin\frac{n\pi}{12}}{\sin\frac{\pi}{12}}$$

LHS

$P(1)=\sin\alpha$

RHS

$P(1)=\sin\alpha$

L.H.S = R.H.S

P(1) is true

P(k) is true.

$$P(K) = \sin\alpha + \sin\left(\alpha + \frac{\pi}{6}\right) + \sin\left(\alpha + \frac{2\pi}{6}\right) + \ldots$$

$$\sin\left[\alpha + \frac{(K-1)\pi}{6}\right] = \frac{\sin\left[\alpha + \frac{(K+1-1)\pi}{12}\right] \times \sin\frac{K\pi}{12}}{\sin\frac{\pi}{12}}$$

To Prove:

$$P(K+1) = \sin(\alpha) + \sin\left(\alpha + \frac{\pi}{6}\right) + \sin\left(\alpha + \frac{2\pi}{6}\right)$$

$$\ldots \sin\left(\alpha + \frac{K\pi}{6}\right)$$

$$= \frac{\sin\left[\alpha + \frac{K\pi}{12}\right] \sin\frac{(K+1)\pi}{12}}{\sin\frac{\pi}{12}}$$

$$= \frac{\sin\left(\alpha + \frac{(K-1)\pi}{12}\right) \sin\left(\frac{K\pi}{12}\right)}{\sin\left(\frac{\pi}{12}\right)} + \sin\left(\alpha + \frac{K\pi}{6}\right)$$

$$= \frac{\sin\left(\alpha + \frac{(K-1)\pi}{12}\right) \sin\left(\frac{K\pi}{12}\right) + \sin\left(\alpha + \frac{K\pi}{6}\right) \sin\left(\frac{\pi}{12}\right)}{\sin\left(\frac{\pi}{12}\right)}$$

$$= \frac{\frac{1}{2}\left[\cos\left(\alpha - \frac{\pi}{12}\right) - \cos\left(\alpha + \frac{2k\pi}{12} - \frac{\pi}{12}\right)\right] + \frac{1}{2}\left[\cos\alpha + \frac{2k\pi}{12} - \frac{\pi}{12}\right] - \cos\left[\left(\alpha + \frac{2k\pi}{12} + \frac{\pi}{12}\right]\right.}{\sin\left(\frac{\pi}{12}\right)}$$

$$= \frac{\frac{1}{2}\left[\cos\left(\alpha - \frac{\pi}{2}\right) - \cos\left(\alpha + \frac{2k\pi}{12} + \frac{\pi}{12}\right)\right]}{\sin\left(\frac{\pi}{12}\right)}$$

$$= \frac{\sin\left(\alpha + \frac{k\pi}{12}\right)\sin\left(\frac{(k+1)\pi}{12}\right)}{\sin\left(\frac{\pi}{12}\right)}$$

$P(k+1)$ is true.

By Mathematical induction we $P(n)$ is true $\forall n \in N$ is true. Proved

Exercise: 4.3

1) If $nC_{12} = nC_9$, find $21C_n$.

Sol:

$$nC_x = nC_y$$

$$x + y = n$$

$$nC_{12} = nC_9$$

$$12 + 9 = n$$

$$21 = n$$

$$21C_n = 21C_{21}$$

$$= \underline{1}$$

$$\boxed{\therefore 21C_n = 1}$$

2) If $15C_{2r-1} = 15C_{2r+4}$, find r.

Sol:

$$nC_x = nC_y$$

$$x + y = n$$

$$(2r-1) + (2r+4) = 15$$

$$4r + 3 = 15$$

$$4r = 15 - 3$$

$$4r = 12$$

$$r = \frac{12}{4}$$

$$r = 3.$$

$$\boxed{\therefore\ r = 3.}$$

3) If ${}^nP_r = 720$, and ${}^nC_r = 120$, find n, r.

