6. APPLICATIONS OF VECTOR ALGEBRA.

Example : 6.1 (Cosine formulae)

With usual notations, in any triangle ABC, prove that

(i) $a^2 = b^2 + c^2 - 2bc\cos A$

(ii) $b^2 = c^2 + a^2 - 2ca\cos B$

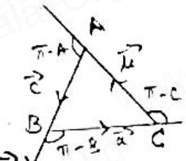
(iii) $c^2 = a^2 + b^2 - 2ab\cos C$. by vector method.

Solution:

In ΔABC,

Let $\vec{BC} = \vec{a}$, $\vec{CA} = \vec{b}$, $\vec{AB} = \vec{c}$

and $\vec{BC} + \vec{CA} + \vec{AB} = \vec{0}$



$\vec{BC} = -\vec{CA} - \vec{AB}$

$\vec{BC}\cdot\vec{BC} = (-\vec{CA} - \vec{AB})\cdot(-\vec{CA} - \vec{AB})$

$|\vec{BC}|^2 = |\vec{CA}|^2 + |\vec{AB}|^2 + 2\vec{CA}\cdot\vec{AB}$

$|\vec{BC}|^2 = |\vec{CA}|^2 + |\vec{AB}|^2 + 2|\vec{CA}||\vec{AB}|\cos(\pi - A)$

$a^2 = b^2 + c^2 + 2bc\cos(\pi - A)$

$a^2 = b^2 + c^2 - 2bc\cos A$

Similarly, we can prove (ii) & (iii).

Example : 6.2. (Projection Formulae)

In any ΔABC, prove that

(i) $a = b\cos C + c\cos B$

(ii) $b = c\cos A + a\cos C$

(iii) $c = a\cos B + b\cos A$. by vector method.

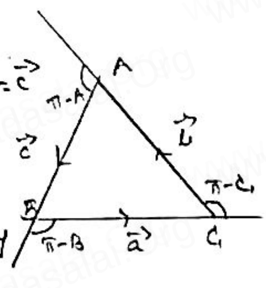
Solution:

In ΔABC,

$\vec{BC} = \vec{a}$, $\vec{CA} = \vec{b}$, $\vec{AB} = \vec{c}$

$\vec{BC} + \vec{CA} + \vec{AB} = \vec{0}$

$\vec{BC} = -\vec{CA} - \vec{AB}$.



Taking dot product by $\vec{BC}$ on both sides.

$\vec{BC}\cdot\vec{BC} = -\vec{BC}\cdot\vec{CA} - \vec{BC}\cdot\vec{AB}$

$|\vec{BC}|^2 = -|\vec{BC}||\vec{CA}|\cos(\pi - C) - |\vec{BC}||\vec{AB}|\cos(\pi - B)$

$a^2 = -ab(-\cos C) - ac(-\cos B)$

$\div$ by 'a' on both sides.

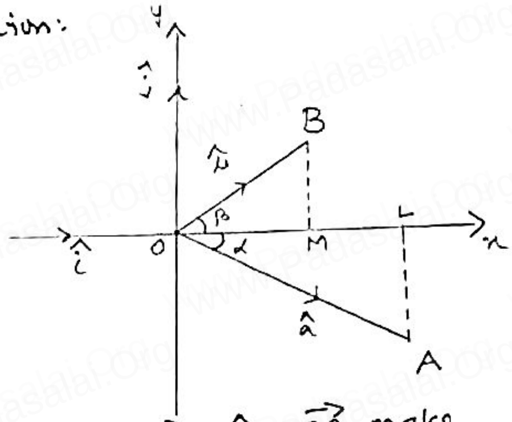
$a = b\cos C + c\cos B$.

III^ly we can prove (ii) & (iii).

Example : 6.3.

$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$

Solution:



* Let $\hat{a} = \vec{OA}$, $\hat{b} = \vec{OB}$ make angles α, β respectively in the positive x-axis.

where A & B as in diagram.

* Draw $AL \perp x$ axis, $BM \perp x$ axis

* In ΔOLA,

$\cos\alpha = \dfrac{|\vec{OL}|}{|\vec{OA}|} = |\vec{OL}|$ $\qquad \sin\alpha = \dfrac{|\vec{LA}|}{|\vec{OA}|} = |\vec{LA}|$

$\vec{OL} = (\cos\alpha)\hat{i}$ $\qquad |\vec{LA}| = (\sin\alpha)(-\hat{j})$

$\therefore \hat{a} = \vec{OA} = \vec{OL} + \vec{LA}$

$\hat{a} = \cos\alpha\,\hat{i} - \sin\alpha\,\hat{j} \longrightarrow ①$

III^ly $\hat{b} = \cos\beta\,\hat{i} + \sin\beta\,\hat{j} \longrightarrow ②$

Angle b/w $\hat{a}$ & $\hat{b}$ is $\alpha+\beta$

$\hat{a}\cdot\hat{b} = |\hat{a}||\hat{b}|\cos(\alpha+\beta) = \cos(\alpha+\beta) \longrightarrow ③$

From ① & ②

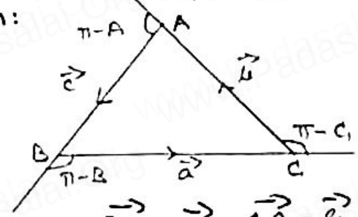
$\hat{a}\cdot\hat{b} = \cos\alpha\cos\beta - \sin\alpha\sin\beta \longrightarrow ④$

From ③ & ④

$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$.

Example : 6.4 $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$

Solution:



In ΔABC, $\vec{BC} = \vec{a}$, $\vec{CA} = \vec{b}$, $\vec{AB} = \vec{c}$

Area of the triangle ABC is

$\frac{1}{2}|\vec{AB}\times\vec{BC}| = \frac{1}{2}|\vec{BC}\times\vec{CA}| = \frac{1}{2}|\vec{CA}\times\vec{AB}|$

$|\vec{c}\times\vec{a}| = |\vec{a}\times\vec{b}| = |\vec{b}\times\vec{c}|$

$ca\sin(\pi - B) = ab\sin(\pi - C) = bc\sin(\pi - A)$

$ca\sin B = ab\sin C = bc\sin A$

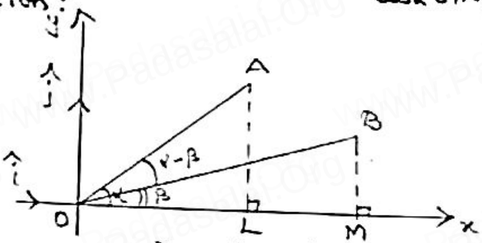
$\div$ by abc on all the sides

$\dfrac{\sin B}{b} = \dfrac{\sin C}{c} = \dfrac{\sin A}{a}$

Taking reciprocal $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$

Example : 6.5 $\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$

Solution :



Let $\hat{a} = \vec{OA}$. $\hat{b} = \vec{OB}$ makes angles α, β respectively in the positive x-axis.

where A & B as in diagram

Draw $AL \perp$ x-axis, $BM \perp$ x-axis

In ΔOLA

$\cos\alpha = \dfrac{|\vec{OL}|}{|\vec{OA}|} = |\vec{OL}|$ $\sin\alpha = \dfrac{|\vec{LA}|}{|\vec{OA}|} = |\vec{LA}|$

$\vec{OL} = \cos\alpha\, \hat{i}$ $|\vec{LA}| = \sin\alpha\, \hat{j}$

$\vec{OA} = \vec{OL} + \vec{LA}$

$\hat{a} = \cos\alpha\, \hat{i} + \sin\alpha\, \hat{j} \longrightarrow (1)$

$|||^{ly}$ $\hat{b} = \cos\beta\, \hat{i} + \sin\beta\, \hat{j} \longrightarrow (2)$

Angle b/w $\hat{a}$ & $\hat{b}$ is $\alpha-\beta$ and $\hat{a}, \hat{b}$ & $\hat{k}$ are in right handed system

$\hat{b}\times\hat{a} = |\hat{b}||\hat{a}| \sin(\alpha-\beta)\, \hat{k}$.

$= \hat{k} \sin(\alpha-\beta) . \longrightarrow (3)$

From (1) & (2)

$$\hat{b}\times\hat{a} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \cos\beta & \sin\beta & 0 \\ \cos\alpha & \sin\alpha & 0 \end{vmatrix}$$

$= \hat{i}(0) - \hat{j}(0) + \hat{k}(\sin\alpha\cos\beta - \cos\alpha\sin\beta)$

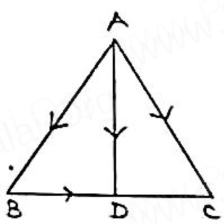
$\hat{b}\times\hat{a} = \hat{k}(\sin\alpha\cos\beta - \cos\alpha\sin\beta) . \longrightarrow (4)$

From (3) & (4)

$\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$.

Example : 6.6

Solution :



$|\vec{AB}|^2 + |\vec{AC}|^2$

$= |\vec{AD} + \vec{DB}|^2 + |\vec{AD} + \vec{DC}|^2$

$= |\vec{AD} - \vec{BD}|^2 + |\vec{AD} + \vec{BD}|^2$

$= |\vec{AD}|^2 + |\vec{BD}|^2 - 2(\vec{AD}\cdot\vec{BD})$

$+ (\vec{AD})^2 + |\vec{BD}| + 2(\vec{AD}\cdot\vec{BD})$

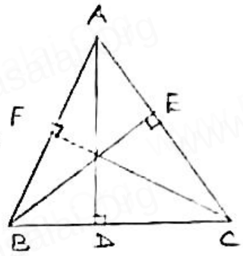
$= 2|\vec{AD}|^2 + 2|\vec{BD}|^2$

$= 2\{|\vec{AD}|^2 + |\vec{BD}|^2\}$.

Hence proved.

Example : 6.7

Solution :



In ΔABC

AD, BE are the altitudes meet at O'

Let CO be produced to meet AB at F.

Let 'O' be the origin then position vectors of A, B, C are $\vec{OA}, \vec{OB}$ and $\vec{OC}$.

Since $\vec{AD} \perp \vec{BC} \Rightarrow \vec{OA} \perp \vec{BC}$

$\vec{OA}\cdot\vec{BC} = 0$

$\vec{OA}\cdot(\vec{OC} - \vec{OB}) = 0$

$\vec{OA}\cdot\vec{OC} - \vec{OA}\cdot\vec{OB} = 0 \longrightarrow (1)$

$\vec{BE} \perp \vec{CA} \Rightarrow \vec{OB} \perp \vec{CA}$

$\vec{OB}\cdot\vec{CA} = 0$

$\vec{OB}\cdot(\vec{OA} - \vec{OC}) = 0$

$\vec{OB}\cdot\vec{OA} - \vec{OB}\cdot\vec{OC} = 0 \longrightarrow (2)$

(1) + (2) $\Rightarrow$

$\vec{OA}\cdot\vec{OC} - \vec{OA}\cdot\vec{OB} + \vec{OB}\cdot\vec{OA} - \vec{OB}\cdot\vec{OC} = 0$

$\vec{OA}\cdot\vec{OC} - \vec{OB}\cdot\vec{OC} = 0$

$\vec{OC}\cdot(\vec{OA} - \vec{OB}) = 0$

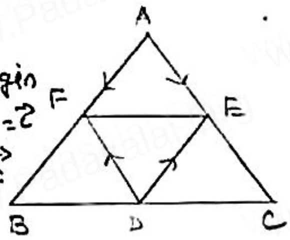
$\vec{OC}\cdot\vec{BA} = 0$

$\vec{OC} \perp \vec{BA} \Rightarrow \vec{CF} \perp \vec{BA}$.

Hence the altitudes of a triangle are concurrent.

Example : 6.8

Solution :



Let 'O' be the origin

$\vec{OA} = \vec{a}$, $\vec{OB} = \vec{b}$, $\vec{OC} = \vec{c}$

$\vec{OD} = \dfrac{\vec{b}+\vec{c}}{2}$, $\vec{OE} = \dfrac{\vec{c}+\vec{a}}{2}$

$\vec{OF} = \dfrac{\vec{a}+\vec{b}}{2}$

Area of $\Delta DEF = \frac{1}{2}|\vec{DE}\times\vec{DF}|$

$= \frac{1}{2}|(\vec{OE} - \vec{OD})\times(\vec{OF} - \vec{OD})|$

$= \frac{1}{2}\left|\left(\dfrac{\vec{c}+\vec{a}}{2} - \left(\dfrac{\vec{b}+\vec{c}}{2}\right)\right) \times \left(\dfrac{\vec{a}+\vec{b}}{2} - \left(\dfrac{\vec{b}+\vec{c}}{2}\right)\right)\right|$

$= \frac{1}{2}\left|\left(\dfrac{\vec{a}}{2} - \dfrac{\vec{b}}{2}\right)\times\left(\dfrac{\vec{a}}{2} - \dfrac{\vec{c}}{2}\right)\right|$

$= \frac{1}{2}\left|\left(\dfrac{\vec{b}-\vec{a}}{2}\right)\times\left(\dfrac{\vec{c}-\vec{a}}{2}\right)\right|$

$= \frac{1}{4}|(\vec{OB} - \vec{OA})\times(\vec{OC} - \vec{OA})|$

$= \frac{1}{4}|\vec{AB}\times\vec{AC}| = \frac{1}{4}$(Area of ΔABC)

WORK DONE BY THE FORCE

Work done, $W = \vec{F}\cdot\vec{d}$

where $\vec{F}$ – Force

$\vec{d}$ – displacement.

Example : 6.9

Solution: $\vec{F_1} = 2\hat{i}+5\hat{j}+6\hat{k}$

$\vec{F_2} = -\hat{i}-2\hat{j}-\hat{k}$

$\vec{F} = \vec{F_1}+\vec{F_2} = \hat{i}+3\hat{j}+5\hat{k}$

$\vec{OA} = 4\hat{i}-3\hat{j}-2\hat{k}$.

$\vec{OB} = 6\hat{i}+\hat{j}-3\hat{k}$

$\vec{d} = \vec{AB} = \vec{OB}-\vec{OA} = 2\hat{i}+4\hat{j}-\hat{k}$.

Work done $W = \vec{F}\cdot\vec{d}$

$W = (\hat{i}+3\hat{j}+5\hat{k})\cdot(2\hat{i}+4\hat{j}-\hat{k})$

$W = 2+12-5 = 9$ units.

Example : 6.10

Solution: $\vec{F_1} = 3\hat{i}-2\hat{j}+2\hat{k}$

$\vec{F_2} = 2\hat{i}+\hat{j}-\hat{k}$.

$\vec{F} = \vec{F_1}+\vec{F_2} = 5\hat{i}-\hat{j}+\hat{k}$.

$\vec{OA} = \hat{i}+3\hat{j}-\hat{k}$

$\vec{OB} = 4\hat{i}-\hat{j}+\lambda\hat{k}$.

$\vec{d} = \vec{AB} = \vec{OB}-\vec{OA} = 3\hat{i}-4\hat{j}+(\lambda+1)\hat{k}$.

Given that $W = \vec{F}\cdot\vec{d} = 16$

$15+4+\lambda+1 = 16$

$20+\lambda = 16$

$\boxed{\lambda = -4}$

TORQUE (or) MOMENT.

$\vec{t} = \vec{r}\times\vec{F}$

where $\vec{F}$ – Force

$\vec{r}$ = through (or) at the point – about the point.

Example : 6.11

Solution: $\vec{F} = 2\hat{i}+\hat{j}-\hat{k}$

$\vec{r} = 0\hat{i}+0\hat{j}+0\hat{k} - 2\hat{i}+0\hat{j}+\hat{k}$

$\vec{r} = -2\hat{i}+\hat{k}$

$\vec{t} = \vec{r}\times\vec{F} = \begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ -2 & 0 & 1\\ 2 & 1 & -1\end{vmatrix} = -\hat{i}-2\hat{k}$

Magnitude: $|\vec{t}| = \sqrt{1+4} = \sqrt{5}$

Direction cosines:

$\left(-\frac{1}{\sqrt{5}}, 0, -\frac{2}{\sqrt{5}}\right)$.

Exercise : 6.1

(1) solution:

$\vec{OA} = \vec{a}$, $\vec{OB} = \vec{b}$

$\vec{OD} = \frac{\vec{a}+\vec{b}}{2}$

$|\vec{OA}| = |\vec{OB}|$

$|\vec{a}| = |\vec{b}|$.

To prove: $\vec{OD}\perp\vec{AB}$

$\vec{OD}\cdot\vec{AB} = \vec{OD}\cdot(\vec{OB}-\vec{OA})$

$= \left(\frac{\vec{a}+\vec{b}}{2}\right)\cdot(\vec{b}-\vec{a})$

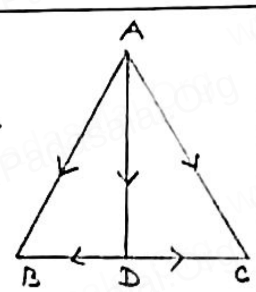
$= \frac{1}{2}(\vec{b}+\vec{a})\cdot(\vec{b}-\vec{a})$

$= \frac{1}{2}(|\vec{b}|^2-|\vec{a}|^2) = 0$

$\vec{OD}\cdot\vec{AB} = 0$

$\therefore \vec{OD}\perp\vec{AB}$.

(2) solution:



Let ABC be an Isosceles triangle with AB = AC.

Let D be the midpoint of BC.

$|\vec{AB}|^2 = |\vec{AC}|^2$

$|\vec{AD}+\vec{DB}|^2 = |\vec{AD}+\vec{DC}|^2$

$\because \vec{DC} = -\vec{DB}$

$|\vec{AD}+\vec{DB}|^2 = |\vec{AD}-\vec{DB}|^2$

$\vec{AD}^2+\vec{DB}^2+2\vec{AD}\cdot\vec{DB} = \vec{AD}^2+\vec{DB}^2-2\vec{AD}\cdot\vec{DB}$

$4\vec{AD}\cdot\vec{DB} = 0$

$\vec{AD}\cdot\vec{DB} = 0$

$\therefore \vec{AD}\perp\vec{DB}$, |||ly $\vec{AD}\perp\vec{DC}$

∴ Median to the base of the isosceles triangle is ⊥r to the base.

③ Solution:

AB - diameter of the semi-circle centre at 'O'.

Let 'P' be any point on the semi-circle

To prove: $\angle APB = 90°$

We have $OA = OB = OP$ (radii)

$\vec{PA} = \vec{PO} + \vec{OA}$

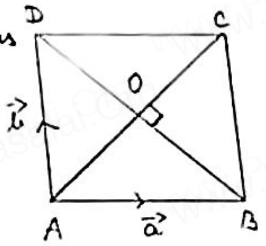
$\vec{PB} = \vec{PO} + \vec{OB} = \vec{PO} - \vec{OA}$

$\vec{PA} \cdot \vec{PB} = (\vec{PO} + \vec{OA}) \cdot (\vec{PO} - \vec{OA})$

$= \vec{PO}^2 - \vec{OA}^2 = 0$.

$\therefore \vec{PA} \perp \vec{PB}$, $\therefore \angle APB = 90°$.

④ Solution:
ABCD be a Rhombus



Let $\vec{AB} = \vec{a}$

$\vec{AD} = \vec{b}$

$AB = BC = CD = DA$

$|\vec{a}| = |\vec{b}|$

$\vec{AC} \cdot \vec{BD} = (\vec{AB} + \vec{BC}) \cdot (\vec{BC} + \vec{CD})$

$= (\vec{BC} + \vec{AB}) \cdot (\vec{BC} - \vec{AB})$ $\quad \boxed{\because \vec{CD} = -\vec{AB}}$

$= \vec{BC}^2 - \vec{AB}^2$

$= |\vec{BC}|^2 - |\vec{AB}|^2 = 0$.

$\therefore \vec{AC} \perp \vec{BD} \longrightarrow ①$

Take $|\vec{AB}| = |\vec{BC}| = |\vec{CD}|$

$|\vec{AB}| = |\vec{BC}|$ $\qquad |\vec{BC}| = |\vec{CD}|$

$|\vec{AB}|^2 = |\vec{BC}|^2$ $\qquad |\vec{BC}|^2 = |\vec{CD}|^2$

From the right triangles,
ΔOAB, ΔOBC & ΔOCD,

$|\vec{OA}|^2 + |\vec{OB}|^2 = |\vec{OB}|^2 + |\vec{OC}|^2$ $\qquad |\vec{OB}|^2 + |\vec{OC}|^2 = |\vec{OC}|^2 + |\vec{OD}|^2$

$|\vec{OA}|^2 = |\vec{OC}|^2$ $\qquad |\vec{OB}|^2 = |\vec{OD}|^2$

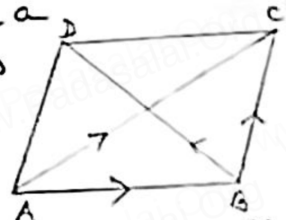
$\therefore |\vec{OA}| = |\vec{OC}| \to ②$ $\qquad |\vec{OB}| = |\vec{OD}| \to ③$

From ①, ② & ③

Diagonals of a rhombus are bisect at right angles.

