12th Standard

Maths

Reg.No. :

Exam Time : 00:50:00 Hrs

Total Marks : 50

50 x 1 = 50

1) If |adj(adj A)| = $|A|^9$, then the order of the square matrix A is

(a) 3 (b) 4 (c) 2 (d) 5

2) If A = $\begin{bmatrix} 2 & 0 \\ 1 & 5 \end{bmatrix}$ and B = $\begin{bmatrix} 1 & 4 \\ 2 & 0 \end{bmatrix}$ then |adj (AB)| =

(a) -40 (b) -80 (c) -60 (d) -20

3) If A, B and C are invertible matrices of some order, then which one of the following is not true?

(a) adj A = $|A|A^{-1}$ (b) adj(AB) = (adj A)(adj B) (c) det A^{-1} = $(\det A)^{-1}$ (d) $(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$

4) If A = $\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$ and A(adj A) = $\begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}$ then adj (AB) is

(a) 0 (b) sin θ (c) cos θ (d) 1

5) If $x^a y^b = e^m$, $x^c y^d = e^n$, $\Delta_1 = \begin{vmatrix} m & b \\ n & d \end{vmatrix}$, $\Delta_2 = \begin{vmatrix} a & m \\ c & n \end{vmatrix}$, $\Delta_3 = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$, then the values of x and y are respectively,

(a) $e^{(\Delta_2/\Delta_1)}$, $e^{(\Delta_3/\Delta_1)}$ (b) log (Δ_1/Δ_3), log (Δ_2/Δ_3) (c) log (Δ_2/Δ_1), log(Δ_3/Δ_1) (d) $e^{(\Delta_1/\Delta_3)}$,$e^{(\Delta_2/\Delta_3)}$

6) Which of the following is/are correct?

(i) Adjoint of a symmetric matrix is also a symmetric matrix.

(ii) Adjoint of a diagonal matrix is also a diagonal matrix.

(iii) If A is a square matrix of order n and λ is a scalar, then adj(λA) = λ^n adj(A).

(iv) A(adjA) = (adjA)A = |A| I

(a) Only (i) (b) (ii) and (iii) (c) (iii) and (iv) (d) (i), (ii) and (iv)

7) If ρ(A) = ρ([A | B]), then the system AX = B of linear equations is

(a) consistent and has a unique solution (b) consistent (c) consistent and has infinitely many solution (d) inconsistent

8) If 0 ≤ θ ≤ π and the system of equations x + (sinθ)y - (cosθ)z = 0, (cosθ)x - y +z = 0, (sinθ)x + y - z = 0 has a non-trivial solution then θ is

(a) $\frac{2\pi}{3}$ (b) $\frac{3\pi}{4}$ (c) $\frac{5\pi}{6}$ (d) $\frac{\pi}{4}$

9) If the system of equations x = cy + bz, y = az + cx and z = bx + ay has a non - trivial solution then

(a) $a^2 + b^2 + c^2 = 1$ (b) abc ≠ 1 (c) a + b + c =0 (d) $a^2 + b^2 + c^2 + 2abc = 1$

10) The number of solutions of the system of equations 2x+y = 4, x - 2y = 2, 3x + 5y = 6 is

(a) 0 (b) 1 (c) 2 (d) infinitely many

11) Every homogeneous system ______

(a) Is always consistent (b) Has only trivial solution (c) Has infinitely many solution (d) Need not be consistent

12) The value of $\sum_{i=1}^{13} \left(i^n + i^{n-1}\right)$ is

(a) 1+ i (b) i (c) 1 (d) 0

13) The area of the triangle formed by the complex numbers z,iz, and z+iz in the Argand's diagram is

(a) $\frac{1}{2}|z|^2$ (b) $|z|^2$ (c) $\frac{3}{2}|z|^2$ (d) $2|z|^2$

14) The solution of the equation |z|-z=1+2i is

(a) $\frac{3}{2} - 2i$ (b) $-\frac{3}{2} + 2i$ (c) $2 - \frac{3}{2}i$ (d) $2 + \frac{3}{2}i$

15) z_1, z_2 and z_3 are complex number such that $z_1+z_2+z_3=0$ and $|z_1|=|z_2|=|z_3|=1$ then $z_1^2+z_2^2+z_3^3$ is

(a) 3 (b) 2 (c) 1 (d) 0

(a) (1,0) (b) (−1,1) (c) (0,1) (d) (1,1)

17) If $\omega \neq 1$ is a cubic root of unity and $\begin{vmatrix} 1 & 1 & 1 \\ 1 & -\omega^2 & \omega^2 \\ 1 & \omega^2 & \omega^2 \end{vmatrix}$ =3k, then k is equal to

(a) 1 (b) -1 (c) $\sqrt{3i}$ (d) $-\sqrt{3i}$

18) If, i^2 = -1, then $i^1 + i^2 + i^3 + ...+$ up to 1000 terms is equal to

(a) 1 (b) -1 (c) i (d) 0

19) The amplitude of $\frac{1}{i}$ is equal to

(a) 0 (b) $\frac{\pi}{2}$ (c) -$\frac{\pi}{2}$ (d) π

20) If ω is the cube root of unity, then the value of $(1-\omega)(1-\omega^2)(1-\omega^4)(1-\omega^8)$ is

(a) 9 (b) -9 (c) 16 (d) 32

21) A polynomial equation in x of degree n always has

(a) n distinct roots (b) n real roots (c) n imaginary roots (d) at most one root

22) The polynomial x^3-kx^2+9x has three real zeros if and only if, k satisfies

(a) |k|≤6 (b) k=0 (c) |k| > 6 (d) |k|≥6

23) The number of real numbers in [0,2π] satisfying $\sin^4x-2\sin^2x+1$ is

(a) 2 (b) 4 (c) 1 (d) ∞

24) If $x^3+12x^2+10ax+1999$ definitely has a positive zero, if and only if

(a) a≥0 (b) a > 0 (c) a< 0 (d) a≤0

25) The number of positive zeros of the polynomial $\sum_{j=0}^{n} n_{C_r}$ $(-1)^r x^r$ is

(a) 0 (b) n (c) < n (d) r

26) The quadratic equation whose roots are ∝ and β is

(a) (x - ∝)(x -β) =0 (b) (x - ∝)(x + β) =0 (c) ∝+β=$\frac{b}{a}$ (d) ∝.β=$\frac{-c}{a}$

27) Let a > 0, b > 0, c > 0. h n both th root of th quatlon $ax^2+b+C=0$ are

(a) real and negative (b) real and positive (c) rational numb rs (d) none

28) If ∝, β,ɤ are the roots of 9x3-7x+6=0, then ∝ β ɤ is __________

(a) $\frac{-7}{9}$ (b) $\frac{7}{9}$ (c) 0 (d) $\frac{-2}{3}$

29) If $\sin^{-1}x+\sin^{-1}y=\frac{2\pi}{3}$; then $\cos^{-1}x+\cos^{-1}y$ is equal to

(a) $\frac{2\pi}{3}$ (b) $\frac{\pi}{3}$ (c) $\frac{\pi}{6}$ (d) π

30) $\sin^{-1}(\cos x)= \frac{\pi}{2} - x$ is valid for

(a) $-\pi \leq x \leq 0$ (b) $0\pi \leq x \leq 0$ (c) $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$ (d) $-\frac{\pi}{4} \leq x \leq \frac{3\pi}{4}$

31) If $\sin^{-1}x+\sin^{-1}y+\sin^{-1}z=\frac{3\pi}{2}$, the value of $x^{2017}+y^{2018}+z^{2019}-\frac{9}{x^{101}+y^{101}+z^{101}}$ is

(a) 0 (b) 1 (c) 2 (d) 3

32) If the function $f(x)\sin^{-1}(x^2-3)$, then x belongs to

(a) [-1,1] (b) [$\sqrt{2}$,2] (c) $[-2,-\sqrt{2}] \cup [\sqrt{2},2]$ (d) $[-2,-\sqrt{2}] \cap [\sqrt{2},2]$

33) If $\sin^{-1}\frac{x}{5} + cosec^{-1}\frac{5}{4} = \frac{\pi}{2}$, then the value of x is

(a) 4 (b) 5 (c) 2 (d) 3

34) $\sin(\tan^{-1}x)$, |x|< 1 ia equal to

(a) $\frac{x}{\sqrt{1-x^2}}$ (b) $\frac{1}{\sqrt{1-x^2}}$ (c) $\frac{1}{\sqrt{1+x^2}}$ (d) $\frac{x}{\sqrt{1+x^2}}$

35) If $\tan^{-1}(3)+\tan^{-1}(x)=\tan^{-1}(8)$then x=

(a) 5 (b) $\frac{1}{5}$ (c) $\frac{5}{14}$ (d) $\frac{14}{5}$

36)

the distance between the foci is

(a) $\frac{4}{3}$ (b) $\frac{4}{\sqrt{3}}$ (c) $\frac{2}{\sqrt{3}}$ (d) $\frac{3}{2}$

37) The length of the diameter of the circle which touches the x -axis at the point (1,0) and passes through the point (2,3) .

(a) $\frac{6}{5}$ (b) $\frac{5}{3}$ (c) $\frac{10}{5}$ (d) $\frac{3}{5}$

38) If P(x, y) be any point on $16x^2+25y^2=400$ with foci F1 (3,0) and F2 (-3,0) then PF_1 PF_2 + is

(a) 8 (b) 6 (c) 10 (d) 12

39) If the normals of the parabola $y^2 = 4x$ drawn at the end points of its latus rectum are tangents to the circle $(x-3)^2+(y+2)^2=r^2$, then the value of r^2 is

(a) 2 (b) 3 (c) 1 (d) 4

40) The eccentricity of the ellipse $(x-3)^2 +(y-4)^2 =\frac{y^2}{9}$ is

(a) $\frac{\sqrt{3}}{2}$ (b) $\frac{1}{3}$ (c) $\frac{1}{3\sqrt{2}}$ (d) $\frac{1}{\sqrt{3}}$

41) If the two tangents drawn from a point P to the parabolay2 = 4xare at right angles then the locus of P is

(a) 2x+1=0 (b) x = −1 (c) 2x−1=0 (d) x =1

42) If a parabolic reflector is 20 em in diameter and 5 em deep, then its focus is

(a) (0,5) (b) (5,0) (c) (10,0) (d) (0, 10)

43) The locus of the point of intersection of perpendicular tangents of the parabola $y^2 = 4ax$ is

(a) latus rectum (b) directrix (c) tangent at the vertex (d) axis of the parabola

44) If $\vec{a}$ and $\vec{b}$ are parallel vectors, then $[\vec{a}, \vec{c}, \vec{b}]$ is equal to

(a) 2 (b) -1 (c) 1 (d) 0

45) $\vec{a}.\vec{b} = \vec{b}.\vec{c} = \vec{c}.\vec{a} = 0$, then the value of $[\vec{a}, \vec{b}, \vec{c}]$ is

(a) $|\vec{a}|\,|\vec{b}|\,|\vec{c}|$ (b) $\frac{1}{3}|\vec{a}|\,|\vec{b}|\,|\vec{c}|$ (c) 1 (d) -1

46) If $\vec{a}, \vec{b}, \vec{c}$ are three unit vectors such that $\vec{a}$ is perpendicular to $\vec{b}$ and is parallel to $\vec{c}$ then $\vec{a} \times (\vec{b} \times \vec{c})$ is equal to

(a) $\vec{a}$ (b) $\vec{b}$ (c) $\vec{c}$ (d) $\vec{0}$

47) If $\vec{a}, \vec{b}, \vec{c}$ are non-coplanar, non-zero vectors such that $[\vec{a}, \vec{b}, \vec{c}] = 3$, then $\{[\vec{a} \times \vec{b}, \vec{b} \times \vec{c}, \vec{c} \times \vec{a}]\}^2$ is equal to

(a) 81 (b) 9 (c) 27 (d) 18

48) If $\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \times \vec{b}) \times \vec{c}$ where $\vec{a}, \vec{b}, \vec{c}$ are any three vectors such that $\vec{a}, \vec{b} \neq 0$ and $\vec{a}.\vec{b} \neq 0$ then $\vec{a}$ and $\vec{c}$ are

(a) perpendicular (b) parallel (c) inclined at an angle $\frac{\pi}{3}$ (d) inclined at an angle $\frac{\pi}{6}$

49) If the length of the perpendicular from the origin to the plane $2x + 3y + \lambda z =1, \lambda > 0$ is $\frac{1}{5}$ then the value of λ is

(a) $2\sqrt{3}$ (b) $3\sqrt{2}$ (c) 0 (d) 1

50) The p.v, OP of a point P make angles 60^o and 45^o with X and Y axis respectively. The angle of inclination of $\overrightarrow{OP}$ with z-axis is

(a) 75^o (b) 60^o (c) 45^o (d) 3

****

50 x 1 = 50

1) (b) 4
2) (b) -80
3) (b) adj(AB) = (adj A)(adj B)
4) (d) 1
5) (d) $e^{(\Delta_1/\Delta_3)}, e^{(\Delta_2/\Delta_3)}$

7) (b) consistent

8) (d) $\frac{\pi}{4}$

9) (d) $a^2 + b^2 + c^2 + 2abc = 1$

10) (a) 0

11) (a) Is always consistent

12) (a) 1+ i

13) (a) $\frac{1}{2}|z|^2$

14) (a) $\frac{3}{2} - 2i$

15) (d) 0

16) (d) (1,1)

17) (d) $-\sqrt{3i}$

18) (d) 0

19) (c) -$\frac{\pi}{2}$

20) (a) 9

21) (a) n distinct roots

22) (d) |k|≥6

23) (a) 2

24) (c) a<0

25) (b) n

26) (a) (x - ∝)(x -β) =0

27) (b) real and positive

28) (d) $\frac{-2}{3}$

29) (b) $\frac{\pi}{3}$

30) (b) $0\pi \leq x \leq 0$

31) (a) 0

32) (c) $\left[-2, -\sqrt{2}\right] \cup \left[\sqrt{2}, 2\right]$

33) (d) 3

34) (d) $\frac{x}{\sqrt{1+x^2}}$

35) (b) $\frac{1}{5}$

36) (c) $\frac{2}{\sqrt{3}}$

37) (c) $\frac{10}{5}$

38) (c) 10

39) (a) 2

40) (b) $\frac{1}{3}$

41) (b) x = −1

42) (b) (5,0)

43) (b) directrix

44) (d) 0

45) (a) $|\vec{a}| \left|\vec{b}\right| |\vec{c}|$

46) (b) $\vec{b}$

48) (b) parallel

49) (a) $2\sqrt{3}$

50) (b) 60^{o}

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Quarterly important 2 marks with answers

12th Standard

Maths

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Total Marks : 100

49 x 2 = 98

1) If A = $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is non-singular, find A^{-1}.

We first find adj A.By definition, we get adj A = $\begin{bmatrix} +M_{11} & -M_{12} \\ -M_{21} & +M_{22} \end{bmatrix}^T = \begin{bmatrix} d & -c \\ -b & a \end{bmatrix}^T = \begin{bmatrix} d & -c \\ -c & a \end{bmatrix}$.

Since A is non-singular, |A| = ad - bc ≠ 0.

As $A^{-1} = \frac{1}{|A|}$ adj A, we get A_{-1} = $\frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

2) If A is a non-singular matrix of odd order, prove that |adj A| is positive

Let A be a non-singular matrix of order 2m+1, where m=0,1,2,... Then, we get |A| ≠ 0 and, by property (ii), we have |adj A| = $|A|^{(2m+1)} - 1 = |A|^{2m}$.

Since $|A|^{2m}$ is always positive, we get that |adj A| is positive.

3) If A is symmetric, prove that then adj Ais also symmetric.

Suppose A is symmetric. Then, $A^T = A$ and so, by property (vi), we get

$adj(A^T) = (adj\ A)^T \Rightarrow adj\ A = (adj\ A)^T \Rightarrow$ adj A is symmetric.

4) Find the inverse (if it exists) of the following:

$\begin{bmatrix} -2 & 4 \\ 1 & -3 \end{bmatrix}$

$\begin{bmatrix} -2 & 4 \\ 1 & -3 \end{bmatrix}$

Let A= $\begin{bmatrix} -2 & 4 \\ 1 & -3 \end{bmatrix}$

|A|= $\begin{vmatrix} -2 & 4 \\ 1 & -3 \end{vmatrix}$ =6-4=2≠0

Since A is non singular, A^{-1} exists

$A^{-1} = \frac{1}{|A|}$

Now, adj A= $\begin{bmatrix} -3 & -4 \\ -1 & -2 \end{bmatrix}$

[Inter change the entries in leading diagonal and change the sign of elements in the off diagonal]

∴ $A_{-1} = \frac{1}{2}\begin{bmatrix} -3 & -4 \\ -1 & -2 \end{bmatrix}$.

5) Reduce the matrix $\begin{bmatrix} 3 & -1 & 2 \\ -6 & 2 & 4 \\ -3 & 1 & 2 \end{bmatrix}$ to a row-echelon form.

$$\begin{bmatrix} 3 & -1 & 2 \\ -6 & 2 & 4 \\ -3 & 1 & 2 \end{bmatrix} \xrightarrow{\substack{R_2 \longrightarrow R_2+2R_1, \\ R_3 \longrightarrow R_3+R_1}} \begin{bmatrix} 3 & -1 & 2 \\ 0 & 0 & 8 \\ 0 & 0 & 4 \end{bmatrix} \xrightarrow{R_3 \longrightarrow R_3-\frac{1}{2}R_2} \begin{bmatrix} 3 & -1 & 2 \\ 0 & 0 & 8 \\ 0 & 0 & 0 \end{bmatrix}$$

Note

$\begin{bmatrix} 3 & -1 & 2 \\ 0 & 0 & 8 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_2 \longrightarrow R_2/8} \begin{bmatrix} 3 & -1 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$. This is also a row-echelon form of the given matrix.

So, a row-echelon form of a matrix is not necessarily unique.

6) Find the rank of the following matrices by minor method:

$\begin{bmatrix} 2 & -4 \\ -1 & 2 \end{bmatrix}$

$\begin{bmatrix} 2 & -4 \\ -1 & 2 \end{bmatrix}$

Let A= $\begin{bmatrix} 2 & -4 \\ -1 & 2 \end{bmatrix}$

A is a matrix of order 2 x 2

$\therefore \rho$(A) ≤ min(2,2)=2

The highest order of minor of A is 2

it is $\begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix}$ =4-4=0

So, ρ(A)< 2

Next consider the minor of order 1 |2|=2≠0

$\therefore \rho$(A)=1

7) Find the rank of the following matrices by minor method:

$\begin{bmatrix} 1 & -2 & -1 & 0 \\ 3 & -6 & -3 & 1 \end{bmatrix}$

$\begin{bmatrix} 1 & -2 & -1 & 0 \\ 3 & -6 & -3 & 1 \end{bmatrix}$

Let A= $\begin{bmatrix} 1 & -2 & -1 & 0 \\ 3 & -6 & -3 & 1 \end{bmatrix}$

A is a matrix of order ()2 x 4)

$\therefore \rho$(A) ≤ min(2,4)=2

The highest order of minor of A is 2

It is $\begin{vmatrix} 1 & -2 \\ 3 & -6 \end{vmatrix}$ =-6+6=0

Also, $\begin{vmatrix} -1 & 0 \\ -3 & 1 \end{vmatrix}$ =-1+0=-1≠0

$\therefore \rho$(A)=2

8) Find the rank of the following matrices which are in row-echelon form :

$$\begin{bmatrix} 6 & 0 & -9 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Let A = $\begin{bmatrix} 6 & 0 & -9 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. Then A is a matrix of order 4 × 3 and ρ(A) ≤ 3.

The last two rows are zero rows. There are several second order minors. We find that there is a second order minor, for example, $\begin{vmatrix} 6 & 0 \\ 0 & 2 \end{vmatrix}$ = (6)(2) = 12 ≠ 0. So. ρ(A) = 2.

Note that there are two non-zero rows. The third and fourth rows are zero rows.

9) Solve 6x - 7y = 16, 9x - 5y = 35 using (Cramer's rule).

Δ = $\begin{vmatrix} 6 & -7 \\ 9 & -5 \end{vmatrix}$ =-30 +63=33

Δ_1 = $\begin{vmatrix} 16 & -7 \\ 35 & -5 \end{vmatrix}$ =-80+245 =165

Δ_2 = $\begin{vmatrix} 6 & 16 \\ 9 & 35 \end{vmatrix}$ =210 - 144 =66

∴ x = $\frac{\triangle_1}{\triangle} = \frac{165}{33}$ =5

y= $\frac{\triangle_2}{\triangle} = \frac{66}{33}$ =2

∴ Solution set is { 5, 2}

10) Simplify the following

$i^{1947}+i^{1950}$

$i^{1947}+i^{1950}$

$i^{1947}+i^{1950}=i^{1944}.i^3+i^{1948}.i^2$

[∴ 1944 is a multiple of 4 of 1948 is also a multiple of 4]

$=(i^4)486.i^2.i^1+(i^4)487.i^2$ $[i^4=1]$

$=(1^{486})(-1)(i)+(1)^{487}(-1)$ $[i^2=-1]$

=-i-1

=-1-i

11) Write $\dfrac{3+4i}{5-12i}$ in the x+iy form, hence find its real and imaginary parts.

To find the real and imaginary parts of $\dfrac{3+4i}{5-12i}$ first it should be expressed in the rectangular formx+iy .To simplify the quotient of two complex numbers, multiply the numerator and denominator by the conjugate of the denominator to eliminate i in the denominator $\dfrac{3+4i}{5-12i} = \dfrac{(3-4i)(5+12i)}{(5-12i)(5+12i)}$

= $\dfrac{(15-48)+(20+36)}{5^2+12^2}$

= $\dfrac{-33+56i}{169} = \dfrac{33}{169} + i\dfrac{56}{169}$

Therefore, $\frac{3+4i}{5-12i}=\frac{33}{169}+i\frac{56}{169}$ This is in the x+iy form.

Hence real part is $-\frac{33}{169}$ and imaginary part is $\frac{56}{169}$

12) Find z^{-1}, if z=(2+3i)(1− i).

We find that z = (2+3i)(1−i)=(2+3)+(3−2)i=5+i

$\Rightarrow z^{-1}=\frac{1}{z}=\frac{1}{5+i}$

Multiplying the numerator and denominator by the conjugate of the denominator, we get

$z^{-1}=\frac{(5-1)}{(5+i)(5-i)}=\frac{5-i}{5^2+I^2}=\frac{5}{26}-i\frac{1}{26}$

$\Rightarrow z^{-1}=\frac{5}{26}-i\frac{1}{26}$

13) Find the square root of 6−8i .

We compute $|6-8i|=\sqrt{6^2+(-8)^2}=10$

and applying the formula for square root, we get

$\sqrt{6-8i}=\pm\left(\sqrt{\frac{10+6}{2}}-i\sqrt{\frac{10-6}{2}}\right)$

= $\pm\left(\sqrt{8}+i\sqrt{2}\right)$

= $\pm\left(2\sqrt{2}-i\sqrt{2}\right)$

14) Show that the following equations represent a circle, and, find its centre and radius|

$|z-2-i|=3$

$|z-2-i|=3$

$\Rightarrow$ |z-(2+i)|=3

It is of the form $|z-z_0|$ =r and so it represents a circle.