Sol:

$${}^nP_r = {}^nC_r \times r!$$

$$720 = 120 \times r!$$

$$\frac{72\cancel{0}}{12\cancel{0}} = r!$$

$$\frac{72}{12} = r!$$

$$r! = \frac{72}{12}$$

$$r! = 6$$

$$r! = 3!$$

$r! = 3!$

$r = 3$

$^nP_r = 720$

$$\frac{n!}{(n-r)!} = 720$$

$$\frac{n!}{(n-3)!} = 720$$

$$\frac{n(n-1)(n-2)\cancel{(n-3)!}}{\cancel{(n-3)!}} = 720$$

$n(n-1)(n-2) = 720.$

$n(n-1)(n-2) = 10 \times 9 \times 8$

$n = 10.$

Prove that $^{15}C_3 + 2 \times {}^{15}C_4 + {}^{15}C_5 = {}^{17}C_5.$

Sol: L.H.S

$^{15}C_3 + 2 \times {}^{15}C_4 + {}^{15}C_5.$

$= ({}^{15}C_3 + {}^{15}C_4) + ({}^{15}C_4 + {}^{15}C_5)$

$= ({}^{15}C_4 + {}^{15}C_3) + ({}^{15}C_5 + {}^{15}C_4)$

$nC_r + nC_{r-1} = (n+1)C_r$

$= (15+1)C_4 + (15+1)C_5$

$= 16C_4 + 16C_5$

$= (16+1)C_5 =$

$= 17C_5$

$\therefore$ It is proved.

Prove that ${}^{35}C_5 + \sum_{r=0}^{4}(39-r)C_4 = {}^{40}C_5$.

$\cancel{{}^{35}C_5 + 2\times{}^{15}C_4 + {}^{15}C_5 = {}^{17}C_5}$

Sol

L.H.S

$= {}^{35}C_5 + \sum_{r=0}^{4}(39-r)C_4$

$= {}^{35}C_5 + {}^{39}C_4 + {}^{38}C_4 + {}^{37}C_4 + {}^{36}C_4 + {}^{35}C_4$

$= ({}^{35}C_5 + {}^{35}C_4) + {}^{36}C_4 + {}^{37}C_4 + {}^{38}C_4 + {}^{39}C_4$

${}^{n}C_r + {}^{n}C_{r-1} = (n+1)C_r$.

$= (35+1)C_5 + {}^{36}C_4 + {}^{37}C_4 + {}^{38}C_4 + {}^{39}C_4$

$= (36C_5 + {}^{36}C_4) + {}^{37}C_4 + {}^{38}C_4 + {}^{39}C_4$

$= {}^{(36+1)}C_5 + {}^{37}C_4 + {}^{38}C_4 + {}^{39}C_4$

$= \left[{}^{37}C_5 + {}^{37}C_4\right] + {}^{38}C_4 + {}^{39}C_4$

$= \left[{}^{38}C_5 + {}^{38}C_4\right] + {}^{39}C_4$

$= {}^{(38+1)}C_5 + {}^{39}C_4$

$= {}^{39}C_5 + {}^{39}C_4$

$= {}^{(39+1)}C_5$

$= {}^{40}C_5$

$\therefore$ It is Proved.

L.H.S = R.H.S

If ${}^{(n+1)}C_8 : {}^{(n-3)}P_4 = 57:16$ find the values of n.

Sol:

$$\frac{{}^{(n+1)}C_8}{{}^{(n-3)}P_4} = \frac{57}{16}$$

$${}^{n}C_r = \frac{n!}{(n-r)!\,r!} \quad . \quad {}^{n}P_r = \frac{n!}{(n-r)!}$$

$$\Rightarrow \frac{\dfrac{(n+1)!}{[(n+1)-8]!\,8!}}{\dfrac{(n-3)}{(n-3)-4!}} = \frac{57}{16}$$

$$\frac{(n+1)!}{(n-7)!\,8!} \times \frac{(n-7)!}{(n-3)!} = \frac{57}{16}$$

$$\frac{(n+1)n(n-1)(n-2)(n-3)!}{8!\,(n-3)!} = \frac{57}{16}$$

$$\frac{(n+1)n(n-1)(n-2)}{8!} = \frac{57}{16}$$

$$(n+1)n(n-1)(n-2) = \frac{57 \times 8!}{16}$$

$$(n+1)n(n-1)(n-2) = \frac{59 \times \cancel{8} \times 7 \times 6 \times 5 \times \cancel{4} \times 3 \times 2}{\cancel{16}}$$

$$= 57 \times 30 \times 12$$

$$= 57 \times 2520$$

$$(n+1)n(n-1)(n-2) = 19 \times 3 \times 2 \times 2 \times 3 \times 3 \times 5 ...$$

$$= (3 \times 7) \times (2 \times 2 \times 5) \times$$

$$(2 \times 3 \times 3) \times 19$$

$$(n+1)n(n-2) = 21 \times 20 \times 19 \times 18$$

$n+1 = 21$

$n = 21-1$

$n = 20.$

$\therefore$ Value of $n = 20.$

Prove that $^{2n}C_n = \frac{2^n \times 1 \times 3 \times \ldots (2n-1)}{n!}$

Sol:

$n = 2, \quad r = n$

$$^{2n}C_n = \frac{2n!}{(2n-n)!\,n!} \qquad nC_r = \frac{n!}{(n-r)!\,r!}$$

$$= \frac{2n!}{n!\,n!}$$

$$= \frac{2n(2n-1)(2n-2)(2n-3)\ldots 4,3,2,1}{n!\,n!}$$

$$= \frac{(1,3,5\ldots 2n-1)(2,4,6\ldots(2n-2)\,2n)}{n!\,n!}$$

$$= \frac{(1,3,5\ldots 2n-1)(2,4,6\ldots(2n-2)\,2n)}{n!\,n!}$$

$$= \frac{(1,3,5,\ldots(2n-1)\cdot 2^n(1,2,3\ldots n)}{n!\,n!}$$

$$= \frac{[1,3,5\ldots(2n-1)]\,2^n n!}{n!\,n!}$$

$$= \frac{2^n(1,3,5\ldots(2n-1))}{n!}$$

L.H.S = R.H.S

$\therefore$ It is Proved.

Prove that if $1 \le r \le$ then $n \times {}^{(n-1)}C_{r-1} = (n-r+1)\,{}^{n}C_{r-1}$.

Sol:

$$n \times {}^{n-1}C_{r-1} = n\left[\frac{(n-1)!}{(n-1)-(r-1)!\,(r-1)!}\right]$$

$$= \left[\frac{n(n-1)!}{(n-1-r+1)!\,(r-1)!}\right]$$

$$= \frac{n(n-1)!}{(n-r)!\,(r-1)!}$$

$\frac{x}{y}$ and divided by $(n-r+1)$

$$= \frac{(n-r+1)\, n(n-1)!}{(n-r+1)(n-r)!\,(r-1)!}$$

$$= \frac{(n-r+1)\, n!}{(n-r+1)!\,(r-1)!}$$

$$= \frac{(n-r+1)\, n!}{[n-(r-1)]!\,(r-1)!}$$

$$= (n-r+1)\, {}^{n}C_{r-1}$$

$\therefore$ It is Proved.

---

i) A kabaddi coach has 14 players ready to play. How many different teams of 7 players could the coach put on the court?

Sol: Here, 7 players must be selected from 14 players.

This can be done in ${}^{14}C_7$

$${}^{14}C_7 = \frac{14\times13\times12\times11\times10\times9\times8}{1\times2\times3\times4\times5\times6\times7}$$

$$= \frac{14 \times 13 \times 12 \times 11 \times 10 \times 9 \times 8}{1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 7}$$

$$= 13 \times 11 \times 2 \times 2 \times 3 \times 2$$

$$= 13 \times 22 \times 12$$

$$= 3432.$$

$\therefore$ 3432 different teams of 7 players could the coach put on th court.

---

ii) There are 15 persons in a party and if each 2 of them shak hands with each other how many hands shakes happen in the party

Sol:

The total number of handshakes in same on the numb of ways of selecting 2 persons among 15 persons.

This can be done in

$15C_2$ ways.

$15C_2 = \frac{15 \times \cancel{14}^{7}}{1 \times \cancel{2}}$

$= 15 \times 7$

$= 105.$

$\therefore$ 105 handshakes happen in the party.

---

iii) How many chords can be drawn through 20 points on a circle.

Sol:

A chord is obtained by joining any two points on a circle.

Number of chords drawn through 20 points in same as the number of ways of selecting 2 points out of 20 points.

This can be done in $20C_2$ ways, $20C_2 = \frac{\cancel{20}^{10} \times 19}{1 \times \cancel{2}}$

$= 10 \times 19$

$= 190$ chords

∴ 190 chords can be drawn through 20 points on a circle.

iv) In a parking lot one hundred, one year old cars are parked. Out of them five are to be chosen at random for to check its pollution devices. How many different set of five cars be chosen?

Sol:

5 cars can be chosen out of 100 cars in $100C_5$ ways.

v) How many ways can a team of 3 boys 2 girls and 1 transgender be selected from 5 boys 4 girls and 2 transgenders?

Sol:

3 boys can be selected from 5 boys in $5C_3$ ways.

2 girls can be selected from 4 girls in $4C_2$ ways.

1 transgender can be selected from 2 transgenders in $2C_1$ ways.