⑤ Solution:

Let ABCD be a parallelogram



Let $\vec{AC} = \vec{BD}$

$\vec{AB} + \vec{BC} = \vec{BC} + \vec{CD}$

$\vec{AB} + \vec{BC} = \vec{BC} - \vec{AB}$ $\quad \boxed{\vec{CD} = -\vec{AB}}$

$(\vec{AB} + \vec{BC})^2 = (\vec{BC} - \vec{AB})^2$

$\vec{AB}^2 + \vec{BC}^2 + 2\,\vec{AB}\cdot\vec{BC}$
$= \vec{BC}^2 + \vec{AB}^2 - 2\,\vec{BC}\cdot\vec{AB}$

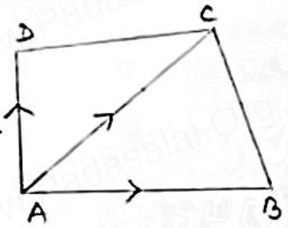
$4\,\vec{AB}\cdot\vec{BC} = 0$

$\vec{AB}\cdot\vec{BC} = 0$

$\therefore \vec{AB} \perp \vec{BC}$.

Hence ABCD is a rectangle.

⑥ Solution:



Vector area of a quadrilateral ABCD

= Vector area of ΔABC + vector area of ΔACD.

$= \frac{1}{2}(\vec{AB} \times \vec{AC}) + \frac{1}{2}(\vec{AC} \times \vec{AD})$.

$= \frac{1}{2}(-\vec{AC} \times \vec{AB}) + \frac{1}{2}(\vec{AC} \times \vec{AD})$

$= \frac{1}{2}\vec{AC} \times (-\vec{AB} + \vec{AD})$

$= \frac{1}{2}\vec{AC} \times (\vec{BA} + \vec{AD})$

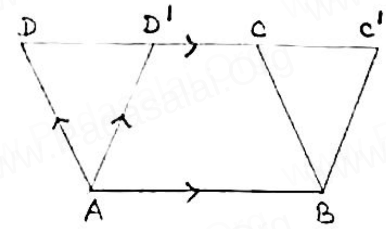
$= \frac{1}{2}(\vec{AC} \times \vec{BD})$.

$\therefore$ Area of a quadrilateral ABCD is $\frac{1}{2}|\vec{AC} \times \vec{BD}|$

RESULT:

If $\vec{d_1}$ & $\vec{d_2}$ are the two diagonals of the parallelogram then Area of the parallelogram $= \frac{1}{2}|\vec{d_1} \times \vec{d_2}|$.

(7) Solution:



Area of a parallelogram ABCD
$= |\vec{AB} \times \vec{AB}|$

Area of a parallelogram ABC'D'
$= |\vec{AB} \times \vec{AD'}|$

$= |\vec{AB} \times (\vec{AD} + \vec{DD'})|$

$= |(\vec{AB} \times \vec{AD}) + (\vec{AB} \times \vec{DD'})|$

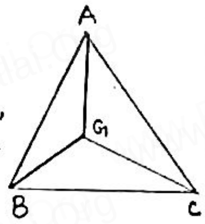
$= |(\vec{AB} \times \vec{AD}) + \vec{0}|$ $(\because \vec{AB} \parallel \vec{DD'})$

$= |\vec{AB} \times \vec{AD}|$

= Area of a parallelogram ABCD.

(8) Solution:

In ΔABC,



Let 'O' be the origin,
$\vec{OA} = \vec{a}, \vec{OB} = \vec{b}, \vec{OC} = \vec{c}$

and let 'G' be the centroid of the ΔABC.

$\therefore \vec{OG} = \frac{\vec{a}+\vec{b}+\vec{c}}{3}$

Also Area of a ΔABC

$= \frac{1}{2}|(\vec{a}\times\vec{b}) + (\vec{b}\times\vec{c}) + (\vec{c}\times\vec{a})| \longrightarrow (1)$

Area of a $\Delta GBC = \frac{1}{2}|\vec{GB}\times\vec{BC}|$

$= \frac{1}{2}|(\vec{OB}-\vec{OG})\times(\vec{OC}-\vec{OB})|$

$= \frac{1}{2}\left|\left(\vec{b} - \left(\frac{\vec{a}+\vec{b}+\vec{c}}{3}\right)\right) \times (\vec{c}-\vec{b})\right|$

$= \frac{1}{2}\left|\left(\frac{3\vec{b}-\vec{a}-\vec{b}-\vec{c}}{3}\right) \times (\vec{c}-\vec{b})\right|$

$= \frac{1}{6}|(2\vec{b}-\vec{a}-\vec{c}) \times (\vec{c}-\vec{b})|$

$= \frac{1}{6}|2(\vec{b}\times\vec{c}) - 2(\vec{b}\times\vec{b}) - (\vec{a}\times\vec{c}) + (\vec{a}\times\vec{b}) - (\vec{c}\times\vec{c}) + (\vec{c}\times\vec{b})|$

$= \frac{1}{6}|2(\vec{b}\times\vec{c}) - \vec{0} + (\vec{c}\times\vec{a}) + (\vec{a}\times\vec{b}) - \vec{0} - (\vec{b}\times\vec{c})|$

Area of a ΔGBC

$= \frac{1}{6}|(\vec{a}\times\vec{b}) + (\vec{b}\times\vec{c}) + (\vec{c}\times\vec{a})|$

III^{ly}

Area of a ΔGCA

= Area of a ΔGAB

$= \frac{1}{6}|(\vec{a}\times\vec{b}) + (\vec{b}\times\vec{c}) + (\vec{c}\times\vec{a})|$

From (1)

Area of a ΔGAB

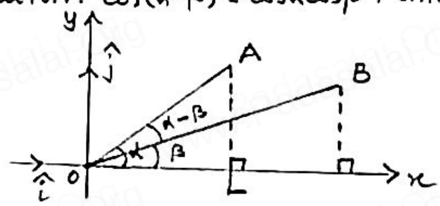
= Area of a ΔGBC

= Area of a ΔGCA

$= \frac{1}{3}$(Area of a ΔABC).

Hence proved.

(9) Solution: $\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$



* Let $\hat{a} = \vec{OA}$, $\hat{b} = \vec{OB}$ makes angles α, β respectively in the positive x-axis.

where A & B as in diagram.

* Draw $AL \perp$ x-axis, $BM \perp$ x-axis

* In ΔOLA

$\sin\alpha = \frac{|\vec{LA}|}{|\vec{OA}|} = |\vec{LA}|$ $\cos\alpha = \frac{|\vec{OL}|}{|\vec{OA}|} = |\vec{OL}|$

$\vec{LA} = \sin\alpha(\hat{j})$ $\vec{OL} = \cos\alpha(\hat{i})$

$\hat{a} = \vec{OA} = \vec{OL} + \vec{LA}$

$\hat{a} = \cos\alpha\,\hat{i} + \sin\alpha\,\hat{j} \longrightarrow (1)$

III^{ly} $\hat{b} = \cos\beta\,\hat{i} + \sin\beta\,\hat{j} \longrightarrow (2)$

Angle b/w $\hat{a}$ & $\hat{b}$ is $\alpha-\beta$

$\hat{a}\cdot\hat{b} = |\hat{a}||\hat{b}|\cos(\alpha-\beta)$

$\hat{a}\cdot\hat{b} = \cos(\alpha-\beta) \longrightarrow (3)$

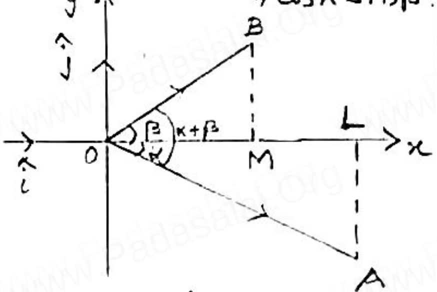
From (1) & (2)

$\hat{a}\cdot\hat{b} = \cos\alpha\cos\beta + \sin\alpha\sin\beta \longrightarrow (4)$

From (3) & (4)

$\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$.

(10) solution: $\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$.



* Let $\hat{a} = \vec{OA}$, $\hat{b} = \vec{OB}$ make angles α, β respectively in the positive x-axis where A & B as in diagram

* Draw AL ⊥ x-axis, BM ⊥ x-axis.

* In ΔOLA,

$\sin\alpha = \frac{|\vec{LA}|}{|\vec{OA}|} = |\vec{LA}|$ $\cos\alpha = \frac{|\vec{OL}|}{|\vec{OA}|} = |\vec{OL}|$

$\vec{LA} = \sin\alpha(-\hat{j})$ $\vec{OL} = \cos\alpha(\hat{i})$

$\therefore \hat{a} = \vec{OA} = \vec{OL} + \vec{LA}$

$\hat{a} = \cos\alpha\,\hat{i} - \sin\alpha\,\hat{j} \rightarrow (1)$

$|||^{ly}$ $\hat{b} = \cos\beta\,\hat{i} + \sin\beta\,\hat{j} \rightarrow (2)$

Angle b/w $\hat{a}$ & $\hat{b}$ is $\alpha+\beta$

$\therefore \hat{a}\times\hat{b} = |\hat{a}||\hat{b}|\sin(\alpha+\beta)\hat{k}$.

$\hat{a}\times\hat{b} = \hat{k}\sin(\alpha+\beta) \rightarrow (3)$

From (1) & (2)

$$\hat{a}\times\hat{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \cos\alpha & -\sin\alpha & 0 \\ \cos\beta & \sin\beta & 0 \end{vmatrix}$$

$= 0 - 0 + \hat{k}(\cos\alpha\sin\beta + \sin\alpha\cos\beta)$

$\hat{a}\times\hat{b} = \hat{k}(\sin\alpha\cos\beta + \cos\alpha\sin\beta) \rightarrow (4)$

From (3) & (4)

$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$.

(11) solution: $\vec{F_1} = 8\hat{i} + 2\hat{j} - 6\hat{k}$

$\vec{F_2} = 6\hat{i} + 2\hat{j} - 2\hat{k}$.

$\vec{F} = \vec{F_1} + \vec{F_2} = 14\hat{i} + 4\hat{j} - 8\hat{k}$.

$\vec{OA} = \hat{i} + 2\hat{j} + 3\hat{k}$, $\vec{OB} = 5\hat{i} + 4\hat{j} + \hat{k}$

$\vec{d} = \vec{AB} = \vec{OB} - \vec{OA} = 4\hat{i} + 2\hat{j} - 2\hat{k}$.

work done, $W = \vec{F}\cdot\vec{d}$

$W = (14\hat{i} + 4\hat{j} - 8\hat{k})\cdot(4\hat{i} + 2\hat{j} - 2\hat{k})$

$= 56 + 8 + 16 = 80$ units.

(12) Solution: $\vec{a} = 3\hat{i} + 4\hat{j} + 5\hat{k}$, $\vec{b} = 10\hat{i} + 6\hat{j} - 8\hat{k}$

$\vec{F} = 5\sqrt{2}\,\hat{a} + 10\sqrt{2}\,\hat{b}$.

$\vec{F} = 5\sqrt{2}\left(\frac{\vec{a}}{|\vec{a}|}\right) + 10\sqrt{2}\left(\frac{\vec{b}}{|\vec{b}|}\right)$.

$= 5\sqrt{2}\left(\frac{3\hat{i}+4\hat{j}+5\hat{k}}{\sqrt{9+16+25}}\right) + 10\sqrt{2}\left(\frac{10\hat{i}+6\hat{j}-8\hat{k}}{\sqrt{100+36+64}}\right)$

$= 5\sqrt{2}\left(\frac{3\hat{i}+4\hat{j}+5\hat{k}}{5\sqrt{2}}\right) + 10\sqrt{2}\left(\frac{10\hat{i}+6\hat{j}-8\hat{k}}{10\sqrt{2}}\right)$

$\vec{F} = 3\hat{i} + 4\hat{j} + 5\hat{k} + 10\hat{i} + 6\hat{j} - 8\hat{k}$

$\vec{F} = 13\hat{i} + 10\hat{j} - 3\hat{k}$.

$\vec{OA} = 4\hat{i} - 3\hat{j} - 2\hat{k}$, $\vec{OB} = 6\hat{i} + \hat{j} - 3\hat{k}$

$\vec{d} = \vec{AB} = \vec{OB} - \vec{OA} = 2\hat{i} + 4\hat{j} - \hat{k}$.

Work done, $W = \vec{F}\cdot\vec{d}$

$W = (13\hat{i} + 10\hat{j} - 3\hat{k})\cdot(2\hat{i} + 4\hat{j} - \hat{k})$

$W = 26 + 40 + 3 = 69$ units.

(13) Solution: $\vec{F} = 3\hat{i} + 4\hat{j} - 5\hat{k}$

$\vec{r}$ = through − about

$\vec{r} = 4\hat{i} + 2\hat{j} - 3\hat{k} - 2\hat{i} + 3\hat{j} - 4\hat{k}$

$\vec{r} = 2\hat{i} + 5\hat{j} - 7\hat{k}$.

Torque, $\vec{t} = \vec{r}\times\vec{F}$

$$\vec{t} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 5 & -7 \\ 3 & 4 & -5 \end{vmatrix}$$

$= \hat{i}(-25+28) - \hat{j}(-10+21) + \hat{k}(8-15)$

$\vec{t} = 3\hat{i} - 11\hat{j} - 7\hat{k}$.

Magnitude, $|\vec{t}| = \sqrt{9+121+49}$

$|\vec{t}| = \sqrt{179}$.

Direction cosines,

$\frac{3}{\sqrt{179}}, \frac{-11}{\sqrt{179}}, \frac{-7}{\sqrt{179}}$.

(14) Solution: $\vec{F_1} = -3\hat{i} + 6\hat{j} - 3\hat{k}$.

$\vec{F_2} = 4\hat{i} - 10\hat{j} + 12\hat{k}$, $\vec{F_3} = 4\hat{i} + 7\hat{j}$

$\vec{F} = \vec{F_1} + \vec{F_2} + \vec{F_3} = 5\hat{i} + 3\hat{j} + 9\hat{k}$

$\vec{r}$ = at the point − about the point

$\vec{r} = 8\hat{i} - 6\hat{j} - 4\hat{k} - 18\hat{i} - 3\hat{j} + 9\hat{k}$

$\vec{r} = -10\hat{i} - 9\hat{j} + 5\hat{k}$. $\boxed{\vec{t} = \vec{r}\times\vec{F}}$

$$\vec{t} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -10 & -9 & 5 \\ 5 & 3 & 9 \end{vmatrix} = \hat{i}(-81-15) - \hat{j}(-90-25) + \hat{k}(-30+45)$$

$\vec{t} = -96\hat{i} + 115\hat{j} + 15\hat{k}$.

Example : 6.12

Solution: $\vec{a} = -3\hat{i} - \hat{j} + 5\hat{k}$

$\vec{b} = \hat{i} - 2\hat{j} + \hat{k}$

$\vec{c} = 4\hat{j} - 5\hat{k}$.

$$\vec{a}\cdot(\vec{b}\times\vec{c}) = \begin{vmatrix} -3 & -1 & 5 \\ 1 & -2 & 1 \\ 0 & 4 & -5 \end{vmatrix}$$

$= -3(10-4) + 1(-5-0) + 5(4-0)$

$= -18 - 5 + 20$

$= -23 + 20 = -3$.

Example : 6.13.

Solution: ~~Whet~~ $\vec{a} = 2\hat{i} - 3\hat{j} + 4\hat{k}$

$\vec{b} = \hat{i} + 2\hat{j} - \hat{k}$, $\vec{c} = 3\hat{i} - \hat{j} + 2\hat{k}$.

are the coterminus edges of the parallelopiped.

$$[\vec{a}\ \vec{b}\ \vec{c}] = \begin{vmatrix} 2 & -3 & 4 \\ 1 & 2 & -1 \\ 3 & -1 & 2 \end{vmatrix}$$

$= 2(4-1) + 3(2+3) + 4(-1-6)$

$= 6 + 15 - 28$

$= 21 - 28 = -7$.

volume of the parallelopiped $= |[\vec{a}\ \vec{b}\ \vec{c}]| = |-7| = 7$.

Example : 6.14.

Solution: $\vec{a} = \hat{i} + 2\hat{j} - 3\hat{k}$

$\vec{b} = 2\hat{i} - \hat{j} + 2\hat{k}$.

$\vec{c} = 3\hat{i} + \hat{j} - \hat{k}$.

$$[\vec{a}\ \vec{b}\ \vec{c}] = \begin{vmatrix} 1 & 2 & -3 \\ 2 & -1 & 2 \\ 3 & 1 & -1 \end{vmatrix}$$

$= 1(1-2) - 2(-2-6) - 3(2+3)$

$= -1 + 16 - 15$

$[\vec{a}\ \vec{b}\ \vec{c}] = 0$.

$\therefore$ $\vec{a}, \vec{b}$ & $\vec{c}$ are coplanar.

Example : 6.15

Solution: $\vec{a} = 2\hat{i} - \hat{j} + 3\hat{k}$

$\vec{b} = 3\hat{i} + 2\hat{j} + \hat{k}$, $\vec{c} = \hat{i} + m\hat{j} + 4\hat{k}$

Since $\vec{a}, \vec{b}$ & $\vec{c}$ are coplanar

$\therefore [\vec{a}\ \vec{b}\ \vec{c}] = 0$

$$\begin{vmatrix} 2 & -1 & 3 \\ 3 & 2 & 1 \\ 1 & m & 4 \end{vmatrix} = 0$$

$2(8-m) + 1(12-1) + 3(3m-2) = 0$

$16 - 2m + 12 - 1 + 9m - 6 = 0$

$21 + 7m = 0$

$7m = -21$

$\boxed{m = -3}$

Example : 6.16

Solution: Let 'O' be the origin and let $\vec{OA} = 6\hat{i} - 7\hat{j}$

$\vec{OB} = 16\hat{i} - 19\hat{j} - 4\hat{k}$

$\vec{OC} = 3\hat{j} - 6\hat{k}$

$\vec{OD} = 2\hat{i} - 5\hat{j} + 10\hat{k}$.

$\vec{AB} = \vec{OB} - \vec{OA} = 10\hat{i} - 12\hat{j} - 4\hat{k}$

$\vec{AC} = \vec{OC} - \vec{OA} = -6\hat{i} + 10\hat{j} - 6\hat{k}$

$\vec{AD} = \vec{OD} - \vec{OA} = -4\hat{i} + 2\hat{j} + 10\hat{k}$.

$$[\vec{AB}\ \vec{AC}\ \vec{AD}] = \begin{vmatrix} 10 & -12 & -4 \\ -6 & 10 & -6 \\ -4 & 2 & 10 \end{vmatrix}$$

$= 10(100+12) + 12(-60-24) - 4(-12+40)$.

$= 1120 + 12(-84) - 4(28)$.

$= 1120 - 1008 - 112$

$= 1120 - 1120$

$= 0$.

$[\vec{AB}\ \vec{AC}\ \vec{AD}] = 0$

$\therefore \vec{AB}, \vec{AC}$ & $\vec{AD}$ are coplanar

$\therefore$ A, B, C & D lie on a plane.

Example : 6.17

Solution:

Since $\vec{a}, \vec{b}$ & $\vec{c}$ are coplanar.

$\therefore [\vec{a}\ \vec{b}\ \vec{c}] = 0$

$[\vec{a}+\vec{b}\quad \vec{b}+\vec{c}\quad \vec{c}+\vec{a}]$

$= [\vec{a}\quad \vec{b}+\vec{c}\quad \vec{c}+\vec{a}]$

$+ [\vec{b}\quad \vec{b}+\vec{c}\quad \vec{c}+\vec{a}]$

$= [\vec{a}\ \vec{b}\ \ \vec{c}+\vec{a}] + [\vec{a}\ \vec{c}\ \ \vec{c}+\vec{a}]$

$+ [\vec{b}\ \vec{b}\ \ \vec{c}+\vec{a}] + [\vec{b}\ \vec{c}\ \ \vec{c}+\vec{a}]$

$= [\vec{a}\ \vec{b}\ \vec{c}] + [\vec{a}\ \vec{b}\ \vec{a}] + [\vec{a}\ \vec{c}\ \vec{c}]$

$+ [\vec{a}\ \vec{c}\ \vec{a}] + [\vec{b}\ \vec{b}\ \vec{c}] + [\vec{b}\ \vec{b}\ \vec{a}]$

$+ [\vec{b}\ \vec{c}\ \vec{c}] + [\vec{b}\ \vec{c}\ \vec{a}]$

$= [\vec{a}\ \vec{b}\ \vec{c}] + [\vec{a}\ \vec{b}\ \vec{c}]$

$= 2\,[\vec{a}\ \vec{b}\ \vec{c}] = 2(0) = 0$

$\therefore \vec{a}+\vec{b}, \vec{b}+\vec{c}, \vec{c}+\vec{a}$ are coplanar.