15) Write in polar form of the following complex numbers

$2+i2\sqrt{3}$

2+i2$\sqrt{3}$

Let 2+i2$\sqrt{3}$ =x+iy=r (cosθ+i sinθ)

r =modulus =$\sqrt{x^2+y^2}$

=$\sqrt{2^2+(2\sqrt{3})^2}$

=$\sqrt{4+12}=\sqrt{16}$ =4

α=tan-1$\left|\frac{y}{x}\right|=tan^{-1}\left|\frac{2\sqrt{3}}{2}\right|$

=$tan^{-1}(\sqrt{3})=\frac{\pi}{3}$

Since the complex number 2+i2 $\sqrt{3}$ lies in the I quadrant, [x,y both +ve] its principal value

θ=α=$\frac{\pi}{3}$

∴ Its polar form is 2+i2$\sqrt{3}$

=4$\left[cos\left(2k\pi+\frac{\pi}{3}\right)+isin\left(2k\pi+\frac{\pi}{3}\right)\right],k\in Z$.

16) Find the value of $\sum_{k=1}^{8}\left(cos\frac{2k\pi}{9}+isin\frac{2k\pi}{9}\right)$.

$cis\frac{2\pi}{9}+cis\frac{4\pi}{9}+cis\frac{6\pi}{9}+cis\frac{8\pi}{9}+cis\left(\frac{10\pi}{9}\right)+cis\frac{12\pi}{9}+cis\frac{14\pi}{9}+cis\frac{16\pi}{9}$

=$cis\left(\frac{2\pi}{9}+\frac{4\pi}{9}+\frac{6\pi}{9}+\frac{8\pi}{9}+\frac{10\pi}{9}+\frac{12\pi}{9}+\frac{14\pi}{9}+\frac{16\pi}{9}\right)$

$= \text{cis}\frac{2\pi}{9}(1+2+3+...+8) = \text{cis}\frac{2\pi}{9}\times\frac{8\times9}{2}$

= $\left[\because 1+2+3+....+n = \frac{n(n+1)}{2}\right]$

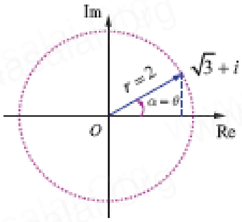
=cis8π=[cos(8π)+i sin8π]

=-1+i(0) [∴ cos8π=-1 and sin 8π=0=-1

17) Find the modulus and principal argument of the following complex numbers.

$\sqrt{3}+i.$

$\sqrt{3}+i$



Modulus = $\sqrt{x^2+y^2} = \sqrt{(\sqrt{3})^2+1^2} = \sqrt{3+1} = 2$

$\alpha = tan^{-1}\left|\frac{y}{x}\right| = tan^{-1}\frac{1}{\sqrt{3}} = \frac{\pi}{6}$

Since the complex number $\sqrt{3+i}$ lies in the first quadrant, has the principal value

$\theta = \alpha = \frac{\pi}{6}$

Therefore, the modulus and principal argument of $\sqrt{3+i}$ are 2 and $\frac{\pi}{6}$ respectively.

18) Simplify the following

$i^{59}+\frac{1}{i^{59}}$

$i^{59}+\frac{1}{i^{59}}$

$i^{4\times14+3}+i^{-(4\times14+3)}$

$=(i^4)^{14}.i^3+(i^4)^{-14}.i^{-3}$

$=1.i^3+1.i^{-3}$ [∵ $i^4=1$]

=-i+i [∴ $i^3=-i$ and $i^{-3}=i$]

=0

19) Simplify the following

$i\, i^2 i^3...i^{2000}$

$i\, i^2\, i^3 ....i^{2000}$

$=i^{1+2+3+.....+2000}$

$=i^{\frac{2000\times2001}{2}}$

[∴ 1+2+3+....n= $\frac{n(n+1)}{2}$]

$=i^{1000\times2001}$

$=i^{2001000}$

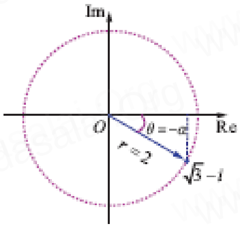
=1

[∴ 2001000 is divisible by 4 as its last two digits are divisible by 4]

20) Find the modulus and principal argument of the following complex numbers.

$\sqrt{3}$-i

$\sqrt{3}-i$



r = 2 and $-\alpha = \frac{\pi}{6}$

Since the complex number lies in the fourth quadrant, has the principal value

$$\theta = -\alpha = -\frac{\pi}{6}$$

Therefore, the modulus and principal argument of

$-\sqrt{3}-i$ are 2 and $\frac{\pi}{6}$

In all the four cases, modulus are equal, but the arguments are depending on the quadrant in which the complex number lies.

21) If α, β, and γ are the roots of the equatio $x^3+pz^2+qx+r=0$, find the value of $\Sigma\frac{1}{\beta\gamma}$ in terms of the coefficients.

Since α, β, and γ are the roots of the equation $x^3+px^2+qx+r=0$, we have

Σ_1 α + β + γ =-p amd Σ_3 αβγ = -r

$\Sigma\frac{1}{\beta\gamma} = \frac{1}{\beta\gamma} + \frac{1}{\gamma\alpha} + \frac{1}{\alpha\beta} = \frac{\alpha+\beta+\gamma}{\alpha\beta\gamma} = \frac{-p}{-r} = \frac{p}{r}$.

22) Construct a cubic equation with roots 1,2, and 3

Given roots are 1,2 and 3

Here a = 1, β = 2 and ɤ = 3

A cubic polynomial equation whose roots are

α,β,ɤ is

x^3-(α+β+ɤ)+x^2(αβ+βɤ+ɤα)x-∝βɤ=0

⇒ x^3-(1+1+2)x^2(2+6+3)x-6=0

⇒ $x^3-6x^2+11x-6=0$

23) If $x^2+2(k+2)x+9k=0$ has equal roots, find k.

Here $\Delta=b^2-4ac=0$ for equal roots. This implies $4(k+2)^2=4(9)k$. This implies k = 4 or 1.

24) Obtain the condition that the roots of $x^3+px^2+qx+r=0$ are in A.P.

Let the roots be in A.P. Then, we can assume them in the form α-d, α, α+d

Applying the Vieta's formula (α-d)+α+(α+d) = $\frac{p}{1}$=p ⇒ 3α =-p ⇒ α=-$\frac{p}{3}$.

But, we note that αis a root of the given equation. Therefore, we get

$\left(\frac{p}{3}\right)^3 + p\left(\frac{p}{3}\right)^3 + q\left(\frac{p}{3}\right)^3 + r$ =0 ⇒ 9 pq =2p3+27r.

25) Examine for the rational roots of $x^8-3x+1=0$

$x^8-3x+1=0$

Here a_n = 1, a_o = 1

If $\frac{p}{q}$ is a root of the polynomial, then as

(p, q) = 1p is a factor of a_o = 1 and q is a factor of a_n= 1

Since 1 has no factors, the given equation has no rational roots.

26) Find the principal value of

$Sin^{-1}\left(\frac{1}{\sqrt{2}}\right)$

We know that sin^{-1}: [-1,1] $\rightarrow \left[-\frac{\pi}{2},\frac{\pi}{2}\right]$ is given by

sin^{-1}x=y if and only if x=sin y for $-1\leq x\leq$ and $-\frac{\pi}{2}\leq y\leq\frac{\pi}{2}$. Thus,

$Sin^{-1}\left(\frac{1}{\sqrt{2}}\right)=\frac{\pi}{4}$, Since $\frac{\pi}{4}\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$ and sin $\frac{\pi}{4}=\frac{1}{\sqrt{2}}$

.

27) Find all the values of x such that

$-10\pi\leq x\leq 10\pi$ and sin x=0

Given sin x = 0

$\Rightarrow$ sin x = sin 0

$\Rightarrow\ x=n\pi, n\varepsilon z$

Since $-10\pi\leq x\leq 10\pi$ n can take the values only from -10 to +10.

$\therefore x=n\pi$, When $n=0,\pm,\pm2,\pm3,\pm4,\pm5,\pm6,\pm7,\pm8,\pm9,\pm10$

28) Find the period and amplitude of

y=sin 7x

The amplitude of sin x is 1 [Max of sinx curve is 1]

$\Rightarrow$ amplitude of sin 7x is also 1.

If p is the period of the function,

then f(x+p)=f(x).

Since the period of sine function is 2π. The period.of sin is $\frac{2\pi}{7}$ Since sin $7\left(\frac{2\pi}{7}\right)=sin2\pi$

29) Find the value of

$tan(tan^{-1}(\frac{7\pi}{4}))$

$tan(tan^{-1}(\frac{7\pi}{4}))$

= $tan(tan^{-1}(\frac{7\pi}{4}))$

= $\frac{7\pi}{4}$[$\therefore$ [tan (tan$_{-1}$(x)- x for any real number]

30) Find the value of

$\tan^{-1}(tan(-\frac{\pi}{6}))$

$\tan^{-1}(tan(-\frac{\pi}{6}))$

= $\frac{-\pi}{6}$

Since $\frac{-\pi}{6}\epsilon\left(\frac{-\pi}{2},\frac{\pi}{2}\right)$

31) Find the value of

tan(tan^{-1}(-0.2021)).

tan(tan-1(0.2021))

= -0.2021[$\because$ tan (tan^{-1}x)=x for any real number]

32) Find the value of

$cos\left[\frac{1}{2}cos^{-1}\left(\frac{1}{8}\right)\right]$

Consider cos$\left[\frac{1}{2}cos^{-1}\left(\frac{1}{8}\right)\right] Let\quad cos^{-1}\left(\frac{1}{8}\right)=\theta.\quad Then,\quad cos\theta=\frac{1}{8}and\quad \theta\in[0,\pi]$

Now, cos$\theta=\frac{1}{8}$ implies 2cos$^2\frac{\theta}{2}-1=\frac{1}{8}$. Thus, cos$(\frac{\theta}{2})$ is positive

Thus, $cos\left[\frac{1}{2}cos^{-1}\left(\frac{1}{8}\right)\right]=cos\left(\frac{\theta}{2}\right)=\frac{3}{4}$

33) Determine whether x+y−1=0 is the equation of a diameter of the circle $x^2+y^2-6x+4y+c = 0$ for all possible values of c .

Centre of the circle is (3,-2) which lies on x+y−1=0. So the line x+y−1=0 passes through the centre and therefore the linex+y−1=0 is a diameter of the circle for all possible values of c .

34) Find the general equation of the circle whose diameter is the line segment joining the points (−4,−2)and (1,1).

Equation of the circle with end points of the diameter as x y 1 1 (,)and x y 2 2 (,)given in theorem 5.2 is

$(x-x_1)(x-x_2)+(y-y_1)(y-y_2) = 0$

$(x+4)(x-1)+(y+2)(y-1) = 0$

$x^2+y^2+3x+y-6= 0$ is the required equation of the circle.

35) The line 3x+4y−12 = 0 meets the coordinate axes at A and B . Find the equation of the circle drawn on AB as diameter.

Writing the line 3x+4y=12, in intercept form yields .Hence the points A and B are (4,0) and(0,3) .

Equation of the circle in diameter form is

$(x-x_1)(x-x_2)+(y-y_1)(y-y_2) = 0$

$(x-4)(x-0)+(y-0)(y-3) = 0$

$x^2+y^2-4x-3y = 0$.

36) If y=4x+c is a tangent to the circle $x^2+y^2=9$, find c .

The condition for the line y=mx+c to be a tangent to the circle $x^2+y^2=a^2$ is $c^2=a^2(1+m^2)$ from 5.2.3.

Then $c=\pm\sqrt{9\left(1+16\right)}$

$c=\pm3\sqrt{17}$

37) Obtain the equation of the circle for which (3,4) and (2,-7) are the ends of a diameter.

Given ends of diameter are (3, 4)(2, -7)

∴ Equation of the circle is

$(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0$

$\Rightarrow (x - 3)(x - 2) + (y - 4)(y + 7) =0$

$\Rightarrow x^2 - 2x - 3x + 6 + y^2 + 7y - 4y - 28 = 0$

$\Rightarrow x^2+y^2-5x+3y-22=0$

38) Find centre and radius of the following circles.

$x_2+ (y+2)^2 =0$

Equation of the circle is $x^2 + (y + 2)^2 = 0$

Centre is (0, -2) and radius is 0.

39) Find centre and radius of the following circles.

$2x^2+2y^2-6x+4y+2=0$

Equation of the circle is

$2x^2 + 2y^2 - 6x + 4y + 2 = 0$

Dividing by 2, we get

$x^2 + y^2 - 3x + 2y + 1 = 0$

Here 2g = -3 $\Rightarrow$ g = $\frac{-3}{2}$

2f = 2 $\Rightarrow$ f = 1

and c = 1

∴ Centre is (-g, -f) = $\left(\frac{3}{2}, -1\right)$

and r = $\sqrt{g^2+f^2-c}$ = $\sqrt{\left(\frac{3}{2}\right)^2+1^2-1}$

= $\sqrt{\frac{9}{4}} = \frac{3}{2}$ units.

40) If $\hat{a} = -\hat{3}i - \hat{j} + 5\hat{k}$, $\hat{b} = \hat{i} - \hat{2}j + \hat{k}$, $\hat{c} = \hat{4}i - 4\hat{k}$and $\hat{a}.(\hat{b}\times\hat{c})$

We find,

$\hat{a}.(\hat{b}\times\hat{c})$ = $\begin{vmatrix} -3 & -1 & 5 \\ 1 & -2 & 1 \\ 0 & 4 & -5 \end{vmatrix} = -3$

41) If $\hat{2i} - \hat{j} + \hat{3k}, \hat{3i} + \hat{2j} + \hat{k}, \hat{i} + \hat{mj} + \hat{4k}$ are coplanar, find the value of m.

Since the given three vectors are coplanar, we have $\begin{vmatrix} 2 & -1 & 3 \\ 3 & 2 & 1 \\ 1 & m & 4 \end{vmatrix}$ = 0 ⇒ m = -3

42) If $\vec{a}, \vec{b}, \vec{c}$ are three vectors, prove that $[\vec{a}+\vec{c}, \vec{a}+\vec{b}, \vec{a}+\vec{b}+\vec{c}]$ = $[\vec{a}, \vec{b}, \vec{c}]$

We get

$[\vec{a}+\vec{c}, \vec{a}+\vec{b}, \vec{a}+\vec{b}+\vec{c}]$ = $\begin{vmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{vmatrix} [\vec{a}, \vec{b}, \vec{c}]$

= - $[\vec{a}, \vec{b}, \vec{c}]$

43) Find the volume of the parallelepiped whose coterminous edges are represented by the vectors $-6\hat{i} + 14\hat{j} + 10\hat{k}, 14\hat{i} - 10\hat{j} - 6\hat{k}$ and $2\hat{i} + 4\hat{j} - 2\hat{k}$

Let $\vec{a} = -6\hat{i} + 14\hat{j} + 10\hat{k}$, $\vec{b} = 14\hat{i} - 10\hat{j} - 6\hat{k}$ and $\vec{c} = 2\hat{i} + 4\hat{j} - 2\hat{k}$

Volume ofthe parallelepiped having $\vec{a}, \vec{b}$ and $\vec{c}$ as its co-terminus edges is $\vec{a}.(\vec{b}\times\vec{c})$.

∴ $\vec{a}.(\vec{b}\times\vec{c}) = \begin{vmatrix} -6 & 14 & 10 \\ 14 & -10 & -6 \\ 2 & 4 & -2 \end{vmatrix}$

= $-6\begin{vmatrix} -10 & -6 \\ 4 & -2 \end{vmatrix} - 14\begin{vmatrix} 14 & -6 \\ 2 & -2 \end{vmatrix} + 10\begin{vmatrix} 14 & -10 \\ 2 & 4 \end{vmatrix}$

=-6(20 + 24) - 14(-28 + 12) + 10(56 + 20)

= -6(44) -14(-16) + 10(76)

= -264 + 224 + 760 = 720.

∴ Volume of the required parallelepiped = 720 cubic units.

44) The volume of the parallelepiped whose coterminus edges are $7\hat{i} + \lambda\hat{j} - 3\hat{k}, \hat{i} + 2\hat{j} - \hat{k}, -3\hat{i} + 7\hat{j} + 5\hat{k}$ is 90 cubic units. Find the value of λ.

Let $\vec{a} = 7\hat{i} + \lambda\hat{j} - 3\hat{k}, \vec{b} = \hat{i} + 2\hat{j} - \hat{k}$ and $\vec{c} = -3\hat{i} + 7\hat{j} - 5\hat{k}$

∴ volume of the parallelepiped

=$\vec{a}.(\vec{b}\times\vec{c})$

Given $\vec{a}.(\vec{b}\times\vec{c})$ =90

⇒ $\begin{vmatrix} 7 & \lambda & -3 \\ 1 & 2 & -1 \\ -3 & 7 & 5 \end{vmatrix}$ =90

⇒ $-6\begin{vmatrix} 2 & -1 \\ 7 & 5 \end{vmatrix} - \lambda\begin{vmatrix} 1 & -1 \\ -3 & 5 \end{vmatrix} - 3\begin{vmatrix} 1 & 2 \\ -3 & 7 \end{vmatrix}$ =90

⇒ 7(10+7)-λ(5-3)-3(7+6)=90

⇒ 7(17)-λ(2)-3(13)=90

⇒ 119-2λ-39=90

⇒ 119-39-90=2λ

⇒ -10=2λ

⇒ λ=-5

45) Show that the points (2, 3, 4),(−1, 4, 5) and (8,1, 2) are collinear.

Let the points be A (2,3,4), B (-1, 4, 5) and C (8, 1,2)

Equation of the line joining A and B is

$\frac{x-x_1}{x_2-x_1}=\frac{y-y_1}{y_2-y_1}=\frac{z-z_1}{z_2-z_1}$

⇒ $\frac{x-2}{-1-2}=\frac{y-3}{4-}=\frac{z-4}{5-4}$

⇒ $\frac{x-2}{-3}=\frac{y-3}{1}=\frac{z-4}{1}$

Substitute the point C (8, 1, 2) in line (1),

$\frac{8-2}{-3}=\frac{1-3}{1}=\frac{2-4}{1}$

⇒ -2=-2=-2

Since the point C satisfies the equation of line joining A and B, all the three points lie on the same line.

Hence the given points are collinear.

46) Find the direction cosines and length of the perpendicular from the origin to the plane $\vec{r}.(3\hat{i}-4\hat{j}+12\hat{k})=5$

Let $\vec{d}=3\hat{i}-4\hat{j}+12\hat{k}$ and q = 5.

If $\vec{d}$ is the unit vector in the direction of the vector $3\hat{i}-4\hat{j}+12\hat{k}$, then $\vec{d}=\frac{1}{13}(3\hat{i}-4\hat{j}+12\hat{k})$

Now, dividing the given equation by 13, we get

$\hat{r}.\left(\frac{3}{13}\hat{i}-\frac{4}{13}\hat{j}+\frac{12}{13}\hat{k}\right)=\frac{5}{13}$

which is the equation of the plane in the normal form $\hat{r}.\hat{d}=p$

From this equation, we infer that $\hat{d}=\frac{1}{3}(3\hat{i}-4\hat{j}+12\hat{k})$ is a unit vector normal to the plane from the origin.

Therefore, the direction cosines of $\frac{3}{13},\frac{-4}{13},\frac{12}{13}$ and the length of the perpendicular from the origin to the plane is $\frac{5}{13}$

47) Find the angle between the straight line $\vec{r}=(2\hat{i}+\hat{j}+\hat{k})+t(\hat{i}-\hat{j}+\hat{k})$ and the plane 2x-y+z=5

The angle between a line $\vec{r}=\vec{a}+t\vec{b}$ and a plane $\vec{r}.\vec{n}$=p with normal $\vec{n}$ is θ= $sin^{-1}\left(\frac{|\vec{b}.\vec{n}|}{|\vec{b}.\vec{n}|}\right)$

Here, $\vec{b}=\hat{i}-\hat{j}+\hat{k}$ and $\vec{n}=2\hat{i}\hat{j}+\hat{k}$

So,we get θ= $sin^{-1}\left(\frac{|\vec{b}.\vec{n}|}{|\vec{b}.\vec{n}|}\right)=sin^{-1}\left(\frac{|(\hat{i}-\hat{j}+\hat{k}).(2\hat{i}-\hat{j}+\hat{k})|}{|\hat{i}-\hat{j}+\hat{k}||2\hat{i}-\hat{j}+\hat{k}|}\right)=sin^{-1}\left(\frac{2\sqrt{3}}{3}\right)$.

48) Find the distance between the planes $\vec{r}.(2\hat{i}-\hat{j}-\hat{k})$=6 and $\vec{r}.(6\hat{i}-\hat{j}-2\hat{k})$= 27

Let $\vec{\mu}$ the position vector of an arbitrary point on the plane $\vec{r}.(2\hat{i}-\hat{j}-\hat{k})$ = 6. Then, we have

$\vec{\mu}.(2\hat{i}-\hat{j}-2\hat{k})=6$ .....(1)

If δ is the distance between the given planes, then δ is the perpendicular distance from $\vec{\mu}$ to the plane

$\vec{r}.(6\hat{i}-\hat{j}-2\hat{k})$ = 27

Therefore, δ = $\frac{|\vec{u}.\vec{n}-p|}{|\vec{n}|}=\left|\frac{\vec{u}.(6\hat{i}-3\hat{j}-6\hat{k})-27}{\sqrt{6^2+(-3)^2+(-6)^2}}\right|=\left|\frac{3(\vec{u}.(2\hat{i}-\hat{j}-2\hat{k}))-27}{9}\right|=\left|\frac{(3(6)-27}{9}\right|=1$ unit

49) Find the angle between the following lines.

2x = 3y = −z and 6x = − y = −4z.