Total number of Selections

$= 5C_3 \times 4C_2 \times 2C_1$

$= \frac{5 \times \cancel{4}^{2} \times \cancel{3}}{1 \times \cancel{2} \times \cancel{3}} \times \frac{\cancel{4}^{2} \times 3}{1 \times \cancel{2}} \times \frac{2}{1}$

$= 5 \times 2 \times 2 \times 3 \times 2$

$= 10 \times 6 \times 2$

$= 60 \times 2$

$= 120$

∴ In 120 ways can a team of 3 boys 2 girls and 1 transgender be selected from 5 boys 4 girls and 2 transgenders.

Find the total number of subsets of a set with [Hint: $^nC_0 + {}^nC_1 + {}^nC_2 + \dots {}^nC_n = 2^n$].

ii) 4 elements.

Sol

Number of subsets with no element

Number of subsets with 1 element is $= 4C_1$

Number of subsets with 2 element is $4C_2$

Number of subsets with 3 element is $4C_3$

Number of subsets with 4 element is $4C_4$

$\therefore$ Total number of Subsets,

$= 4C_0 + 4C_1 + 4C_2 + 4C_3 + 4C_4$

$= 1 + 4 + \frac{\overset{2}{\cancel{4}} \times 3}{1 \times \cancel{2}} + \frac{4 \times \cancel{3 \times 2}}{1 \times \cancel{2 \times 3}} + \frac{4 \times 3 \times 2 \times}{1 \times \cancel{2 \times 3} \times 4}$

$= 1 + 4 + 2 \times 3 + 4 + 1$

$= 1 + 4 + 6 + 4 + 1$

$= 11 + 5$

$= 16.$

$2^4 = 2 \times 2 \times 2 \times 2$

$= 4 \times 2 \times 2$

$= 16$

$\boxed{2^4 = 16}$

Total number of subsets, $2^4 = 16.$

iii) 5 elements.

Sol

Number of subsets with no element $= 5C_0$.

Number of subsets with 1 element $= 5C_1$

Number of subsets with 2 elements $= 5C_2$

Number of subsets with 3 elements $= 5C_3$

Number of subsets with 4 elements $= 5C_4$

Number of subsets with 5 elements $= 5C_5$

$\therefore$ Total number of subsets,

$$= 5C_0 + 5C_1 + 5C_2 + 5C_3 + 5C_4 + 5C_5$$

$$= 1 + 5 + \frac{5\times \cancel{4}^{2}}{1\times \cancel{2}} + \frac{5\times \cancel{4}^{2}\times \cancel{3}}{1\times \cancel{2}\times \cancel{3}} + \frac{5\times \cancel{4}\times \cancel{3\times 2}}{1\times \cancel{2\times 3}\times \cancel{4}} + \frac{\cancel{5\times 4\times 3\times 2\times 1}}{\cancel{1\times 2\times 3\times 4\times 5}}$$

$$= 1 + 5 + (5\times 2) + (5\times 2) + 5 + 1$$

$$= 1 + 5 + 10 + 10 + 6$$

$$= 6 + 20 + 6$$

$$= 32$$

$2^5 = 2\times 2\times 2\times 2\times 2$

$= 8\times 2\times 2$

$= 32$

$\boxed{2^5 = 32}$

$\therefore$ Total number of Subsets is $2^5 = 32$.

iii) n elements.

Sol:

Number of subsets with no elements $= nC_0$

Number of subsets with 1, 2, 3, 4 elements are $nC_0 + nC_1 + nC_2 + \dots + nC_n$ respectively.

$\therefore$ Total number of subsets $= 2^n$.

) A trust has 25 members.

i) How many ways 3 officers can be selected?

Sol:

3 officers can be selected from 25 members in $25C_3$ ways.

ii) In how many ways can a president, vice president and a secretary be selected?

Sol:

A President can be selected from 25 members in 25 ways.

Vice President can be

selected from remaining 24 ways.

Secretary can be selected from remaining 23 ways.

$\therefore$ Total number of ways $= 25P_3$.

12) How many ways a committee of six persons from 10 persons can be chosen along with a chair person and a secretary?

Sol. A committee must have a chair person and a secretary.

These 2 can be selected in $10C_2$ ways.

Remaining 4 persons for a committee can be selected from 8 persons in $8C_4$ ways.