Aliter: 1

Since $\vec{a}, \vec{b}$ & $\vec{c}$ are coplanar,

$[\vec{a}\ \vec{b}\ \vec{c}] = 0$.

$[\vec{a}+\vec{b}\quad \vec{b}+\vec{c}\quad \vec{c}+\vec{a}]$

$= (\vec{a}+\vec{b}) \cdot \{(\vec{b}+\vec{c}) \times (\vec{c}+\vec{a})\}$

$= (\vec{a}+\vec{b}) \cdot \{(\vec{b}\times\vec{c}) + (\vec{b}\times\vec{a}) + (\vec{c}\times\vec{c}) + (\vec{c}\times\vec{a})\}$.

$= \vec{a}\cdot(\vec{b}\times\vec{c}) + \vec{a}\cdot(\vec{b}\times\vec{a}) + \vec{a}\cdot(\vec{c}\times\vec{a})$

$+ \vec{b}\cdot(\vec{b}\times\vec{c}) + \vec{b}\cdot(\vec{b}\times\vec{a}) + \vec{b}\cdot(\vec{c}\times\vec{a})$

$\because \vec{c}\times\vec{c} = \vec{0}$

$= [\vec{a}\ \vec{b}\ \vec{c}] + 0 + 0 + 0 + 0 + [\vec{b}\ \vec{c}\ \vec{a}]$

$= [\vec{a}\ \vec{b}\ \vec{c}] + [\vec{a}\ \vec{b}\ \vec{c}]$

$= 2\,[\vec{a}\ \vec{b}\ \vec{c}] = 2(0) = 0$.

$\therefore \vec{a}+\vec{b}, \vec{b}+\vec{c}$ & $\vec{c}+\vec{a}$ are coplanar.

Aliter: 2

Since $\vec{a}, \vec{b}$ & $\vec{c}$ are coplanar

$[\vec{a}\ \vec{b}\ \vec{c}] = 0$

$[\vec{a}+\vec{b}\quad \vec{b}+\vec{c}\quad \vec{c}+\vec{a}]$

$= \begin{vmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{vmatrix} [\vec{a}\ \vec{b}\ \vec{c}]$

$= \{1(1-0) - 1(0-1) + 0\} [\vec{a}\ \vec{b}\ \vec{c}]$

$= 2\,[\vec{a}\ \vec{b}\ \vec{c}]$

$= 2(0)$

$[\vec{a}+\vec{b}\quad \vec{b}+\vec{c}\quad \vec{c}+\vec{a}] = 0$

$\therefore \vec{a}+\vec{b}\quad \vec{b}+\vec{c}\quad \vec{c}+\vec{a}$ are coplanar.

Example : 6.18

Solution:

$[\vec{a}+\vec{c}\quad \vec{a}+\vec{b}\quad \vec{a}+\vec{b}+\vec{c}]$

$= \begin{vmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{vmatrix} [\vec{a}\ \vec{b}\ \vec{c}]$

$= \{1(1-0) - 0 + 1(1-1)\} [\vec{a}\ \vec{b}\ \vec{c}]$

$= \{1 - 0 + 0\} [\vec{a}\ \vec{b}\ \vec{c}]$.

$= [\vec{a}\ \vec{b}\ \vec{c}]$.

Exercise 6.2.

① Solution:

$\vec{a} = \hat{i} - 2\hat{j} + 3\hat{k}$

$\vec{b} = 2\hat{i} + \hat{j} - 2\hat{k}$

$\vec{c} = 3\hat{i} + 2\hat{j} + \hat{k}$

$$\vec{a}\cdot(\vec{b}\times\vec{c}) = \begin{vmatrix} 1 & -2 & 3 \\ 2 & 1 & -2 \\ 3 & 2 & 1 \end{vmatrix}$$

$= 1(1+4) + 2(2+6) + 3(4-3)$

$= 5 + 16 + 3$

$= 24$.

② Solution:

$\vec{a} = -6\hat{i} + 14\hat{j} + 10\hat{k}$

$\vec{b} = 14\hat{i} - 10\hat{j} - 6\hat{k}$

$\vec{c} = 2\hat{i} + 4\hat{j} - 2\hat{k}$

Volume of the parallelopiped

$$= [\vec{a}\ \vec{b}\ \vec{c}] = \begin{vmatrix} -6 & 14 & 10 \\ 14 & -10 & -6 \\ 2 & 4 & -2 \end{vmatrix}$$

$$= (2)(2)(2)\begin{vmatrix} -3 & 7 & 5 \\ 7 & -5 & -3 \\ 1 & 2 & -1 \end{vmatrix}$$

$= 8\{-3(5+6) - 7(-7+3) + 5(14+5)\}$

$= 8[-33 + 28 + 95]$

$= 8(90) = 720$ cu. units.

③ Solution: $\vec{a} = 7\hat{i} + \lambda\hat{j} - 3\hat{k}$

$\vec{b} = \hat{i} + 2\hat{j} - \hat{k}$, $\vec{c} = -3\hat{i} + 7\hat{j} + 5\hat{k}$

Volume of a parallelopiped

$= [\vec{a}\ \vec{b}\ \vec{c}] = 90$ cu. units

$$\begin{vmatrix} 7 & \lambda & -3 \\ 1 & 2 & -1 \\ -3 & 7 & 5 \end{vmatrix} = 90$$

$7(10+7) - \lambda(5-3) - 3(7+6) = 90$

$119 - 2\lambda - 39 = 90$

$-2\lambda = 90 + 39 - 119$

$-2\lambda = 10$

$\boxed{\lambda = -5}$

④ Solution:

Volume of a parallelopiped

$= [\vec{a}\ \vec{b}\ \vec{c}] = \pm 4$

$(\vec{a}+\vec{b})\cdot(\vec{b}\times\vec{c}) + (\vec{b}+\vec{c})\cdot(\vec{c}\times\vec{a}) + (\vec{c}+\vec{a})\cdot(\vec{a}\times\vec{b})$

$= \vec{a}\cdot(\vec{b}\times\vec{c}) + \vec{b}\cdot(\vec{b}\times\vec{c}) + \vec{b}\cdot(\vec{c}\times\vec{a}) + \vec{c}\cdot(\vec{c}\times\vec{a}) + \vec{c}\cdot(\vec{a}\times\vec{b}) + \vec{a}\cdot(\vec{a}\times\vec{b})$

$= [\vec{a}\ \vec{b}\ \vec{c}] + 0 + [\vec{a}\ \vec{b}\ \vec{c}] + 0 + [\vec{a}\ \vec{b}\ \vec{c}] + 0$

$= 3[\vec{a}\ \vec{b}\ \vec{c}] = 3(\pm 4)$

$= \pm 12$.

⑤ Solution: $\vec{a} = -2\hat{i} + 5\hat{j} + 3\hat{k}$

$\vec{b} = \hat{i} + 3\hat{j} - 2\hat{k}$, $\vec{c} = -3\hat{i} + \hat{j} + 4\hat{k}$

Volume of the parallelopiped

$$= [\vec{a}\ \vec{b}\ \vec{c}] = \begin{vmatrix} -2 & 5 & 3 \\ 1 & 3 & -2 \\ -3 & 1 & 4 \end{vmatrix}$$

$= 12$

Vector area of the base parallelogram $= \vec{b}\times\vec{c}$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 3 & -2 \\ -3 & 1 & 4 \end{vmatrix}$$

$= 14\hat{i} + 2\hat{j} + 10\hat{k}$

Area of the parallelogram $= |\vec{b}\times\vec{c}|$

$= \sqrt{196 + 4 + 100}$

$= \sqrt{300} = 10\sqrt{3}$

Volume of the parallelopiped = base area × height (altitude)

$12 = 10\sqrt{3} \times$ altitude

altitude $= \frac{12}{10\sqrt{3}} = \frac{6}{5\sqrt{3}} = \frac{2\sqrt{3}}{5}$.

⑥ solution:

Let $\vec{a} = 2\hat{i} + 3\hat{j} + \hat{k}$

$\vec{b} = \hat{i} - 2\hat{j} + 2\hat{k}$

$\vec{c} = 3\hat{i} + \hat{j} + 3\hat{k}$.

$$[\vec{a}\ \vec{b}\ \vec{c}] = \begin{vmatrix} 2 & 3 & 1 \\ 1 & -2 & 2 \\ 3 & 1 & 3 \end{vmatrix}$$

$= 2(-6-2) - 3(3-6) + 1(1+6)$

$= -16 + 9 + 7$

$[\vec{a}\ \vec{b}\ \vec{c}] = 0$

$\therefore \vec{a}, \vec{b}$ & $\vec{c}$ are coplanar.

⑦ solution: Given that $c_1 = 1, c_2 = 2$

and $\vec{a} = \hat{i} + \hat{j} + \hat{k}$

$\vec{b} = \hat{i}$

$\vec{c} = c_1\hat{i} + c_2\hat{j} + c_3\hat{k}$

Since $\vec{a}, \vec{b}$ & $\vec{c}$ are coplanar

$[\vec{a}\ \vec{b}\ \vec{c}] = 0$

$$\begin{vmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ c_1 & c_2 & c_3 \end{vmatrix} = 0$$

Expand through R_2

$-1(c_3 - c_2) + 0 - 0 = 0$

$-c_3 + c_2 = 0$

$-c_3 + 2 = 0$

$-c_3 = -2$

$\boxed{c_3 = 2}$

⑧ solution: $\vec{a} = \hat{i} - \hat{k}$

$\vec{b} = x\hat{i} + \hat{j} + (1-x)\hat{k}$

$\vec{c} = y\hat{i} + x\hat{j} + (1+x-y)\hat{k}$.

$$[\vec{a}\ \vec{b}\ \vec{c}] = \begin{vmatrix} 1 & 0 & -1 \\ x & 1 & 1-x \\ y & x & 1+x-y \end{vmatrix}$$

$$= \begin{vmatrix} 0 & 0 & -1 \\ 0 & 1 & 1-x \\ 1 & x & 1+x-y \end{vmatrix} \quad c_1 \to c_1 - c_2 + c_3$$

$= 0 - 0 - 1(0-1) = 1$.

$[\vec{a}\ \vec{b}\ \vec{c}] = 1$.

$\therefore [\vec{a}\ \vec{b}\ \vec{c}]$ depends on neither x nor y.

⑨ solution:

Let $\vec{p} = a\hat{i} + a\hat{j} + c\hat{k}$

$\vec{q} = \hat{i} + \hat{k}$

$\vec{r} = c\hat{i} + c\hat{j} + b\hat{k}$.

Since $\vec{p}, \vec{q}$ and $\vec{r}$ are coplanar

$[\vec{p}\ \vec{q}\ \vec{r}] = 0$.

$$\begin{vmatrix} a & a & c \\ 1 & 0 & 1 \\ c & c & b \end{vmatrix} = 0$$

$a(0-c) - a(b-c) + c(c-0) = 0$

$-ac - ab + ac + c^2 = 0$

$-ab + c^2 = 0$

$c^2 = ab$

$c = \pm\sqrt{ab}$

'c' is a geometric mean of a and b

Hence proved.

⑩ solution: $|\vec{c}| = 1$

$\vec{c}$ is $\perp r$ to both $\vec{a}$ and $\vec{b}$

$\vec{a} \times \vec{b}$ is also $\perp r$ to both $\vec{a}$ and $\vec{b}$

$\therefore \vec{c}$ and $\vec{a} \times \vec{b}$ are parallel.

$\therefore$ angle b/w $\vec{c}$ and $\vec{a} \times \vec{b}$ is zero

Given that angle b/w $\vec{a}$ and $\vec{b}$ is $\frac{\pi}{6}$

$[\vec{a}\ \vec{b}\ \vec{c}] = (\vec{a} \times \vec{b}) \cdot \vec{c}$

$= |\vec{a} \times \vec{b}|\,|\vec{c}| \cos 0$

$= |\vec{a} \times \vec{b}|\,(1)\,(1)$

$= |\vec{a}|\,|\vec{b}| \sin\frac{\pi}{6}$

$[\vec{a}\ \vec{b}\ \vec{c}] = \frac{1}{2}|\vec{a}|\,|\vec{b}|$.

$[\vec{a}\ \vec{b}\ \vec{c}]^2 = \frac{1}{4}|\vec{a}|^2|\vec{b}|^2$

Hence proved.

EXERCISE - 6.3

VECTOR TRIPLE PRODUCT:

$\vec{a}\times(\vec{b}\times\vec{c}) = (\vec{a}\cdot\vec{c})\vec{b} - (\vec{a}\cdot\vec{b})\vec{c}$

$(\vec{a}\times\vec{b})\times\vec{c} = (\vec{a}\cdot\vec{c})\vec{b} - (\vec{b}\cdot\vec{c})\vec{a}$

LAGRANGE'S IDENTITY:

$$(\vec{a}\times\vec{b})\cdot(\vec{c}\times\vec{d}) = \begin{vmatrix} \vec{a}\cdot\vec{c} & \vec{a}\cdot\vec{d} \\ \vec{b}\cdot\vec{c} & \vec{b}\cdot\vec{d} \end{vmatrix}$$

EX: 6.3 - 1

Given, $\vec{a} = \hat{i} - 2\hat{j} + 3\hat{k}$

$\vec{b} = 2\hat{i} + \hat{j} - 2\hat{k}$

$\vec{c} = 3\hat{i} + 2\hat{j} + \hat{k}$

(i) $(\vec{a}\times\vec{b})\times\vec{c}$

$$\vec{a}\times\vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -2 & 3 \\ 2 & 1 & -2 \end{vmatrix}$$

$= \hat{i}(4-3) - \hat{j}(-2-6) + \hat{k}(1+4)$

$= \hat{i} + 8\hat{j} + 5\hat{k}$

$$(\vec{a}\times\vec{b})\times\vec{c} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 8 & 5 \\ 3 & 2 & 1 \end{vmatrix}$$

$= \hat{i}(8-10) - \hat{j}(1-15) + \hat{k}(2-24)$

$= -2\hat{i} + 14\hat{j} - 22\hat{k}$

(ii) $\vec{a}\times(\vec{b}\times\vec{c})$

$$\vec{b}\times\vec{c} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 1 & -2 \\ 3 & 2 & 1 \end{vmatrix}$$

$= \hat{i}(1+4) - \hat{j}(2+6) + \hat{k}(4-3)$

$= 5\hat{i} - 8\hat{j} + \hat{k}$

$$\vec{a}\times(\vec{b}\times\vec{c}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -2 & 3 \\ 5 & -8 & 1 \end{vmatrix}$$

$= \hat{i}(-2+24) - \hat{j}(1-15) + \hat{k}(-8+10)$

$= 22\hat{i} + 14\hat{j} + 2\hat{k}$

2. Soln:

Let $\vec{a} = x\hat{i} + y\hat{j} + z\hat{k}$

$\hat{i}\times(\vec{a}\times\hat{i}) = (\hat{i}\cdot\hat{i})\vec{a} - (\hat{i}\cdot\vec{a})\hat{i}$

$= \vec{a} - x\hat{i}$

$\hat{j}\times(\vec{a}\times\hat{j}) = (\hat{j}\cdot\hat{j})\vec{a} - (\hat{j}\cdot\vec{a})\hat{j}$

$= \vec{a} - y\hat{j}$

$\hat{k}\times(\vec{a}\times\hat{k}) = (\hat{k}\cdot\hat{k})\vec{a} - (\hat{k}\cdot\vec{a})\hat{k}$

$= \vec{a} - z\hat{k}$

LHS $= \hat{i}\times(\vec{a}\times\hat{i}) + \hat{j}\times(\vec{a}\times\hat{j}) + \hat{k}\times(\vec{a}\times\hat{k})$

$= \vec{a} - x\hat{i} + \vec{a} - y\hat{j} + \vec{a} - z\hat{k}$

$= 3\vec{a} - (x\hat{i} + y\hat{j} + z\hat{k})$

$= 3\vec{a} - \vec{a}$

$= 2\vec{a}$

LHS = RHS.

Hence proved.

3. $[\vec{a}-\vec{b} \quad \vec{b}-\vec{c} \quad \vec{c}-\vec{a}]$

$= (\vec{a}-\vec{b})\cdot[(\vec{b}-\vec{c})\times(\vec{c}-\vec{a})]$

$= (\vec{a}-\vec{b})\cdot[(\vec{b}\times\vec{c}) - (\vec{b}\times\vec{a}) - (\vec{c}\times\vec{c}) + (\vec{c}\times\vec{a})]$

$= (\vec{a}-\vec{b})\cdot[(\vec{b}\times\vec{c}) - (\vec{b}\times\vec{a}) + (\vec{c}\times\vec{a})] \quad \because \vec{c}\times\vec{c} = \vec{0}$

$= \vec{a}\cdot(\vec{b}\times\vec{c}) - \vec{a}\cdot(\vec{b}\times\vec{a}) + \vec{a}\cdot(\vec{c}\times\vec{a}) - \vec{b}\cdot(\vec{b}\times\vec{c}) + \vec{b}\cdot(\vec{b}\times\vec{a}) - \vec{b}\cdot(\vec{c}\times\vec{a})$

$= [\vec{a}\ \vec{b}\ \vec{c}] - [\vec{a}\ \vec{b}\ \vec{a}] + [\vec{a}\ \vec{c}\ \vec{a}] - [\vec{b}\ \vec{b}\ \vec{c}] + [\vec{b}\ \vec{b}\ \vec{a}] - [\vec{b}\ \vec{c}\ \vec{a}]$

$= [\vec{a}\ \vec{b}\ \vec{c}] - 0 + 0 - 0 + 0 - [\vec{a}\ \vec{b}\ \vec{c}]$

$= 0$

$\therefore [\vec{a}-\vec{b} \quad \vec{b}-\vec{c} \quad \vec{c}-\vec{a}] = 0$

Hence proved.

4. Soln:

Given, $\vec{a} = 2\hat{i} + 3\hat{j} - \hat{k}$

$\vec{b} = 3\hat{i} + 5\hat{j} + 2\hat{k}$

$\vec{c} = -\hat{i} - 2\hat{j} + 3\hat{k}$

(i) To verify

$(\vec{a}\times\vec{b})\times\vec{c} = (\vec{a}\cdot\vec{c})\vec{b} - (\vec{b}\cdot\vec{c})\vec{a}$.

$$\vec{a}\times\vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 3 & -1 \\ 3 & 5 & 2 \end{vmatrix}$$

$= \hat{i}(6+5) - \hat{j}(4+3) + \hat{k}(10-9)$

$= 11\hat{i} - 7\hat{j} + \hat{k}$

$$(\vec{a}\times\vec{b})\times\vec{c} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 11 & -7 & 1 \\ -1 & -2 & 3 \end{vmatrix}$$

$= \hat{i}(-21+2) - \hat{j}(33+1) + \hat{k}(-22-7)$

$= -19\hat{i} - 34\hat{j} - 29\hat{k} \longrightarrow (1)$

$(\vec{a}.\vec{c})\vec{b} = (-2-6-3)(3\hat{i}+5\hat{j}+2\hat{k})$

$= -11(3\hat{i}+5\hat{j}+2\hat{k})$

$= -33\hat{i}-55\hat{j}-22\hat{k}$

$(\vec{b}.\vec{c})\vec{a} = (-3-10+6)(2\hat{i}+3\hat{j}-\hat{k})$

$= -7(2\hat{i}+3\hat{j}-\hat{k})$

$= -14\hat{i}-21\hat{j}+7\hat{k}$

$(\vec{a}.\vec{c})\vec{b}-(\vec{b}.\vec{c})\vec{a} = -33\hat{i}-55\hat{j}-22\hat{k}+14\hat{i}+21\hat{j}-7\hat{k}$

$= -19\hat{i}-34\hat{j}-29\hat{k} \longrightarrow (2)$

from (1) and (2), we get

$(\vec{a}\times\vec{b})\times\vec{c} = (\vec{a}.\vec{c})\vec{b}-(\vec{b}.\vec{c})\vec{a}$

Hence Verified.

ii). Soln:

$\vec{b}\times\vec{c} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 5 & 2 \\ -1 & -2 & 3 \end{vmatrix}$

$= \hat{i}(15+4)-\hat{j}(9+2)+\hat{k}(-6+5)$

$= 19\hat{i}-11\hat{j}-\hat{k}$

$\vec{a}\times(\vec{b}\times\vec{c}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 3 & -1 \\ 19 & -11 & -1 \end{vmatrix}$

$= \hat{i}(-3-11)-\hat{j}(-2+19)+\hat{k}(-22-57)$

$= -14\hat{i}-17\hat{j}-79\hat{k} \longrightarrow (1)$

$(\vec{a}.\vec{c})\vec{b} = -33\hat{i}-55\hat{j}-22\hat{k}$

$(\vec{a}.\vec{b})\vec{c} = (6+15-2)(-\hat{i}-2\hat{j}+3\hat{k})$

$= -19\hat{i}-38\hat{j}+57\hat{k}$

$(\vec{a}.\vec{c})\vec{b}-(\vec{a}.\vec{b})\vec{c} = -33\hat{i}-55\hat{j}-22\hat{k}+19\hat{i}+38\hat{j}-57\hat{k}$

$= -14\hat{i}-17\hat{j}-79\hat{k} \longrightarrow (2)$

from (1) and (2), we get

$\vec{a}\times(\vec{b}\times\vec{c}) = (\vec{a}.\vec{c})\vec{b}-(\vec{a}.\vec{b})\vec{c}$

Hence Verified.