Given lines are 2x = 3y = -2 and 6x = -y - 4z

2x=3y=-2 ⇒ $\frac{x-0}{\frac{1}{2}}=\frac{y-0}{-1}=\frac{z-0}{-1}$

⇒ $\vec{b}=\frac{1}{2}\hat{i}+\frac{1}{3}\hat{j}-\hat{k}$

6x=-y=-4z ⇒ $\frac{x-0}{\frac{1}{6}}=\frac{y-0}{\frac{1}{-1}}=\frac{z-0}{\frac{1}{-4}}$

⇒ $\hat{d}=\frac{1}{6}-\hat{i}-\hat{j}-\frac{1}{4}\hat{k}$

$\vec{b}.\vec{d}=\left(\frac{1}{2}\hat{i}+\frac{1}{3}\hat{j}-\hat{k}\right).\left(\frac{1}{6}\hat{i}-\hat{j}-\frac{1}{4}\hat{k}\right)$

1 x 3 = 3

50) Find the inverse of each of the following by Gauss-Jordan method:

$\begin{bmatrix}2 & -1\\5 & -2\end{bmatrix}$

$\begin{bmatrix}2 & -1\\5 & -2\end{bmatrix}$

Let A = $\begin{bmatrix}2 & -1\\5 & -2\end{bmatrix}$

Applying Gauss - Jordan method, we get

$[A|I_2]=\left[\begin{array}{cc|cc}2 & -1 & 1 & 0\\5 & -2 & 0 & 1\end{array}\right]$

$\xrightarrow{R_1\to R_1\div 2}\left[\begin{array}{cc|cc}1 & -\frac{1}{2} & \frac{1}{2} & 0\\5 & -2 & 0 & 1\end{array}\right]$

$\xrightarrow{R_2\to R_2-5R_1}\left[\begin{array}{cc|cc}1 & -\frac{1}{2} & \frac{1}{2} & 0\\5 & -2 & -\frac{5}{2} & 1\end{array}\right]$

$\xrightarrow{R_2\to R_2\times 2}\left[\begin{array}{cc|cc}1 & -\frac{1}{2} & \frac{1}{2} & 0\\0 & 1 & -5 & 2\end{array}\right]$

$\xrightarrow{R_1\to R_1+\frac{1}{2}R_2}\left[\begin{array}{cc|cc}1 & 0 & -2 & 1\\0 & 1 & -5 & 2\end{array}\right]$

∴ We get $A_{-1}=\begin{bmatrix}-2 & 1\\-5 & 2\end{bmatrix}$

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70 x 3 = 210

1) Verify $(AB)^{-1} = B^{-1}A^{-1}$ with $A = \begin{bmatrix} 0 & -3 \\ 1 & 4 \end{bmatrix}$, $B = \begin{bmatrix} -2 & -3 \\ 0 & -1 \end{bmatrix}$.

We get $AB = \begin{bmatrix} 0 & -3 \\ 1 & 4 \end{bmatrix}\begin{bmatrix} -2 & -3 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 0+0 & 0+3 \\ -2+0 & -3-4 \end{bmatrix} = \begin{bmatrix} 0 & 3 \\ -2 & -7 \end{bmatrix}$

$(AB)^{-1} = \frac{1}{(0+6)}\begin{bmatrix} -7 & -3 \\ 2 & 0 \end{bmatrix} = \frac{1}{6}\begin{bmatrix} -7 & -3 \\ 2 & 0 \end{bmatrix}$ ....(1)

$A^{-1} = \frac{1}{(0+3)}\begin{bmatrix} 4 & 3 \\ -1 & 0 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 4 & 3 \\ -1 & 0 \end{bmatrix}$

$B^{-1} = \frac{1}{(2-0)}\begin{bmatrix} -1 & 3 \\ 0 & -2 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} -1 & 3 \\ 0 & -2 \end{bmatrix}$

$B^{-1}A^{-1} = \frac{1}{2}\begin{bmatrix} -1 & 3 \\ 0 & -2 \end{bmatrix}\frac{1}{3}\begin{bmatrix} 4 & 3 \\ -1 & 0 \end{bmatrix} = \frac{1}{6}\begin{bmatrix} -7 & -3 \\ 2 & 0 \end{bmatrix}$ ....(2).

As the matrices in (1) and (2) are same, $(AB)^{-1}=B^{-1}A^{-1}$ is verified.

2) If $A = \frac{1}{9}\begin{bmatrix} -8 & 1 & 4 \\ 4 & 4 & 7 \\ 1 & -8 & 4 \end{bmatrix}$, prove that $A^{-1} = A^{T}$.

Given $A = \frac{1}{9}\begin{bmatrix} -8 & 1 & 4 \\ 4 & 4 & 7 \\ 1 & -8 & 4 \end{bmatrix}$

$A^{T} = \frac{1}{9}\begin{bmatrix} -8 & 4 & 1 \\ 1 & 4 & 7 \\ 4 & 7 & 4 \end{bmatrix}$ ....(1)

We know that $(\lambda A)^{-1} = \frac{1}{\lambda}.A^{-1}$

A-1 = $\left\{\frac{1}{9}\begin{bmatrix}-8 & 1 & 4\\ 4 & 4 & 7\\ 1 & -8 & 4\end{bmatrix}\right\}^{-1} = \frac{1}{9}\begin{bmatrix}-8 & 1 & 4\\ 4 & 4 & 7\\ 1 & -8 & 4\end{bmatrix}^T$

where λ=$\frac{1}{9}$

A -1 =9B -1 where B= $\begin{bmatrix}-8 & 1 & 4\\ 4 & 4 & 7\\ 1 & -8 & 4\end{bmatrix}$ ....(2)

Now, |B|= $-8\begin{vmatrix}4 & 7\\ -8 & 4\end{vmatrix} - 1\begin{vmatrix}4 & 7\\ 1 & 4\end{vmatrix} + 4\begin{vmatrix}4 & 4\\ 1 & -8\end{vmatrix}$

=-8(16+56)-1(16-7)+4(-32-4)

=-8(72)-1(9)+4(-36)=-576-9144

=-729

adj B= $\begin{bmatrix}+\begin{vmatrix}4 & 7\\ -8 & 4\end{vmatrix} & -\begin{vmatrix}4 & 7\\ 1 & 4\end{vmatrix} & +\begin{vmatrix}4 & 4\\ 1 & -8\end{vmatrix}\\ -\begin{vmatrix}1 & 4\\ -8 & 4\end{vmatrix} & +\begin{vmatrix}-8 & 4\\ 1 & 4\end{vmatrix} & -\begin{vmatrix}-8 & 1\\ 1 & -8\end{vmatrix}\\ +\begin{vmatrix}1 & 4\\ 4 & 7\end{vmatrix} & -\begin{vmatrix}-8 & 4\\ 4 & 7\end{vmatrix} & +\begin{vmatrix}-8 & 1\\ 4 & 1\end{vmatrix}\end{bmatrix}$

= $\begin{bmatrix}+(16+56)-(16-7)+(-32-3)\\ -(4+32)+(-32-4)+(64-1)\\ +(7-16)-(-56-16)+(-3-4)\end{bmatrix}$

= $\begin{bmatrix}72 & -9 & -36\\ -36 & -36 & -63\\ -9 & 72 & -36\end{bmatrix}^T = \begin{bmatrix}72 & -36 & -9\\ -9 & -36 & 72\\ -36 & -63 & -36\end{bmatrix}$

=9 $\begin{bmatrix}8 & -4 & -1\\ -1 & -4 & +8\\ -4 & -7 & -4\end{bmatrix}$

∴ B -1 = $\frac{1}{|B|}adjB = \frac{-9}{729}\begin{bmatrix}8 & -4 & -1\\ -1 & -4 & +8\\ -4 & -7 & -4\end{bmatrix}$

=$\frac{1}{81}\begin{bmatrix}-8 & 4 & 1\\ 1 & 4 & -8\\ 4 & 7 & 4\end{bmatrix}$

Substituting this in (2) we get,

A -1 = $9.\frac{1}{81}\begin{bmatrix}-8 & 4 & 1\\ 1 & 4 & -8\\ 4 & 7 & 4\end{bmatrix} = \frac{1}{9}\begin{bmatrix}-8 & 4 & 1\\ 1 & 4 & -8\\ 4 & 7 & 4\end{bmatrix}$

From (1) and (3)we get,

A T=A -1

Processing math: 100%

3)

If adj(A) = $\begin{bmatrix} 2 & -4 & 2 \\ -3 & 12 & -7 \\ -2 & 0 & 2 \end{bmatrix}$, find A.

Given adj A= $\begin{bmatrix} 2 & -4 & 2 \\ -3 & 12 & -7 \\ -2 & 0 & 2 \end{bmatrix}$

We know that A = $\pm \frac{1}{\sqrt{|adjA|}}$.adj (adj A) ....(1)

|adj A| = $2\begin{vmatrix} 12 & -7 \\ 0 & 2 \end{vmatrix} + 4\begin{vmatrix} -3 & -7 \\ -2 & 2 \end{vmatrix} + 2\begin{vmatrix} -3 & 12 \\ -2 & 0 \end{vmatrix}$

[Expanded along R_1]

=2(24-0)+4(-6-14)+2(0+24)

=2(24)+4(-20)+2(24)=48-80+48

=96-80=16

Now, adj (adj A)

= $\begin{bmatrix} +\begin{vmatrix} 12 & -7 \\ 0 & 2 \end{vmatrix} & -\begin{vmatrix} -3 & -7 \\ -2 & 2 \end{vmatrix} & +\begin{vmatrix} -3 & 12 \\ -2 & 0 \end{vmatrix} \\ -\begin{vmatrix} -4 & 2 \\ 0 & 2 \end{vmatrix} & +\begin{vmatrix} 2 & 2 \\ -2 & 2 \end{vmatrix} & -\begin{vmatrix} 2 & -4 \\ -2 & 0 \end{vmatrix} \\ +\begin{vmatrix} -4 & 2 \\ 12 & -7 \end{vmatrix} & -\begin{vmatrix} 2 & 2 \\ -3 & -7 \end{vmatrix} & +\begin{vmatrix} 2 & -4 \\ -3 & 12 \end{vmatrix} \end{bmatrix}^T$

= $\begin{bmatrix} +(24-0)-(6-14)+(0+24) \\ -(-8-0)+(4+4)-(0-8) \\ +(28-24)-(-14+6)+(24-12) \end{bmatrix}^T$

= $\begin{bmatrix} 24 & 20 & 24 \\ 8 & 8 & 8 \\ 4 & 8 & 12 \end{bmatrix}^T = \begin{bmatrix} 24 & 8 & 4 \\ 20 & 8 & 8 \\ 24 & 8 & 12 \end{bmatrix}$

= $4\begin{bmatrix} 6 & 2 & 1 \\ 5 & 2 & 2 \\ 6 & 2 & 3 \end{bmatrix}$

Substituting (2) and (3) in (1) we get,

A= $\frac{1}{\sqrt{16}}.4\begin{bmatrix} 6 & 2 & 1 \\ 5 & 2 & 2 \\ 6 & 2 & 3 \end{bmatrix}$

A= $\pm\frac{4}{4}\begin{bmatrix} 6 & 2 & 1 \\ 5 & 2 & 2 \\ 6 & 2 & 3 \end{bmatrix} = \pm\begin{bmatrix} 6 & 2 & 1 \\ 5 & 2 & 2 \\ 6 & 2 & 3 \end{bmatrix}$

4)

A = $\begin{bmatrix} 1 & \tan x \\ -\tan x & 1 \end{bmatrix}$, show that $A^TA^{-1} = \begin{bmatrix} \cos 2x & -\sin 2x \\ \sin 2x & \cos 2x \end{bmatrix}$

Given A= $\begin{bmatrix} 1 & \tan x \\ -\tan x & 1 \end{bmatrix}$

|A| =1+tan²x

$\therefore A^{-1} = \frac{1}{|A|} adjA$

= $\frac{1}{1+tan^2} \begin{bmatrix} 1 & -tanx \\ tanx & 1 \end{bmatrix}$

[Interchange the elements in the leading diagonal and change the sign of elementsin the off diagonal]

$A^T = \begin{bmatrix} 1 & -tanx \\ tanx & 1 \end{bmatrix}$

$\therefore A^T A^{-1}$

$= \begin{bmatrix} 1 & -tanx \\ tanx & 1 \end{bmatrix} \frac{1}{1+tan^2x} \begin{bmatrix} 1 & -tanx \\ tanx & 1 \end{bmatrix}$

$= \frac{1}{1+tan^2x} \begin{bmatrix} 1 & -tanx \\ tanx & 1 \end{bmatrix} \begin{bmatrix} 1 & -tanx \\ tanx & 1 \end{bmatrix}$

$= \frac{1}{1+tan^2x} \begin{bmatrix} 1 - tan^2x & -tanx - tanx \\ tanx + tanx & -tan^2x + 1 \end{bmatrix}$

$= \begin{bmatrix} \frac{1-tan^2x}{1+tan^2x} & \frac{-2tanx}{1+tan^2x} \\ \frac{2tanx}{1+tan^2x} & \frac{1-tan^2x}{1+tan^2x} \end{bmatrix}$

$A_T A_{-1} = \begin{bmatrix} cos2x & -sin2x \\ sin2x & cos2x \end{bmatrix}$

[$\because$ $sin2x = \frac{2tanx}{1+tan^2x}$ and $cos2x = \frac{1-tan^2x}{1+tan^2x}$].

5) Given A = $\begin{bmatrix} 1 & -1 \\ 2 & 0 \end{bmatrix}$, B = $\begin{bmatrix} 3 & -2 \\ 1 & 1 \end{bmatrix}$ and C = $\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$, find a matrix X such that AXB = C.

Given A= $\begin{bmatrix} 1 & -1 \\ 2 & 0 \end{bmatrix}$, B= $\begin{bmatrix} 3 & -2 \\ 1 & 1 \end{bmatrix}$ and C= $\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$

Also, A X B=C

Premultiply by A^{-1} we get,

$(A^{-1} A) X B = A^{-1} C$

$\Rightarrow XB=A^{-1}C$ [$\because A^{-1}a=1$]

Post Multiply by B^{-1} we get

$(X B) B^{-1} = (A^{-1} C) B^{-1}$

$\Rightarrow X = (A^{-1} C) B^{-1}$

$|A| = \begin{vmatrix} 1 & -1 \\ 2 & 0 \end{vmatrix} = 0+2=2\neq0$

$\therefore A_{-1} = \frac{1}{|A|} adjA = \frac{1}{2} \begin{bmatrix} 0 & 1 \\ -2 & 1 \end{bmatrix}$

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$|B| = \begin{vmatrix} 3 & -2 \\ 1 & 1 \end{vmatrix} = 3+2 = 5 \neq 0$

$\therefore B^{-1} = \frac{1}{|B|} adjB = \frac{1}{5}\begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}$

$A^{-1}C = \frac{1}{2}\begin{bmatrix} 0 & 1 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$

$= \frac{1}{2}\begin{bmatrix} 0+2 & 0+2 \\ -2+2 & -2+2 \end{bmatrix}$

$= \frac{1}{2}\begin{bmatrix} 2 & 2 \\ 0 & 0 \end{bmatrix} = \frac{1}{2}(2)\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$

$= \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$

$\therefore X = Z(A^{-1}C).B^{-1}$

$= \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}\frac{1}{5}\begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 1-1 & 2+3 \\ 0+0 & 0+0 \end{bmatrix}$

$= \frac{1}{5}\begin{bmatrix} 0 & 5 \\ 0 & 0 \end{bmatrix} = \frac{1}{5}(5)\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

$\therefore X = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$.

6) Decrypt the received encoded message $\begin{bmatrix} 2 & -3 \end{bmatrix}\begin{bmatrix} 20 & 4 \end{bmatrix}$ with the encryption matrix $\begin{bmatrix} -1 & -1 \\ 2 & 1 \end{bmatrix}$ and the decryption matrix as its inverse, where the system of codes are described by the numbers 1 - 26 to the letters A - Z respectively, and the number 0 to a blank space.

Let the encryption matrix be $A = \begin{bmatrix} -1 & -1 \\ 2 & 1 \end{bmatrix}$

$|A| = -1+2 = 1 \neq 0$

$\therefore A^{-1} = \frac{1}{|A|} adjA = \frac{1}{1}\begin{bmatrix} 1 & 1 \\ -2 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -2 & -1 \end{bmatrix}$

Hence the decryption matrix is $\begin{bmatrix} 1 & 1 \\ -2 & -1 \end{bmatrix}$

Coded row matrix	Decoding matrix	Decoded row matrix
[2 -3]	$\begin{bmatrix} 1 & 1 \\ -2 & -1 \end{bmatrix}$	=[2+6 2+3]=[8 5]
[20 4]	$\begin{bmatrix} 1 & 1 \\ -2 & -1 \end{bmatrix}$	=[20-8 20-4] =[12 16]

So, the sequence of decoded row matrices is

[8 5], [12 16]

Now the 8^{th} English alphabet is H.

5^{th} English alphabet is E.

12^{th} English alphabet is L.

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and the 16th. English alphabet is P.

Thus the receiver reads the message as "HELP".

7) Find the rank of the matrix $\begin{bmatrix} 2 & -2 & 4 & 3 \\ -3 & 4 & -2 & -1 \\ 6 & 2 & -1 & 7 \end{bmatrix}$ by reducing it to an echelon form.

Let A be the matrix. Performing elementary row operations, we get

A = $\begin{bmatrix} 2 & -2 & 4 & 3 \\ -3 & 4 & -2 & -1 \\ 6 & 2 & -1 & 7 \end{bmatrix} \xrightarrow{R_2 \to 2R_2} \begin{bmatrix} 2 & -2 & 4 & 3 \\ -6 & 8 & -4 & -2 \\ 6 & 2 & -1 & 7 \end{bmatrix} \xrightarrow[R_3 \to R_3 - 3R_1]{R_2 \to R_2 + 3R_1} \begin{bmatrix} 2 & -2 & 4 & 3 \\ 0 & 2 & 8 & 7 \\ 0 & 8 & -13 & -2 \end{bmatrix}$.

$\xrightarrow{R_3 \to R_3 - 4R_2} \begin{bmatrix} 2 & -2 & 4 & 3 \\ 0 & 2 & 8 & 7 \\ 0 & 0 & -45 & -30 \end{bmatrix} \xrightarrow{R_3 \to R_3 \div (-15)} \begin{bmatrix} 2 & -2 & 4 & 3 \\ 0 & 2 & 8 & 7 \\ 0 & 0 & 3 & 2 \end{bmatrix}$.

The last equivalent matrix is in row-echelon form. It has three non-zero rows. So,ρ(A) = 3.

8) 4 men and 4 women can finish a piece of work jointly in 3 days while 2 men and 5 women can finish the same work jointly in 4 days. Find the time taken by one man alone and that of one woman alone to finish the same work by using matrix inversion method.

Let the time by one man alone be x days and one woman alone be y days

∴ By the given data,

$\frac{4}{x} + \frac{4}{y} = \frac{1}{3}$

and $\frac{2}{x} + \frac{5}{y} = \frac{1}{4}$

put $\frac{1}{x}$=s and $\frac{1}{y}$=t

∴ 4s+4t=$\frac{1}{3}$

and 2x+5t=$\frac{1}{4}$

The matrix form of the system of equation is

⇒ AX=B where

$\begin{bmatrix} 4 & 4 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} s \\ t \end{bmatrix} = \begin{bmatrix} \frac{1}{3} \\ \frac{2}{4} \end{bmatrix}$

A= $\begin{bmatrix} 4 & 4 \\ 2 & 5 \end{bmatrix}$ and B= $\begin{bmatrix} \frac{1}{3} \\ \frac{2}{4} \end{bmatrix}$

X=A^{-1}B

Now |A|= $\begin{vmatrix} 4 & 4 \\ 2 & 5 \end{vmatrix}$ =20-8 =12≠0

∴ A^{-1} = $\frac{1}{|A|} adjA = \frac{1}{12}\begin{bmatrix} 5 & -4 \\ -2 & 4 \end{bmatrix}$

∴ X=A $^{-1}$ B=

$$\frac{1}{12}\begin{bmatrix}5 & -4\\ -2 & 4\end{bmatrix}\begin{bmatrix}\frac{1}{3}\\ \frac{1}{4}\end{bmatrix}$$

=

$$\frac{1}{12}\begin{bmatrix}\frac{5}{3} & -1\\ \frac{-2}{3} & +1\end{bmatrix}$$

=

$$\frac{1}{12}\begin{bmatrix}\frac{2}{3}\\ \frac{1}{3}\end{bmatrix} = \begin{bmatrix}\frac{2}{3}\times\frac{1}{12}\\ \frac{1}{3}\times\frac{1}{12}\end{bmatrix} = \begin{bmatrix}\frac{1}{18}\\ \frac{1}{36}\end{bmatrix}$$

∴ $\frac{1}{18} \Rightarrow \frac{1}{x} = \frac{1}{18} \Rightarrow$ x=18

t= $\frac{1}{36} \Rightarrow \frac{1}{y} = \frac{1}{36} \Rightarrow$ y=36

9) A chemist has one solution which is 50% acid and another solution which is 25% acid. How much each should be mixed to make 10 litres of a 40% acid solution ? (Use Cramer's rule to solve the problem).

Let the amount of 50%acid be xl and the amount of 25% acid be y litre

By the given data, x + y = 10 ....(1)

and $x\left(\frac{50}{100}\right) + y\left(\frac{25}{100}\right) = 10\left(\frac{40}{100}\right)$

$\Rightarrow$ 50x+25y=400 $\Rightarrow$ 2x+y=16 ....(2)

The matrix from ofthe equation is $\begin{bmatrix}1 & 1\\ 2 & 1\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix} = \begin{bmatrix}10\\ 16\end{bmatrix}$

$\Rightarrow$ AX=B where A= $\begin{bmatrix}1 & 1\\ 2 & 1\end{bmatrix}$

$X = \begin{bmatrix}x\\ y\end{bmatrix}, B = \begin{bmatrix}10\\ 16\end{bmatrix}$

$\Rightarrow$ X=A $^{-1}$ N |A|= $\begin{vmatrix}1 & 1\\ 2 & 1\end{vmatrix}$ =1-2=-1

$\Rightarrow$ X= $\frac{1}{|A|}$ adj A.B

$\Rightarrow$ X= $-1\begin{bmatrix}1 & -1\\ -2 & 1\end{bmatrix}\begin{bmatrix}10\\ 16\end{bmatrix}$

=- $\begin{bmatrix}10-16\\ -20+16\end{bmatrix}$

$\Rightarrow$ X=- $\begin{bmatrix}-6\\ -4\end{bmatrix} = \begin{bmatrix}6\\ 4\end{bmatrix}$

Thus, the amount of 50% acid is 6 litre and the l amount of 25% acid is 4 litre.

10) A fish tank can be filled in 10 minutes using both pumps A and B simultaneously. However, pump B can pump water in or out at the same rate. If pump B is inadvertently run in reverse, then the tank will be filled in 30 minutes. How long would it take each pump to fill the tank by itself ? (Use Cramer's rule to solve the problem).

Let the pump A can fill the tank in x minutes, and the pump B can fill. the tank in y minutes In 1 minute A can fill $\frac{1}{x}$ units and in 1 minute B can fill $\frac{1}{y}$ units.

$\therefore \frac{1}{x}+\frac{1}{y}=10$

and $\frac{1}{x}-\frac{1}{y}=30$

Put $\frac{1}{x}$=a and $\frac{1}{y}$=b

$\Rightarrow a+b=\frac{1}{10}$ ....(1)

and $a-b=\frac{1}{30}$ ......(2)

$\Delta=\begin{vmatrix}1 & 1\\ 1 & -1\end{vmatrix}=-1-1=-2$

$\Delta_1=\begin{vmatrix}\frac{1}{10} & 1\\ \frac{1}{30} & -1\end{vmatrix}=\frac{-1}{10}-\frac{1}{30}$

$=\frac{-3-1}{30}=\frac{-4}{30}=\frac{-2}{15}$

$\Delta_2=\begin{vmatrix}1 & \frac{1}{10}\\ 1 & \frac{1}{30}\end{vmatrix}=\frac{1}{30}-\frac{1}{10}=\frac{1-3}{30}$

$=\frac{-2}{30}=\frac{-1}{15}$

$\therefore a=\frac{\Delta_1}{\Delta}=\frac{\frac{-2}{15}}{-2}=\frac{1}{15}\Rightarrow\frac{1}{x}=\frac{1}{15}\Rightarrow$ =x=15

$b=\frac{\Delta_2}{\Delta}=\frac{\frac{-1}{15}}{-2}=\frac{1}{30}\Rightarrow\frac{1}{y}=\frac{1}{30}\Rightarrow$ y=30

Hence the pump A can fill the tank in 15 minutes and the pump B can fill the tank in 30 minutes.