Number of ways of Selection of the committee $\Big\} = 10C_2 \times 8C_4$

$$= 10C_2 \times 8C_4$$

$$= \frac{10 \times 9}{1 \times 2} \times \frac{8 \times 7 \times 6 \times 5}{1 \times 2 \times 3 \times 4}$$

$$= \frac{\cancel{10}^{5} \times 9}{1 \times \cancel{2}} \times \frac{8 \times 7 \times \cancel{6} \times 5}{1 \times \cancel{2} \times \cancel{3} \times \cancel{4}}$$

$= 5 \times 9 \times 2 \times 7 \times 5$

$= 45 \times 2 \times 35$

$= 90 \times 35$

$= 3150$ ways.

$\therefore$ In 3150 ways, a committee of six persons from 10 persons can be chosen.

13) How many different selections of 5 books can be made from 12 different books

i) If Two particular books are always selected?

Sol: Two Particular books are always selected, the remaining 3 books can be selected from 10 books in $10C_3$ ways.

$$10C_3 = \frac{10 \times 9 \times 8}{1 \times 2 \times 3}$$

$$= \frac{10 \times \cancel{9}^{3} \times \cancel{8}^{4}}{1 \times \cancel{2} \times \cancel{3}}$$

$$= 10 \times 3 \times 4$$

$$= 30 \times 4$$

$$= 120 \text{ ways.}$$

$$\therefore 10C_3 = 120 \text{ ways.}$$

ii) Two Particular books are never selected?

Sol: Since two books are never to be selected, the selection of 5 books from 10 books are done in $10C_5$ ways.

$$10C_5 = \frac{\cancel{10}^{2} \times \cancel{9}^{3} \times \cancel{8}^{4} \times 7 \times \cancel{6}^{3}}{1 \times \cancel{2} \times \cancel{3} \times \cancel{4} \times \cancel{5}}$$

$$= 3 \times 4 \times 7 \times 3$$

$$= 12 \times 7 \times 3$$

$$= 36 \times 7$$

$$= 252 \text{ ways.}$$

$$\therefore 10C_5 = 252 \text{ ways.}$$

There are 5 teachers and 20 students. Out of them a committee of 2 teachers and 3 students is to be formed. Find the number of ways in which this can be done. Further find in how many of these committees.

Sol:

There are 5 teachers and 20 students, 2 teachers out of 5 teachers can be selected in $5C_2$ ...

Hence, total number of committees $= 20C_3 \times 5C_2$

$= 5C_2 \times 20C_3$

$= \frac{5 \times \cancel{4}^{2}}{1 \times \cancel{2}} \times \frac{\cancel{20}^{10} \times 19 \times \cancel{18}^{6}}{1 \times \cancel{2} \times \cancel{3}}$

$= 5 \times 2 \times 10 \times 19 \times 6$

$= 10 \times 10 \times 114$

$= 100 \times 14$

$= 11400$ ways.

$5C_2 \times 20C_3 = 11400$ ways.

i) particular teacher is included.

Sol:

A Particular teacher is included from 5 teachers is $4C_1$

$= 4C_1 \times 20C_3$

$= 4 \times \frac{\cancel{20}^{10} \times 19 \times \cancel{18}_{6}}{1 \times \cancel{2} \times \cancel{3}}$

$= 4 \times 10 \times 19 \times 6$

$= 40 \times 114$

$= 4560$ ways.

$4C_1 \times 20C_3 = 4560$ ways.

ii) A Particular student is excluded.

Sol:

2 teachers can be selected from 5 teachers in $5C_2$ ways.

Since a particular student is excluded, 3 students can be selected from 19 students $19C_3$ ways.

Hence, required number of committees $= 19C_3 \times 5C_2$

$$= \frac{19 \times \cancel{18}^{3} \times 17}{1 \times \cancel{2} \times \cancel{3}} \times \frac{5 \times \cancel{4}^{2}}{1 \times \cancel{2}}$$

$$= 19 \times 3 \times 17 \times 5 \times 2$$

$$= 57 \times 17 \times 10$$

$$= 57 \times 170$$

$$= 9690$$

$\therefore 5C_2 \times 19C_3 = 9690.$

In an examination a student has to answer 5 questions, out of 9 questions in which 2 are compulsory. In how many ways a student can answer the questions?

Sol:

Since 2 questions are compulsory, a student must answer 3 questions out of 7 questions.

This can be done in $7C_3$ way

No. of ways of answering 15 question

$$7C_3 = \frac{7 \times 6 \times 5}{1 \times \cancel{2 \times 3}}$$

$$7C_3 = \frac{7 \times 6 \times 5}{1 \times \cancel{2 \times 3}}$$

$$= 7 \times 5$$

$$= 35$$

$$7C_3 = 35$$

$\therefore$ In 35 ways, a student can answer the questions.