(5). Soln:

$\vec{a} = 2\hat{i}+3\hat{j}-\hat{k}$

$\vec{b} = -\hat{i}+2\hat{j}-4\hat{k}$

$\vec{c} = \hat{i}+\hat{j}+\hat{k}$

$(\vec{a}\times\vec{b}).(\vec{a}\times\vec{c}) = \begin{vmatrix} \vec{a}.\vec{a} & \vec{a}.\vec{c} \\ \vec{b}.\vec{a} & \vec{b}.\vec{c} \end{vmatrix}$

$= \begin{vmatrix} 4+9+1 & 2+3-1 \\ -2+6+4 & -1+2-4 \end{vmatrix}$

$= \begin{vmatrix} 14 & 4 \\ 8 & -3 \end{vmatrix}$

$= -42-32$

$= -74$

(6). Soln:

Given $\vec{a},\vec{b},\vec{c}$ and $\vec{d}$ are coplanar vectors.

$\therefore \vec{a},\vec{b},\vec{c}$ are coplanar or $\vec{a},\vec{b},\vec{d}$ are also coplanar.

$\Rightarrow [\vec{a}\ \vec{b}\ \vec{c}] = 0$, $[\vec{a}\ \vec{b}\ \vec{d}] = 0$

$(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d}) = [\vec{a}\ \vec{b}\ \vec{d}]\vec{c}-[\vec{a}\ \vec{b}\ \vec{c}]\vec{d}$

$= (0)\vec{c}-(0)\vec{d}$

$(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d}) = \vec{0}$

Hence proved.

(7). Soln:

Given, $\vec{a} = \hat{i}+2\hat{j}+3\hat{k}$

$\vec{b} = 2\hat{i}-\hat{j}+\hat{k}$

$\vec{c} = 3\hat{i}+2\hat{j}+\hat{k}$

$\vec{a}\times(\vec{b}\times\vec{c}) = (\vec{a}.\vec{c})\vec{b}-(\vec{a}.\vec{b})\vec{c}$

$= (3+4+3)\vec{b}-(2-2+3)\vec{c}$

$= 10\vec{b}-3\vec{c}$

$l\vec{a}+m\vec{b}+n\vec{c} = 10\vec{b}-3\vec{c}$

Equating like terms, we get

$l=0$, $m=10$, $n=-3$

(8). Soln:

Let $\hat{a},\hat{b},\hat{c}$ be the three unit vectors

$\therefore |\hat{a}|=|\hat{b}|=|\hat{c}|=1$

Given $\hat{a}\times(\hat{b}\times\hat{c}) = \frac{1}{2}\hat{b}$

$(\hat{a}.\hat{c})\hat{b}-(\hat{a}.\hat{b})\hat{c} = \frac{1}{2}\hat{b}+0\hat{c}$

Equating like terms.

$\vec{a}.\vec{c} = \frac{1}{2}$ $\quad \vec{a}.\vec{b} = 0$

$|\vec{a}||\vec{c}|\cos\theta = \frac{1}{2}$ $\quad \because |\hat{a}|=1$, $|\hat{c}|=1$

$\cos\theta = \frac{1}{2}$

$\theta = \cos^{-1}(\frac{1}{2})$

$\theta = \frac{\pi}{3}$

$\therefore$ The angle between $\hat{a}$ and $\hat{b}$ is $\frac{\pi}{3}$

Example : 6.19

$[\vec a\times\vec b \quad \vec b\times\vec c \quad \vec c\times\vec a]$

$=\{(\vec a\times\vec b)\times(\vec b\times\vec c)\}\cdot(\vec c\times\vec a)$

$=\{[\vec a\ \vec b\ \vec c]\vec b-[\vec a\ \vec b\ \vec b]\vec c\}\cdot(\vec c\times\vec a)$

$=\{[\vec a\ \vec b\ \vec c]\vec b-0\}\cdot(\vec c\times\vec a)$

$=[\vec a\ \vec b\ \vec c]\ \vec b\cdot(\vec c\times\vec a)$

$=[\vec a\ \vec b\ \vec c][\vec b\ \vec c\ \vec a]$

$=[\vec a\ \vec b\ \vec c][\vec a\ \vec b\ \vec c]$

$=[\vec a\ \vec b\ \vec c]^2$

$\therefore [\vec a\times\vec b \quad \vec b\times\vec c \quad \vec c\times\vec a]=[\vec a\ \vec b\ \vec c]^2$

Hence proved.

Eg : 6.20

$(\vec a\times\vec b)\times(\vec a\times\vec c)=[\vec a\ \vec b\ \vec c]\vec a-[\vec a\ \vec b\ \vec a]\vec c$

$=[\vec a\ \vec b\ \vec c]\vec a-0$

$=(\vec a\cdot(\vec b\times\vec c))\vec a$

Hence proved.

Eg : 6.21

Soln: Let $\vec p=\vec a\times\vec b$

$(\vec a\times\vec b)\times(\vec c\times\vec d)=\vec p\times(\vec c\times\vec d)$

$=(\vec p\cdot\vec d)\vec c-(\vec p\cdot\vec c)\vec d$

$=((\vec a\times\vec b)\cdot\vec d)\vec c-((\vec a\times\vec b)\cdot\vec c)\vec d$

$=[\vec a\ \vec b\ \vec d]\vec c-[\vec a\ \vec b\ \vec c]\vec d$

Let $\vec q=\vec c\times\vec d$.

$(\vec a\times\vec b)\times(\vec c\times\vec d)=(\vec a\times\vec b)\times\vec q$

$=(\vec a\cdot\vec q)\vec b-(\vec b\cdot\vec q)\vec a$

$=(\vec a\cdot(\vec c\times\vec d))\vec b-(\vec b\cdot(\vec c\times\vec d))\vec a$

$=[\vec a\ \vec c\ \vec d]\vec b-[\vec b\ \vec c\ \vec d]\vec a$

$\therefore (\vec a\times\vec b)\times(\vec a\times\vec c)=[\vec a\ \vec b\ \vec d]\vec c-[\vec a\ \vec b\ \vec c]\vec d=[\vec a\ \vec c\ \vec d]b-[\vec b\ \vec c\ \vec d]\vec a$

Eg : 6.22

Given, $\vec a=-2\hat i+3\hat j-2\hat k$

$\vec b=3\hat i-\hat j+3\hat k$

$\vec c=2\hat i-5\hat j+\hat k$

$\vec a\times\vec b=\begin{vmatrix}\hat i&\hat j&\hat k\\-2&3&-2\\3&-1&3\end{vmatrix}$

$=\hat i(9-2)-\hat j(-6+6)+\hat k(2-9)$

$=7\hat i-7\hat j$

$(\vec a\times\vec b)\times\vec c=\begin{vmatrix}\hat i&\hat j&\hat k\\7&0&-7\\2&-5&1\end{vmatrix}$

$=\hat i(0-35)-\hat j(7+14)+\hat k(-35-0)$

$=-35\hat i-21\hat j-35\hat k \longrightarrow (1)$

$\vec b\times\vec c=\begin{vmatrix}\hat i&\hat j&\hat k\\3&-1&3\\2&-5&1\end{vmatrix}$

$=\hat i(-1+15)-\hat j(3-6)+\hat k(-15+2)$

$=14\hat i+3\hat j-13\hat k$

$\vec a\times(\vec b\times\vec c)=\begin{vmatrix}\hat i&\hat j&\hat k\\-2&3&-2\\14&3&-13\end{vmatrix}$

$=\hat i(-39+6)-\hat j(26+28)+\hat k(-6-42)$

$=-33\hat i-54\hat j-48\hat k \longrightarrow (2)$

from (1) & (2) we get

$(\vec a\times\vec b)\times\vec c\neq\vec a\times(\vec b\times\vec c)$

Eg : 6.23

Soln:

Given, $\vec a=\hat i-\hat j$

$\vec b=\hat i-\hat j-4\hat k$

$\vec c=3\hat j-\hat k$

$\vec d=2\hat i+5\hat j+\hat k$

(i) $\vec a\times\vec b=\begin{vmatrix}\hat i&\hat j&\hat k\\1&-1&0\\1&-1&-4\end{vmatrix}$

$=\hat i(4-0)-\hat j(-4-0)+\hat k(-1+1)$

$=4\hat i+4\hat j$

$\vec c\times\vec d=\begin{vmatrix}\hat i&\hat j&\hat k\\0&3&-1\\2&5&1\end{vmatrix}$

$=\hat i(3+5)-\hat j(0+2)+\hat k(0+6)$

$=8\hat i-2\hat j-6\hat k$

$(\vec a\times\vec b)\times(\vec c\times\vec d)=\begin{vmatrix}\hat i&\hat j&\hat k\\4&4&0\\8&-2&-6\end{vmatrix}$

$=\hat i(-24+0)-\hat j(-24-0)+\hat k(-8-32)$

$=-24\hat i+24\hat j-40\hat k \longrightarrow (1)$

$[\vec a\ \vec b\ \vec d]=\begin{vmatrix}1&-1&0\\1&-1&-4\\2&5&1\end{vmatrix}$

$=1(-1+20)+1(1+8)+0$

$=19+9=28$

$$[\vec{a}\ \vec{b}\ \vec{c}] = \begin{vmatrix} 1 & -1 & 0 \\ 1 & -1 & -4 \\ 0 & 3 & -1 \end{vmatrix}$$

$$= 1(1+12)+1(-1-0)+0$$
$$= 13-1$$
$$= 12$$

$$[\vec{a}\ \vec{b}\ \vec{d}]\vec{c} - [\vec{a}\ \vec{b}\ \vec{c}]\vec{d} = 28(3\hat{j}-\hat{k}) - 12(2\hat{i}+5\hat{j}+\hat{k})$$
$$= 84\hat{j} - 28\hat{k} - 24\hat{i} - 60\hat{j} - 12\hat{k}$$
$$= -24\hat{i} + 24\hat{j} - 40\hat{k} \longrightarrow ②$$

from ① & ② we get

$$(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d}) = [\vec{a}\ \vec{b}\ \vec{d}]\vec{c} - [\vec{a}\ \vec{b}\ \vec{c}]\vec{d}$$

Hence verified.

(ii)

$$\vec{a}\times\vec{b} = 4\hat{i}+4\hat{j} \qquad \text{Refer (i)}$$
$$\vec{c}\times\vec{d} = 8\hat{i}-2\hat{j}-6\hat{k}$$
$$(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d}) = -24\hat{i}+24\hat{j}-40\hat{k}$$

$$[\vec{a}\ \vec{c}\ \vec{d}] = \begin{vmatrix} 1 & -1 & 0 \\ 0 & 3 & -1 \\ 2 & 5 & 1 \end{vmatrix}$$
$$= 1(3+5)+1(0+2)-0$$
$$= 10$$

$$[\vec{b}\ \vec{c}\ \vec{d}] = \begin{vmatrix} 1 & -1 & -4 \\ 0 & 3 & -1 \\ 2 & 5 & 1 \end{vmatrix}$$
$$= 1(3+5)+1(0+2)-4(0-6)$$
$$= 8+2+24$$
$$= 34$$

$$[\vec{a}\ \vec{c}\ \vec{d}]\vec{b} - [\vec{b}\ \vec{c}\ \vec{d}]\vec{a} = 10(\hat{i}-\hat{j}-4\hat{k}) - 34(\hat{i}-\hat{j})$$
$$= 10\hat{i} - 10\hat{j} - 40\hat{k} - 34\hat{i} + 34\hat{j}$$
$$= -24\hat{i}+24\hat{j}-40\hat{k} \longrightarrow ②$$

from ① & ②, we get

$$(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d}) = [\vec{a}\ \vec{c}\ \vec{d}]\vec{b} - [\vec{b}\ \vec{c}\ \vec{d}]\vec{a}$$

Hence Verified.

Example : 6.32

Soln: Let $\vec{b} = 4\hat{i}-6\hat{j}+12\hat{k}$

$$\vec{d} = -2\hat{i}+3\hat{j}-6\hat{k}.$$
$$\vec{b} = -2(2\hat{i}+3\hat{j}-6\hat{k})$$
$$\vec{b} = -2\vec{d}.$$
$$\therefore \vec{b} \parallel \vec{d}$$

∴ The given straight lines are parallel. Hence proved.

EXERCISE 6.4

A point on the straight line and the direction of the straight line are given,

(a). Parametric form:

$$\vec{r} = \vec{a}+t\vec{b}$$

(b). Non - parametric form:

$$(\vec{r}-\vec{a})\times\vec{b} = \vec{0}$$

(c). Cartesian form:

$$\frac{x-x_1}{b_1} = \frac{y-y_1}{b_2} = \frac{z-z_1}{b_3}$$

Straight line passing through two given points:

(a). Parametric form:

$$\vec{r} = \vec{a}+t(\vec{b}-\vec{a}),\ t\in R$$

(b). Non - parametric form:

$$(\vec{r}-\vec{a})\times(\vec{b}-\vec{a}) = \vec{0}$$

(c). Cartesian form:

$$\frac{x-x_1}{x_2-x_1} = \frac{y-y_1}{y_2-y_1} = \frac{z-z_1}{z_2-z_1}$$

Angle between two straight line

(a) Vector form

$$\vec{r} = \vec{a}+s\vec{b}, \quad \vec{r} = \vec{c}+t\vec{d}$$

$$\theta = \cos^{-1}\left[\frac{|\vec{b}\cdot\vec{d}|}{|\vec{b}||\vec{d}|}\right]$$

(b). cartesian form:

$$\frac{x-x_1}{b_1} = \frac{y-y_1}{b_2} = \frac{z-z_1}{b_3} \text{ and}$$

$$\frac{x-x_1}{d_1} = \frac{y-y_1}{d_2} = \frac{z-z_2}{d_3}$$

$$\therefore \theta = \cos^{-1}\left[\frac{|b_1d_1+b_2d_2+b_3d_3|}{\sqrt{b_1^2+b_2^2+b_3^2}\sqrt{d_1^2+d_2^2+d_3^2}}\right]$$

Ex: 6.4 - ①

Soln: Let $\vec{a} = 4\hat{i}+3\hat{j}-7\hat{k}$

$\vec{b} = 2\hat{i}-6\hat{j}+7\hat{k}$

Non parametric form of vector eqn:

$(\vec{r}-\vec{a})\times\vec{b} = \vec{0}$

$\left(\vec{r}-(4\hat{i}+3\hat{j}-7\hat{k})\right)\times(2\hat{i}-6\hat{j}+7\hat{k}) = \vec{0}$

cartesian form:

$(x_1, y_1, z_1) = (4, 3, -7)$

$(b_1, b_2, b_3) = (2, -6, 7)$

$$\frac{x-x_1}{b_1} = \frac{y-y_1}{b_2} = \frac{z-z_1}{b_3}$$

$$\frac{x-4}{2} = \frac{y-3}{-6} = \frac{z+7}{7}$$

②

The given eqn can be written as

$$\frac{x-1}{-4} = \frac{y+3}{5} = \frac{z-8}{-6}$$

Soln: Let $\vec{a} = -2\hat{i}+3\hat{j}+4\hat{k}$

$\vec{b} = -4\hat{i}+5\hat{j}-6\hat{k}$

Parametric form of vector eqn:

$\vec{r} = \vec{a}+t\vec{b}$

$\vec{r} = (-2\hat{i}+3\hat{j}+4\hat{k})+t(-4\hat{i}+5\hat{j}-6\hat{k})$

Cartesian form:

$(x_1, y_1, z_1) = (-2, 3, 4)$

$(b_1, b_2, b_3) = (-4, 5, -6)$

$$\frac{x+2}{-4} = \frac{y-3}{5} = \frac{z-4}{-6}$$

③

Soln:

The direction ratios of the straight line joining two points are $(2, -3, 5)$

Cartesian equation:

$$\frac{x-6}{2} = \frac{y-7}{-3} = \frac{z-4}{5} \quad \text{(or)}$$

$$\frac{x-8}{2} = \frac{y-4}{-3} = \frac{z-9}{5}$$

The straight line cuts the xz plane.

ie, $y=0$

$$\frac{x-6}{2} = \frac{7}{3} = \frac{z-4}{5}$$

$\frac{x-6}{2} = \frac{7}{3}$

$x-6 = \frac{14}{3}$

$x = \frac{14}{3}+6$

$x = \frac{32}{3}$

$\frac{z-4}{5} = \frac{7}{3}$

$z-4 = \frac{35}{3}$

$z = \frac{35}{3}+4$

$= \frac{47}{3}$

$\therefore$ The required point is $\left(\frac{32}{3}, 0, \frac{47}{3}\right)$

(ii) The straight line cuts yz plane:

ie, $x=0$

$$-3 = \frac{y-7}{-3} = \frac{z-4}{5}$$

$\frac{y-7}{-3} = -3$

$y-7 = 9$

$y = 16$

$\frac{z-4}{5} = -3$

$z-4 = -15$

$z = -11$

$\therefore$ The required point is $(0, 16, -11)$

④. The straight line passes through the points $(5, 6, 7)$ and $(7, 9, 13)$

The direction ratio's are 2, 3, 6

Let $\vec{r} = 2\hat{i}+3\hat{j}+6\hat{k}$

$|\vec{r}| = \sqrt{4+9+36}$

$= 7$

d.c.s $= \left\{\frac{2}{7}, \frac{3}{7}, \frac{6}{7}\right\}$

Parametric form:

Let $\vec{a} = 5\hat{i}+6\hat{j}+7\hat{k}$

$\vec{b} = 7\hat{i}+9\hat{j}+13\hat{k}$

$\vec{r} = \vec{a}+t(\vec{b}-\vec{a})$

$\vec{r} = (5\hat{i}+6\hat{j}+7\hat{k})+t(2\hat{i}+3\hat{j}+6\hat{k})$

cartesian form:

$(x_1, y_1, z_1) = (5, 6, 7)$

$(x_2, y_2, z_2) = (7, 9, 13)$

$$\frac{x-x_1}{x_2-x_1} = \frac{y-y_1}{y_2-y_1} = \frac{z-z_1}{z_2-z_1}$$

$$\frac{x-5}{2} = \frac{y-6}{3} = \frac{z-7}{6}$$

(5) (i) Soln:

Let $\vec{b} = \hat{i}+2\hat{j}-2\hat{k}$

$\vec{d} = -\hat{i}-2\hat{j}+2\hat{k}$

$$\theta = \cos^{-1}\left[\frac{|\vec{b}\cdot\vec{d}|}{|\vec{b}||\vec{d}|}\right]$$

$$= \cos^{-1}\left[\frac{|-1-4-4|}{\sqrt{1+4+4}\sqrt{1+4+4}}\right]$$

$$= \cos^{-1}\left[\frac{9}{9}\right]$$

$$= \cos^{-1}(1)$$

$$\theta = 0^\circ$$

ii) Let $\vec{b} = 3\hat{i}+4\hat{j}+5\hat{k}$

$\vec{d} = 2\hat{i}+\hat{j}+\hat{k}$

$$\theta = \cos^{-1}\left[\frac{|\vec{b}\cdot\vec{d}|}{|\vec{b}||\vec{d}|}\right]$$

$$= \cos^{-1}\left[\frac{6+4+5}{\sqrt{9+16+25}\sqrt{4+1+1}}\right]$$

$$= \cos^{-1}\left[\frac{15}{\sqrt{50}\sqrt{6}}\right]$$

$$= \cos^{-1}\frac{15}{5\sqrt{2}\sqrt{6}}$$

$$= \cos^{-1}\left(\frac{3}{\sqrt{12}}\right)$$

$$\theta = \cos^{-1}\left[\frac{3}{2\sqrt{3}}\right]$$

iii) $2x = 3y = -z$

$$\frac{x}{1/2} = \frac{y}{1/3} = \frac{z}{-1}$$

$$\frac{x-0}{(1/2)} = \frac{y-0}{(1/3)} = \frac{z-0}{-1}$$

$$\vec{b} = \frac{1}{2}\hat{i}+\frac{1}{3}\hat{j}-\hat{k}$$

$6x = -y = -4z$

$$\frac{x-0}{(1/6)} = \frac{y-0}{-1} = \frac{z-0}{(-1/4)}$$

$$\vec{d} = \frac{1}{6}\hat{i}-\hat{j}-\frac{1}{4}\hat{k}$$

$$\theta = \cos^{-1}\left[\frac{|\vec{b}\cdot\vec{d}|}{|\vec{b}||\vec{d}|}\right]$$

$$= \cos^{-1}\left[\frac{\frac{1}{12}-\frac{1}{3}+\frac{1}{4}}{\sqrt{\frac{1}{4}+\frac{1}{9}+1}\sqrt{\frac{1}{36}+1+\frac{1}{16}}}\right]$$

$$= \cos^{-1}(0)$$

$$\frac{1}{12}-\frac{1}{3}+\frac{1}{4} = \frac{1-4+3}{12} = 0$$

$$\theta = \pi/2$$

(6) Soln:

Let $\vec{OA} = 7\hat{i}+2\hat{j}+\hat{k}$

$\vec{OB} = 6\hat{i}+3\hat{k}$

$\vec{OC} = 4\hat{i}+2\hat{j}+4\hat{k}$

$\vec{AB} = \vec{OB}-\vec{OA}$

$= -\hat{i}-2\hat{j}+2\hat{k}$

$\vec{BC} = \vec{OC}-\vec{OB}$

$= -2\hat{i}+2\hat{j}+\hat{k}$

$\vec{AB}\cdot\vec{BC} = 2-4+2$

$\vec{AB}\cdot\vec{BC} = 0$

$\therefore \vec{AB} \perp^{er} \vec{BC}$

$\therefore \angle ABC = \pi/2$

(7) Soln:

Let $\vec{OA} = 2\hat{i}+\hat{j}+4\hat{k}$ $\vec{OC} = 2\hat{j}+(b-1)\hat{k}$

$\vec{OB} = (a-1)\hat{i}+4\hat{j}-\hat{k}$ $\vec{OD} = 5\hat{i}+3\hat{j}-2\hat{k}$

$\vec{AB} = \vec{OB}-\vec{OA}$

$= (a-3)\hat{i}+3\hat{j}-5\hat{k}$

$\vec{CD} = \vec{OD}-\vec{OC}$

$= 5\hat{i}+\hat{j}-(b+1)\hat{k}$

Given, $\vec{AB}$ is $\parallel^{el}$ to $\vec{CD}$

$\therefore 3\vec{CD} = 15\hat{i}+3\hat{j}-3(b+1)\hat{k}$

($\because \hat{j}$ component are same)

Now, $\vec{AB} = 3\vec{CD}$

$(a-3)\hat{i}+3\hat{j}-5\hat{k} = 15\hat{i}+3\hat{j}-3(b+1)\hat{k}$

Equating like terms,

$a-3=15$	$5 = 3(b+1)$
$a=18$	$= 3b+3$
	$3b = 2$
	$b = 2/3$

$\therefore$ The value of $a=18$ and $b=2/3$.