11) Solve the following system of linear equations, by Gaussian elimination method :

4x + 3y + 6z = 25, x + 5y + 7z = 13, 2x + 9y + z = 1.

Transforming the augmented matrix to echelon form, we get

$$\left[\begin{array}{ccc|c}4 & 3 & 6 & 25\\ 1 & 5 & 7 & 13\\ 2 & 9 & 1 & 1\end{array}\right]\xrightarrow{R_1\leftrightarrow R_2}\left[\begin{array}{ccc|c}1 & 5 & 7 & 13\\ 4 & 3 & 6 & 25\\ 2 & 9 & 1 & 1\end{array}\right]\xrightarrow[R_3\to R_3-2R_1]{R_2\to R_2-4R_1}\left[\begin{array}{ccc|c}1 & 5 & 7 & 13\\ 0 & -17 & -22 & -27\\ 0 & -1 & -13 & -25\end{array}\right]$$

$$\xrightarrow[R_3\to R_3\div(-1)]{R_2\to R_2\div(-1)}\left[\begin{array}{ccc|c}1 & 5 & 7 & 13\\ 0 & 17 & 22 & 27\\ 0 & 1 & 13 & 25\end{array}\right]\xrightarrow{R_3\to 17R_3-R_2}\left[\begin{array}{ccc|c}1 & 5 & 7 & 13\\ 0 & 17 & 22 & 27\\ 0 & 0 & 199 & 398\end{array}\right].$$

The equivalent system is written by using the echelon form:

x + 5y + 7 = 13, ... (1)

17y + 22z = 27, ... (2)

199z = 398. ...... (3)

From (3), we get z = $\frac{398}{199}$ = 2.

Substituting z = 2 in (2), we get y = $\frac{27-22\times 2}{17}=\frac{-17}{17}$ = -1

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Substituting z = 2, y = -1, in (1), we get x = 13 - 5 x (-1) - 7 x 2 = 4.

So, the solution is (x = 4, y = -1, z = 2).

12) Find the rank of the following matrices by row reduction method:

$$\begin{bmatrix} 1 & 2 & -1 \\ 3 & -1 & 2 \\ 1 & -2 & 3 \\ 1 & -1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & -1 \\ 3 & -1 & 2 \\ 1 & -2 & 3 \\ 1 & -1 & 1 \end{bmatrix}$$

Let A=
$$\begin{bmatrix} 1 & 2 & -1 \\ 3 & -1 & 2 \\ 1 & -2 & 3 \\ 1 & -1 & 1 \end{bmatrix}$$

A=
$$\begin{bmatrix} 1 & 2 & -1 \\ 3 & -1 & 2 \\ 1 & -2 & 3 \\ 1 & -1 & 1 \end{bmatrix} \overset{R_2 \to R_2 3R_1 R_3 \to R_3 - R_1}{\longrightarrow R_4 \to R_4 - R_1} \begin{bmatrix} 1 & 2 & -1 \\ 0 & -7 & 5 \\ 0 & -4 & 4 \\ 0 & -3 & 2 \end{bmatrix}$$

$$\overset{R_3 \to R_3 \div 4}{\longrightarrow} \begin{bmatrix} 1 & 2 & -1 \\ 0 & -7 & 5 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix}$$

$$\overset{R_4 \to R_4 - 3R_3}{\longrightarrow} \begin{bmatrix} 1 & 2 & -1 \\ 0 & -7 & 5 \\ 0 & 0 & 1 \\ 0 & 0 & -1 \end{bmatrix}$$

$$\overset{R_3 \to 7R_3 - R_2}{\longrightarrow} \begin{bmatrix} 1 & 2 & -1 \\ 0 & -7 & 5 \\ 0 & 0 & 2 \\ 0 & 0 & -1 \end{bmatrix}$$

$$\overset{R_4 \to 2R_4 - R_3}{\longrightarrow} \begin{bmatrix} 1 & 2 & -1 \\ 0 & -7 & 5 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

The last equivalent matrix is in row echelon form It has three non-zero rows.

$\therefore \rho(A)=3$

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13) Find the value of the real numbers x and y, if the complex number (2+i)x+(1−i)y+2i −3 and x+(−1+2i)y+1+iare equal

Let z_1 = (2+i)x+(1−i)y+2i−3=(2x+y−3)+i(x−y+ 2)and

z_2 = x+(−1+2i)y+1+i=(x−y+1)+i(2y+1)

Given that $z_1 = z_2$.

Therefore (2x+y−3)+i(x−y+2)=(x−y+1)+i(2y+1).

Equating real and imaginary parts separately, gives

2x+y−3 = x−y+1 ⇒ x+2y=4

x−y+2 = 2y +1 ⇒ x−3y= −1

Solving the above equations, gives

x = 2 and y =1.

14) The complex numbers u,v, and w are related by $\frac{1}{u}=\frac{1}{v}+\frac{1}{w}$ If v=3−4i and w=4+3i, find u in rectangular form.

Given v=3-4i, w=4+3i and $\frac{1}{u}=\frac{1}{v}+\frac{1}{w}$

$\therefore \frac{1}{u}=\frac{1}{3-4i}+\frac{1}{4+3i}$

$=\frac{3+4i}{(3-4i)(3+4i)}+\frac{4-3i}{(4+3i)(4-3i)}$

$=\frac{3+4i}{9-(4i)^2}+\frac{4-3i}{16-(3i)^2}=\frac{3+4i}{9+16}+\frac{4-3i}{16+9}$

$=\frac{3+4i}{25}+\frac{4-3i}{25}=\frac{3+4i+4-3i}{25}$

$\frac{1}{u}=\frac{7+i}{25}$

$\therefore u=\frac{25}{7+i}\times\frac{7-i}{7-i}=\frac{25(7-i}{7^2-(i^2)}$

$=\frac{25(7-i)}{49+1}=\frac{25(7-i)}{50}=\frac{1}{2}$(7-i)

$\therefore u=\frac{1}{2}$(7-i)

15) If z_1,z_2 and z_3 ,are complex numbers such that $|z_1|=|z_2|=|z_3|=|z_1+z_2+z_3|=1$ find the value of $\left|\frac{1}{z_1}+\frac{1}{z_2}+\frac{1}{z_3}\right|$

Since, $|z_1|=|z_2|=|z_3|=1$

$|z_1|^2=1 \Rightarrow z\bar{z}_1=1,\ |z_2|^2=1 \Rightarrow z_2\bar{z}_2=1$

Therefore, $\bar{z}_1=\frac{1}{z_1}, \bar{z}_2=\frac{1}{z_3}$ and hence

$\left|\frac{1}{z_1}+\frac{1}{z_2}+\frac{1}{z_3}\right|=\left|\bar{z}_1+\bar{z}_2+\bar{z}_3\right|$

$=\left|\overline{z_1+|z_2+z_3|}\right|=z_1+z_2+z_3=1$

16) If |z|=2 show that $3 \le |z+3+4i| \le 7$

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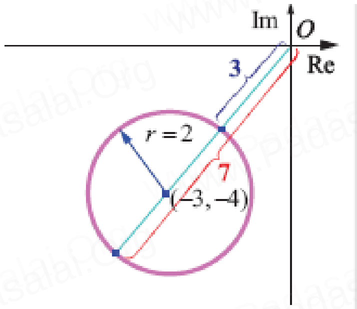

$|z + 3 + 4i| \le |z| + |3 + 4i| = 2 + 5 = 7$

$|z + 3 + 4i| \le 7$

$|z + 3 + 4i| \ge ||z| - |3 + 4i|| = |2 - 5| = 3$

$|z + 3 + 4i| \ge 3$

From (1) and (2) we get, $3 \le |z + 3 + 4i| \le 7$

17) For any two complex number z_1 and z_2 such that $|z_1|=|z_2|=1$ and z1z2 ≠ , then show that $\frac{z_1 + z_2}{1 + z_1 z_2}$ is real number.

Given $|z_1|=|z_2|=1$

$\Rightarrow z_1 \bar{z}=1$

$\Rightarrow z_1 = \frac{1}{\bar{z}_1}$

and $z_1 z_2 \ne -1$

Also $z_2 \bar{z}_2 = 1$

$\Rightarrow z_2 = \frac{1}{\bar{z}_2}$

Consider $\frac{z_1+z_2}{1+z_1z_2}$

∴

$$\frac{z_1+z_2}{1+z_1z_2} = \frac{\frac{1}{\bar{z}_1}+\frac{1}{\bar{z}_2}}{1+\frac{1}{\bar{z}1}\cdot\frac{1}{\bar{z}_2}} = \frac{\frac{\bar{z}_2+\bar{z}_1}{\bar{z}_1\bar{z}_2}}{\frac{\bar{z}_1\bar{z}_2+1}{\bar{z}_1\bar{z}_2}}$$

$$= \frac{\bar{z}_2+\bar{z}_1}{\bar{z}_1\bar{z}_2+1} = \frac{\bar{z_1 1_2}}{1+z_1z_2}$$

$$= \left(\frac{\bar{z_1 1_2}}{1+z_1z_2}\right)$$

∴

$$\frac{z_1+z_2}{1+z_1z_2} = \frac{\frac{1}{\bar{z}_1}+\frac{1}{\bar{z}_2}}{1+\frac{1}{\bar{z}1}\cdot\frac{1}{\bar{z}_2}} = \frac{\frac{\bar{z}_2+\bar{z}_1}{\bar{z}_1\bar{z}_2}}{\frac{\bar{z}_1\bar{z}_2+1}{\bar{z}_1\bar{z}_2}}$$

is real [∵ z= $\bar{z}$ ⇒ is real]

18) Which one of the points10 − 8i , 11+ 6i is closest to1+ i .

Let the points b A (10 - 8i), B (11 + 6i) and C(1-i)

Distance between A and C is |(10-8i)-(1+i)|

=|10-8i-1-i|=|9-9i|

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$=\sqrt{9^2+(-9)^2} = \sqrt{81+81} = \sqrt{2\times 81} = \sqrt{162}$

$9\sqrt{2}$

Distance between Band Cis s |(11+6i)-(1+i)|

=|11+6i-1-i|=|10+5i|

$=\sqrt{10^2+5^2} = \sqrt{100+25} = \sqrt{125}$

$=\sqrt{25\times 5} = 5\sqrt{5}$

Since $\sqrt{125} < \sqrt{162}$, B is closest to C.

∴ 11+6i is closet to 1+i.

19) If $(x_1+iy_1)(x_2+iy_2)(x_3+iy_3)...(x_n+iy_n)$ =a+ib, show that

i) $(x_1^2+y_1^2)(x_2^2+y_2^2)(x_3^2+y_3^2)...(x_n^2+y_n^2)=a^2+b^2$

ii) $\sum_{r=1}^{n} tan^{-1}\left(\frac{y_r}{x_r}\right) = tan^{-1}\left(\frac{b}{a}\right) + 2k\pi, k\epsilon Z$

i) $\sum_{r=1}^{n} tan^{-1}\left(\frac{y_r}{x_r} + \right) = tan^{-1}\left(\frac{b}{a}\right) + 2k\pi, k\epsilon Z$

$(x_1+iy_1)(x_2+iy_2)....(x_n+iy_n)$ =a+ib

$arg(x_1+iy_1)(x_2+iy_2)....(x_n+iy_n))$ =arg(a+ib)

$\Rightarrow arg(x_1+iy_1)+arg(x_2+iy_2)+...+arg(x_n+iy_n)=arg(a+ib)$

$(\because arg(z_1z_2....z_n)=argz_1+arg\ z_2+...+argz_n)$

$\Rightarrow$

$tan^{-1}\left(\frac{y_1}{x_1}\right) + tan^{-1}\left(\frac{y_2}{x_2}\right) + ... + tan^{-1}\left(\frac{y_n}{x_n}\right)$

=

$tan^{-1}\left(\frac{b}{a}\right) + 2k\pi\ k \in Z$

$\Rightarrow$

$\Sigma_{r=1}^{n} tan^{-1}\left(\frac{y_r}{x_r}\right) = tan^{-1}\left(\frac{b}{a}\right)2k\pi \quad k \in Z$

ii)

$\sum_{r=1}^{n} tan^{-1}\left(\frac{y_r}{x_r} + \right) = tan^{-1}\left(\frac{b}{a}\right) + 2k\pi, k\epsilon Z$

$(x_1+iy_1)(x_2+iy_2)....(x_n+iy_n)$ =a+ib

$arg(x_1+iy_1)(x_2+iy_2)....(x_n+iy_n))$ =arg(a+ib)

$\Rightarrow arg(x_1+iy_1)+arg(x_2+iy_2)+...+arg(x_n+iy_n)=arg(a+ib)$

$(\because arg(z_1z_2....z_n)=argz_1+arg\ z_2+...+argz_n)$

$\Rightarrow$

$tan^{-1}\left(\frac{y_1}{x_1}\right) + tan^{-1}\left(\frac{y_2}{x_2}\right) + ... + tan^{-1}\left(\frac{y_n}{x_n}\right)$

$=tan^{-1}\left(\frac{b}{a}\right) + 2k\pi\ k \in Z$

$\Rightarrow \Sigma_{r=1}^{n} tan^{-1}\left(\frac{y_r}{x_r}\right) = tan^{-1}\left(\frac{b}{a}\right)2k\pi \quad k \in Z.$

20) If $\frac{1+z}{1-z} = cos2\theta + isin2\theta$, show that z=itan$\theta$

Processing math: 100%

Let z=x+iy

Then $\frac{1+z}{1-z}$=cos2θ+isin2θ

$\Rightarrow \frac{1+x+iy}{1-x-iy}$ =cos2θ+isin2θ .....(1)

Taking modulus,

$\left|\frac{1+x+iy}{1-x-iy}\right|$=|cos2θ+isin2θ| $\Rightarrow \frac{|1+x+iy|}{|1-x-iy|}$

$=\sqrt{cos^2 2\theta + sin^2 2\theta}$=1

$\Rightarrow$ |1+x+iy|=|1-x-iy|

$\Rightarrow \sqrt{(1+x)^2+y^2} = \sqrt{(1+x)^2+y^2}$

$\Rightarrow (1+x)^2+y^2=(1-x)^2+y^2$

$\Rightarrow 1+x^2+2x+y^2 = 1+x^2-2x+y^2$

$\Rightarrow$ 4x=0 $\Rightarrow$ x=0

From (1) $\frac{(1+x)+iy}{(1-x)-iy} \times \frac{(1-x)+iy}{(1-x)+iy}$

=cos2θ+isin2θ

Choosing the imaginary part alone we get,

$\frac{y(1+x)+y(1-x)}{(1-x)^2+y^2}$=sin2θ

$\Rightarrow \frac{y+xy+y-xy}{(1-x)^2+y^2} = \sin 2\theta$

$\frac{2y}{1+y^2}$=sin2θ

$\Rightarrow \frac{2tan\theta}{1+tan^2\theta}$=sin2θ

∴ y must be equal to tanθ $\Rightarrow$ y=tanθ

∴ z=0+itanθ $\Rightarrow$ z=tanθ

21) If $cos\alpha + cos\beta + cos\gamma = sin\alpha + sin\beta + sin\gamma = 0$ then show that

(i) $cos3\alpha + cos3\beta + cos3\gamma = 3cos(\alpha+\beta+\gamma)$

(ii) $sin3\alpha + sin3\beta + sin3\gamma + sin3\gamma = 3sin(\alpha+\beta+\gamma)$

Given cosα+cosβ+cosγ=sinα+sinβ+sinγ

∴ (cosα+cosβ+cosγ)+i(sinα+sinβ+sinγ)=0

$\Rightarrow$ (cosα+i sinα) + (cosβ+i sinβ)+(cosγ+i sinγ)=0

$\Rightarrow$ a+b+c=0 where a=cosα+i sinα, b=cosβ+i sinβ, c=cosγ+i sinγ

If a+b+c=0, then $a^3+b^3+c^3=3abc$

∴ $(cos\alpha+i\ sin\alpha)^3 + (cos\beta+i\ sin\beta)^3 + (cos\gamma+i\ sin\gamma)^3$ = 3[(cosα+i sinα) + (cosβ+i sinβ) + (cosγ+i sinγ)

=3[(cos(α+β+γ) + i sin(α+β+γ)]

$\Rightarrow$ (cos3α+cosβ+cosγ) + i[sin3α+sin3β+sin3γ)]

=3(cos(α+β+γ) + i sin(α+β+γ)

Equating the real and imaginary parts, we get cosα+cosβ+cosγ = 3sin(α+β+γ) And sinα+sinβ+sinγ=sin(α+β+γ).

22)

Find the value of $\left(\frac{1+sin\frac{\pi}{10}+icos\frac{\pi}{10}}{1+sin\frac{\pi}{10}-icos\frac{\pi}{10}}\right)^{10}$

Processing math: 100%

LHS = $\left(\dfrac{1+sin\frac{\pi}{10}+icos\frac{\pi}{10}}{1+sin\frac{\pi}{10}-icos\frac{\pi}{10}}\right)^{10}$

Let z=$sin\frac{\pi}{10}+icos\frac{\pi}{10}$

$\therefore \frac{1}{z}=sin\frac{\pi}{10}-icos\frac{\pi}{10}$

$\therefore$ LHS = $\left[\dfrac{1+z}{1+\frac{1}{z}}\right]^{10}=\left[\dfrac{1+z}{\frac{z+1}{z}}\right]^{10}$ =z ^{10}

= $\left[sin\frac{\pi}{10}+icos\frac{\pi}{10}\right]^{10}$

= $i^{10}\left[cos\frac{\pi}{10}-isin\frac{\pi}{10}\right]^{10}$

= $i^{10}\left[cos10\frac{\pi}{10}-isin10\frac{\pi}{10}\right]$ [By De Moivers theorum]

=i^{10}[cosπ-i sinπ]=-1(-1-i(0))=1

23) If $\omega\neq 1$ is a cube root of unity, show that the roots of the equation $(z-1)^3+8=0$ are $-1, 1-2\omega, 1-2\omega^2$.

Given $(z-1)^3+8=0$

$(z-2)^3=-8=-1\times 2^3$

$\Rightarrow$ z-1 $=(-1)^{1/3}\times 2$

$\Rightarrow$ z-1=2[cosπ+i sinπ]$^{1/3}$

z-1=2

[cos$\frac{1}{3}$(2kπ+π)+i sin$\frac{1}{3}$(2kπ+π)], k=0,1,2

When k=0

z-1=2 $\left[cos\frac{\pi}{3}+isin\frac{\pi}{3}\right]$

=2 $\left[\frac{1}{2}+\frac{-\sqrt{3}}{2}\right]$=1+i c $\sqrt{3}$

=-2ω^2 ....(1)

[$\because$ ω $_2$ = $\left[\dfrac{-1i-\sqrt{3}}{2}\right]$ $\Rightarrow$ 2ω $_2$ =-1-i $\sqrt{3}$

$\Rightarrow$ 2ω^2=-1+i$\sqrt{3}$

$\Rightarrow$ z=1-2ω^2 ....(2)

When k=1, z-1=2=2 $\left[cos\frac{3\pi}{3}+isin\frac{3\pi}{3}\right]$

=2[cosπ+isinπ]=-2

$\Rightarrow$ z=-2+1=-1 ....(2)

z-1=2 $\left[cos\frac{5\pi}{3}+isin\frac{5\pi}{3}\right]$

Processing math: 100%

$=2\left[\cos\left(2\pi-\frac{\pi}{3}\right)+i\sin\left(2\pi-\frac{\pi}{3}\right)\right]$

$=2\left[\cos\frac{\pi}{3}-i\sin\frac{\pi}{3}\right]=2\left[\frac{1}{2}+\frac{-\sqrt{3}}{2}\right]$

$=1-i\sqrt{3}=-2\omega$

$[\because \omega=\frac{-1+i\sqrt{3}}{2} \Rightarrow 2\omega=-1+i\sqrt{3}$

$\Rightarrow -2\omega=1-i\sqrt{3}$

$\Rightarrow z=1-2\omega]$ ....(3)

From (1), (2) and (3), the roots are -1, $1-2\omega^2$ and $1-2\omega^2$.

24) If $\omega \neq 1$ is a cube root of unity, show that $\left(1-\omega+\omega^2\right)^6+\left(1+\omega-\omega^2\right)^6=128$

$(1-\omega+\omega^2)^6+(1+\omega-\omega^2)^6=128$

LHS $=(1-\omega+\omega^2)6+(1+\omega-\omega^2)^6$

$=(1+\omega^2-\omega)^6+(-\omega^2+\omega^2)^6$

$[\because 1+\omega+\omega^2=0$

$\Rightarrow 1+\omega=-\omega^2$

$\Rightarrow 1+\omega^2=-\omega$

$=(-\omega-\omega)^6+(-2\omega^2)^6$

$=(-2\omega)^6+(-2\omega^2)^6$

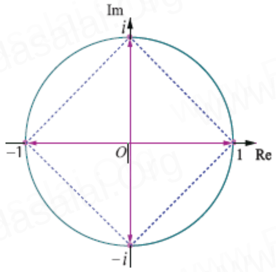
$=2^6.\omega^6+2^6.\omega^{12}$

$=2^6[(\omega^3)^2+(\omega^3)^4]$

$=2^6[1+1]$ $[\because \omega^3=1]$

$2^6 \times 2^1=2^7=128$ =RHS

25) Find the fourth roots of unity.



Fourth roots of unity

Let $z^4=1$

In polar form, the equation z =1can be written as

$z=\cos(0+2k\pi)+i\sin(0+2k\pi)=e^{12k\pi}$, k=0,1,2,...

Therefore, $(z)^{\frac{1}{4}}=\cos\left(\frac{2k\pi}{4}\right)+i\sin\left(\frac{2k\pi}{4}\right)=e^{i\frac{2k\pi}{4}}$, k=0,1,2,3.