Determine the number of 5 card combinations out of a deck of 52 cards if there is exactly three aces in each combination?

Sol:

In a Pack of 52 cards, there are four aces.

3 aces will be selected from 4 cards and remaining 2 cards will be selected from rest of 48 cards.

$\therefore$ Number of ways of Selecting 3 aces from 4 aces $= 4C_3$

Number of ways of selecting 2 cards out of 48 cards = $48C_2$

∴ Required number of ways,

$= 4C_3 \times 48C_2$

$= \frac{\cancel{4}^{2} \times \cancel{3} \times 2}{1 \times \cancel{2} \times \cancel{3}} \times \frac{\cancel{48}^{24} \times 47}{1 \times \cancel{2}}$

$= 2 \times 2 \times 24 \times 47$

$= 4 \times 24 \times 47$

$= 96 \times 47$

$= 4512$

∴ Number of ways = 4512.

7) Find the number of ways of forming a committee of 5 member out of 7 Indians and 5 Americans so that always Indians will be the majority in the committee?

Sol: Given,

A Committee of 5 member

out of 7 Indians and 5 Americans,

Indians(7)	Americans (5)	Combination	
5	0	$7C_5 \times 5C_0$	21
4	1	$7C_4 \times 5C_1$	175
3	2	$7C_3 \times 5C_2$	350
			546

Total number of ways are forming committee is,

$= 7C_5 \times 5C_0 + 7C_4 \times 5C_1 + 7C_3 \times 5C_2$

$= 21 + 175 + 350$

$= 546.$

$\therefore$ Number of ways $= 546.$

18) A committee of 7 peoples has to be formed from 8 men and 4 women. In how many ways can this be done when the committee consists of

i) exactly 3 women.

Sol:

Since the committee must have exactly 3 women remaining 4 men can be selected from 8 men.

$$= 8C_4 \times 4C_3$$

$$= \frac{\cancel{8}^{4} \times 7 \times \cancel{6}^{2} \times 5}{1 \times \cancel{2} \times \cancel{3} \times \cancel{4}} \times \frac{4 \times \cancel{3} \times \cancel{2}}{1 \times \cancel{2} \times \cancel{3}}$$

$$= 7 \times 2 \times 5 \times 4$$

$$= 14 \times 20$$

$$= 280.$$

$\therefore 8C_4 \times 4C_3 = 280.$

ii) atleast 3 women.

Sol:

The following are the choices to select atleast 3 women.

	Men (8)	Women (4)	Combination
a)	4	3	$8C_4 \times 4C_3$
b)	3	4	$8C_3 \times 4C_4$

Required number of ways of forming committee,

$= 8C_4 \times 4C_3 + 8C_3 \times 4C_4$

$$= \frac{\overset{2}{\cancel{8}} \times 7 \times \overset{2}{\cancel{6}} \times 5}{1 \times \cancel{2} \times \cancel{3} \times \cancel{4}} \times \frac{4 \times \cancel{3 \times 2}}{1 \times \cancel{2 \times 3}} + \frac{\overset{4}{\cancel{8}} \times 7 \times \overset{2}{\cancel{6}}}{1 \times \cancel{2} \times \cancel{3}} \times \frac{\cancel{4 \times 3 \times 2 \times 1}}{\cancel{1 \times 2 \times 3 \times 4}}$$

$= 7 \times 2 \times 5 \times 4 + 4 \times 7 \times 2$

$= 14 \times 20 + 28 \times 2$

$= 280 + 56$

$= 336$

$\therefore$ Required Number of ways is 336.

---

iii) Atmost 3 women.

Sol: The following are the choices to select atmost 3 women.

	Men	Women	Combinations
a)	4	3	$8C_4 \times 4C_3$
b)	5	2	$8C_5 \times 4C_2$
c)	6	1	$8C_6 \times 4C_1$
d)	7	0	$8C_7 \times 4C_0$

Hence, required number of ways of forming the 49 committee is

$= 8C_4 \times 4C_3 + 8C_5 \times 4C_2 + 8C_6 \times 4C_1 + 8C_7 \times 4C_0$

$= 8C_4 \times 4C_1 + 8C_3 \times 4C_2 + 8C_2 \times 4C_1 + 8C_1 \times 1$

$= \left(\frac{8 \times 7 \times \cancel{6} \times 5}{1 \times \cancel{2 \times 3} \times \cancel{4}} \times 4\right) + \left(\frac{8 \times 7 \times \cancel{6}}{1 \times \cancel{2 \times 3}} \times \frac{\cancel{4} \times 3}{1 \times \cancel{2}}\right) +$

$\left(\frac{\cancel{8} \times 7}{1 \times \cancel{2}} \times 4\right) + 8 \times 1$

$= 2 \times 7 \times 5 \times 4 + 8 \times 7 \times 2 \times 3 + 4 \times 7 \times 4 + 8 \times 1$

$= 14 \times 20 + 56 \times 6 + 28 \times 4 + 8$

$= 280 + 336 + 112 + 8$

$= 280 + 336 + 120$

$= 736$

$\therefore$ Number of ways $= 736$.