(8) Soln:

The given eqn can be written as

$\frac{x-5}{5m+2} = \frac{y-2}{-5} = \frac{z-1}{1}$ and $\frac{x}{1} = \frac{y+\frac{1}{2}}{2m} = \frac{z-1}{3}$

Let $\vec{b} = (5m+2)\hat{i} - 5\hat{j} + \hat{k}$

$\vec{d} = \hat{i} + 2m\hat{j} + 3\hat{k}$

Given, The straight lines are perendicular to each other.

$\therefore \vec{b}\cdot\vec{d} = 0$

$(5m+2) - 10m + 3 = 0$

$-5m + 5 = 0$

$5m = 5$

$m = 1$

(9). Soln:

Let $\vec{OA} = 2\hat{i} + 3\hat{j} + 4\hat{k}$

$\vec{OB} = -\hat{i} + 4\hat{j} + 5\hat{k}$

$\vec{OC} = 8\hat{i} + \hat{j} + 2\hat{k}$

$\vec{AB} = \vec{OB} - \vec{OC}$

$= -3\hat{i} + \hat{j} + \hat{k}$

$\vec{AC} = \vec{OC} - \vec{OA}$

$= 6\hat{i} - 2\hat{j} - 2\hat{k}$

$\vec{AB} = -2(-3\hat{i} + \hat{j} + \hat{k})$

$\vec{AC} = -2(\vec{AB})$

$\therefore \vec{AB} \parallel \vec{AC}$

A is a common point

$\therefore$ The given points are collinear.

Example 6.28

Soln: Let $\vec{b} = 2\hat{i} + 2\hat{j} - \hat{k}$

If $\hat{b}$ is a unit vector parallel to the given line,

$\hat{b} = \frac{\vec{b}}{|\vec{b}|}$

$= \frac{2\hat{i} + 2\hat{j} - \hat{k}}{\sqrt{4+4+1}}$

$= \frac{2\hat{i} + 2\hat{j} - \hat{k}}{3}$

The definition of D.c.s of $\hat{b}$

$\cos\alpha = \frac{2}{3}$ $\cos\beta = \frac{2}{3}$ $\cos\gamma = -\frac{1}{3}$

$\alpha = \cos^{-1}\left(\frac{2}{3}\right),\ \beta = \cos^{-1}\left(\frac{2}{3}\right),\ \gamma = \cos^{-1}\left(-\frac{1}{3}\right)$

Example : 6.29.

Soln: Let $\vec{b} = 2\hat{i} + 2\hat{j} + \hat{k}$

Direction ratios are $(4, 1, 8)$

$\therefore \vec{d} = 4\hat{i} + \hat{j} + 8\hat{k}$

$\theta = \cos^{-1}\frac{|\vec{b}\cdot\vec{d}|}{|\vec{b}||\vec{d}|}$

$= \cos^{-1}\left[\frac{8+2+8}{\sqrt{4+4+1}\sqrt{16+1+64}}\right]$

$= \cos^{-1}\left[\frac{18}{\sqrt{9}\sqrt{81}}\right]$

$= \cos^{-1}\left[\frac{18}{3\times 9}\right]$

$\theta = \cos^{-1}\left(\frac{2}{3}\right)$

Example : 6.30

Soln: Let $\vec{b} = 2\hat{i} + \hat{j} - 2\hat{k}$

$\vec{d} = 4\hat{i} - 4\hat{j} + 2\hat{k}$

The angle between the straight line is

$\theta = \cos^{-1}\left[\frac{|\vec{b}\cdot\vec{d}|}{|\vec{b}||\vec{d}|}\right]$

$= \cos^{-1}\left[\frac{8-4-4}{\sqrt{4+1+4}\sqrt{16+16+4}}\right]$

$\theta = \cos^{-1}(0)$

$\theta = \pi/2$

Example : 6.31

Soln:

The straight line passing through the pts $A(6,7,5)$ and $B(8,10,6)$

$\vec{OA} = 6\hat{i} + 7\hat{j} + 5\hat{k}$

$\vec{OB} = 8\hat{i} + 10\hat{j} + 6\hat{k}$

$\vec{AB} = \vec{OB} - \vec{OA}$

$= 2\hat{i} + 3\hat{j} + \hat{k}$

The straight line passing through the pts $C(10,2,-5)$ and $D(8,3,-4)$

$\vec{OC} = 10\hat{i} + 2\hat{j} - 5\hat{k}$

$\vec{OD} = 8\hat{i} + 3\hat{j} - 4\hat{k}$

$\vec{CD} = \vec{OD} - \vec{OC}$

$= -2\hat{i} + \hat{j} + \hat{k}$

Now, $\vec{AB}\cdot\vec{CD} = -4 + 3 + 1$

$= 0$

$\vec{AB} \perp \vec{CD}$.

Hence, The two straight lines are perpendicular.

Hence proved.

Example : 6.24

Soln:

Let $\vec{a} = \hat{i}+2\hat{j}-3\hat{k}$

$\vec{b} = 4\hat{i}+5\hat{j}-7\hat{k}$

(i) Parametric form:

$$\vec{r} = \vec{a} + t\vec{b}, \quad t \in R$$

$$\vec{r} = (\hat{i}+2\hat{j}-3\hat{k}) + t(4\hat{i}+5\hat{j}-7\hat{k})$$

(ii) Non- Parametric form:

$$(\vec{r}-\vec{a})\times\vec{b} = \vec{0}$$

$$(\vec{r}-(\hat{i}+2\hat{j}-3\hat{k}))\times(4\hat{i}+5\hat{j}-7\hat{k}) = \vec{0}$$

iii). Cartesian form:

$(x_1, y_1, z_1) = (1, 2, -3)$

$(b_1, b_2, b_3) = (4, 5, -7)$

$$\frac{x-x_1}{b_1} = \frac{y-y_1}{b_2} = \frac{z-z_1}{b_3}$$

$$\frac{x-1}{4} = \frac{y-2}{5} = \frac{z+3}{-7}$$

Example : 6.25

Soln:

Given, Parametric form of a line is $\vec{r} = (3\hat{i}-2\hat{j}+6\hat{k}) + t(2\hat{i}-\hat{j}+3\hat{k})$

Let $\vec{a} = 3\hat{i}-2\hat{j}+6\hat{k}$

$\vec{b} = 2\hat{i}-\hat{j}+3\hat{k}$

(i) Direction ratio are $(2, -1, 3)$

Let $\vec{b} = 2\hat{i}-\hat{j}+3\hat{k}$

$|\vec{b}| = \sqrt{4+1+9} = \sqrt{14}$

$$\text{D.c.s} = \left\{\frac{2}{\sqrt{14}}, \frac{-1}{\sqrt{14}}, \frac{3}{\sqrt{14}}\right\}$$

(ii). Non- Parametric form:

$$(\vec{r}-\vec{a})\times\vec{b} = \vec{0}$$

$$(\vec{r}-(3\hat{i}-2\hat{j}+6\hat{k}))\times(2\hat{i}-\hat{j}+3\hat{k}) = \vec{0}$$

(iii). Cartesian form:

$(x_1, y_1, z_1) = (3, -2, 6)$

$(b_1, b_2, b_3) = (2, -1, 3)$

$$\frac{x-3}{2} = \frac{y+2}{-1} = \frac{z-6}{3}$$

Example : 6.26

The eqn can be written as $\frac{x+2}{-4} = \frac{y+3}{-2} = \frac{z-3}{(3/2)}$

Soln: Let $\vec{a} = -4\hat{i}+2\hat{j}-3\hat{k}$

$\vec{b} = -4\hat{i}-2\hat{j}+\frac{3}{2}\hat{k}$

Parametric form:

$$\vec{r} = \vec{a} + t\vec{b}, \quad t \in R$$

$$\vec{r} = (-4\hat{i}+2\hat{j}-3\hat{k}) + t(-4\hat{i}-2\hat{j}+\frac{3}{2}\hat{k})$$

$$\vec{r} = (-4\hat{i}+2\hat{j}-3\hat{k}) - \frac{1}{2}t(8\hat{i}+4\hat{j}-3\hat{k})$$

$$\vec{r} = (-4\hat{i}+2\hat{j}-3\hat{k}) + t(8\hat{i}+4\hat{j}-3\hat{k})$$

$\because \vec{b}$ is $\parallel^{el}$ to $8\hat{i}+4\hat{j}-3\hat{k}$

Cartesian form:

$(x_1, y_1, z_1) = (-4, 2, -3)$

$(b_1, b_2, b_3) = (8, 4, -3)$

$$\frac{x+4}{8} = \frac{y-2}{4} = \frac{z+3}{-3}$$

Example : 6.27

Soln:

The straight line passes through the points $(-5, 7, -4)$ and $(13, -5, 2)$

Let $\vec{a} = -5\hat{i}+7\hat{j}-4\hat{k}$

$\vec{b} = 13\hat{i}-5\hat{j}+2\hat{k}$

Parametric form:

$$\vec{r} = \vec{a} + t(\vec{b}-\vec{a})$$

$$= (-5\hat{i}+7\hat{j}-4\hat{k}) + t(18\hat{i}-12\hat{j}+6\hat{k})$$

Cartesian form:

$(x_1, y_1, z_1) = (-5, 7, -4)$

$(x_2, y_2, z_2) = (13, -5, 2)$

$$\frac{x-x_1}{x_2-x_1} = \frac{y-y_1}{y_2-y_1} = \frac{z-z_1}{z_2-z_1}$$

$$\frac{x+5}{18} = \frac{y-7}{-12} = \frac{z+4}{6}$$

The line meet at xy-plane

ie, $z = 0$.

$$\frac{x+5}{18} = \frac{y-7}{-12} = \frac{4}{6}$$

$\frac{x+5}{18} = \frac{4}{6}$ $\qquad$ $\frac{y-7}{-12} = \frac{4}{6}$

$x+5 = \frac{72}{6}$ $\qquad$ $y-7 = -48/6$

$x+5 = 12$ $\qquad$ $y-7 = -8$

$x = 7$ $\qquad$ $y = -1$

$\therefore$ The required pt $(7, -1, 0)$

Ex 6.5

1) let $\vec{a} = 5\hat{i}+2\hat{j}+8\hat{k}$

$\vec{b} = 2\hat{i}+\hat{j}-2\hat{k}$, $\vec{d} = \hat{i}+2\hat{j}+2\hat{k}$

$$\vec{b}\times\vec{d} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -2 & 1 \\ 1 & 2 & 2 \end{vmatrix}$$

$= \hat{i}(-4-2)-\hat{j}(4-1)+\hat{k}(4+2)$

$= -6\hat{i}-3\hat{j}+6\hat{k}$

$\vec{b}\times\vec{d} = -3(2\hat{i}+\hat{j}-2\hat{k})$

Here let $\vec{c} = 2\hat{i}+\hat{j}-2\hat{k}$

VECTOR EQUATION:

$\vec{r} = \vec{a}+t\vec{c}$

$\vec{r} = (5\hat{i}+2\hat{j}+8\hat{k})+t(2\hat{i}+\hat{j}-2\hat{k})$

CARTESIAN EQUATION:

$$\frac{x-x_1}{c_1} = \frac{y-y_1}{c_2} = \frac{z-z_1}{c_3}$$

$$\Rightarrow \frac{x-5}{2} = \frac{y-2}{1} = \frac{z-8}{-2}$$

2) Soln:

let $\vec{a} = 6\hat{i}+\hat{j}+2\hat{k}$, $\vec{b} = \hat{i}+2\hat{j}-3\hat{k}$

$\vec{c} = 3\hat{i}+2\hat{j}-2\hat{k}$, $\vec{d} = 2\hat{i}+4\hat{j}-5\hat{k}$

$\vec{c}-\vec{a} = -3\hat{i}+\hat{j}-4\hat{k}$

$$[(\vec{c}-\vec{a})\ \vec{b}\ \vec{d}] = \begin{vmatrix} -3 & 1 & -4 \\ 1 & 2 & -3 \\ 2 & 4 & -5 \end{vmatrix}$$

$= -3(-10+12)-1(-5+6)+4(4-4)$

$= -6-1-0 = -7 \neq 0$

$\therefore$ Given lines are skew lines.

$$\vec{b}\times\vec{d} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & -3 \\ 2 & 4 & -5 \end{vmatrix}$$

$= \hat{i}(-10+12)-\hat{j}(-5+6)+\hat{k}(4-4)$

$\vec{b}\times\vec{d} = 2\hat{i}-\hat{j}$

$|\vec{b}\times\vec{d}| = \sqrt{4+1} = \sqrt{5}$

The shortest distance between the skew lines

$$d = \frac{|[(\vec{c}-\vec{a})\ \vec{b}\ \vec{d}]|}{|\vec{b}\times\vec{d}|}$$

$$= \frac{|-7|}{\sqrt{5}}$$

$d = 7/\sqrt{5}$ units

3) condition is

$$\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ b_1 & b_2 & b_3 \\ d_1 & d_2 & d_3 \end{vmatrix} = 0$$

Here $(x_1, y_1, z_1) = (1, -1, 1)$

$(x_2, y_2, z_2) = (3, m, 0)$

$(b_1, b_2, b_3) = (2, 3, 4)$

$(d_1, d_2, d_3) = (1, 2, 1)$

$$\therefore \begin{vmatrix} 2 & m+1 & -1 \\ 2 & 3 & 4 \\ 1 & 2 & 1 \end{vmatrix} = 0$$

$\Rightarrow 2(3-8)-(m+1)(2-4)-1(4-3)=0$

$2(-5)-(m+1)(-2)-1(1)=0$

$-10+2m+2-1=0$

$2m-9=0$

$2m=9$

$m = 9/2$

4) Solution.

$\frac{x-3}{3} = \frac{y-3}{-1}$, $z-1=0 \Rightarrow z=1$

$\frac{x-6}{2} = \frac{z-1}{3}$, $y-2=0 \Rightarrow y=2$

Here $(x_1, y_1, z_1) = (3, 3, 1)$

$(x_2, y_2, z_2) = (6, 2, 1)$

$(b_1, b_2, b_3) = (3, -1, 0)$

$(d_1, d_2, d_3) = (2, 0, 3)$

Condition for intersecting

$$\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ b_1 & b_2 & b_3 \\ d_1 & d_2 & d_3 \end{vmatrix} = 0$$

$$\therefore \begin{vmatrix} 3 & -1 & 0 \\ 3 & -1 & 0 \\ 2 & 0 & 3 \end{vmatrix} = 0 \quad (R_1 \equiv R_2)$$

$\therefore$ The given lines are intersecting

Let $\frac{x-3}{3} = \frac{y-3}{-1} = \lambda$ and $z=1$

Any point on this line is

$(3\lambda+3, -\lambda+3, 1)$

Let $\frac{x-6}{2} = \frac{z-1}{3} = \mu$ and $y=2$.

Any point on this line is

$(2\mu+6, 2, 3\mu+1)$

For some λ, μ

$(3\lambda+3, -\lambda+3, 1) = (2\mu+6, 2, 3\mu+1)$

$\therefore -\lambda+3 = 2$

$-\lambda = -1 \Rightarrow \lambda = 1$

and $3\mu+1=1$

$3\mu=0 \Rightarrow \mu=0$

∴ The req. point of intersection is $(6,2,1)$.

5) $x+1=2y=-12z$ | $x=y+2=6z-6$

$\frac{x+1}{1}=\frac{y}{\frac{1}{2}}=\frac{z}{-\frac{1}{12}}$ | $\frac{x}{1}=\frac{y+2}{1}=\frac{z-1}{\frac{1}{6}}$

Let $(x_1,y_1,z_1)=(-1,0,0)$

$(x_2,y_2,z_2)=(0,-2,1)$

$(b_1,b_2,b_3)=(1,\frac{1}{2},-\frac{1}{12})$

$(d_1,d_2,d_3)=(1,1,\frac{1}{6})$

$$\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ b_1 & b_2 & b_3 \\ d_1 & d_2 & d_3 \end{vmatrix} = \begin{vmatrix} +1 & -2 & 1 \\ 1 & \frac{1}{2} & -\frac{1}{12} \\ 1 & 1 & \frac{1}{6} \end{vmatrix}$$

$$=1\left[\frac{1}{12}+\frac{1}{12}\right]+2\left[\frac{1}{6}+\frac{1}{12}\right]+1\left[1-\frac{1}{2}\right]$$

$$=\frac{2}{12}+2\left(\frac{2+1}{12}\right)+1\left(\frac{1}{2}\right)$$

$$=\frac{1}{6}+\frac{3}{6}+\frac{1}{2}$$

$$=\frac{1+3+3}{6}=\frac{7}{6}\neq 0$$

∴ The given lines are skew lines.

$$\vec{b}\times\vec{d}=\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & \frac{1}{2} & -\frac{1}{12} \\ 1 & 1 & \frac{1}{6} \end{vmatrix}$$

$$=\hat{i}\left(\frac{1}{12}+\frac{1}{12}\right)-\hat{j}\left(\frac{1}{6}+\frac{1}{12}\right)+\hat{k}\left(1-\frac{1}{2}\right)$$

$$=\hat{i}\left(\frac{2}{12}\right)-\hat{j}\left(\frac{2+1}{12}\right)+\hat{k}\left(\frac{1}{2}\right)$$

$$\vec{b}\times\vec{d}=\frac{1}{6}\hat{i}-\frac{1}{4}\hat{j}+\frac{1}{2}\hat{k}$$

$$|\vec{b}\times\vec{d}|=\sqrt{\frac{1}{36}+\frac{1}{16}+\frac{1}{4}}=\sqrt{\frac{4+9+36}{144}}$$

$$=\sqrt{\frac{49}{144}}=\frac{7}{12}.$$

The shortest distance b/w skew lines

$$d=\frac{|[(\vec{c}-\vec{a})\ \vec{b}\ \vec{d}]|}{|\vec{b}\times\vec{d}|}$$

$$=\frac{7/6}{7/12}=\frac{7}{6}\times\frac{12}{7}=2\text{ units}$$

$d=2$ units.