Taking k = 0,1,2,3, we get

k=0, z=cos0+isin0=1

k=1, $z=\cos\left(\frac{\pi}{2}\right)+i\sin\left(\frac{\pi}{2}\right)=i$

k=2, $z=\cos\pi+i\sin\pi=-1$

Processing math: 100%

k=3,

$$z = cos\frac{3\pi}{2} + isin\frac{3\pi}{2} = -cos\frac{\pi}{2} - isin\frac{\pi}{2} = i$$

Fourth roots of unity are 1,i, −1, −i $\Rightarrow$ 1, ω, ω^2 and ω^3, where $\omega = e^{i\frac{2\pi}{4}} = i$

26) Simplify $\left(\frac{1 + cos2\theta + isin2\theta}{1 + cos2\theta - isin2\theta}\right)^{30}$

Let $z = cos2\theta + isin2\theta$

As $|z|=|z|^2 = z\bar{z} = 1$, we get $\bar{z} = \frac{1}{z} = cos2\theta - isin2\theta$

Therefore, $\frac{1 + cos2\theta + isin2\theta}{1 + cos2\theta - isin2\theta} = \frac{1 + z}{1 + \frac{1}{z}} = \frac{(1 + z)z}{z + 1} = z$

Therefore, $\left(\frac{1 + cos2\theta + isin2\theta}{1 + cos2\theta - isin2\theta}\right)^{30} = z^{30} = (cos2\theta + isin2\theta)^{30}$

$= cos60\theta + isin60\theta$

27) If $z = (cos\theta + isin\theta)$, show that $z^n + \frac{1}{z^n} = 2cosn\theta$ and $z^n - \frac{1}{z^n} = 2isinn\theta$

Let $z = (cos\theta + isin\theta)$

By de Moivre's theorem ,

$zn = (cos\theta + isin\theta)^n = cosn\theta + isinn\theta$

$\frac{1}{z^n} = z^{-n} = cosn\theta - isinn\theta$

Therefore,

$$z^n + \frac{1}{z^n} = (cosn\theta + isinn\theta) + (cosn\theta - isinn\theta)$$

$$z^n + \frac{1}{z^n} = 2cosn\theta$$

Similarly,

$$z^n - \frac{1}{z^n} = (cosn\theta + isinn\theta) - (cosn\theta - isinn\theta)$$

$$z^n - \frac{1}{z^n} = 2isinn\theta$$

28) Show that $\left(\frac{19 - 7i}{9 + i}\right)^{12} + \left(\frac{20 - 5i}{7 - 6i}\right)^{12}$ is real

Consider $\frac{19-7i}{9+i} = \frac{19-7i}{9+i} \times \frac{9-i}{9-i}$

$= \frac{171-19i-63i+7i^2}{(9)^2-i^2} = \frac{171-82i-7}{81+1}$

$= \frac{164-82i}{82} = \frac{82(2-i)}{82}$ =2-i

Also $\frac{20-5i}{7-6i} = \frac{20-5i}{7-6i} \times \frac{7+6i}{7+6i}$

$= \frac{140+120i-35i-30^2}{7^2-(6i)^2}$

Processing math: 100%

$= \frac{140+85i+30}{49+36} = \frac{170+85i}{85} = \frac{85(2+i)}{85} = 2+i$

$\therefore \left(\frac{19-7i}{9+i}\right)^{12} + \left(\frac{20-5i}{7-6i}\right)^{12}$=(2-i)12+(2+i)12

Let z=$(2-i)^{12}+(2+i)^{12}$

$\therefore \bar{z} = \overline{(2-i)^{12}+(2+i)^{12}} = \overline{(2-i)^{12}} + \overline{(2+i)^{12}}$

$[\because \overline{z_1+z_2} = \overline{z_1}+\overline{z_2}]$

$=(2+i)^{12}+(2-i)^{12}=z$

$\therefore \bar{z}=z \Rightarrow$ z is purely real

$\therefore \left(\frac{19-7i}{9+i}\right)^{12} + \left(\frac{20-5i}{7-6i}\right)^{12}$ is real.

29) Obtain the Cartesian equation for the locus of z=x+iy in

$|z-4|^2-|z-1|^2=16$

$|z-4|^2-|z-1|^2=16$

$|x+iy-4|^2 - |x+iy-1|^2=16$

$\Rightarrow |(x-4)+iy|^2 - |(x-1)+iy2|^2=16$

$\Rightarrow [(x-4)^2+y^2] - [(x-1)^2+y^2]=16$

$\Rightarrow x^2-8x+16+y^2-[x^2-2x+1+y^2]=16$

$\Rightarrow x^2-8x+16+y^2-x^2+2x-1-y^2=16$

$\Rightarrow -6x+15-16=0$

$\Rightarrow -6x-1=0$

$\Rightarrow$ 6x+1=0 Which is the required Cartesian equation.

30) If $\omega \neq 1$ is a cube root of unity, show that $(1+\omega)\left(1+\omega^2\right)\left(1+\omega^4\right)\left(1+\omega^8\right)\ldots\left(1+\omega^{2^{11}}\right) = 11.$

$(1+\omega)(1+\omega^2)(1+\omega^4)(1+\omega^8)....(1+\omega^2)^{11}=1$

LHS=$(1+\omega)(1+\omega^2)(1+\omega^4)(1+\omega^8)..2(1+\omega^2)^{11}$

$(1+\omega^{2^0})^6((1+\omega^{2^1})(1+\omega^{2^2})((1+\omega^{2^{11}})$

$[\because \omega^4=\omega^3.\omega\omega^8=\omega^6.\omega^2]$

There are 12 terms

$=(1+\omega)(1+\omega^2)(1+\omega)(1+\omega^{2^1}$... upto 12 terms

$=(1+\omega)^6(1+\omega^2)^6(-\omega)^6(-\omega^2)^6$

$=(\omega^3)6$ =16=1 =RHS.

31) Obtain the Cartesian form of the locus of z in

|2z-3-i|=3

|2z-3-i|=3

|2(x+iy)-3-i|=3

Squaring on both sides, we get

$|(2x-3)+(2y-1)i|^2=9$

$\Rightarrow (2x-3)^2+(2y-1)^2=9$

$\Rightarrow$ 4x2+4y2-12x-4y+1=0, the locus of z in Cartesian form

32) If α and β are the roots of the quadratic equation $17x^2+43x-73 = 0$, construct a quadratic equation whose roots are α + 2 and β + 2.

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Since α and β are the roots of 17x2+43x−73 = 0 , we have $\alpha + \beta = \frac{-43}{17}$ and $\alpha\beta = \frac{-73}{17}$.

We wish to construct a quadratic equation with roots are α + 2 and β + 2 .Thus, to construct such a quadratic equation, calculate,

the sum of the roots = $\alpha + \beta + 4 = \frac{-4}{17} + 4 = \frac{25}{17}$ and

the product of the roots $=\alpha\beta+2(\alpha+\beta)+4= \frac{-73}{17} + 2\left(\frac{-43}{17}\right) + 4 = \frac{-91}{17}$

Hence a quadratic equation with required roots is $x^2 - \frac{25}{17}x - \frac{91}{17} = 0$

Multiplying this equation by 17, gives $17x^2-25x-91 = 0$

which is also a quadratic equation having roots α + 2 and β + 2

33) If α, β and γ are the roots of the cubic equation $x^3+2x^2+3x+4=0$, form a cubic equation whose roots are, 2α, 2β, 2γ

The roots of $x^3+2x2+3x+4=0$ are ∝,β,ɣ

∴ ∝+β+ɣ=-co-efficient of x^2=-2 ...(1)

∝β+βɣ+ɣ∝=co-effficient of x=3 ...(2)

-∝βɣ=+4 ⇒ ∝βɣ=-4 ...(3)

Form a cubic equation whose roots are 2∝,2β,2ɣ

2∝+2β+2ɣ=2(∝+β+ɣ)=2(-2)=-4 [from (1)]

4∝β+4βɣ+4ɣ∝=4(∝β+βɣ+ɣ∝)=4(3)=12 [from (2)]

(2∝)(2β)(2ɣ)=8(∝βɣ)=8(-4)=-32 [from (3)]

∴ The required cubic equation is

x3-(2∝+2β+2ɣ)x^2+(2∝β+2βɣ+2ɣ∝)x-(2∝)(2β)(2ɣ)=0

⇒ $x^3+(-4)x^2+12x+32=0$

⇒ $x^3+4x^2+12x+32=0$

34) Find the sum of squares of roots of the equation $2x^4-8x+6x^2-3=0$.

Given equation is $2x^4-8x+6x^2-3=0$

Here a=2, b =-8, c= 6, d= 0, e=-3

Let ∝, β,ɣ yand δ be the roots of eqn (1)

Then by Vieta's formula,

$$\sum_1 = \alpha + \beta + \gamma + \delta = \frac{-b}{a} = \frac{-(-8)}{2} = 4$$

$$\sum_2 = \alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta = \frac{c}{a} = \frac{6}{2} = 3$$

$$\sum_3 = \alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta = \frac{-d}{a} = \frac{0}{a}$$

$$\sum_4 = \alpha\beta\gamma\delta = \frac{e}{a} = \frac{-3}{2}$$

Now, $(a+b+c+d)^2=a^2+b^2+c^2+d^2+2(ab+ac+ad+bc+cd)$

⇒ n$\propto^2+\beta^2+ɣ^2+\delta^2=(\propto+\beta+ɣ+\delta)^2-2(\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta)$

$\propto^2+\beta^2+ɣ^2=4^2-2(3)=16-6=10$

35) If p and q are the roots of the equation $lx^2+nx+n = 0$, show that $\sqrt{\frac{p}{q}} + \sqrt{\frac{q}{p}} + \sqrt{\frac{n}{l}}=0$.

Given p,q are the roots of $lx^2+nx+n=000$

$$p + q = \frac{-b}{a} = \frac{-n}{l} \quad ...(1)$$

$$pq = \frac{c}{a} = \frac{n}{l} \quad ...(2)$$

Processing math: 100%

and $\left(\sqrt{\frac{p}{q}}+\sqrt{\frac{q}{p}}+\sqrt{\frac{n}{l}}\right)$

consider $\left(\sqrt{\frac{p}{q}}+\sqrt{\frac{q}{p}}+\sqrt{\frac{n}{l}}\right)$

$=\frac{p}{q}+\frac{q}{p}+\frac{n}{l}+2\sqrt{\frac{pq}{qb}}+2\sqrt{\frac{qn}{pl}}+2\sqrt{\frac{np}{ql}}$

$[\because (a+bc)^2=a^2+b^2+c^2+2ab+2bc+2ca]$

$\Rightarrow \frac{p^2+q^2}{pq}+\frac{n}{l}+2+2\sqrt{\frac{p}{q}.pq}+2\sqrt{\frac{q}{p}.pq}+2\sqrt{\frac{p}{q}.pq}$

$=\frac{(p+q)^2-2pq}{pq}+\frac{n}{l}+2+2\sqrt{p^2}+2\sqrt{q^2}$

$=\frac{(p+q)^2}{pq}-\frac{2pq}{pq}+\frac{n}{l}+2+2p+2q$

$\Rightarrow \frac{\left(\frac{-n}{l}\right)^2}{\frac{n}{l}}-\not{2}+\frac{n}{l}+\not{2}+2(p+q)$

$\Rightarrow \frac{n^2}{l^2.\frac{n}{l}}+\frac{n}{l}-\frac{2n}{l}$ $[\because p+q=\frac{-n}{l}]$

$\Rightarrow \frac{n}{l}+\frac{n}{l}-\frac{2n}{l}=0 \therefore \left(\sqrt{\frac{p}{q}}+\sqrt{\frac{q}{p}}+\sqrt{\frac{n}{l}}\right)^2=0$

Talking square root both sides, we get

$\sqrt{\frac{p}{q}}+\sqrt{\frac{q}{p}}+\sqrt{\frac{n}{l}}=0$

36) Form a polynomial equation with integer coefficients with $\sqrt{\frac{\sqrt{2}}{\sqrt{3}}}$ as a root.

Since $\sqrt{\frac{\sqrt{2}}{\sqrt{3}}}$ is a root, x- $\sqrt{\frac{\sqrt{2}}{\sqrt{3}}}$ is a factor. To remove the outermost square root, we take x+ $\sqrt{\frac{\sqrt{2}}{\sqrt{3}}}$ as another factor and find their product.

$\left(x+\sqrt{\frac{\sqrt{2}}{\sqrt{3}}}\right)\left(x-\sqrt{\frac{\sqrt{2}}{\sqrt{3}}}\right)=x^2-\frac{\sqrt{2}}{\sqrt{3}}$

Still we didn't achieve our goal. So we include another factor $x_2+\frac{\sqrt{2}}{\sqrt{3}}$ and get the product.

$\left(x^2-\frac{\sqrt{2}}{\sqrt{3}}\right)\left(x^2+\frac{\sqrt{2}}{\sqrt{3}}\right)=x^4-\frac{2}{3}$

So, $3x^4-2=0$ is a required polynomial equation with the integer coefficients.

Now we identify the nature of roots of the given equation without solving the equation. The idea comes from the negativity, equality to 0, positivity of $\Delta=b^2-4ac$.

37) Solve the equation $x^4-9x^2+20=0$.

The given equation is

$x^4-9x^2+20=0$

This is a fourth degree equation. If we replace x^2 by y then we get the quadratic equation

$y^2-9y+20=0$

Processing math: 100%

It is easy to see that 4 and 5 as solutions for y^2-9y+20=0. Now taking x^2=4 and x^2=5, we get 2,-2,$\sqrt{5}$,-$\sqrt{5}$ as solutions of the given equation.

We note that the technique adopted above can be applied to polynomial equations like $x^6-17x^3+30=0$, $ax^{2k}+bx^k+c=0$ and in general polynomial equations of the form $a_nx^{kn} + a_{n-1}x^{k(n-1)} + ... + a_1x^k+a_0=0$ where k is any positive integer.

38) If the roots of $x^3+px^2+qx+r=0$ are in H.P. prove that $9pqr = 27r^3+2p$.

Let the roots be in H.P. Then, their reciprocals are in A.P. and roots of the equation

$\left(\frac{1}{x}\right)^3+p\left(\frac{1}{x}\right)^2+q\left(\frac{1}{x}\right)+r=0 \Leftrightarrow rx_3+qx_2+px+1=0$ ....(1)

Since the roots of (1) are in A.P., we can assume them as α-d,α,α+d.

Applying the Vieta's formula, we get

$\Sigma_1=(\alpha-d)+\alpha+(\alpha+d)=-\frac{q}{r} \Rightarrow 3\alpha=-\frac{q}{r} \Rightarrow \alpha=-\frac{q}{3r}$.

But, we note that α is a root of (1). Therefore, we get

$\left(-\frac{q}{3r}\right)^2+q\left(-\frac{q}{3r}\right)^2+p\left(-\frac{q}{3r}\right)+1=0 \Rightarrow q_3+3q_3-9pqr+27r_2=0 \Rightarrow 2q_3+27r_2$.

39) Solve the following equations,

$\sin^2x-5\sin x+4=0$

$\sin^2x-5\sin x+4=0$

put y=sin x

⇒ y2-5y+4=0

⇒ (y-4)(y-1)=0

⇒ y=4,1

Case(i)

When y=4, sin x=4 and no solution for sin x = 4 since the range thesine function is [-1, 1]

Case (ii)

When

y = 1, sin x = 1

$\Rightarrow \sin x=\sin\frac{\pi}{2}\ [\because \sin\frac{\pi}{2}=1]$

$\Rightarrow x=2n\pi+\frac{\pi}{2}, n\in Z$

$[\because \sin\quad x=\sin\quad \alpha \Rightarrow x=2n\pi+n\in Z]$.

40) Find the domain of $\sin^{-1}(2-3x^2)$

We know that the domain of sin−1(x) is [-1,1].

This leads to $-1 \le 2-3x^2 \le 1$, Which implies $-3 \le -3x^2 \le -1$

Now, $-3 \le -3x^2$, gives $x^2 \le 1$ and ....(1)

$-3 \le -3x2 \le -1$, gives $x2 \ge \frac{1}{2}$ ....(2)

Combining the equations (1) and (2), we get $\frac{1}{3} \le x^2 \le 1$. This is $\frac{1}{\sqrt{3}} \le |x| \le 1$, Which gives $x\in[-1,-1\frac{1}{\sqrt{3}}]\cup[\frac{1}{\sqrt{3}},1]$

since $a \le |x| \le b$ implies $x \in[-b,-a]\cup[a,b]$.

41) Find the domain of $\cos^{-1}(\frac{2+\sin x}{3})$

By definition, the domain of yx=cos-1 x is -1 This leads to $-1 \le \frac{2+\sin x}{3} \le 1$ which is same as $-3 \le 2+\sin x \le 3$

so, $-5 \le \sin x \le 1$ reduces to $-1 \le \sin x \le 1$, which gives

$-\sin_{-1}(1) \le x \le \sin^{-}1(1) or -\frac{\pi}{2} \le x \le \frac{\pi}{2}$

Thus, the domain of cos-1$(\frac{2+\sin x}{3}) is [-\frac{\pi}{2},\frac{\pi}{2}]$.

Processing math: 100%

42) Find the value of

$$cos\left(cos^{-1}\left(\frac{4}{5}\right)+sin^{-1}\left(\frac{4}{5}\right)\right)$$

$$cos\left(cos^{-1}\left(\frac{4}{5}\right)+sin^{-1}\left(\frac{4}{5}\right)\right)$$

$$cos^{-1}\left(\frac{4}{5}\right)=\theta \Rightarrow \frac{4}{5}=cos\theta$$

Also [using (1)]

$$sin^{-1}\left(\frac{4}{5}\right)=sin^{-1}(cos\theta)$$

= $sim^{-1}\left(sin\left(\frac{\pi}{2}-\theta\right)\right)\left[\because cos\theta=sin\left(\frac{\pi}{2}-\theta\right)\right]$

= $\frac{\pi}{2}-\theta$

$$\therefore cos\left(cos^{i1}\left(\frac{4}{5}\right)+sin^{-1}\left(\frac{4}{5}\right)\right)$$

= $cos\left(\theta+\frac{\pi}{2}-\theta\right)$ [using (1) & (2)]

= $cos\frac{\pi}{2}$ =0

43) Prove that

$\tan^{-1}\frac{1}{2}+tan^{-1}\frac{1}{3}=\frac{\pi}{4}$

(We know that tan-1 x+tan-1 y=tan-1 $\frac{x+y}{1-xy}$, xy< 1

Thus, tan−1 $\frac{1}{2}+tan^{-1}\frac{1}{3}=tan^{-1}\frac{\frac{1}{2}+\frac{1}{3}}{1-\left(\frac{1}{2}\right)\left(\frac{1}{3}\right)}=tan^{-1}(1)=\frac{\pi}{4}$

44) If $\cos^{-1}x+\cos^{-1}y+\cos^{-1}z$ = and 0< x,y,z< 1, show that $x^2+y^2+z^2+2xyz=1$

Let $\cos^{-1}x=\alpha$ and cos-1 $y=\beta$. Then, $x=\cos\alpha$ and y $\cos=\beta$

$\cos^{-1}x+x+\cos^{-1}y+\cos^{-1}x=\pi$ gives $\alpha+\beta=\pi-\cos^{-1}z$.

Now, $\cos(\alpha+\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta=xy-\sqrt{1-x^2}\sqrt{1-y^2}$

$-\cos(\cos^{-1}z)=xy\sqrt{1-x^2}\sqrt{1-y^2}$

so, $-z=xy-\sqrt{1-x^2}\sqrt{1-y^2}$, Which gives $-xy-z=-\sqrt{1-x^2}\sqrt{1-y^2}$

Squaring on both sides and simplifying, we get $x^2+y^2+z^2+2xyz=1$.

45) Find the value of

$$sin^{-1}\left(cos\left(sin^{-1}\left(\frac{\sqrt{3}}{2}\right)\right)\right)$$

$$sin^{-1}\left(cos\left(sin^{-1}\left(\frac{\sqrt{3}}{2}\right)\right)\right)$$

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Let $sin^{-1}\left(\frac{\sqrt{3}}{2}\right) = x$

$\Rightarrow \frac{\sqrt{3}}{2} = sinx$

$\Rightarrow sinx = sin\frac{\pi}{3} \left[\because \frac{\pi}{3} \varepsilon \left[\frac{-\pi}{2}, \frac{\pi}{2}\right]\right]$

$\Rightarrow x = \frac{\pi}{3}$

$\therefore sin^{-1}\left(cbs\left(sin^{-1}\left(\frac{\sqrt{3}}{2}\right)\right)\right) = sin^{-1}\left(cos\left(\frac{\pi}{3}\right)\right)$

$= sin^{-1}\left(\frac{1}{2}\right)$

Let $sin^{-1}\frac{\pi}{6} = siny$

$\Rightarrow y = \frac{\pi}{6} \left[\because \frac{\pi}{6} \varepsilon \left[\frac{-\pi}{2}, \frac{\pi}{2}\right]\right]$

$\therefore sin^{-1}\left(cos\left(sin^{-1}\left(\frac{\sqrt{3}}{2}\right)\right)\right) = \frac{\pi}{6}$

46) Prove that $tan^{-1}x + tan^{-1}\frac{2x}{1-x^2} = tan^{-1}\frac{3x-x^2}{1-3x^2}, \ |x| < \frac{1}{\sqrt{3}}$

$LHS = tan^{-1}x + tan^{-1}\frac{2x}{1-x^2}$

$= tan^{-1}\left(\frac{x+\frac{2x}{1-x^2}}{1-x\left(\frac{2x}{1-x^2}\right)}\right)$

$= tan^{-1}\left(\frac{\frac{x(1-x^2)+2x}{1-x^2}}{\frac{1-x^2-2x^2}{1-x^2}}\right)$

$= tan^{-1}\left(\frac{\frac{x-x^3+2x}{1-x^2}}{\frac{1-3x^2}{1-x^2}}\right) \left[\because |x| < \frac{1}{\sqrt{3}}\right]$

$= tan^{-1}\left(\frac{3x-x^3}{1-x^2} \times \frac{1-x^2}{1-3x^2}\right)$

Processing math: 100%

$$\left[\because \ |x| < \frac{1}{\sqrt{3}} \Rightarrow x^2 < \frac{1}{3} \Rightarrow 3x^3 < 1\right]$$

$$= tan^{-1}\left(\frac{3x - x^3}{1 - 3x^2}\right)$$

47) Simplify: $tan^{-1}\frac{x}{y} - tan^{-1}\frac{x-y}{x+y}$

$$tan^{-1}\left(\frac{x}{y}\right) - tan^{-1}\left(\frac{x-y}{x+y}\right)$$

$$= tan^{-1}\left(\frac{\frac{x}{y} - \frac{x-y}{x+y}}{1 + \frac{x}{y}\left(\frac{x-y}{x+y}\right)}\right)$$

$$\left[\because tan^{-1}x - tan^{-1}y = tan^{-1}\left(\frac{x-y}{1+xy}\right)\right]$$

$$= tan^{-1}\left(\frac{\frac{x(x+y) - y(x-y)}{y(x+y)}}{\frac{y(x+y) + x(x-y)}{y(x+y)}}\right)$$

$$= \tan^{-1}\left(\frac{\frac{x^2 + xy - xy + y^2}{y(x+y)}}{\frac{xy + y^2 + x^2 - xy}{y(x+y)}}\right)$$

$$= \tan^{-1}\left(\frac{x^2 + y^2}{y(x+y)} \times \frac{y(x+y)}{x^2 + y^2}\right)$$

$= \tan^{-1}(1)$

$= \frac{\pi}{4}$

48) Solve:

$$2tan^{-1}x = cos^{-1}\frac{1-a^2}{1+a^2} - cos^{-1}\frac{1-b^2}{1+b^2}, a > 0, b > 0$$

$$2tan^{-1}x = cos^{-1}\frac{1-a^2}{1+a^2} - cos^{-1}\frac{1-b^2}{1+b^2}, a > 0, b > 0$$

Let $a = tan\theta \quad b = tan\phi$

$$\therefore cos^{-1}\left(\frac{1 - a^2}{1 + a^2}\right) = cos^{-1}\left(\frac{1 - tan^2\theta}{1 + tan^2\theta}\right)$$

$= cos^{-1}(cos2\theta) = 2\theta \quad ...(1)$

$$\left[\because cos2\theta = \frac{1 - tan^2\theta}{1 + tan^2\theta}\right]$$

Also

$$cos^{-1}\left(\frac{1 - b^2}{1 + b^2}\right) = cos^{-1}\left(\frac{1 - tan^2\phi}{1 + tan^2}\right)$$

$= cos^{-1}(cos2\Phi)$

$\therefore 2tan^{-1}x = 2\theta - 2\phi = 2(\theta - \phi)$

[using (1) and (2)]

$\Rightarrow tan^{-1}x = \theta - \phi$

$\Rightarrow x = tan(\theta - \phi) = \frac{tan\theta - tan\phi}{1 + tan\theta tan\phi}$

$\left[\because tan(A - B) = \frac{tanA - tanB}{1 + tanAtanB}\right]$

$\Rightarrow x = \frac{a - b}{1 + ab}$ which is the solution.