19) 7 relatives of a man comprises 4 ladies and 3 gentlemen, his wife also has 7 relatives; 3 of them are ladies and 4 gentlemen. In how many ways can they

invite a dinner party of 3 ladies and 3 gentleman so that there are 3 of man's relative and 3 of the wife's relatives?

Sol:

The different possible arrangements are given in the following table.

S.No	Husband		Wife		Test	Test
	Boy 3	Girl 4	Boy 4	Girl 3	Test	Test
1	-	3	3	-	$4C_3 \times 4C_3$	16
2	3	-	-	3	$3C_3 \times 3C_3$	1
3	2	1	1	2	$3C_2 \times 4C_1 \times 3C_1 \times 3C_2$	144
4	1	2	2	1	$4C_2 \times 3C_1 \times 3C_1 \times 4C_2$	324
						485

Hence the required number of ways,

$$= 4C_3 \times 4C_3 + 4C_2 \times 3C_1 \times 3C_1 \times 4C_2 + 4C_1 \times 3C_2 \times 3C_2 \times 4C_1 + 3C_3 \times 3C_3$$

$$= (4)(4) + (6)(3)(3)(6) + 4(3)(3)(4) + 1(1)$$

$$= 16 + (18)(18) + (12)(12) + 1$$

$$= 16 + 324 + 144 + 1$$

$= 16 + 324 + 144 + 1$

$= 340 + 145$

$= 485$

20) A box contains two white balls, three black balls and four red balls. In how many ways can three balls be drawn from the box, if at least one black ball is to be included in the draw?

Sol. Three balls must be drawn with atleast one black bal

Black ball (3)	White + Red (2+4) =6	Selections	No. of Selections
1	2	$3C_1 \times 6C_2$	45
2	1	$3C_2 \times 6C_1$	18
3	0	$3C_3 \times 6C_0$	1
			64

Required number of ways of drawing 3 balls.

$= (3C_1 \times 6C_2) + (3C_2 \times 6C_1) + (3C_3 \times 6C_0)$

$= \left(3 \times \frac{\cancel{6}^{3} \times 5}{1 \times \cancel{2}}\right) + \left(\frac{3 \times \cancel{2}}{1 \times \cancel{2}} \times 6\right) + (1 \times 1)$

$= (3 \times 3 \times 5) + (3 \times 1 \times 6) + (1)$

$= (9 \times 5) + (18) + (1)$

$= 45 + 18 + 1$

$= 46 + 18$

$= 64$ ways.

Find the number of strings of 4 letters that can be formed with the letters of the word EXAMINATION?

Sol:

In the word of EXAMINATION.

There are 11 letters.

Repeated letters = AA, II, NN

Remaining letters = E, X, M, T, O.

S.No	Method of written Test	No. of Selection	Arranging in orders based on Test.
1	Four different letters E, X, M, T, O, A, I, N	$8C_4$	$8C_4 \times 4! = 1680$

2	1 pair in 3 pairs 2 letters selecting from remaining 7 letters	$3C_1 \times 7C_2$	$\frac{3C_1 \times 7C_2 \times 4!}{2!} = 756$
3	2 Set selecting from 3 letters	$3C_2$	$\frac{3C_2 \times 4!}{2!\,2!} = 18$
			2454

$\therefore$ Required number of strings

$= 1680 + 756 + 18$

$= 2454.$

22) How many triangles can be formed by joining 15 points on the plane, in which no line joining any three points?

Sol: To form a triangle we need minimum 3 non-collinear points.

3 points can be selected from 15 non-collinear points in $15C_3$ ways,

$$15C_3 = \frac{\cancel{15}^{5} \times \cancel{14}^{7} \times 13}{1 \times \cancel{2} \times \cancel{3}}$$

$$= 5 \times 7 \times 13$$

$$= 35 \times 13$$

$$= 455$$

$$15C_3 = 455.$$

How many triangles can be formed by 15 points in which 7 of them lie on one line and the remaining 8 on another parallel line?