6) Let $\vec{a}=-\hat{i}+2\hat{j}+\hat{k}$,

$\vec{b}=\hat{i}-2\hat{j}+\hat{k}$

∴ $\vec{r}=\vec{a}+t\vec{b}$

$=(-\hat{i}+2\hat{j}+\hat{k})+t(\hat{i}-2\hat{j}+\hat{k})$

parallel to the st. line

$\vec{r}=(2\hat{i}+3\hat{j}-\hat{k})+t(\hat{i}-2\hat{j}+\hat{k})$

Here $\vec{c}=2\hat{i}+3\hat{j}-\hat{k}$

$\vec{c}-\vec{a}=3\hat{i}+\hat{j}-2\hat{k}$

$$\therefore (\vec{c}-\vec{a})\times\vec{b}=\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 1 & -2 \\ 1 & -2 & 1 \end{vmatrix}$$

$$=\hat{i}(1-4)-\hat{j}(3+2)+\hat{k}(-6-1)$$

$$=-3\hat{i}-5\hat{j}-7\hat{k}$$

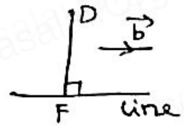
$|(\vec{c}-\vec{a})\times\vec{b}|=\sqrt{9+25+49}=\sqrt{83}$

$|\vec{b}|=\sqrt{1+4+1}=\sqrt{6}$

∴ The distance b/w the parallel lines

$$d=\frac{|(\vec{c}-\vec{a})\times\vec{b}|}{|\vec{b}|}=\frac{\sqrt{83}}{\sqrt{6}}\text{ units.}$$

7) Soln:



$\vec{b}=2\hat{i}+3\hat{j}-\hat{k}$

$\vec{OD}=5\hat{i}+4\hat{j}+2\hat{k}$

Let $\frac{x+1}{2}=\frac{y-3}{3}=\frac{z-1}{-1}=\lambda$

$\therefore F[(2\lambda-1),(3\lambda+3),(-\lambda+1)] \rightarrow (1)$

$\Rightarrow \vec{OF}=(2\lambda-1)\hat{i}+(3\lambda+3)\hat{j}+(-\lambda+1)\hat{k}$

$\vec{DF}=\vec{OF}-\vec{OD}$

$=(2\lambda-1-5)\hat{i}+(3\lambda+3-4)\hat{j}+(-\lambda+1-2)\hat{k}$

$\vec{DF}=(2\lambda-6)\hat{i}+(3\lambda-1)\hat{j}+(-\lambda-1)\hat{k}$

$\vec{DF}\perp\vec{b}\Rightarrow\vec{DF}\cdot\vec{b}=0$

$\Rightarrow 2(2\lambda-6)+3(3\lambda-1)-1(-\lambda-1)=0$

$4\lambda-12+9\lambda-3+\lambda+1=0$

$14\lambda-14=0\Rightarrow\boxed{\lambda=1}$

From (1), F is $(1,6,0)$

Eqn. of DF is $\frac{x-x_1}{x_2-x_1}=\frac{y-y_1}{y_2-y_1}=\frac{z-z_1}{z_2-z_1}$

$\Rightarrow \frac{x-5}{-4}=\frac{y-4}{2}=\frac{z-2}{-2}.$

Eg 6.33.

Let $\frac{x-1}{2}=\frac{y-2}{3}=\frac{z-3}{4}=s$

Any point on this line is of the form

$(2s+1, 3s+2, 4s+3)$

Let $\frac{x-4}{5}=\frac{y-1}{2}=\frac{z}{1}=t$

Any point on this line

$(5t+4, 2t+1, t)$

For some s, t

$(2s+1, 3s+2, 4s+3)=(5t+4, 2t+1, t)$.

$2s+1=5t+4$
$2s-5t=4-1$
$2s-5t=3 \rightarrow ①$

$3s+2=2t+1$
$3s-2t=1-2$
$3s-2t=-1 \rightarrow ②$

$4s+3=t$
$4s-t=-3 \rightarrow ③$

$① \times 3 \Rightarrow 6s-15t=9$
$② \times 2 \Rightarrow 6s-4t=-2$
$-11t=11 \Rightarrow t=-1$

$① \Rightarrow 2s-5(-1)=3$
$2s+5=3$
$2s=-2 \Rightarrow \boxed{s=-1}$

$\therefore s=-1, t=-1$

$③ \Rightarrow 4(-1)-(-1)=-3$
$-4+1=-3$
$-3=-3$

$\therefore$ The third eqn. Satisfied

The given lines intersect.

when $s=-1$

pt. of intersection $\}=(-2+1, -3+2, -4+3)$
$=(-1, -1, -1)$

Eg 6.34)

The cartesian eqn. of the st. line $\vec{r}=(\hat{i}+3\hat{j}-\hat{k})+t(2\hat{i}+3\hat{j}+2\hat{k})$ is

$\frac{x-1}{2}=\frac{y-3}{3}=\frac{z+1}{2}=s$ (say)

Any point on this line

$(2s+1, 3s+3, 2s-1)$

Let $\frac{x-2}{1}=\frac{y-4}{2}=\frac{z+3}{4}=t$

Any point on this line

$(t+2, 2t+4, 4t-3)$.

For some s, t

$(2s+1, 3s+3, 2s-1)=(t+2, 2t+4, 4t-3)$

$2s+1=t+2$
$2s-t=1 \rightarrow ①$

$3s+3=2t+4$
$3s-2t=1 \rightarrow ②$

$2s-1=4t-3$
$2s-4t=-2 \rightarrow ③$

$① \times 2 \Rightarrow 4s-2t=2$
$② \Rightarrow 3s-2t=1$
$s=1$

$① \Rightarrow 2-t=1 \Rightarrow 2-1=t$
$\boxed{t=1}$

$③ \Rightarrow 2(1)-4(1)=-2$
$2-4=-2$
$-2=-2$

$\therefore$ The third eqn. satisfied

The given lines intersect.

$\therefore$ The point of intersection $\}=(3, 6, 1)$

Take $\vec{b}=2\hat{i}+3\hat{j}+2\hat{k}$, $\vec{d}=\hat{i}+2\hat{j}+4\hat{k}$

$\vec{b}\times\vec{d}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 2 & 3 & 2\\ 1 & 2 & 4\end{vmatrix}$

$=\hat{i}(12-4)-\hat{j}(8-2)+\hat{k}(4-3)$

$=8\hat{i}-6\hat{j}+\hat{k}$

$\therefore$ The req. line passing through $(3,6,1)$ and parallel to $8\hat{i}-6\hat{j}+\hat{k}$ is

$\vec{r}=(3\hat{i}+6\hat{j}+\hat{k})+m(8\hat{i}-6\hat{j}+\hat{k})$.

Eg 6.35)

Let $\vec{a}=2\hat{i}+6\hat{j}+3\hat{k}$, $\vec{b}=2\hat{i}+3\hat{j}+4\hat{k}$

$\vec{c}=2\hat{j}-3\hat{k}$, $\vec{d}=\hat{i}+2\hat{j}+3\hat{k}$

Clearly $\vec{b}$ is not a Scalar Multiple of $\vec{d}$

$\therefore$ The given lines are not parallel

$\therefore \vec{c}-\vec{a}=-2\hat{i}-4\hat{j}-6\hat{k}$

$\vec{b}\times\vec{d}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 2 & 3 & 4\\ 1 & 2 & 3\end{vmatrix}$

$=\hat{i}(9-8)-\hat{j}(6-4)+\hat{k}(4-3)$

$\vec{b}\times\vec{d}=\hat{i}-2\hat{j}+\hat{k}$

$|\vec{b}\times\vec{d}|=\sqrt{1+4+1}=\sqrt{6}$

$(\vec{c}-\vec{a})\cdot(\vec{b}\times\vec{d})=(-2\hat{i}-4\hat{j}-6\hat{k})\cdot(\hat{i}-2\hat{j}+\hat{k})$

$=-2+8-6=0$

The shortest distance b/w two lines

$$d = \frac{|[(\vec{c}-\vec{a})\ \vec{b}\ \vec{d}]|}{|\vec{b}\times\vec{d}|}$$

$$= \frac{|(\vec{c}-\vec{a})\cdot(\vec{b}\times\vec{d})|}{|\vec{b}\times\vec{d}|} = \frac{0}{\sqrt{6}} = 0.$$

$d = 0$

Eg. 6.36.

Let $\vec{a} = 2\hat{i}+3\hat{j}+4\hat{k}$, $\vec{b} = -2\hat{i}+\hat{j}-2\hat{k}$

$\vec{c} = 3\hat{i}-2\hat{k}$, $\vec{d} = 2\hat{i}-\hat{j}+2\hat{k}$

$\vec{c}-\vec{a} = \hat{i}-3\hat{j}-6\hat{k}$

$$(\vec{c}-\vec{a})\times\vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -3 & -6 \\ -2 & 1 & -2 \end{vmatrix}$$

$$= \hat{i}(6+6) - \hat{j}(-2-12) + \hat{k}(1-6)$$

$(\vec{c}-\vec{a})\times\vec{b} = 12\hat{i}+14\hat{j}-5\hat{k}$

$|(\vec{c}-\vec{a})\times\vec{b}| = \sqrt{144+196+25} = \sqrt{365}$

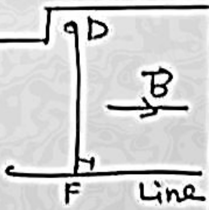
$|\vec{b}| = \sqrt{4+1+4} = \sqrt{9} = 3$

∴ The distance b/w the parrallel lines

$$d = \frac{|(\vec{c}-\vec{a})\times\vec{b}|}{|\vec{b}|}$$

$d = \frac{\sqrt{365}}{3}$ units.

Example 6.37



$\vec{b} = 2\hat{i}+3\hat{j}+\hat{k}$

$\vec{OB} = -\hat{i}+2\hat{j}+3\hat{k}$

Let the C.E. of the Line

$\vec{r} = (\hat{i}-4\hat{j}+3\hat{k}) + t(2\hat{i}+3\hat{j}+\hat{k})$ is

$$\frac{x-1}{2} = \frac{y+4}{3} = \frac{z-3}{1} = \lambda(\text{say})$$

$\therefore F[(2\lambda+1), (3\lambda-4), (\lambda+3)] \to (1)$

$\vec{OF} = (2\lambda+1)\hat{i}+(3\lambda-4)\hat{j}+(\lambda+3)\hat{k}$

$\vec{DF} = \vec{OF}-\vec{OB}$

$= (2\lambda+1+1)\hat{i}+(3\lambda-4-2)\hat{j}+(\lambda+3-3)\hat{k}$

$\vec{DF} = (2\lambda+2)\hat{i}+(3\lambda-6)\hat{j}+(\lambda)\hat{k} \to (2)$

$\vec{DF}\perp\vec{b} \Rightarrow \vec{DF}\cdot\vec{b} = 0$

$2(2\lambda+2)+3(3\lambda-6)+\lambda = 0$

$4\lambda+4+9\lambda-18+\lambda = 0$

$14\lambda-14 = 0 \to \boxed{\lambda = 1}$

From (1),

F is $(3, -1, 4)$

(2) ⇒ $\vec{DF} = 4\hat{i}-3\hat{j}+\hat{k}$

$|\vec{DF}| = \sqrt{4^2+3^2+1}$

$= \sqrt{16+9+1}$

$DF = \sqrt{26}$ units.

Ex 6.6

1) Given $p = 7$

$\vec{d} = 3\hat{i}-4\hat{j}+5\hat{k}$

$|\vec{d}| = \sqrt{9+16+25} = \sqrt{50} = 5\sqrt{2}$

Vector equation

$\vec{r}\cdot\hat{d} = p$

$\vec{r}\cdot\left(\frac{\vec{d}}{|\vec{d}|}\right) = p$

$\vec{r}\cdot\left(\frac{3\hat{i}-4\hat{j}+5\hat{k}}{5\sqrt{2}}\right) = 7$

$\vec{r}\cdot(3\hat{i}-4\hat{j}+5\hat{k}) = 35\sqrt{2}.$

2) Eqn. of the plane is

$12x+3y-4z = 65$

Parrametric form

$\vec{r}\cdot(12\hat{i}+3\hat{j}-4\hat{k}) = 65$

$[\vec{r}\cdot\vec{d} = q]$

Let $\vec{d} = 12\hat{i}+3\hat{j}-4\hat{k}$, $q = 65$

∴ The req. Length is

$$p = \frac{q}{|\vec{d}|} = \frac{65}{\sqrt{144+9+16}} = \frac{65}{\sqrt{169}}$$

$$= \frac{65}{13}$$

$p = 5$

Example 6.39

$3x - 4y + 3z = -8$

$-3x + 4y - 3z = 8$

$\Rightarrow (x\hat{i} + y\hat{j} + z\hat{k}) \cdot (-3\hat{i} + 4\hat{j} - 3\hat{k}) = 8$

$\vec{r} \cdot (-3\hat{i} + 4\hat{j} - 3\hat{k}) = 8$

which is the vector eqn. of the plane.

Example 6.40)

let $\vec{d} = 3\hat{i} - 4\hat{j} + 12\hat{k}$

and $q = 5$

$\hat{d} = \frac{\vec{d}}{|\vec{d}|} = \frac{3\hat{i} - 4\hat{j} + 12\hat{k}}{\sqrt{9+16+144}}$

$= \frac{3\hat{i} - 4\hat{j} + 12\hat{k}}{\sqrt{169}}$

$\hat{d} = \frac{3\hat{i} - 4\hat{j} + 12\hat{k}}{13}$

Given: $\vec{r} \cdot (3\hat{i} - 4\hat{j} + 12\hat{k}) = 5$

$\vec{r} \cdot \frac{(3\hat{i} - 4\hat{j} + 12\hat{k})}{13} = \frac{5}{13}$

Compare $\vec{r} \cdot \hat{d} = p$

$\therefore \hat{d} = \frac{3}{13}\hat{i} - \frac{4}{13}\hat{j} + \frac{12}{13}\hat{k}$

$\therefore$ The direction cosines are

$\left(\frac{3}{13}, \frac{-4}{13}, \frac{12}{13}\right)$ and

Length of the perpendicular from the origin to the plane is $\frac{5}{13}$.

Example 6.41)

Let $\vec{a} = 4\hat{i} + 2\hat{j} - 3\hat{k}$

$\vec{n} = 2\hat{i} - \hat{j} + \hat{k}$

Vector equation:

$\vec{r} \cdot \vec{n} = \vec{a} \cdot \vec{n}$

$\vec{r} \cdot (2\hat{i} - \hat{j} + \hat{k}) = (4\hat{i} + 2\hat{j} - 3\hat{k}) \cdot (2\hat{i} - \hat{j} + \hat{k})$

$= 8 - 2 - 3$

$\vec{r} \cdot (2\hat{i} - \hat{j} + \hat{k}) = 3$

Cartesian equation:

$(x\hat{i} + y\hat{j} + z\hat{k}) \cdot (2\hat{i} - \hat{j} + \hat{k}) = 3$

$2x - y + z = 3$

Example 6.42)

Let a, b, c be the x, y, z intercepts of the plane respectively.

$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$

Given: $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = K$

$\frac{1}{aK} + \frac{1}{bK} + \frac{1}{cK} = 1$

$\frac{1}{a}\left(\frac{1}{K}\right) + \frac{1}{b}\left(\frac{1}{K}\right) + \frac{1}{c}\left(\frac{1}{K}\right) = 1$

$\therefore$ The plane $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$ passes through the fixed point $\left(\frac{1}{K}, \frac{1}{K}, \frac{1}{K}\right)$.

Eqn of a plane passing through three given non-collinear points

a) Parametric form of vector eqn:

$$\vec{r} = \vec{a} + s(\vec{b}-\vec{a}) + t(\vec{c}-\vec{a}),\ s,t \in R$$

or $$\vec{r} = (1-s-t)\vec{a} + s\vec{b} + t\vec{c},\ s,t \in R$$

b) Non-Parametric form of vector eqn:

$$(\vec{r}-\vec{a})\cdot[(\vec{b}-\vec{a})\times(\vec{c}-\vec{a})] = 0 \text{ or}$$

$$[\vec{r}-\vec{a},\ \vec{b}-\vec{a},\ \vec{c}-\vec{a}] = 0.$$

c) Cartesian form of eqn:

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ x_2-x_1 & y_2-y_1 & z_2-z_1 \\ x_3-x_1 & y_3-y_1 & z_3-z_1 \end{vmatrix} = 0$$

Eqn of a plane passing through a given point and $\parallel^{l}$ to two given non-$\parallel^{l}$ vectors:

a) Parametric form:

$$\vec{r} = \vec{a} + s\vec{b} + t\vec{c},\ s,t \in R.$$

b) Non-Parametric form:

$$(\vec{r}-\vec{a})\cdot(\vec{b}\times\vec{c}) = 0$$

c) Cartesian form

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = 0$$

Eqn of a plane passing through two given distinct points and is $\parallel^{l}$ to a non-zero vector

a) Parametric form:

$$\vec{r} = \vec{a} + s(\vec{b}-\vec{a}) + t\vec{c},\ s,t \in R$$

(or) $$\vec{r} = (1-s)\vec{a} + s\vec{b} + t\vec{c},\ s,t \in R$$

b) Non-Parametric form

$$(\vec{r}-\vec{a})\cdot[(\vec{b}-\vec{a})\times\vec{c}] = 0$$

c) Cartesian form:

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ x_2-x_1 & y_2-y_1 & z_2-z_1 \\ c_1 & c_2 & c_3 \end{vmatrix} = 0$$

Ex: 6.7

① Sol:

Let $\vec{a} = 2\hat{i}+3\hat{j}+6\hat{k}$, $\vec{b} = 2\hat{i}+3\hat{j}+\hat{k}$, $\vec{c} = 2\hat{i}-5\hat{j}-3\hat{k}$. $\vec{r} = x\hat{i}+y\hat{j}+z\hat{k}$

Non-parametric form:

$$(\vec{r}-\vec{a})\cdot(\vec{b}\times\vec{c}) = 0 \rightarrow ①$$

$$\vec{b}\times\vec{c} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 3 & 1 \\ 2 & -5 & -3 \end{vmatrix}$$

$$= \hat{i}(-9+5) - \hat{j}(-6-2) + \hat{k}(-10-6)$$

$\vec{b}\times\vec{c} = -4\hat{i}+8\hat{j}-16\hat{k}$. Cartesian form:

① ⇒ $$[(x\hat{i}+y\hat{j}+z\hat{k}) - (2\hat{i}+3\hat{j}+6\hat{k})]\cdot(-4\hat{i}+8\hat{j}-16\hat{k}) = 0$$

$$[(x-2)\hat{i}+(y-3)\hat{j}+(z-6)\hat{k}]\cdot(-4\hat{i}+8\hat{j}-16\hat{k}) = 0$$

$$-4(x-2)+8(y-3)-16(z-6) = 0$$

$$-4x+8+8y-24-16z+96 = 0$$

$$-4x+8y-16z+80 = 0$$

$\div -4$ $$x-2y+4z-20 = 0.$$

② Sol:

Let $\vec{a} = 2\hat{i}+2\hat{j}+\hat{k}$, $\vec{b} = 9\hat{i}+3\hat{j}+6\hat{k}$. $\vec{c} = 2\hat{i}+6\hat{j}+6\hat{k}$

Parametric form:

$$\vec{r} = (1-s)\vec{a} + s\vec{b} + t\vec{c}$$

$$\vec{r} = (1-s)(2\hat{i}+2\hat{j}+\hat{k}) + s(9\hat{i}+3\hat{j}+6\hat{k}) + t(2\hat{i}+6\hat{j}+6\hat{k})$$

Cartesian form:

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ x_2-x_1 & y_2-y_1 & z_2-z_1 \\ c_1 & c_2 & c_3 \end{vmatrix} = 0$$

$$\begin{vmatrix} x-2 & y-2 & z-1 \\ 7 & 1 & 5 \\ 2 & 6 & 6 \end{vmatrix} = 0$$

$$(x-2)(6-30) - (y-2)(42-10) + (z-1)(42-2) = 0$$

$$(x-2)(-24) - 32(y-2) + 40(z-1) = 0$$

$$-24x+48-32y+64+40z-40 = 0$$

$$-24x-32y+40z+72 = 0$$

$\div -8$ $$3x+4y-5z-9 = 0$$

3. Sol:

Let $\vec{a} = 2\hat{i}+2\hat{j}+\hat{k}$, $\vec{b} = \hat{i}-2\hat{j}+3\hat{k}$

Eqn of line passing through (x_1,y_1,z_1) (x_2,y_2,z_2) is

$$\frac{x-x_1}{x_2-x_1} = \frac{y-y_1}{y_2-y_1} = \frac{z-z_1}{z_2-z_1}$$

$(x_1,y_1,z_1) = (2,1,-3)$ $(x_2,y_2,z_2) = (-1,5,-8)$

$$\frac{x-2}{-3} = \frac{y-1}{4} = \frac{z+3}{-5}$$

Let $\vec{c} = -3\hat{i}+4\hat{j}-5\hat{k}$

Parametric form:

$$\vec{r} = (1-s)\vec{a}+s\vec{b}+t\vec{c}$$

$$\vec{r} = (1-s)(2\hat{i}+2\hat{j}+\hat{k})+s(\hat{i}-2\hat{j}+3\hat{k})+t(-3\hat{i}+4\hat{j}-5\hat{k})$$

Cartesian form:

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ x_2-x_1 & y_2-y_1 & z_2-z_1 \\ c_1 & c_2 & c_3 \end{vmatrix} = 0$$

$$\begin{vmatrix} x-2 & y-2 & z-1 \\ -1 & -4 & 2 \\ -3 & 4 & -5 \end{vmatrix} = 0$$

$(x-2)(20-8)-(y-2)(5+6)+(z-1)(-4-12) = 0$

$12(x-2)-11(y-2)-16(z-1) = 0$

$12x-24-11y+22-16z+16 = 0$

$12x-11y-16z+14 = 0$

4. Sol:

Let $\vec{a} = \hat{i}-2\hat{j}+4\hat{k}$, $\vec{b} = \hat{i}+2\hat{j}-3\hat{k}$, $\vec{c} = 3\hat{i}-\hat{j}+\hat{k}$

Non-Parametric form:

$$(\vec{r}-\vec{a})\cdot(\vec{b}\times\vec{c}) = 0$$

$$\vec{b}\times\vec{c} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & -3 \\ 3 & -1 & 1 \end{vmatrix}$$

$= \hat{i}(2-3)-\hat{j}(1+9)+\hat{k}(-1-6)$

$\vec{b}\times\vec{c} = -\hat{i}-10\hat{j}-7\hat{k}$

$\vec{r}\cdot(\vec{b}\times\vec{c})-\vec{a}\cdot(\vec{b}\times\vec{c}) = 0$

$\vec{r}\cdot(-\hat{i}-10\hat{j}-7\hat{k})-(-1+20-28) = 0$

$\vec{r}\cdot(-\hat{i}-10\hat{j}-7\hat{k})+9 = 0$

$\vec{r}\cdot(-\hat{i}-10\hat{j}-7\hat{k}) = -9$

Cartesian form:

Let $\vec{r} = x\hat{i}+y\hat{j}+z\hat{k}$.