49) Prove that

$2tan^{-1}\frac{1}{2} + tan^{-1}\frac{1}{7} = tan^{-1}\frac{31}{17}$

We know that 2tan^-1 x=tan^-1\(\frac{2x}{1-x^{2}}, -1

So,

$$2tan^{-1}\frac{1}{2} = tan^{-1}\frac{2\left(\frac{1}{2}\right)}{1 - \left(\frac{1}{2}\right)^2} = tan^{-1}\left(\frac{4}{3}\right)$$

Hence,

$$2tan^{-1}\frac{1}{2} + tan^{-1}\frac{1}{7} = tan^{-1}\frac{4}{3} + tan^{-1}\frac{1}{7} = tan^{-1}\left(\frac{\frac{4}{3} + \frac{1}{7}}{1 - \left(\frac{4}{3}\right)\left(\frac{1}{7}\right)}\right) = tan^{-1}\left(\frac{31}{17}\right)$$

50) A circle of area 9π square units has two of its diameters along the lines x+y=5 and x−y=1. Find the equation of the circle.

Area of the circle = 9π sq.units

$\pi r^2 = 9\pi \Rightarrow r^2 \Rightarrow 9 \Rightarrow r \Rightarrow 3$

Diameters are x + y = 5 (1) and x - y = 1 (2)

We know that centre is the point of intersection of diameters.

∴ To find the centre, solve (1) and (2).

$$\Rightarrow \begin{aligned} x + y &= 5 \\ x - y &= 1 \end{aligned}$$

2x = 6

$\Rightarrow$ x = 3

∴ (1) $\Rightarrow$ 3 + y = 5

$\Rightarrow$ y = 5 - 3 = 2

∴ Centre is (3, 2)

Hence, equation of the circle is

(x - h)2 + (y - k)2 = r2

$\Rightarrow$ (x - 3)2 + (y - 2)2 = 32

$\Rightarrow x^2 - 6x + 9 + y^2 - 4y + 4 = 9$

$\Rightarrow x^2 + y^2 - 6x - 4y + 4 = 0$

51) Find the equation of the ellipse with foci (±2,0) , vertices (±3,0) .

SS' = 2c and 2c =4 ; A'A=2a=6

c = 2 and a = 3,

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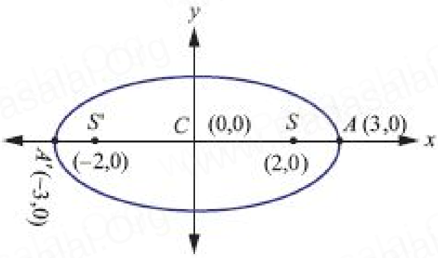
$b^2 = a^2 - c^2 = 9 - 4 = 5$.

Major axis is along x -axis, since a > b.

Centre (0, 0) and Foci are (±2,0) .

Therefore, equation of the ellipse is $\frac{x^2}{9}\frac{y^2}{5} = 1$



52) Find the equation of the hyperbola with vertices (0,±4) and foci(0,±6).

the midpoint of line joining foci is the centre C(0,0).

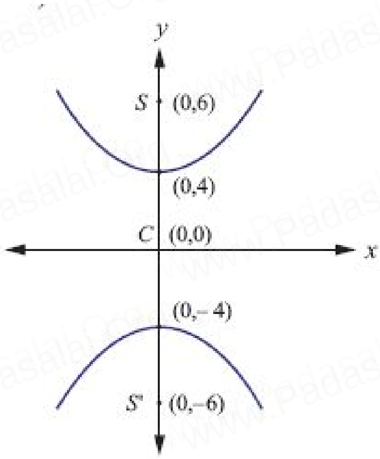
Transverse axis is y -axis AA' = 2a ⇒ 2a=8,

SS' = 2c=12,c=6

a = 4

$b^2 = c^2 - a^2 = 36 - 16 = 20$

Hence the equation of the required hyperbola is $\frac{y^2}{16}\frac{x^2}{20} = 1$



53) Find the vertex, focus, equation of directrix and length of the latus rectum of the following:

$y^2 = 16x$

$y^2 = 16x$

The given parabola is right open parabola and 4a = 16 ⇒ a = 4.

(a) Vertex is (0, 0)

⇒ h = 0, k= 0

(b) focus is (h + a, 0 + k)

⇒ (0 + 4, 0 + 0) = (4, 0)

(c) Equation of directrix is x = h - a

⇒ x=0-4 ⇒ x=-4

(d) Length of latus rectum is 4a = 16.

54) Identify the type of conic and find centre, foci, vertices, and directrices of each of the following:

$\frac{x^2}{25} + \frac{y^2}{9} = 1$

$\frac{x^2}{25} + \frac{y^2}{9} = 1$

This is an equation of the ellipse.

$a^2 = 25$ and $b^2 = 9$ and $c^2 = a^2 + b^2$

$\Rightarrow c^2 = 25 + 9 = 16 \Rightarrow c = 4$

(a) Center is (0, 0) $\Rightarrow$ h = 0, k = 0

(b) foci are (h - c, k), (h + c, k)

$\Rightarrow$ (0 - 4, 0), (0 + 4, 0)

$\Rightarrow$ (-4, 0) and (4, 0)

(c) Vertices are (h - a, k) and (h + a, k)

$\Rightarrow$ (0 - 5, 0) and (0 + 5, 0)

$\Rightarrow$ (-5, 0) and (5, 0)

(d) Directrices are x = $\pm\frac{a}{e}$

$\Rightarrow x = \pm\frac{5}{e}$

$$e = \sqrt{1 - \frac{b^2}{a^2}} = \sqrt{1 - \frac{9}{25}} = \sqrt{\frac{16}{25}} = \frac{4}{5}$$

∴ Directrice are x= $\pm\frac{5}{\frac{4}{5}} \Rightarrow x = \pm\frac{25}{4}$

55) Identify the type of conic and find centre, foci, vertices, and directrices of each of the following :

$$\frac{(x-3)^2}{225} + \frac{(y-4)^2}{289} = 1$$

$$\frac{(x-3)^2}{225} + \frac{(y-4)^2}{289} = 1$$

Given equation is $\frac{(x-3)^2}{225} + \frac{(y-4)^2}{289} = 1$

This is an equation of the ellipse a^2 = 289, b^2 = 225 and $c^2 = a^2 + b^2 \Rightarrow$ 289 - 225 = 64 $\Rightarrow$ c = 8.

$$e = \sqrt{\frac{b^2}{a^2}} = \sqrt{1 - \frac{225}{289}} = \sqrt{\frac{289-225}{289}}$$

$$= \sqrt{\frac{64}{289}} = \frac{8}{17}$$

(a) Center is (3, 4) $\Rightarrow$ h = 3, k= 4

(b) fociare (h,k+c), (h,k-c)

$\Rightarrow$ (3, 4 + 8), (3, 4 - 8) $\Rightarrow$ (3, 12), (3,-4)

(c) Verticesare (h, k - a), (h, k + a)

$\Rightarrow$ (3, 4 -17), (3, 4 + 17) $\Rightarrow$ (3, -13), (3, 21)

(d) Equations of directrices are y - 4 = $\pm\frac{a}{e}$

$$\Rightarrow y - 4 = \pm\frac{17}{\frac{8}{17}} + 4 \Rightarrow y - 4 = \pm\frac{-289}{8} + 4$$

$$\Rightarrow y = \frac{289}{8} + 4 \text{ and } y = \frac{-289}{8} + 4$$

$$\Rightarrow y = \frac{289+32}{2} \text{ and } y = \frac{-289+32}{8}$$

$$\Rightarrow y = \frac{321}{8} \text{ and } y = \frac{-257}{8}$$

56) A semielliptical archway over a one-way road has a height of 3m and a width of 12m. The truck has a width of 3m and a height of 2.7m. Will the truck clear the opening of the archway?

Since the truck's width is 3m, to determine the clearance, we must find the height of the archway 1 5 . m from the centre. If this height is 2 7 . m or less the truck will not clear the archway. From the diagram a = 6 and b =1 5 . yielding the equation of ellipse as $\frac{x^2}{6^2} + \frac{y^2}{3^2} = 1$

The edge of the 3m wide truck corresponds tox=1.5m. We will find the height of the

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archway 1 5 . m from the centre by substituting x =1 5 . and solving for y $\frac{\left(\frac{3}{2}\right)^2}{36} - \frac{y^2}{9} = 1$

$y^2 = 9\left(1 - \frac{9}{144}\right)$

$\frac{9(135)}{144} = \frac{135}{16}$

$y = \frac{\sqrt{135}}{4}$

= 2.90

Thus the height of arch way 1.5m from the centre is approximately2.90m . Since the truck's height is 2 7 . m, the truck will clear the archway.

57) The parabolic communication antenna has a focus at 2m distance from the vertex of the antenna. Find the width of the antenna 3m from the vertex.

Let the parabola be y2 = 4ax .

Since focus is 2m from the vertex a = 2

Equation of the parabola is y2 = 8x

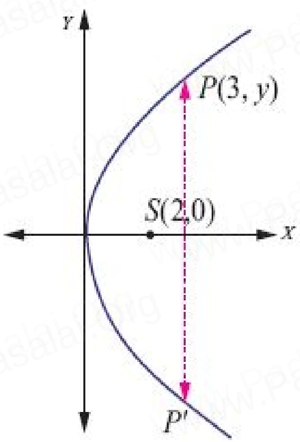
Let P be a point on the parabola whose x -coordinate is 3m from the vertex P (3, y)

$y^2 = 8\times3$

$y = \sqrt{8\times3}$

$=2\sqrt{6}$

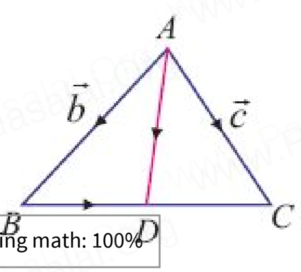
The width of the antenna 3m from the vertex is $4\sqrt{6}$ m.



58) If D is the midpoint of the side BC of a triangle ABC, then show by vector method that

$$\left|\vec{AB}\right|^2 + \left|\vec{AC}\right|^2 = 2(\left|\vec{AD}\right|^2 + \left|\vec{BD}\right|^2)$$

Let A be the origin, $\vec{b}$ be the position vector of B and $\vec{c}$ be the position vector of C . Now D is the midpoint of BC , and so the position vector of D $\frac{\hat{b}+\hat{c}}{2}$. There, we get



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$$\left|\vec{AD}\right|^2 = \vec{AD}.\vec{AD} = \left(\frac{\vec{b}+\vec{c}}{2}\right).\left(\frac{\vec{b}+\vec{c}}{2}\right) = \frac{1}{4}\left(\left|\vec{b}\right|^2 + \left|\vec{c}\right|^2 + 2\vec{b}.\vec{c}\right) \quad \text{..........(1)}$$

Now, $\vec{BD} = \vec{AD} - \vec{AB} = \frac{\vec{b}+\vec{c}}{2} - \vec{b} = \frac{\vec{b}-\vec{c}}{2}$

Then, we get, = $\left|\vec{BD}\right|^2 = \vec{BD}.\vec{BD} = \left(\frac{\vec{c}+\vec{b}}{2}\right).\left(\frac{\vec{c}+\vec{b}}{2}\right) = \frac{1}{4}\left(\left|\vec{b}\right|^2 + \left|\vec{c}\right|^2 + 2\vec{b}.\vec{c}\right)$ ....(2)

Now, adding (1) and (2), we get

Therefore, $\left|\vec{AD}\right|^2 + \left|\vec{BD}\right|^2 = \frac{1}{4}\left(\left|\vec{b}\right|^2 + \left|\vec{c}\right|^2 + 2\vec{b}.\vec{c}\right) + \frac{1}{4}\left(\left|\vec{b}\right|^2 + \left|\vec{c}\right|^2 - 2\vec{b}.\vec{c}\right) = \frac{1}{2}\left(\left|\vec{b}\right|^2 + \left|\vec{c}\right|^2\right)$

$\Rightarrow \left|\vec{AD}\right|^2 + \left|\vec{BD}\right|^2 = \frac{1}{2}\left(\left|\vec{AB}\right|^2 + \left|\vec{AC}\right|^2\right)$

Hence, $\left|\vec{AD}\right|^2 + \left|\vec{BD}\right|^2 = 2\left(\left|\vec{AB}\right|^2 + \left|\vec{AC}\right|^2\right)$

59) A particle acted upon by constant forces $2\hat{j} + 5\hat{j} + 6\hat{k}$ and $-\hat{i} - 2\hat{j} - \hat{k}$ is displaced from the point (4, −3, −2) to the point (6, 1, −3) . Find the total work done by the forces.

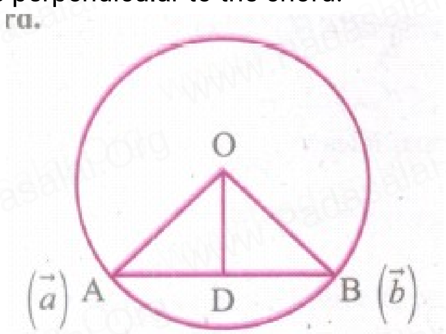
Resultant of the given forces is $\hat{F} = 2\hat{j} + 5\hat{j} + 6\hat{k} + -\hat{i} - 2\hat{j} - \hat{k} = \hat{i} + 3\hat{j} + 5\hat{k}$

Let A and B be the points (4, −3, −2) and (6, 1, −3) respectively. Then the displacement vector of the particle is

$\vec{d} = \vec{AB} = \vec{OB} - \vec{OA} = (6\hat{i} + 4\hat{j} - 3\hat{k}) - (4\hat{i} - 3\hat{j} - 2\hat{k}) = 2\hat{i} + 4\hat{j} - \hat{k}$

Therefore the work done w = $\vec{f}.\vec{d} = (\hat{i} + 3\hat{j} - 5\hat{k}).(2\hat{j} + 4\hat{j} - \hat{k})$ = 9 units.

60) Prove by vector method that if a line is drawn from the centre of a circle to the midpoint of a chord, then the line is perpendicular to the chord.



Let the position vectors of the parts A and B on the circle lie a and $\vec{b}$ respectively.

Since O is the centre of the circle

$|\vec{OA}| = |\vec{OB}| \Rightarrow |a| = |\vec{b}|$ ....(1)

Also D is the mid-point of AB,

$\Rightarrow \vec{OD} = \frac{\vec{a}+\vec{b}}{2}$ (mid-point formula)

$\left(\frac{\vec{a}+\vec{b}}{2}\right).(\vec{OB} - \vec{OA})$

$= \left(\frac{\vec{a}+\vec{b}}{2}\right).(\vec{Ob} - \vec{Oa})$

$= \frac{1}{2}\left[|\vec{b}|^2 - |\vec{a}|^2 \right]$

$= \left[\because (\vec{a}+\vec{b}).(\vec{b}-\vec{a}) = |\vec{b}|^2 - |\vec{a}|^2 \right]$

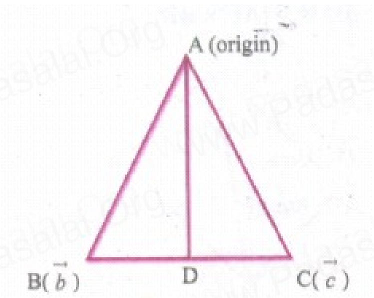
$= \frac{1}{2}\left[|\vec{b}|^2 - |\vec{b}|^2 \right]$ (using (1))

$= \frac{1}{2}(0) = 0$

$\Rightarrow \vec{OD}.\vec{AB} = 0 \Rightarrow \vec{OD} \perp \vec{AB}$

Hence, if a line is drawn from the centre of a to the mid-point of a chord, then that line is perpendicular to the chord.

61) Prove by vector method that the median to the base of an isosceles triangle is perpendicular to the base.



Let ABC be an isosceles triangle with AB = AC and let D be the mid-point of BC.

Taking A as the origin, let the position vectors of B and C be $\vec{b}$ and $\vec{c}$ respectively.

The p.v of D= $\frac{\vec{b}+\vec{c}}{2}, \vec{AB} = \vec{b}, \vec{AC} = \vec{c}$

Now $\vec{AD} = p.v\vec{D}$ - p.v. of $\frac{\vec{b}+\vec{c}}{2} - \vec{0}$

$= \frac{1}{2}(\vec{b} + c)$

and $\vec{BC}$ =p.v. of C - p.v. of B= $c - \vec{b}$

$\therefore \vec{AD}.\vec{BC} = \frac{1}{2}(\vec{b} + c).(c - \vec{b})$

$\Rightarrow \vec{AD}.\vec{BC} = \frac{1}{2}\{(\vec{c} + \vec{b}).(\vec{c} - \vec{b})\}$

$= \frac{1}{2}\{ |c|^2 - |b^2| \}$ [∵ AB=BC]

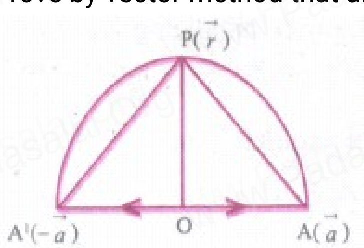
$= \frac{1}{2}(AC^2 - AB^2) = \frac{1}{2}(AC^2 - AC^2)$

=0 [∵ AB=AC]

$\therefore \vec{AD} \perp \vec{BC} \Rightarrow AD \perp BC$

Hence the median AD is perpendicular to the base BC of ΔABC.

62) Prove by vector method that an angle in a semi-circle is a right angle.



Let O be the centre of the semi-circle and AA[1] be the diameter. Let P be any point on the circumference of the semi circle. Taking O as the origin, let the position vectors of A and P be a and r respectively.

Then, p.v. of A1 is $-\vec{a}$

Now, $\vec{AP}$ =(p.v. of P) - Cp.v. of A) = $\vec{r}-\vec{a}$

and $\vec{A^1P}$ =(p.v. of P) - (p.v. of A1)

$=r-(-a)$

$=\vec{r}+\vec{a}$

$\therefore \vec{AP}.\vec{A^1P}=(r-a).(r+a)=|r|^2-|a|^2$

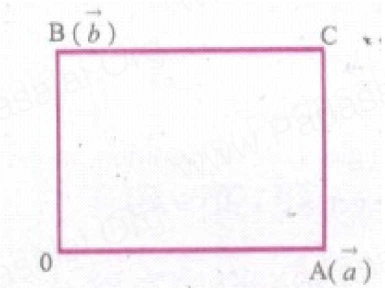
$\Rightarrow \vec{AP}.\vec{A^1P}=OP^2-OA^2=(\text{radius})^2-(\text{radius})^2$

=0

$\Rightarrow \vec{AP}.\vec{A^1P} \Rightarrow \angle APA_1=\frac{\pi}{2}$

Hence, angle in a semi-circle is a right angle.

63) Prove by vector method that the diagonals of a rhombus bisect each other at right angles.



Let OACB be a rhombus. Taking O as the origin, let the position vectors of A and B be a and $\vec{b}$ respectively.

Then $\vec{OA}=a$ and $\vec{OB}=\vec{b}$ $[\because \vec{AC}=\vec{OB}]$

So, the p.v. of C is $a+\vec{b}$

$\therefore$ Position vector O f the miid-point of OC is $\frac{\vec{a}+\vec{b}}{2}$

Similarly, the position vector of mid-point of AB is $\frac{\vec{a}+\vec{b}}{2}$.

Hence, the mid-point of OC coincides with the mid-point of AB.

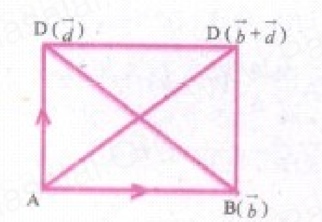
Now, $\vec{OC}.\vec{AB}=(a+\vec{b}).(\vec{b}-a)=|\vec{b}|^2-|a|^2$

$=OB^2-OA^2=0$ $[\because OB=OA]$

$\Rightarrow \vec{OC}\perp\vec{AB}$.

Hence, the diagonals of a rhombus bisect each other at right angles.

64) Using vector method, prove that if the diagonals of a parallelogram are equal, then it is a rectangle



Let ABCD be a parallelogram such that its diagonals AC and BD are equal. Taking A as the origin, let the p.v. of B and D be $\vec{b}$ and $\vec{d}$ respectively.

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Then $\vec{AB} = \vec{b}$ and $\vec{AD} = \vec{d}$

Using triangle law of addition of vectors is ΔaBC, we get

$\vec{AB} + \vec{BC} = \vec{AC}$

$\Rightarrow \vec{AB} + \vec{AD} = \vec{AC}$

$\Rightarrow \vec{b} + \vec{d} = \vec{AC}$

Using triangle law of addition of vectors in ΔABD, we get $\vec{AB} + \vec{BD} = \vec{AD}$

$\Rightarrow \vec{b} + \vec{BD} = \vec{d}$

$\Rightarrow \vec{BD} = \vec{d} - \vec{b}$

In parallelogram ABCD we have AC = BD

$\Rightarrow |\vec{AC}| = |\vec{BD}|$

$\Rightarrow |\vec{AC}|^2 = |\vec{BD}|^2$

$\Rightarrow |\vec{b} + \vec{d}|^2 = |\vec{d} - \vec{b}|^2$

$\Rightarrow |\vec{b}|^2 + |\vec{d}|^2 + 2(\vec{b}\cdot\vec{d}) = |\vec{d}|^2 + |\vec{b}|^2 - 2(\vec{b}\cdot\vec{d})$

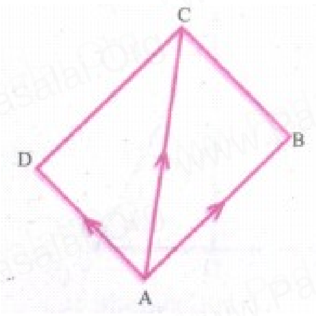
$\Rightarrow 4(\vec{b}.\vec{d}) = 0 \Rightarrow \vec{b}.\vec{d} = 0$=0

$\Rightarrow \vec{b} \perp \vec{d}$

$\Rightarrow \vec{AB} \perp \vec{AD}$

Hence, ABCD is a rectangle.