Sol: To form a triangle we need 3 points.

From 15 points,

$$15C_3 = \frac{\cancel{15}^{5} \times \cancel{14}^{7} \times 13}{1 \times \cancel{2} \times \cancel{3}}$$

$$= 5 \times 7 \times 13$$

$$= 35 \times 13$$

$$= 455$$

$$\boxed{15C_3 = 455}$$

From 15 points we can made 455 triangles.

7 points are in the same line

So, $7C_3 = \frac{7\times\cancel{6}\times5}{1\times\cancel{2\times3}}$

$= 7\times5$

$\boxed{7C_3 = 35}$

$\therefore$ 7 points are in the same line

So, we cannot able to form 35 triangles.

8 points $8C_3 = \frac{8\times7\times\cancel{6}}{1\times\cancel{2\times3}}$

$= 8\times7$

$\boxed{8C_3 = 56}$

Subtracting $15C_3$, $7C_3$ and $8C_3$

$= 15C_3 - 8C_3 - 7C_3$

$= 455 - 56 - 35$

$= 455 - 91$

$= 364.$

24) There are 11 points in a plane. No three of these lies in the same straight line except 4 points, which are collinear. Find

i) the number of straight lines that can be obtained from the pairs of these points?

Sol: Given,

11 points in a plane.

Pair of straight lines.

$$11C_2 = \frac{11 \times \cancel{10}^{5}}{1 \times \cancel{2}}$$

$$= 11 \times 5$$

$$= 55$$

$$11C_2 = 55$$

$$4C_2 = \frac{\cancel{4}^{2} \times 3}{1 \times \cancel{2}}$$

$$= 2 \times 3$$

$$4C_2 = 6$$

$$\Rightarrow 11C_2 - 4C_2$$

$= 11C_2 - 4C_2 + 1$

$= 55 - 6 + 1$

$= 50$

ii) The number of triangles that can be formed for which the points are their vertices?

Sol:

a) The number of triangle that can be formed for which the points are their vertices.

If we take 1 point from 4 collinear points remaining 7 points joining.

$4C_1 \times 7C_2$

$= 4 \times \frac{7 \times \cancel{6}^{3}}{1 \times \cancel{2}}$

$= 4 \times 21$

$= 84.$

$4C_1 \times 7C_2 = 84.$

$\boxed{\therefore 4C_1 \times 7C_2 = 84}$

b) If we take a points from 4 collinear points remaining 7. So this will give.

$$= 4C_2 \times 7C_1$$

$$= \frac{\cancel{4}^{2} \times 3}{1 \times \cancel{2}} \times 7$$

$$= 2 \times 3 \times 7$$

$$= 6 \times 7$$

$$= 42.$$

$$\boxed{\therefore 4C_2 \times 7C_1 = 42.}$$

c) If we take all the forms three points from 7. non-collinear points. So, this will give $7C_3$

$$7C_3 = \frac{7 \times 6 \times 5}{1 \times \cancel{2 \times 3}}$$

$$= 7 \times 5$$

$$= 35$$

$$\boxed{7C_3 = 35}$$

Add,

$4C_1 \times 7C_2$, $4C_2 \times 7C_1$ and $7C_3$

$= (4C_1 \times 7C_2) + (4C_2 \times 7C_1) + 7C_3$

$= 84 + 42 + 35$

$= 126 + 35$

$= 161$

25) A Polygon has 90 diagonals. Find the number of the sides.

Sol: Number of diagonals of n sides polygon,

$$\frac{n(n-3)}{2} = 90$$

$$\frac{n^2 - 3n}{2} = 90$$

$$n^2 - 3n = 90 \times 2$$

$$n^2 - 3n = 180$$

$$n^2 - 3n - 180.$$

-180

-15 12

-3

$$n^2 - 3n - 180 = 0.$$

$$\cancel{n(n-3)} \quad n^2 - 3n - 180 = 0$$

$$\cancel{n^2\{n} \quad n^2 - 15n + 12n - 180 = 0.$$

$$n(n-15) + 12(n-15) = 0$$

$$(n+12)(n-15) = 0$$

$$n + 12 = 0 \qquad \cancel{n+}$$

$$n = -12$$

$$n - 15 = 0$$

$$\boxed{n = 15.}$$

These are 15 sides for the polygon which has 90 sides.

S. Balakrishnan, M.Sc, B.Ed
9159652220, 9944931220