$(x\hat{i}+y\hat{j}+z\hat{k})\cdot(-\hat{i}-10\hat{j}-7\hat{k}) = -9$

$-x-10y-7z = -9$

$x+10y+7z = 9$.

5 Sol:

Let $\vec{a} = \hat{i}-\hat{j}+3\hat{k}$, $\vec{b} = 2\hat{i}-\hat{j}+4\hat{k}$, $\vec{c} = \hat{i}+2\hat{j}+\hat{k}$

Parametric form:

$$\vec{r} = \vec{a}+s\vec{b}+t\vec{c}$$

$$\vec{r} = (\hat{i}-\hat{j}+3\hat{k})+s(2\hat{i}-\hat{j}+4\hat{k})+t(\hat{i}+2\hat{j}+\hat{k})$$

Cartesian form:

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = 0$$

$$\begin{vmatrix} x-1 & y+1 & z-3 \\ 2 & -1 & 4 \\ 1 & 2 & 1 \end{vmatrix} = 0$$

$(x-1)(-1-8)-(y+1)(2-4)+(z-3)(4+1) = 0$

$-9(x-1)+2(y+1)+5(z-3) = 0$

$-9x+9+2y+2+5z-15 = 0$

$-9x+2y+5z-4 = 0$

$9x-2y-5z+4 = 0$

6 Sol:

Let $\vec{a} = 3\hat{i}+6\hat{j}-2\hat{k}$, $\vec{b} = -\hat{i}-2\hat{j}+6\hat{k}$, $\vec{c} = 6\hat{i}-4\hat{j}-2\hat{k}$.

Parametric form:

$$\vec{r} = (1-s-t)\vec{a}+s\vec{b}+t\vec{c}$$

$$\vec{r} = (1-s-t)(3\hat{i}+6\hat{j}-2\hat{k})+s(-\hat{i}-2\hat{j}+6\hat{k})+t(6\hat{i}-4\hat{j}-2\hat{k}).$$

Non-Parametric form:

$(\vec{r}-\vec{a})\cdot[(\vec{b}-\vec{a})\times(\vec{c}-\vec{a})]=0$

$[\vec{r}-(3\hat{i}+6\hat{j}-2\hat{k})][(-4\hat{i}-8\hat{j}+8\hat{k})\times(3\hat{i}-10\hat{j})]=0$

Cartesian form:

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ x_2-x_1 & y_2-y_1 & z_2-z_1 \\ x_3-x_1 & y_3-y_1 & z_3-z_1 \end{vmatrix}=0$$

$$\begin{vmatrix} x-3 & y-6 & z+2 \\ -4 & -8 & 8 \\ 3 & -10 & 0 \end{vmatrix}=0$$

$(x-3)(0+80)-(y-6)(0-24)+(z+2)(40+24)=0$

$80x-240+24y-144+64z+128=0$

$80x+24y+64z-256=0$

$\div 8\quad 10x+3y+8z-32=0.$

7. sol:

Let $\vec{a}=6\hat{i}-\hat{j}+\hat{k}$, $\vec{b}=-\hat{i}+2\hat{j}+\hat{k}$, $\vec{c}=-5\hat{i}-4\hat{j}-5\hat{k}$.

Non-Parametric form:

$(\vec{r}-\vec{a})\cdot(\vec{b}\times\vec{c})=0$

$$\vec{b}\times\vec{c}=\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & 2 & 1 \\ -5 & -4 & -5 \end{vmatrix}$$

$=\hat{i}(-10+4)-\hat{j}(5+5)+\hat{k}(4+10)$

$\vec{b}\times\vec{c}=-6\hat{i}-10\hat{j}+14\hat{k}$

$\vec{r}\cdot(\vec{b}\times\vec{c})-\vec{a}\cdot(\vec{b}\times\vec{c})=0$

$\vec{r}\cdot(-6\hat{i}-10\hat{j}+14\hat{k})-[-36+10+14]=0$

$\vec{r}\cdot(-6\hat{i}-10\hat{j}+14\hat{k})=-12.$

Cartesian form:

Let $\vec{r}=x\hat{i}+y\hat{j}+z\hat{k}$

$(x\hat{i}+y\hat{j}+z\hat{k})\cdot(-6\hat{i}-10\hat{j}+14\hat{k})=-12$

$-6x-10y+14z=-12$

$\div -2\quad 3x+5y-7z=6$

Eg: 6.43

Sol:

Let $\vec{a}=\hat{j}-5\hat{k}$, $\vec{b}=2\hat{i}+3\hat{j}+6\hat{k}$

$\vec{c}=\hat{i}+\hat{j}-\hat{k}.$

Non-Parametric form:

$(\vec{r}-\vec{a})\cdot(\vec{b}\times\vec{c})=0$

$$\vec{b}\times\vec{c}=\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 3 & 6 \\ 1 & 1 & -1 \end{vmatrix}$$

$=\hat{i}(-3-6)-\hat{j}(-2-6)+\hat{k}(2-3)$

$\vec{b}\times\vec{c}=-9\hat{i}+8\hat{j}-\hat{k}$

$[\vec{r}-(\hat{j}-5\hat{k})]\cdot(-9\hat{i}+8\hat{j}-\hat{k})=0$

$\vec{r}\cdot(-9\hat{i}+8\hat{j}-\hat{k})-(0+8+5)=0$

$\vec{r}\cdot(-9\hat{i}+8\hat{j}-\hat{k})=13.$

Cartesian form:

Let $\vec{r}=x\hat{i}+y\hat{j}+z\hat{k}$

$-9x+8y-z-13=0$

$9x-8y+z+13=0$

Eg: 6.44.

Given line $\frac{x-1}{1}=\frac{y+1/2}{1}=\frac{z+1}{-1}$

Let $\vec{a}=-\hat{i}+2\hat{j}$, $\vec{b}=2\hat{i}+2\hat{j}-\hat{k}$

$\vec{c}=\hat{i}+\hat{j}-\hat{k}.$

Parametric form:

$\vec{r}=(1-s)\vec{a}+s\vec{b}+t\vec{c},\ s,t\in R$

$\vec{r}=(1-s)(-\hat{i}+2\hat{j})+s(2\hat{i}+2\hat{j}-\hat{k})+t(\hat{i}+\hat{j}-\hat{k})$

Non-Parametric form:

$(\vec{r}-\vec{a})\cdot[(\vec{b}-\vec{a})\times\vec{c}]=0$

$[\vec{r}-(-\hat{i}+2\hat{j})]\cdot[(3\hat{i}+0\hat{j}-\hat{k})\times(\hat{i}+\hat{j}-\hat{k})]=0$

$$(\vec{b}-\vec{a})\times\vec{c}=\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 0 & -1 \\ 1 & 1 & -1 \end{vmatrix}$$

$=\hat{i}(0+1)-\hat{j}(-3+1)+\hat{k}(3)$

$(\vec{b}-\vec{a})\times\vec{c}=\hat{i}+2\hat{j}+3\hat{k}$

$[\vec{r}-(-\hat{i}+2\hat{j})]\cdot(\hat{i}+2\hat{j}+3\hat{k})=0$

$\vec{r}\cdot(\hat{i}+2\hat{j}+3\hat{k})-(-1+4)=0$

$\vec{r}\cdot(\hat{i}+2\hat{j}+3\hat{k}) = 3$

Cartesian form:

let $\vec{r} = x\hat{i}+y\hat{j}+z\hat{k}$

$x+2y+3z = 3$.

Condition for a line to lie in a plane:

(i) If the line $\vec{r} = \vec{a}+t\vec{b}$ lies in the plane $\vec{r}\cdot\vec{n} = \vec{d}$, then $\vec{a}\cdot\vec{n} = \vec{d}$ and $\vec{b}\cdot\vec{n} = 0$.

(ii) If the line $\frac{x-x_1}{a} = \frac{y-y_1}{b} = \frac{z-z_1}{c}$ lies in the plane $Ax+By+Cz+D=0$ then $Ax_1+By_1+Cz_1+D=0$ and $aA+bB+cC=0$.

Eg: 6.45

Sol:

$(x_1,y_1,z_1) = (3,4,-3)$

Direction ratios of given line are $(a,b,c) = (-4,-7,12)$

Direction ratios of the normal to the given plane are $(A,B,C) = (5,-1,1)$

$(x_1,y_1,z_1) = (3,4,-3)$ satisfies the given plane $5x-y+z=8$.

$aA+bB+cC = (-4)5+(-7)(-1)+12(1)$
$= -20+7+12 = -1 \neq 0$

$\therefore$ The given line doesnot lie in the plane.

Condition for coplanarity of two lines:

(a) vector form:

$(\vec{c}-\vec{a})\cdot(\vec{b}\times\vec{d}) = 0$

(b) Cartesian form:

$\frac{x-x_1}{b_1} = \frac{y-y_1}{b_2} = \frac{z-z_1}{b_3}$ & $\frac{x-x_2}{d_1} = \frac{y-y_2}{d_2} = \frac{z-z_2}{d_3}$ are coplanar, if

$$\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ b_1 & b_2 & b_3 \\ d_1 & d_2 & d_3 \end{vmatrix} = 0$$

Eqn of plane containing two non-||l coplanar lines:

(a) Parametric form:

let $\vec{r} = \vec{a}+s\vec{b}$ & $\vec{r} = \vec{c}+t\vec{d}$ are two non-||l coplanar lines.

$\vec{r} = \vec{a}+t\vec{b}+s\vec{d}$ or $\vec{r} = \vec{c}+t\vec{b}+s\vec{d}$.

(b) Non-parametric form:

let $\vec{r} = \vec{a}+s\vec{b}$, $\vec{r} = \vec{c}+t\vec{d}$ be two non-||l coplanar lines.

$(\vec{r}-\vec{a})\cdot(\vec{b}\times\vec{d}) = 0$ or $(\vec{r}-\vec{c})\cdot(\vec{b}\times\vec{d}) = 0$

(c) Cartesian form:

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ b_1 & b_2 & b_3 \\ d_1 & d_2 & d_3 \end{vmatrix} = 0 \quad \text{(or)}$$

$$\begin{vmatrix} x-x_2 & y-y_2 & z-z_2 \\ b_1 & b_2 & b_3 \\ d_1 & d_2 & d_3 \end{vmatrix} = 0.$$

Eg: 6.46

Sol:

comparing $\vec{r} = \vec{a}+s\vec{b}$, $\vec{r} = \vec{c}+t\vec{d}$

$\vec{a} = -\hat{i}-3\hat{j}-5\hat{k}$, $\vec{b} = 3\hat{i}+5\hat{j}+7\hat{k}$,

$\vec{c} = 2\hat{i}+4\hat{j}+6\hat{k}$, $\vec{d} = \hat{i}+4\hat{j}+7\hat{k}$.

Condition for coplanarity:

$(\vec{c}-\vec{a})\cdot(\vec{b}\times\vec{d}) = 0$

$$\vec{b}\times\vec{d} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 5 & 7 \\ 1 & 4 & 7 \end{vmatrix}$$

$= \hat{i}(35-28)-\hat{j}(21-7)+\hat{k}(12-5)$
$= 7\hat{i}-14\hat{j}+7\hat{k}$.

$(\vec{c}-\vec{a})\cdot(\vec{b}\times\vec{d}) = 0$

$(3\hat{i}+7\hat{j}+11\hat{k})\cdot(7\hat{i}-14\hat{j}+7\hat{k}) = 0$

$= 21-98+77 = 0$.

Non-parametric form:

$(\vec{r}-\vec{a})\cdot(\vec{b}\times\vec{d}) = 0$

$[\vec{r}-(-\hat{i}-3\hat{j}-5\hat{k})]\cdot(7\hat{i}-14\hat{j}+7\hat{k}) = 0$

$\vec{r}\cdot(7\hat{i}-14\hat{j}+7\hat{k})-(-7+42-35) = 0$

$\vec{r}\cdot(7\hat{i}-14\hat{j}+7\hat{k}) = 0$.

Ex : 6.8

① Sol :

Let $\vec{a} = 5\hat{i} + 7\hat{j} - 3\hat{k}$, $\vec{b} = 4\hat{i} + 4\hat{j} - 5\hat{k}$,
$\vec{c} = 8\hat{i} + 4\hat{j} + 5\hat{k}$, $\vec{d} = 7\hat{i} + \hat{j} + 3\hat{k}$.

Condition for coplanarity :

$$(\vec{c} - \vec{a}) \cdot (\vec{b} \times \vec{d}) = 0$$

$$\vec{b} \times \vec{d} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 4 & 4 & -5 \\ 7 & 1 & 3 \end{vmatrix}$$

$= \hat{i}(12 + 5) - \hat{j}(12 + 35) + \hat{k}(4 - 28)$

$= 17\hat{i} - 47\hat{j} - 24\hat{k}$.

$\vec{c} - \vec{a} = 3\hat{i} - 3\hat{j} + 8\hat{k}$.

$(\vec{c} - \vec{a}) \cdot (\vec{b} \times \vec{d}) = (3\hat{i} - 3\hat{j} + 8\hat{k}) \cdot (17\hat{i} - 47\hat{j} - 24\hat{k})$

$= 51 + 141 - 192 = 0$

$(\vec{c} - \vec{a}) \cdot (\vec{b} \times \vec{d}) = 0$.

$\therefore$ The Given two lines are coplanar.

Non-Parametric vector eqn :

$$(\vec{r} - \vec{a}) \cdot (\vec{b} \times \vec{d}) = 0$$

$[\vec{r} - (5\hat{i} + 7\hat{j} - 3\hat{k})] \cdot (17\hat{i} - 47\hat{j} - 24\hat{k}) = 0$

$\vec{r} \cdot (17\hat{i} - 47\hat{j} - 24\hat{k}) - (85 - 329 + 72) = 0$

$\vec{r} \cdot (17\hat{i} - 47\hat{j} - 24\hat{k}) + 172 = 0$.

② Let $\vec{a} = 2\hat{i} + 3\hat{j} + 4\hat{k}$, $\vec{b} = \hat{i} + \hat{j} + 3\hat{k}$
$\vec{c} = \hat{i} + 4\hat{j} + 5\hat{k}$, $\vec{d} = -3\hat{i} + 2\hat{j} + \hat{k}$.

Condition for Coplanarity :

$$(\vec{c} - \vec{a}) \cdot (\vec{b} \times \vec{d}) = 0.$$

$\vec{c} - \vec{a} = -\hat{i} + \hat{j} + \hat{k}$

$$\vec{b} \times \vec{d} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 3 \\ -3 & 2 & 1 \end{vmatrix}$$

$= \hat{i}(1 - 6) - \hat{j}(1 + 9) + \hat{k}(2 + 3)$

$\vec{b} \times \vec{d} = -5\hat{i} - 10\hat{j} + 5\hat{k}$

$(\vec{c} - \vec{a}) \cdot (\vec{b} \times \vec{d}) = 5 - 10 + 5 = 0$

Given lines are coplanar.

Non-Parametric vector eqn :

$$(\vec{r} - \vec{a}) \cdot (\vec{b} \times \vec{d}) = 0$$

$[\vec{r} - (2\hat{i} + 3\hat{j} + 4\hat{k})] \cdot (-5\hat{i} - 10\hat{j} + 5\hat{k}) = 0$

$\vec{r} \cdot (-5\hat{i} - 10\hat{j} + 5\hat{k}) - (-10 - 30 + 20) = 0$

$-5x - 10y + 5z + 20 = 0$

$\div -5 \quad x + 2y - z - 4 = 0$ is the required eqn of plane.

③ Sol :

Let $\vec{a} = \hat{i} + 2\hat{j} + 3\hat{k}$, $\vec{b} = \hat{i} + 2\hat{j} + m^2\hat{k}$,
$\vec{c} = 3\hat{i} + 2\hat{j} + \hat{k}$, $\vec{d} = \hat{i} + m^2\hat{j} + 2\hat{k}$

Condition for coplanarity :

$$(\vec{c} - \vec{a}) \cdot (\vec{b} \times \vec{d}) = 0$$

$\vec{c} - \vec{a} = 2\hat{i} - 2\hat{k}$

$$\vec{b} \times \vec{d} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & m^2 \\ 1 & m^2 & 2 \end{vmatrix}$$

$= \hat{i}(4 - m^4) - \hat{j}(2 - m^2) + \hat{k}(m^2 - 2)$

$(\vec{c} - \vec{a}) \cdot (\vec{b} \times \vec{d}) = 0$

$(2\hat{i} - 2\hat{k}) \cdot [(4 - m^4)\hat{i} - (2 - m^2)\hat{j} + (m^2 - 2)\hat{k}] = 0$

$2(4 - m^4) - 2(m^2 - 2) = 0$

$2[4 - m^4 - m^2 + 2] = 0$

$m^4 + m^2 - 6 = 0$

Let $m^2 = r \quad r^2 + r - 6 = 0$ $\qquad$ $\begin{array}{c} -6 \\ \hline 3 \mid -2 \\ 1 \end{array}$

$(r + 3)(r - 2) = 0$

$r = -3, \quad r = 2$

$m^2 = -3$, $m^2 = 2$

(not possible) $\quad m = \pm\sqrt{2}$

(4) Sol:

Let $\vec{a} = \hat{i} - \hat{j}$, $\vec{b} = 2\hat{i} + \lambda\hat{j} + 2\hat{k}$,

$\vec{c} = -\hat{i} - \hat{j}$, $\vec{d} = 5\hat{i} + 2\hat{j} + \lambda\hat{k}$

Condition for coplanarity:

$$(\vec{c} - \vec{a}) \cdot (\vec{b} \times \vec{d}) = 0$$

$$\vec{c} - \vec{a} = -2\hat{i}$$

$$\vec{b} \times \vec{d} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & \lambda & 2 \\ 5 & 2 & \lambda \end{vmatrix}$$

$$\vec{b} \times \vec{d} = \hat{i}(\lambda^2 - 4) - \hat{j}(2\lambda - 10) + \hat{k}(4 - 5\lambda)$$

$$(\vec{c} - \vec{a}) \cdot (\vec{b} \times \vec{d}) = 0$$

$$-2(\lambda^2 - 4) = 0$$

$$\lambda^2 - 4 = 0$$

$$\lambda^2 = 4$$

$$\lambda = \pm 2.$$

(Last page).