65) Prove by vector method that the area of the quadrilateral ABCD having diagonals AC and $\frac{1}{2}\left|\vec{AC} \times \vec{BC}\right|$.



Vector area of quadrilateral ABCD

= vector area of ΔABC + vector area of ΔACD

$\frac{1}{2}(\vec{AB} \times \vec{AC}) + \frac{1}{2}(\vec{AC} \times \vec{AD})$

$= -\frac{1}{2}(\vec{AC} \times \vec{AB}) + \frac{1}{2}(\vec{AC} \times \vec{AD})$

$\left[\because \vec{b} \times a = -(a \times \vec{b})\right]$

$= \frac{1}{2}.\vec{AC} \times (-\vec{AB} + \vec{AD})$

$= \frac{1}{2}\vec{AC} \times (\vec{BA} + \vec{AD})$ $[\because \vec{AB} = -\vec{BA}]$

Processing math: 100%

$= \frac{1}{2} \vec{AC} \times \vec{BD}$ [By Δ law of addition]

∴ Area of the quadrilateral ABCD = $\frac{1}{2} \vec{AC} \times \vec{BD}$

66) Find the magnitude and direction cosines of the torque of a force represented by $3\hat{i} + 4\hat{j} - 5\hat{k}$ about the point with position vector $2\hat{i} - 3\hat{j} + 4\hat{k}$ acting through a point whose position vector is $4\hat{i} + 2\hat{j} - 3\hat{k}$.

Given $\vec{F} = 3\hat{i} + 4\hat{j} - 5\hat{k}$

$\vec{r}$ =(Force acting through the point) - (force acting to the point)

$=(4\hat{i} + 2\hat{j} - 3\hat{k}) - (2\hat{i} - \hat{j} + 4\hat{k})$

$=2\hat{i} + 5\hat{j} - 7\hat{k}$

Torque= $\vec{c} = \hat{r} \times \hat{F}$

$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 5 & -7 \\ 3 & 4 & -5 \end{vmatrix} = \hat{i}\begin{vmatrix} 5 & -7 \\ 4 & -5 \end{vmatrix} - \hat{j}\begin{vmatrix} 2 & -7 \\ 3 & -5 \end{vmatrix} + \hat{k}\begin{vmatrix} 2 & 5 \\ 3 & 4 \end{vmatrix}$

$\hat{i} = \hat{i}(-25 + 28) - \hat{j}(-10 + 21) + \hat{j}(8 - 15)$

$=3\hat{i} - 11\hat{j} - 7\hat{k}$

∴ MagnitudeoftheTorque = $\sqrt{3^2 + (-11)^2 + (-7)^2}$

$=\sqrt{9 + 121 + 49} = \sqrt{179}$

Hence, the direction cosines are $\left(\frac{3}{\sqrt{179}}, \frac{-11}{\sqrt{179}}, \frac{-7}{\sqrt{179}}\right)$.

67) If $\vec{a} = \hat{i} - 2\hat{j} + 3\hat{k}, \vec{b} = 2\hat{i} + \hat{j} + \hat{k}, \vec{c} = 3\hat{i} + 2\hat{j} + \hat{k}$ find

(i) $(\vec{a} \times \vec{b}) \times \vec{c}$

(ii) $\vec{a} \times (\vec{b} \times \vec{c})$

Given $\vec{a} = \hat{i} - 2\hat{j} + 3\hat{k}, \vec{b} = 2\hat{i} + \hat{j} + \hat{k}$ and $\vec{c} = 3\hat{i} + 2\hat{j} + \hat{k}$

(i) $(\vec{a} \times \vec{b}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -2 & 3 \\ 2 & 1 & -2 \end{vmatrix}$

$=\hat{i}\begin{vmatrix} -2 & 3 \\ 1 & -2 \end{vmatrix} - \hat{j}\begin{vmatrix} 1 & 3 \\ 2 & -2 \end{vmatrix} + \hat{k}\begin{vmatrix} 1 & -2 \\ 2 & 1 \end{vmatrix}$

$=\hat{i}(4-3)-\hat{j}(-2-6)+\hat{k}(1+4)$

$=\hat{i} + 8\hat{j} + 5\hat{k}$

$(\vec{a} \times \vec{b}) \times \vec{c} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 8 & 5 \\ 3 & 2 & 1 \end{vmatrix}$

$=\hat{i}\begin{vmatrix} 8 & 5 \\ 2 & 1 \end{vmatrix} - \hat{j}\begin{vmatrix} 1 & 5 \\ 3 & 1 \end{vmatrix} + \hat{k}\begin{vmatrix} 1 & 8 \\ 3 & 2 \end{vmatrix}$

$=\hat{i}(8-10)-\hat{j}(1-15)+\hat{k}(2-24)$

$=-2\hat{i} + 14\hat{j} - 22\hat{k}$

Processing math: 100%

(ii) $a \times (\vec{b} \times c)$

$$\vec{b} \times c = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 1 & -2 \\ 3 & 2 & 1 \end{vmatrix}$$

$$= \hat{i} \begin{vmatrix} 1 & -2 \\ 2 & 1 \end{vmatrix} - \hat{j} \begin{vmatrix} 2 & -2 \\ 3 & 1 \end{vmatrix} + \hat{k} \begin{vmatrix} 2 & 1 \\ 3 & 2 \end{vmatrix}$$

$= \hat{i}(1+4) - \hat{j}(2+6) + \hat{k}(4-3) = 5\hat{i} - 8\hat{j} + \hat{k}$

∴

$$\vec{a} \times (\vec{b} \times \vec{c}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -2 & 3 \\ 5 & -8 & 1 \end{vmatrix}$$

$$= \hat{i} \begin{vmatrix} -2 & -3 \\ -8 & 1 \end{vmatrix} - \hat{j} \begin{vmatrix} 1 & 3 \\ 5 & 1 \end{vmatrix} + \hat{k} \begin{vmatrix} 1 & -2 \\ 5 & -8 \end{vmatrix}$$

$= \hat{i}(-2-24) - \hat{j}(1-15) + \hat{k}(-8-10)$

$22\hat{i} + 14\hat{j} + 2\hat{k}.$

68) Prove that $[\vec{a} - \vec{b}, \vec{b} - \vec{c}, \vec{c} - \vec{a}] = 0$

LHS=$[\vec{a} - \vec{b}, \vec{b} - \vec{c}, \vec{c} - \vec{a}]$=0

[∵ cross product is distributive]

$(a - \vec{b}).[(\vec{b} - c) \times (c - a)]$

$=(a - \vec{b}).[(\vec{b} \times c - \vec{b} \times a - c \times c + c \times a)$

$=(a - \vec{b}).[\vec{b} \times c - \vec{b} \times a - 0 + c \times a]$

$[\because \vec{c} \times \vec{c} = 0]$

$=[\vec{a}\vec{b}\vec{c}] - [\vec{a}\vec{b}\vec{a}] + [\vec{a}\vec{c}\vec{a}] - [\vec{b}\vec{b}\vec{c}] + [\vec{b}\vec{b}\vec{a}] - [\vec{b}\vec{c}\vec{a}]$

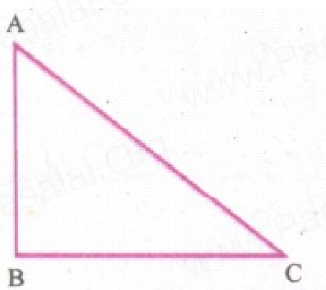
$=[\vec{a}\vec{b}\vec{c}] - 0 + 0 - 0 + 0 - [\vec{b}\vec{c}\vec{a}]$

$=[\because [\vec{a}\vec{b}\vec{a}] = [\vec{b}\vec{b}\vec{c}] = 0]$

$=[\vec{a}\vec{b}\vec{c}] - [\vec{a}\vec{b}\vec{c}]$

=0=RHS.

69) The vertices of ΔABC are A(7, 2, 1),B(6,0,3) , and C(4, 2, 4) . Find ∠ABC .



The direction ratios of AB is

(6-7, 0- 2, 3 - 1) = (-1, - 2, 2)

[∵ $(x_2-x_1), (y_2-y_1), (z_2-z_1)$]

Also direction ratios of BC is

(4 - 6, 2 - 0, 4 - 3) = (- 2, 2, 1)

Product of direction ratios is

(- 1) (- 2) + (- 2) (2) + 2 (1)

=2 - 4 + 2 [∵ if two lines are $\perp^r$ there d.r's d, b, +d_2b_2 + d_3b_3 =0]

=4-4=0

Hence AB $\perp$ BC

$\angle ABC=\frac{\pi}{2}$.

70) Find the equation of the plane passing through the intersection of the planes 2x+3y−z+7=0 and andx+y−2z+5=0 and is perpendicular to the planex+y−3z−5=0.

The equation of the plane passing through the intersection of the planes 2x+3y−z+7=0 and x+y−2z+5=0 is

(2x+3y−z+7)+λ (x+y−2z+5) =0 or

(2+λ)x+ (3+λ)y+ (−1−2λ)z+ (7+5λ) = 0

since this plane is perpendicular to the given plane x+y−3z−5=0, the normals of these two planes are perpendicular to each other. Therefore, we have

(1)(2+λ)+(1)(3+λ)+(−3)(−1−2λ)z = 0

which implies that λ = −1. Thus the required equation of the plane is

(2x+3y−z+7)−(x+y−2z+5) =0 $\Rightarrow$ x+2y+z+2=0 .

RAVI MATHS TUITION CENTER . PH - 8056206308

12TH MATHS IMP 5 MARKS

12th Standard

Date : 20-Aug-19

Maths

Reg.No. :

CHANNEL NAME - SR MATHS TEST PAPERS
FREE VIDEOS AND ANSWERS FOR THIS PAPERS AVAILABLE IN MY YOUTUBE CHANNEL
MORE MODEL PAPERS AVAILABLE IN YOUTUBE
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Time : 04:00:00 Hrs

Total Marks : 500

20X5=100

1) If $(x_1+iy_1)(x_2+iy_2)(x_3+iy_3)\ldots(x_n+iy_n)=a+ib$, show that

i) $(x_1^2+y_1^2)(x_2^2+y_2^2)(x_3^2+y_3^2)\ldots(x_n^2+y_n^2)=a^2+b^2$

ii) $\sum_{r=1}^{n} tan^{-1}\left(\frac{y_r}{x_r}\right) = tan^{-1}\left(\frac{b}{a}\right) + 2k\pi, k\epsilon Z$

i) $\sum_{r=1}^{n} tan^{-1}\left(\frac{y_r}{x_r} + \right) = tan^{-1}\left(\frac{b}{a}\right) + 2k\pi, k\epsilon Z$

$(x_1+iy_1)(x_2+iy_2) \ldots.(x_n+iy_n)=a+ib$

$arg(x_1+iy_1)(x_2+iy_2) \ldots.(x_n+iy_n))=arg(a+ib)$

$\Rightarrow arg(x_1+iy_1)+arg(x_2+iy_2)+\ldots+arg(x_n+iy_n)=arg(a+ib)$

$(\because arg(z_1z_2\ldots.z_n)=argz_1+arg\ z_2+\ldots+argz_n)$

$\Rightarrow$

$tan^{-1}\left(\frac{y_1}{x_1}\right) + tan^{-1}\left(\frac{y_2}{x_2}\right) + \ldots + tan^{-1}\left(\frac{y_n}{x_n}\right)$

$= tan^{-1}\left(\frac{b}{a}\right) + 2k\pi\ k \in Z$

$\Rightarrow \Sigma_{r=1}^{n} tan^{-1}\left(\frac{y_r}{x_r}\right) = tan^{-1}\left(\frac{b}{a}\right)2k\pi \quad k \in Z$

ii)

$\sum_{r=1}^{n} tan^{-1}\left(\frac{y_r}{x_r} + \right) = tan^{-1}\left(\frac{b}{a}\right) + 2k\pi, k\epsilon Z$

$(x_1+iy_1)(x_2+iy_2) \ldots.(x_n+iy_n)=a+ib$

$arg(x_1+iy_1)(x_2+iy_2) \ldots.(x_n+iy_n))=arg(a+ib)$

$\Rightarrow arg(x_1+iy_1)+arg(x_2+iy_2)+\ldots+arg(x_n+iy_n)=arg(a+ib)$

$(\because arg(z_1z_2\ldots.z_n)=argz_1+arg\ z_2+\ldots+argz_n)$

$\Rightarrow$

$tan^{-1}\left(\frac{y_1}{x_1}\right) + tan^{-1}\left(\frac{y_2}{x_2}\right) + ... + tan^{-1}\left(\frac{y_n}{x_n}\right)$

$=tan^{-1}\left(\frac{b}{a}\right) + 2k\pi\, k \in Z$

$\Rightarrow \Sigma_{r=1}^{n} tan^{-1}\left(\frac{y_r}{x_r}\right) = tan^{-1}\left(\frac{b}{a}\right)2k\pi \quad k \in Z.$

2) Find the locus of z if Re $\left(\frac{\bar{z}+1}{\bar{z}-i}\right)$ =0.

Let z = x+iy $\Rightarrow$ $\bar{z}$ =x+iy

$\therefore \frac{\bar{z}+1}{z-1} = \frac{z-iy+1}{x-iy-i} = \frac{(x+1)iy}{x-i(y+1)}$

$=\frac{(x+1)-iy}{x-i(y+1)} \times \frac{x+i(y+1)}{x+i(y+1)}$

Choosing the real part alone we get,

$\frac{x(x+1)+y(y+1)}{x^2+(y+1)^2}$ =0

$\Rightarrow$ x(x+1) + y(y+1) =0

$\Rightarrow$ $x^2+x+y^2+y=0$ which is the locus of z.

3) Solve the cubic equation : $2x^3-x^2-18x+9=0$ if sum of two of its roots vanishes.

Since sum of two of its roots vanishes, let the roots be ∝,-∝ and β

$\alpha - \alpha + \beta = \frac{-b}{a} = \frac{1}{2}$

$\Rightarrow \quad \beta = \frac{1}{2}$

Also, $\alpha\beta\gamma = \frac{-d}{a}$

$\Rightarrow \alpha(-\alpha)(\frac{1}{2}) = \frac{-9}{2}$

$\Rightarrow \frac{\alpha^2}{2} = \frac{9}{2} \Rightarrow \alpha^2 = 9 \Rightarrow \alpha = \pm 3$

$\Rightarrow \alpha = 3$

$\therefore$ The roots are 3,-3 and $\frac{1}{2}$.

4) Solve the equation $9x-36x^2+44x-16=0$ if the roots form an arithmetic progression.

Here, a = 9, b = - 36, c = 44, d= -16

Since the rootsform an arithmetic progression,

Let the roots be a - d, a and a + d

Sum of the roots $= \frac{-b}{a}$

$\Rightarrow (a-d) + (a) + (a+d) = \frac{-(-36)}{9} = 4$

$\Rightarrow \quad 3a = 4 \Rightarrow a = \frac{4}{3}$

and product of the roots $= \frac{-d}{a}$

$= \frac{-(-16)}{9} = \frac{16}{9}$

$\Rightarrow (a-d)(a)(a+d) = \frac{16}{9}$

$\left(a^2 - d^2\right)(a) = \frac{16}{9}$

$\left(\frac{16}{9} - d^2\right)\left(\frac{4}{3}\right) = \frac{16}{9} \qquad [\because a = \frac{4}{3}]$

$\frac{16}{9} - d^2 = \frac{16}{9} \times \frac{3}{4} = \frac{4}{3}$

$\frac{16}{9} - \frac{4}{3} = d^2 = \frac{16}{9} \times \frac{3}{4} = \frac{4}{3}$

$\frac{16}{9} - \frac{4}{3} = d^2$

$\Rightarrow d^2 = \frac{16-12}{9} = \frac{4}{9}$

$\Rightarrow d = \pm\sqrt{\frac{4}{9}} = \frac{2}{3}$

∴ The roots are a-d,a,a+d

$\Rightarrow \frac{4}{3} - \frac{2}{3}, \frac{4}{3}, \frac{4}{3} + \frac{2}{3} \Rightarrow \frac{2}{3}, \frac{4}{3}, 2$

5) Solve the equation $3x^3-26x^2+52x-24=0$ if its roots form a geometric progression.

Given cubic equation is $3x^3-26x^2+52x-24=0$

Here, a = 3, b = -2b, c = 52,d = -24.

Since the roots form an geometric progression,

The roots are $\frac{a}{r}$ a, ar, sum of the roots= $\frac{-b}{a}$

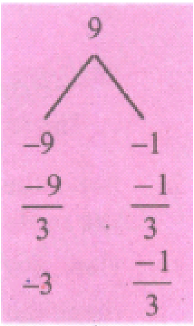
$\Rightarrow \frac{a}{r} + a + ar = \frac{26}{3} \quad \ldots(1)$

and product of the roots $\frac{-d}{a}$

$\Rightarrow \frac{a}{\not r} + a + a\not r = \frac{24}{3} = 8$

$a^3 = 8 = 2^3$

a=2



∴ (1) becomes $\frac{2}{r}$+2+2r=$\frac{26}{3}$

$\Rightarrow \frac{2+2r+2r^2}{r} = \frac{26}{3} \Rightarrow \frac{1+r+r^2}{r} = \frac{13}{3}$

$\Rightarrow 3 + 3r + 3r^2 = 13r \Rightarrow 3r^2 - 10r + 3 = 0$

$\Rightarrow (r-3)(3r-1) = 0 \Rightarrow r = 3$

∴ The roots are $\frac{a}{r}$,a, and ar

$\Rightarrow \frac{2}{3}, 2 \quad and \quad 2(3) \Rightarrow \frac{2}{3}m2 \quad and \quad 6$

6) Solve the following equations,

$\sin^2x-5\sin x+4=0$

$\sin^2x-5\sin x+4=0$

put y=sin x

⇒ y2-5y+4=0

⇒ (y-4)(y-1)=0

⇒ y=4,1

Case(i)

When y=4, sin x=4 and no solution for sin x = 4 since the range thesine function is [-1, 1]

Case (ii)

When

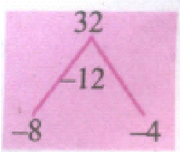
y = 1, sin x = 1

$\Rightarrow$ sinx=sin$\frac{\pi}{2}$ [$\because sin\frac{\pi}{2} = 1$]

$\Rightarrow x = 2n\pi + \frac{\pi}{2}, n \in Z$

[$\because sin \quad x = sin \quad \alpha \Rightarrow x = 2n\pi + n \in Z$].

7) Find all real numbers satisfying $4^x-3(2^{x+2})+2^5=0$



$4^x-3(2^{x+2})+2^5=0$

$\Rightarrow \left(2^2\right)^x - 3(2^x)(2^2) + 2^5 = 0$

$\left(2^x\right)^2 - 3\left(2^x\right)(2^2) = 0$

$\left(2^x\right)^2 - 12\left(2^x\right) + 32 = 0$

Put $2^x = y$

$y^2 - 12y + 32 = 0$

(y-8)(y-4)=0

y=8,4

Case (i) when $y = 8, 2^x = 8 \Rightarrow 2^x = 2^3 \Rightarrow x = 3$

Case (ii) when $y = 4, 2^x = 4 \Rightarrow 2^x = 2^2 x = \pm 2.$

$\therefore$ The roots are 2,3, -2.

8) If α, β and γ are the roots of the cubic equation $x^3+2x^2+3x+4=0$, form a cubic equation whose roots are $\frac{1}{\alpha}, \frac{1}{\beta}, \frac{1}{\gamma}$

The roots of x^3+2x2+3x+4=0 are ∝,β,ɤ

∴ ∝+β+ɤ=-co-efficient of x^2=-2 ...(1)

∝β+βɤ+ɤ∝=co-effficient of x=3 ...(2)

-∝βɤ=+4 ⇒ ∝βɤ=-4 ...(3)

From the cubic equation whose roots are $\frac{1}{\alpha}, \frac{1}{\beta}, \frac{1}{\gamma}$

$\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = \frac{\beta\gamma+\gamma\alpha+\alpha\beta}{\alpha\beta\gamma} = \frac{3}{-4} = \frac{-3}{4}$

$\frac{1}{\alpha\beta} + \frac{1}{\beta\gamma} + \frac{1}{\gamma\alpha} = \frac{\gamma+\alpha+\beta}{\alpha\beta\gamma} = \frac{-2}{-4} = \frac{1}{2}$

$\left(\frac{1}{\alpha}\right)\left(\frac{1}{\beta}\right)\left(\frac{1}{\gamma}\right) = \frac{1}{\alpha\beta\gamma} = \frac{1}{-4} = -\frac{1}{4}$

$\therefore$ The required cubic equation is

$x^3 - \left(\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}\right)x^2 + \left(\frac{1}{\alpha\beta} + \frac{1}{\beta\gamma} + \frac{1}{\gamma\alpha}\right)x - \left(\frac{1}{\alpha}, \frac{1}{\beta}, \frac{1}{\gamma}\right)$

$\Rightarrow x^3 + \frac{3}{4}x^2 + \frac{1}{2}x + \frac{1}{4} = 0$

Multiplying by 4 we get,

$4x^3+3x^2+2x+1=0$

9) Solve the cubic equations:

$8x^3-2x^2-7x+3=0$

Let.f(x) = $8x^3 - 2x^2 - 7x + 3 = 0$

Here sum of the co-efficients of odd terms

= 8-7=1

and sum of the co-efficients of even terms

= -2 +3 = 1

Hence, x = - 1 is a root of f(x)

Let us divide.f{x) by (x + 1)

–1	8	–2	–7	3
	↓	–8	10	–3
	8	–10	3	0

∴ The other factor is $8x^2 - 10x + 3$

$$\Rightarrow x = \frac{10 \pm \sqrt{100 - 4(8)(3)}}{2 \times 8}$$

$$\Rightarrow x = \frac{10 \pm \sqrt{100 - 96}}{16} \quad \Rightarrow x = \frac{10 \pm 2}{16}$$

$$\Rightarrow x = \frac{12}{16} or$$

$$x = \frac{8}{16} \Rightarrow x = \frac{3}{4}, \frac{1}{2}.$$

∴ The roots are -1, $\frac{1}{2}, \frac{3}{4}$

10) Find

i) $\tan^{-1}(-\sqrt{3})$

ii) $\tan^{-1}(tan\frac{3\pi}{5})$

iii) tan(tan^{-1}–(2019))

i) $tan^{-1}(-\sqrt{3}) = tan^{-1}\left(tan\left(-\frac{\pi}{3}\right)\right) = -\frac{\pi}{3}$, since $-\frac{\pi}{3} \in \left(\frac{\pi}{2}, \frac{\pi}{2}\right)$

ii) $tan^{-1}\left(tan\frac{3\pi}{5}\right)$

Let us find $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$ such that tanθ=tan$\frac{3\pi}{5}$

Since the tangent function has period π, $tan\frac{3\pi}{5} = tan(\frac{3\pi}{5} - \pi) = tan(-\frac{2\pi}{5})$

Therefore, $tan^{-1}\left(tan\frac{3\pi}{5}\right) = tan^{-1}\left(tan\left(-\frac{2\pi}{5}\right)\right) = -\frac{2\pi}{5}, since \frac{-2\pi}{5} \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right).$

Since tan(tan^{-1} x)=x,x∈R,We have tan(tan^{-1}(2019))=2019

11) Simplify: $tan^{-1}\frac{x}{y} - tan^{-1}\frac{x-y}{x+y}$

$$tan^{-1}\left(\frac{x}{y}\right) - tan^{-1}\left(\frac{x-y}{x+y}\right)$$

$$= tan^{-1}\left(\frac{\frac{x}{y} - \frac{x-y}{x+y}}{1 + \frac{x}{y}\left(\frac{x-y}{x+y}\right)}\right)$$

$\left[\because \tan^{-1}x - \tan^{-1}y = \tan^{-1}\left(\frac{x-y}{1+xy}\right)\right]$

$= \tan^{-1}\left(\frac{\frac{x(x+y)-y(x-y)}{y(x+y)}}{\frac{y(x+y)+x(x-y)}{y(x+y)}}\right)$

$= \tan^{-1}\left(\frac{\frac{x^2+xy-xy+y^2}{y(x+y)}}{\frac{xy+y^2+x^2-xy}{y(x+y)}}\right)$

$= \tan^{-1}\left(\frac{x^2+y^2}{y(x+y)} \times \frac{y(x+y)}{x^2+y^2}\right)$

$= \tan^{-1}(1)$

$= \frac{\pi}{4}$

12) Solve:

$2\tan^{-1}x = \cos^{-1}\frac{1-a^2}{1+a^2} - \cos^{-1}\frac{1-b^2}{1+b^2}, a > 0, b > 0$

$2\tan^{-1}x = \cos^{-1}\frac{1-a^2}{1+a^2} - \cos^{-1}\frac{1-b^2}{1+b^2}, a > 0, b > 0$

Let $a = \tan\theta \quad b = \tan\phi$

$\therefore \cos^{-1}\left(\frac{1-a^2}{1+a^2}\right) = \cos^{-1}\left(\frac{1-\tan^2\theta}{1+\tan^2\theta}\right)$

$= \cos^{-1}(\cos 2\theta) = 2\theta$ ...(1)

$\left[\because \cos 2\theta = \frac{1-\tan^2\theta}{1+\tan^2\theta}\right]$

Also

$\cos^{-1}\left(\frac{1-b^2}{1+b^2}\right) = \cos^{-1}\left(\frac{1-\tan^2\phi}{1+\tan^2}\right)$

$= \cos^{-1}(\cos 2\Phi)$

$\therefore 2\tan^{-1}x = 2\theta - 2\phi = 2(\theta - \phi)$

[using (1) and (2)]

$\Rightarrow \tan^{-1}x = \theta - \phi$

$\Rightarrow x = \tan(\theta - \phi) = \frac{\tan\theta - \tan\phi}{1 + \tan\theta\tan\phi}$

$\left[\because \tan(A - B) = \frac{\tan A - \tan B}{1 + \tan A\tan B}\right]$

$\Rightarrow x = \frac{a-b}{1+ab}$ which is the solution.