Angle between two planes :

Theorem : 6.18

The acute angle θ between the two planes $\vec{r}\cdot\vec{n_1} = p_1$ and $\vec{r}\cdot\vec{n_2} = p_2$ is given by $\theta = \cos^{-1}\left(\frac{|\vec{n_1}\cdot\vec{n_2}|}{|\vec{n_1}||\vec{n_2}|}\right)$

Remark :

(i) If two planes are ⊥r, then $\vec{n_1}\cdot\vec{n_2} = 0$

(ii) If two planes are $\parallel^l$, then $\vec{n_1} = \lambda\vec{n_2}$

(iii) Eqn of a plane $\parallel^l$ to the plane $\vec{r}\cdot\vec{n} = p$ is $\vec{r}\cdot\vec{n} = k$, $k \in R$.

Theorem : 6.19

The acute angle θ between the planes $a_1x + b_1y + c_1z + d_1 = 0$ & $a_2x + b_2y + c_2z + d_2 = 0$ is given by $\theta = \cos^{-1}\left(\frac{|a_1a_2 + b_1b_2 + c_1c_2|}{\sqrt{a_1^2+b_1^2+c_1^2}\sqrt{a_2^2+b_2^2+c_2^2}}\right)$

Remark :

(i) If the two planes are ⊥r, then $a_1a_2 + b_1b_2 + c_1c_2 = 0$

(ii) If the two planes are $\parallel^l$, then $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$.

(iii) Eqn of the plane $\parallel^l$ to $ax + by + cz = p$ is $ax + by + cz = k$, $k \in R$.

Angle between a line and a plane

Let $\vec{r} = \vec{a} + t\vec{b}$ be a line

$\vec{r}\cdot\vec{n} = p$ be a plane

The acute angle θ between the line and the plane is given by

$$\theta = \sin^{-1}\left(\frac{|\vec{b}\cdot\vec{n}|}{|\vec{b}||\vec{n}|}\right)$$

Cartesian form :

$\frac{x - x_1}{a_1} = \frac{y - y_1}{b_1} = \frac{z - z_1}{c_1}$ be a line.

$ax + by + cz = p$ be a line

$$\theta = \sin^{-1}\left(\frac{|aa_1 + bb_1 + cc_1|}{\sqrt{a^2+b^2+c^2}\sqrt{a_1^2+b_1^2+c_1^2}}\right)$$

Remark :

(i) If the line is ⊥r to the plane, then the line is $\parallel^l$ to normal to the plane so, $\vec{b}$ ⊥r $\vec{n}$. $\vec{b} = \lambda\vec{n} \Rightarrow \frac{a_1}{a} = \frac{b_1}{b} = \frac{c_1}{c}$

(ii) If the line is $\parallel^l$ to the plane, then the line is ⊥r to the normal to the plane.

$\therefore \vec{b}\cdot\vec{n} = 0 \Rightarrow aa_1 + bb_1 + cc_1 = 0.$

Theorem : 6.20

The ⊥r distance from a point with p.v $\vec{u}$ to the plane $\vec{r}\cdot\vec{n} = p$ is given by

$$\delta = \frac{|\vec{u}\cdot\vec{n} - p|}{|\vec{n}|}.$$

The p.v of the foot F of the ⊥r AF is given by

$$\vec{r_1} = \vec{u} + t_1\vec{n} \text{ or}$$

$$\vec{r_1} = \vec{u} + \left(\frac{\vec{u}\cdot\vec{n} - p}{|\vec{n}|^2}\right)\vec{n}$$

Cartesian form :

Let $\vec{u} = x_1\hat{i} + y_1\hat{j} + z_1\hat{k}$, $\vec{n} = a\hat{i} + b\hat{j} + c\hat{k}$

$$\delta = \frac{|ax_1 + by_1 + cz_1 - p|}{\sqrt{a^2+b^2+c^2}}$$

Remark :

The ⊥r distance from the origin to the plane $ax + by + cz + d = 0$ is given by $\delta = \frac{|d|}{\sqrt{a^2+b^2+c^2}}$

Theorem : 6.21

The distance between two $\parallel^l$ planes $ax+by+cz+d_1=0$ and $ax+by+cz+d_2=0$ is given by $\left|\dfrac{d_1-d_2}{\sqrt{a^2+b^2+c^2}}\right|$

Eqn of line of Intersection of two planes:

Let $\vec{r}\cdot\vec{n}=p$ & $\vec{r}\cdot\vec{m}=q$ be two non-$\parallel^l$ planes

$(x_1,y_1,0)$ is a point on the required line, which is $\parallel^l$ to $l_1\hat{i}+l_2\hat{j}+l_3\hat{k}$,

The eqn of the line is

$$\frac{x-x_1}{l_1}=\frac{y-y_1}{l_2}=\frac{z-z_1}{l_3}.$$

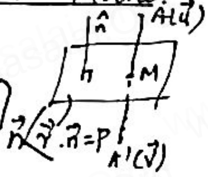
Eqn of plane passing through the line of Intersection of two given planes:

Theorem : 6.22

The vector eqn of a plane which passes through the line of Intersection of the planes $\vec{r}_1\cdot\vec{n}_1=d_1$ and $\vec{r}\cdot\vec{n}_2=d_2$ is given by $(\vec{r}\cdot\vec{n}_1-\vec{d})+\lambda(\vec{r}\cdot\vec{n}_2-d_2)=0$

C.F : $(a_1x+b_1y+c_1z-d_1)+\lambda(a_2x+b_2y+c_2z-d_2)=0$

Image of a Point in a Plane:



The P.V of A' is

$$\vec{v}=\vec{u}+\frac{2[p-(\vec{u}\cdot\vec{n})]}{|\vec{n}|^2}\vec{n}$$

The P.V of the foot M of the $\perp r$ is given by

$$\frac{\vec{u}+\vec{v}}{2}=\frac{\vec{u}}{2}+\frac{1}{2}\left[\vec{u}+\frac{2[p-(\vec{u}\cdot\vec{n})]}{|\vec{n}|^2}\vec{n}\right]$$

Meeting Point of a line and a plane

Theorem : 6.23

The P.V of the Point of Intersection of the st. line $\vec{r}=\vec{a}+t\vec{b}$ and the plane $\vec{r}\cdot\vec{n}=p$ is $\vec{a}+\left(\dfrac{p-(\vec{a}\cdot\vec{n})}{\vec{b}\cdot\vec{n}}\right)\vec{b}$, $\vec{b}\cdot\vec{n}\neq 0$

Eg : 6.47

Sol:

Let $\vec{n_1}=2\hat{i}+2\hat{j}+2\hat{k}$, $\vec{n_2}=4\hat{i}-2\hat{j}+2\hat{k}$

$$\theta=\cos^{-1}\left(\frac{|\vec{n_1}\cdot\vec{n_2}|}{|\vec{n_1}||\vec{n_2}|}\right)$$

$$=\cos^{-1}\left(\frac{8-4+4}{\sqrt{4+4+4}\sqrt{16+4+4}}\right)$$

$$=\cos^{-1}\left(\frac{\cancel{8}\,2}{\cancel{2}\sqrt{3}\times\cancel{2}\sqrt{6}}\right)$$

$$=\cos^{-1}\left(\frac{2}{3\sqrt{2}}\right)=\cos^{-1}\left(\frac{\sqrt{2}}{3}\right)$$

Eg : 6.48

Sol:

Let $\vec{b}=\hat{i}-\hat{j}+\hat{k}$, $\vec{n}=2\hat{i}-\hat{j}+\hat{k}$

$$\theta=\sin^{-1}\left(\frac{|\vec{b}\cdot\vec{n}|}{|\vec{b}||\vec{n}|}\right)$$

$$=\sin^{-1}\left(\frac{2+1+1}{\sqrt{1+1+1}\sqrt{4+1+1}}\right)$$

$$=\sin^{-1}\left(\frac{4}{\sqrt{3}\sqrt{6}}\right)=\sin^{-1}\left(\frac{4}{\sqrt{18}}\right)$$

$$=\sin^{-1}\left(\frac{4}{3\sqrt{2}}\right)=\sin^{-1}\left(\frac{\cancel{4}^{2}\sqrt{2}}{3\times\cancel{2}}\right)$$

$$\theta=\sin^{-1}\left(\frac{2\sqrt{2}}{3}\right)$$

Eg : 6.49

Sol: $(x_1,y_1,z_1)=(2,5,-3)$

Eqn of plane is $6x-3y+2z-5=0$

$$d=\left|\frac{ax_1+by_1+cz_1-d}{\sqrt{a^2+b^2+c^2}}\right|$$

$$=\left|\frac{12-15-6-5}{\sqrt{36+9+4}}\right|=\frac{|-14|}{7}=\frac{\cancel{14}}{\cancel{7}}=2$$

Eg : 6.50

Sol:

The eqn of line joining $A(4,1,2)$, $B(7,5,4)$

$$\frac{x-4}{3}=\frac{y-1}{4}=\frac{z-2}{2}=t$$

$(x,y,z)=(3t+4,\ 4t+1,\ 2t+2)$

It is lies on $x-y+z=5$

$3t+4-4t-1+2t+2=5$

$t=0$, Intersecting Point $(4,1,2)$

Distance between $(5,-5,-10)$ and $(4,1,2)$

$$d=\sqrt{(4-5)^2+(1+5)^2+(2+10)^2}=\sqrt{1+36+144}=\sqrt{181}\text{ units}$$

Eg : 6.51

Sol:

The distance between two $\parallel^{l}$ planes is $\left|\frac{d_1-d_2}{\sqrt{a^2+b^2+c^2}}\right|$

Let $d_1 = 1$, $d_2 = 5/2$

$$d = \frac{|1-5/2|}{\sqrt{1+2^2+2^2}} = \frac{3/2}{\sqrt{1+4+4}} = \frac{3/2}{3} = 1/2$$

Eg : 6.52

Sol:

Cartesian eqn of plane is

$2x-y-2z-6=0$ and $6x-3y-6z-27=0$

$\div 3$ $2x-y-2z-9=0$

$d_1 = -6$, $d_2 = -9$

$$\text{Distance} = \left|\frac{d_1-d_2}{\sqrt{a^2+b^2+c^2}}\right| = \frac{|-6+9|}{\sqrt{4+1+4}} = \frac{3}{3} = 1 \text{ units.}$$

Eg : 6.53

Sol:

The required eqn of the plane is

$(x+y+z+1)+\lambda(2x-3y+5z-2)=0.$

It is passes through $(-1,2,1)$

$(-1+2+1+1)+\lambda(-2-6+5-2)=0$

$3+(-5\lambda)=0$

$5\lambda = 3$

$\lambda = 3/5$

The required eqn of the plane is

$(x+y+z+1)+3/5\,(2x-3y+5z-2)=0$

$11x-4y+20z=1.$

Eg : 6.54

Sol: The required eqn of the plane is

$(2x+3y-z+7)+\lambda(x+y-2z+5)=0$

or $(2+\lambda)x+(3+\lambda)y+(-1-2\lambda)z+(7+5\lambda)=0$

This is $\perp$r to the plane $x+y-3z-5=0$

$(1)(2+\lambda)+1(3+\lambda)-3(-1-2\lambda)=0$

$2+\lambda+3+\lambda+3+6\lambda=0$

$8\lambda+8=0 \Rightarrow \lambda=-1$

The req eqn of the plane is

$x+2y+z+2=0.$

Eg : 6.55

Sol:

Let $\vec{u} = \hat{i}+2\hat{j}+3\hat{k}$, $\vec{n} = \hat{i}+2\hat{j}+4\hat{k}$

$p = 38.$

$$\vec{v} = \vec{u} + \frac{2[p-(\vec{u}\cdot\vec{n})]}{|\vec{n}|^2}\vec{n}$$

$$= (\hat{i}+2\hat{j}+3\hat{k})+\frac{2[38-(1+4+12)]}{(\sqrt{1+4+16})^2}(\hat{i}+2\hat{j}+4\hat{k})$$

$$= (\hat{i}+2\hat{j}+3\hat{k})+2\frac{(21)}{21}(\hat{i}+2\hat{j}+4\hat{k})$$

$$= \hat{i}+2\hat{j}+3\hat{k}+2\hat{i}+4\hat{j}+8\hat{k}$$

$\vec{v} = 3\hat{i}+6\hat{j}+11\hat{k}$

$\therefore$ The image of the point is $(3,6,11)$

Eg : 6.56

Sol:

Cartesian eqn of the line is

$\frac{x-2}{3} = \frac{y+1}{4} = \frac{z-2}{2} = t$

$(x,y,z) = (3t+2,\ 4t-1,\ 2t+2)$

This point lies on the plane

$x-y+z-5=0$

$3t+2-(4t-1)+2t+2-5=0$

$3t+2-4t+1+2t+2-5=0$

$\Rightarrow t=0$

The required point is $(2,-1,2)$.

Ex : 6.9

(1) Sol:

The required eqn of the plane is

$(2x-7y+4z-3)+\lambda(3x-5y+4z+11)=0 \quad \to (1)$

This passes through $(-2,1,3)$

$(-4-7+12-3)+\lambda(-6-5+12+11)=0$

$-2+12\lambda=0$

$12\lambda=2$

$\lambda = 1/6$

$(1) \Rightarrow (2x-7y+4z-3)+\frac{1}{6}(3x-5y+4z+11)=0$

$12x-42y+24z-18+3x-5y+4z+11=0$

$15x-47y+28z-7=0.$

② sol:

The required eqn of the plane is

$(x+2y+3z-2)+\lambda(x-y+z-3)=0 \rightarrow ①$

$(1+\lambda)x+(2-\lambda)y+(3+\lambda)z-2-3\lambda=0$

Distance from $(3,1,-1)$ to the plane $=\frac{2}{\sqrt{3}}$

$$\frac{3(1+\lambda)+1(2-\lambda)-1(3+\lambda)-2-3\lambda}{\sqrt{(1+\lambda)^2+(2-\lambda)^2+(3+\lambda)^2}}=\frac{2}{\sqrt{3}}$$

$$\frac{\cancel{3}+3\cancel{\lambda}+2-\lambda-\cancel{3}-\lambda-\cancel{2}-3\cancel{\lambda}}{\sqrt{1+\lambda^2+2\lambda+4+\lambda^2-4\lambda+9+\lambda^2+6\lambda}}=\frac{2}{\sqrt{3}}$$

$$-2\lambda\sqrt{3}=2\sqrt{3\lambda^2+4\lambda+14}$$

$$12\lambda^2=4(3\lambda^2+4\lambda+14)$$

$$\cancel{12\lambda^2}=\cancel{12\lambda^2}+16\lambda+56$$

$$16\lambda=-56$$

$$\lambda=-\frac{56}{16}=-\frac{7}{2}$$

$① \Rightarrow (x+2y+3z-2)-\frac{7}{2}(x-y+z-3)=0$

$2x+4y+6z-4-7x+7y-7z+21=0$

$-5x+11y-z+17=0$

$5x-11y+z-17=0.$

③ sol:

Let $\vec{b}=\hat{i}+2\hat{j}-2\hat{k}$, $\vec{n}=6\hat{i}+3\hat{j}+2\hat{k}$

$$\theta=\sin^{-1}\left[\frac{|\vec{b}\cdot\vec{n}|}{|\vec{b}||\vec{n}|}\right]$$

$$=\sin^{-1}\left[\frac{|6+6-4|}{\sqrt{1+4+4}\sqrt{36+9+4}}\right]$$

$$=\sin^{-1}\left[\frac{8}{3\times7}\right]$$

$$\theta=\sin^{-1}\left(\frac{8}{21}\right)$$

④ sol:

Let $\vec{n_1}=\hat{i}+\hat{j}-2\hat{k}$, $\vec{n_2}=2\hat{i}-2\hat{j}+\hat{k}$.

Angle between the planes are

$$\theta=\cos^{-1}\left[\frac{|\vec{n_1}\cdot\vec{n_2}|}{|\vec{n_1}||\vec{n_2}|}\right]$$

$$=\cos^{-1}\left[\frac{|2-2-2|}{\sqrt{1+1+4}\sqrt{4+4+1}}\right]$$

$$=\cos^{-1}\left(\frac{2}{\sqrt{6}\times3}\right)$$

$$\theta=\cos^{-1}\left(\frac{2}{3\sqrt{6}}\right)$$

⑤ sol:

Eqn of the $\parallel^{l}$ plane is

$2x-3y+5z+k=0$

It passes through $(3,4,-1)$

$6-12-5+k=0$

$-11+k=0$

$\boxed{k=11}$

$2x-3y+5z+11=0$ is the required eqn of the plane.

Distance between two $\parallel^{l}$ planes

$$=\frac{|d_1-d_2|}{\sqrt{a^2+b^2+c^2}}$$

$$=\left|\frac{7-11}{\sqrt{4+9+25}}\right|$$

$$=\frac{4}{\sqrt{38}}\text{ units}$$

⑥ sol:

Length of $\perp r$ from (x_1,y_1,z_1) to the plane $ax+by+cz-p=0$ is $\left|\frac{ax_1+by_1+cz_1-p}{\sqrt{a^2+b^2+c^2}}\right|$

Length of $\perp r$ from $(1,-2,3)$ to the plane $x-y+z-5=0$ is $\delta=\left|\frac{1+2+3-5}{\sqrt{1+1+1}}\right|$

$$=\frac{1}{\sqrt{3}}\text{ units.}$$

(7) sol:

Let $\frac{x-1}{1} = \frac{y}{2} = \frac{z+1}{1} = t$

$(x,y,z) = (1+t, 2t, t-1) \rightarrow (1)$

It is lies on $2x-y+2z=2$

$2(1+t)-2t+2(t-1)=2$

$2+2t-2t+2t-2=2$

$t=1$

(1) ⇒ The required point is $(2,2,0)$

Let $\vec{b} = \hat{i}+2\hat{j}+\hat{k}$, $\vec{n} = 2\hat{i}-\hat{j}+2\hat{k}$

$\theta = \sin^{-1}\left[\frac{|\vec{b}\cdot\vec{n}|}{|\vec{b}||\vec{n}|}\right]$

$= \sin^{-1}\left[\frac{|2-2+2|}{\sqrt{1+4+1}\sqrt{4+1+4}}\right]$

$\theta = \sin^{-1}\left(\frac{2}{3\sqrt{6}}\right)$

(8) sol:

Cartesian eqn of the line is

$\frac{x-4}{1} = \frac{y-3}{2} = \frac{z-2}{3} = t$

$(x,y,z) = (t+4, 2t+3, 3t+2)$

This point lies on $x+2y+3z=2$

$(t+4)+2(2t+3)+3(3t+2)=2$

$t+4+4t+6+9t+6=2$

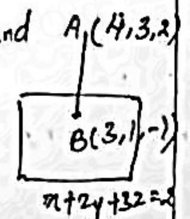
$14t = 2-16$

$14t=-14$

$\boxed{t=-1}$

$(x,y,z) = (3,1,-1)$ is the co-ordinates of the foot of the ⊥r

Distance between $(4,3,2)$ and $(3,1,-1)$



$d = \sqrt{(-1)^2+(-2)^2+(-3)^2}$

$= \sqrt{1+4+9}$

$d = \sqrt{14}$

Ex: 2.8

(4) sol:

$(x_1,y_1,z_1) = (1,-1,0)$, $(x_2,y_2,z_2) = (-1,-1,0)$

$(b_1,b_2,b_3) = (2,\lambda,2)$, $(d_1,d_2,d_3) = (5,2,\lambda)$

Condition for collinearity:

$$\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ b_1 & b_2 & b_3 \\ d_1 & d_2 & d_3 \end{vmatrix} = 0$$

$$\begin{vmatrix} -2 & 0 & 0 \\ 2 & \lambda & 2 \\ 5 & 2 & \lambda \end{vmatrix} = 0$$

$-2(\lambda^2-4)=0$

$\lambda^2-4=0$

$\lambda^2=4$

$\lambda=\pm 2$

(i) If $\lambda=2$

Eqn of plane is

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ b_1 & b_2 & b_3 \\ d_1 & d_2 & d_3 \end{vmatrix} = 0$$

$$\begin{vmatrix} x-1 & y+1 & z \\ 2 & 2 & 2 \\ 5 & 2 & 2 \end{vmatrix} = 0$$

$(x-1)(0)-(y+1)(4-10)+z(4-10)=0$

$6(y+1)-6z=0$

$\div 6$ $\quad y-z+1=0$

(ii) If $\lambda=-2$

Eqn of plane is

$$\begin{vmatrix} x-1 & y+1 & z \\ 2 & -2 & 2 \\ 5 & 2 & -2 \end{vmatrix} = 0$$

$(x-1)(0)-(y+1)(-4-10)+z(4+10)=0$

$14(y+1)+14z=0$

$\div 14$ $\quad y+z+1=0$