13) Find the value of

$$tan\left[\frac{1}{2}sin^{-1}\left(\frac{2a}{1+a^2}\right)+\frac{1}{2}cos^{-1}\left(\frac{1-a^2}{1+a^2}\right)\right]$$

i

$$tan\left[\frac{1}{2}sin^{-1}\left(\frac{2a}{1+a^2}\right)+\frac{1}{2}cos^{-1}\left(\frac{1-a^2}{1+a^2}\right)\right]$$

Let a=tanθ.

Now,

$$tan\left[\frac{1}{2}sin^{-1}\left(\frac{2a}{1+a^2}\right)+\frac{1}{2}cos^{-1}\left(\frac{1-a^2}{1+a^2}\right)\right]=tan\left[\frac{1}{2}sin^{-1}\left(\frac{2tan\theta}{1+tan^2\theta}\right)+\frac{1}{2}cos^{-1}\left(\frac{1-tan^2\theta}{1+tan^2\theta}\right)\right]$$

$$tan\left[\frac{1}{2}sin^{-1}(sin2\theta)+\frac{1}{2}cos^{1}(cos2\theta)\right]=tan[2\theta]=\frac{2tan\theta}{1-tan^2\theta}=\frac{2a}{1-a^2}$$

14) Prove that $tan^{-1}\left(\frac{m}{n}\right)-tan^{-1}\left(\frac{m-n}{m+n}\right)=\frac{\pi}{4}$

LHS = $tan^{-1}\left(\frac{m}{n}\right)-tan^{-1}\left(\frac{m-n}{m+n}\right)$

$$=tan^{-1}\left(\frac{\frac{m}{n}-\frac{m-n}{m+n}}{1+\frac{m}{n}\left(\frac{m-n}{m+n}\right)}\right)$$

$$=tan^{-1}\left(\frac{\frac{m(m+n)-n(m-n)}{m(m+n)}}{\frac{n(m+n)+m(m-n)}{n(m+n)}}\right)$$

$$=tan^{-1}\left(\frac{\frac{m^2+mn-mn+n^2}{n(m+n)}}{\frac{mn+n^2+m^2-mn}{n(m+n)}}\right)$$

$$=tan^{-1}\left(\frac{m^2+n^2}{m^2+n^2}\right)=tan^{-1}(1)$$

$$=\frac{\pi}{4}=RHS$$

Hence proved.

15) Solve: $tan^{-1}\left(\frac{x-1}{x-2}\right)+tan^{-1}\left(\frac{x+1}{x+2}\right)=\frac{\pi}{4}$

$$tan^{-1}\left(\frac{x-1}{x-2}\right)+tan^{-1}\left(\frac{x+1}{x+2}\right)=\frac{\pi}{4}$$

$\Rightarrow tan^{-1}\left(\dfrac{\frac{x-1}{x-2}+\frac{x+1}{x+2}}{1-\left(\frac{x-1}{x-2}\right)\left(\frac{x+1}{x+2}\right)}\right)=\dfrac{\pi}{4}$

$\Rightarrow \left(\dfrac{\dfrac{(x-1)(x+2)+(x+1)(x-2)}{(x-2)(x+2)}}{\dfrac{(x-2)(x+2)-(x-1)(x+1)}{(x-2)(x+2)}}\right)=\tan\dfrac{\pi}{4}=1$

$\Rightarrow \dfrac{x^2+x-2+x^2-x-2}{(x^2-4)-(x^2-1)}=1$

$\Rightarrow \dfrac{2x^2-4}{x^2-4-x^2+1}=1$

$\Rightarrow$ $2x^2$-4 =-3

$\Rightarrow$ $2x^2$-4=-3

$\Rightarrow$ $2x^2$ = -3 + 4 = 1

$\Rightarrow x^2=\dfrac{1}{2}$

$\Rightarrow x=\dfrac{1}{\sqrt{2}}$

16) Solve $tan^{-1}\left(\dfrac{2x}{1-x^2}\right)+cot^{-1}\left(\dfrac{1-x^2}{2x}\right)=\dfrac{\pi}{3}, x>0$

$tan^{-1}\left(\dfrac{2x}{1-x^2}\right)+tan^{-1}\left(\dfrac{2x}{1-x^2}\right)=\dfrac{\pi}{3}$

$\left[\because co^{-1}\left(\dfrac{1}{x}\right)=tan^{-1}(x)\right]$

$\Rightarrow 2tan^{-1}\left(\dfrac{2x}{1-x^2}\right)=\dfrac{\pi}{3}$

$\Rightarrow tan^{-1}\left(\dfrac{2x}{1-x^2}\right)=\dfrac{\pi}{6}$

$\Rightarrow \dfrac{2x}{1-x^2}=tan\left(\dfrac{\pi}{6}\right)=\dfrac{1}{\sqrt{3}}$

$\Rightarrow 2\sqrt{3x}=1-x^2$

$\Rightarrow x^2+2\sqrt{3x}-1=0$

$\Rightarrow x=\dfrac{-2\sqrt{3}\pm\sqrt{12-4(1)(-1)}}{2}$

$\left[\because x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}\right]$

$\Rightarrow x = \dfrac{-2\sqrt{3} \pm \sqrt{16}}{2} \Rightarrow x = \dfrac{-2\sqrt{3} \pm 4}{2}$

$\Rightarrow x = 2\left(\dfrac{-\sqrt{3} \pm 2}{2}\right)$

$\Rightarrow x = -\sqrt{3} \pm 2$

$\Rightarrow x = -2 - \sqrt{3} \quad or \quad -2 - \sqrt{3}$

Since $x > 0, x = -2 - \sqrt{3}$ is not possible

$\therefore x = 2 - \sqrt{3}$

17) Prove that $tan^{-1}\sqrt{x} = \frac{1}{2}cos^{-1} = \frac{1}{2}cos^{-1}\left(\dfrac{1-x}{1+x}\right), x\epsilon|0,1|$

LHS = $tan^{-1}\sqrt{x} = \frac{1}{2}.2tan^{-1}\left(\sqrt{x}\right)$

$= \frac{1}{2}.\left(2tan^{-1}\left(\sqrt{x}\right)\right) = \frac{1}{2}cos^{-1}\left(\dfrac{1-\left(\sqrt{x}\right)^2}{1+\left(\sqrt{x}\right)^2}\right)$

$\left[\because tan^{-1}x = cos^{-1}\left(\dfrac{1-x^2}{1+x^2}\right)\right]$

$= \frac{1}{2}cos^{-1}\left(\dfrac{1-x}{1+x}\right)$

=RHS

Hence proved .

18) Find the equation of the ellipse in each of the cases given below:

(i) foci(±3 0),e =$\frac{1}{2}$

(ii) foci (0,±4)and end points of major axis are(0,±5).

(iii) length of latus rectum 8, eccentricity =$\frac{3}{5}$ and major axis on x -axis.

(iv) length of latus rectum 4 , distance between foci4 $\sqrt{2}$ and major axis as y - axis.

(i) foci (±3 0),e =$\frac{1}{2}$

Since foci are (±ae, 0)

$\Rightarrow$ ae = 3

$\Rightarrow$ a.$\frac{1}{2}$ = 3

$\Rightarrow$ a = 6

Centre is (0, 0)

and $b^2 = a^2 (1 - e^2)$

$\Rightarrow b_2 = 36\left(1 - \frac{1}{4}\right)$

$\Rightarrow b^2 = 36\left(\frac{3}{4}\right)$

$\Rightarrow b^2 = 27$

$\therefore$ Equation of the ellipse is $\frac{x^2}{36} + \frac{y^2}{27} = 1$

$\frac{x^2}{36} + \frac{y^2}{27} = 1$

(ii) Foci (0, ±4) and end points of major axis

are (0, ±5)

Since foci are (0, ±be) $\Rightarrow$ be = 4

End points of major axis are (0, ±5)

$\Rightarrow$ b = 5

$\therefore$ 5(e) = 4

$\Rightarrow e = \frac{4}{5}$

Also, $a^2 = b^2(1 - e^2)$

$a_2 = 25\left(1 - \frac{16}{25}\right) = 25\left(\frac{25-16}{25}\right)$

$\Rightarrow a^2 = 9$

Equation of the ellipse is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$\Rightarrow \frac{x^2}{9} + \frac{y^2}{25} = 1$

(iii) Length of latusectrum = 8, $e = \frac{3}{5}$ and major axis on x-axis .

Given $\frac{2b^2}{a} = 8, e = \frac{3}{5}$

$b^2 = 4a$

$b^2 = a^2(1- e^2)$

$4a = a^2\left(1 - \frac{9}{25}\right)$

$4 = a\left(\frac{25-9}{25}\right)$

100 = a(16)

$a = \frac{100}{16} = \frac{25}{4} \Rightarrow a^2 = \frac{625}{16}$

$b^2 = 4 \times \frac{25}{4} = 25$

Since major axis is on x-axis, equation of the ellipse is

$\Rightarrow \frac{(x-0)^2}{a^2} + \frac{(y-0)^2}{b^2} = 1$

$\Rightarrow \frac{x^2}{\frac{625}{16}} + \frac{y^2}{25} = 1$

$\Rightarrow \frac{x^2}{625} + \frac{y^2}{25} = 1$

(iv) Length of latusrectum = 4, distance between foci = $4\sqrt{2}$ major axis is y-axis

Given $\frac{2b^2}{a} = 4$ and distance between foci = 2ae = $4\sqrt{2}$

$\Rightarrow ae = 2\sqrt{2}$

$\Rightarrow a^2e^2 = 8$ ....(1)

$\frac{2b^2}{a} = 4 \Rightarrow b^2 = 2a$ ...(2)

We know $b^2 = a^2(1 - e^2)$

$b^2 = a^2 - a^2e^2$

2a = a2 - 8

[using (1) and (2)]

$a^2 - 2a - 8 = 0$

On factorising we get

(a - 4)(a + 2) = 0

a = 4 or-2

a = 4

⇒ [∴ a = -2 is not possible]

⇒ ae = 16

∴ From (2), b2 = 2(4) = 8

Hence, the equation of the ellipse is

$\frac{x^2}{8} + \frac{y^2}{16} = 1$ [∵ Major axis is y -axis]

19) Find the equation of the hyperbola in each of the cases given below:

(i) foci(±2,0), eccentricity $=\frac{3}{2}$

(ii) Centre (2,1) , one of the foci (8,1) and corresponding directrix x = 4.

(iii) passing through(5,−2)and length of the transverse axis along x axis and of length 8 units.

(i) $a\left(\frac{3}{2}\right) = 2 \Rightarrow a\frac{4}{3}$

$b^2 = a^2(e^2 - 1)$

$\Rightarrow b^2 = \frac{16}{9}\left(\frac{9}{4}-1\right) = \frac{16}{9}\left(\frac{9-4}{4}\right) = \frac{\overset{4}{\cancel{16}}}{9} \times \frac{5}{4}$

$\Rightarrow b^2 = \frac{4\times5}{9} = \frac{20}{9}$

∴ Equation of the hyperbola is $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

$\Rightarrow \frac{x^2}{\frac{16}{9}} - \frac{y^2}{\frac{20}{9}} = 1$

$\Rightarrow \frac{9x^2}{16} - \frac{9y^2}{20} = 1$

(ii) ae = distance between centre and focus

ae = $\sqrt{(8-2)^2 - (1-1)^2} = \sqrt{6^2} = 6$ ...(1)

Also $\frac{a}{e} = \sqrt{(4-2)^2 + (1-1)^2} = \sqrt{2^2} = 2$

[∴ (4, 1) is a point on the directrix]

$(1) \times (2) \rightarrow a\cancel{e} \times \frac{a}{\cancel{e}} = 6 \times 2$

⇒ $a^2 = 12$

(1) → $a^2e^2 = 36$

12(e2) = 36

⇒ e2 = 3

⇒ e = $\sqrt{3}$

Also, $b^2 = a^2(e^2 - 1) = 12(3 - 1) = 12(2) = 24$

∴ Equation of the hyperbola is

$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$

$\Rightarrow \frac{(x-2)^2}{12} - \frac{(y-1)^2}{24} = 1$

(iii) Passing through (5, -2) length of the transverse axis is a long x-axis and of length 8 units.

2a = 8 ⇒ a=4

Since the transverse axis is along x-axis, centre is (0, 0)

Equation of the hyperbola is

$\frac{x^2}{16} - \frac{y^2}{b^2} = 1$

Since (5, -2)passes through the parabola,

$\frac{25}{16} - \frac{4}{b^2} \Rightarrow \frac{4}{b^2} = \frac{25}{16} - 1 = \frac{25-16}{16} = \frac{9}{16}$

$\therefore b^2 = \frac{16 \times 4}{9} = \frac{64}{9}$

$\therefore$ Equation of the hyperbola is

$\frac{x^2}{16} - \frac{y^2}{\frac{64}{9}} = 1 \Rightarrow \frac{x^2}{16} - \frac{9y^2}{64} = 1$

20) A straight line passes through the point (1, 2, −3) and parallel to $4\hat{i} + 5\hat{j} - 7\hat{k}$. Find

(i) vector equation in parametric form

(ii) vector equation in non-parametric form

(iii) Cartesian equations of the straight line.

The required line passes through (1, 2, −3). So, the position vector of the point is $\hat{i} + 2\hat{j} - 3\hat{k}$.

Let $a = \hat{i} + 2\hat{j} - 3\hat{k}$ and $\vec{b} = 4\hat{i} + 5\hat{j} - 7\hat{k}$. Then, we have

(i) vector equation of the required straight line in parametric form is $r = a + t\vec{b}$, $t \in R$

Therefore, $\vec{r} = (\hat{i} + 2\hat{j} - 3\hat{k}) + t(4\hat{i} + 5\hat{j} - 7\hat{k})$, $t \in R$

(ii) vector equation of the required straight line in non-parametric form is $(\vec{r} - \vec{a}) \times \vec{b} = \vec{0}$

Therefore, $\vec{r} = (\hat{i} + 2\hat{j} - 3\hat{k}) + t(4\hat{i} + 5\hat{j} - 7\hat{k}) = \vec{0}$

(iii) Cartesian equations of the required line are $\frac{x-x_1}{b_1} = \frac{y-y_1}{b_2} = \frac{z-z_1}{b_3}$

Here, (x_1, y_1, z_1) = (1, 2, −3) and direction ratios of the required line are proportional to 4, 5, −7 . Therefore, Cartesian equations of the straight line are $\frac{x-1}{4} = \frac{y-2}{5} = \frac{z+3}{-7}$

80 x 5 = 400

21) If A = $\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & 4 \\ 2 & -4 & 3 \end{bmatrix}$, verify that A(adj A)=(adj A)A = |A| I_3.

We find that |A| = $\begin{vmatrix} 8 & -6 & 2 \\ -6 & 7 & 4 \\ 2 & -4 & 3 \end{vmatrix}$ = 8(21 - 16) + 6(-18 + 8) + 2(24 - 14) = 40 - 60 + 20 = 0

By the definition of adjoint, we get

adj A = $\begin{bmatrix} (21-16) & -(-18+8) & (24-14) \\ -(-18+8) & (24-4) & -(32+12) \\ (24-14) & -(-32+12) & (56-36) \end{bmatrix}^T = \begin{bmatrix} 5 & 10 & 10 \\ 10 & 20 & 20 \\ 10 & 20 & 20 \end{bmatrix}$

So, we get

A(adj A) = $\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix} \begin{bmatrix} 5 & 10 & 10 \\ 10 & 20 & 20 \\ 10 & 20 & 20 \end{bmatrix}$

$= \begin{bmatrix} 40-60+20 & 80-120+40 & 80-120+40 \\ -30+70-40 & -60+140-80 & -60+140-80 \\ 10-40+30 & 20-80+60 & 20-80+60 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0I_3 = |A|I_3$, Similarly, we get

(adj A)A = $\begin{bmatrix} 5 & 10 & 10 \\ 10 & 20 & 20 \\ 10 & 20 & 20 \end{bmatrix}\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$

= $\begin{bmatrix} 40-60+20 & -30+70-40 & 10-40+30 \\ 80-120+40 & -60+140-80 & 20-80+60 \\ 80-120+40 & -60+140-80 & 20-80+60 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0I_3 = |A|I_3$.

Hence, A(adj A) = (adj A)A = $|A|I_3$.

22) If A = $\begin{bmatrix} -4 & 4 & 4 \\ -7 & 1 & 3 \\ 5 & -3 & -1 \end{bmatrix}$ and B = $\begin{bmatrix} 1 & -1 & 1 \\ 1 & -2 & -2 \\ 2 & 1 & 3 \end{bmatrix}$, find the productsAB and BAand hence solve the system of equations x - y + z = 4, x - 2y - 2z = 9, 2x + y + 3z = 1.

We find AB = $\begin{bmatrix} -4 & 4 & 4 \\ -7 & 1 & 3 \\ 5 & -3 & -1 \end{bmatrix}\begin{bmatrix} 1 & -1 & 1 \\ 1 & -2 & -2 \\ 2 & 1 & 3 \end{bmatrix} = \begin{bmatrix} -4+4+8 & 4-8+4 & -4-8+12 \\ -7+1+6 & 7-2+3 & -7-2+9 \\ 5-3-2 & -5+6-1 & 5+6-3 \end{bmatrix} = \begin{bmatrix} 8 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 8 \end{bmatrix} = 8I_3$

and BA = $\begin{bmatrix} 1 & -1 & 1 \\ 1 & -2 & -2 \\ 2 & 1 & 3 \end{bmatrix}\begin{bmatrix} -4 & 4 & 4 \\ -7 & 1 & 3 \\ 5 & -3 & -1 \end{bmatrix} = \begin{bmatrix} -4+7+5 & 4-1-3 & 4-3-1 \\ -4+14-10 & 4-2+6 & 4-6+2 \\ -8-7+15 & 8+1-9 & 8+3-3 \end{bmatrix} = \begin{bmatrix} 8 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 8 \end{bmatrix} = 8I_3$

So we get AB = BA = $8I_3$. That is, $(\frac{1}{8}A)B = B(\frac{1}{8}A) = I_3$. Hence, $B^{-1} = \frac{1}{8}A$.

Writing the given system of equations in matrix form, we get

$\begin{bmatrix} 1 & -1 & 1 \\ 1 & -2 & -2 \\ 2 & 1 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 9 \\ 1 \end{bmatrix}$. That is B $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 9 \\ 1 \end{bmatrix}$.

So, $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = B^{-1}\begin{bmatrix} 4 \\ 9 \\ 1 \end{bmatrix} = \left(\frac{1}{8}A\right)\begin{bmatrix} 4 \\ 9 \\ 1 \end{bmatrix} = \frac{1}{8}\begin{bmatrix} -4 & 4 & 4 \\ -7 & 1 & 3 \\ 5 & -3 & -1 \end{bmatrix}\begin{bmatrix} 4 \\ 9 \\ 1 \end{bmatrix} = \frac{1}{8}\begin{bmatrix} -16+36+4 \\ -28+9+3 \\ 20-27-1 \end{bmatrix} = \frac{1}{8}\begin{bmatrix} 24 \\ -16 \\ -8 \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \\ -1 \end{bmatrix}$

Hence, the solution is (x = 3, y = -2, z = -1).

23) The prices of three commodities A, B and C are Rs.x, y and z per units respectively. A person P purchases 4 units of B and sells two units of A and 5 units of C . Person Q purchases 2 units of C and sells 3 units of A and one unit of B . Person R purchases one unit of A and sells 3 unit of B and one unit of C . In the process, P, Q and R earn Rs.15,000, Rs.1,000 and Rs.4,000 respectively. Find the prices per unit of A, B and C . (Use matrix inversion method to solve the problem.)

Let the prices per unit for the commodities A, B and C be Rs.x, Rs.y and Rs.z.

By the given data,

2x - 4y + 5z = 15000

3x + y - 2z· = 1000

-x + 3y + z = 4000

The matrix form of the system of equations is

$\begin{bmatrix} 2 & -4 & 5 \\ 3 & 1 & -2 \\ -1 & 3 & 12 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 15000 \\ 1000 \\ 4000 \end{bmatrix}$

AX=B where A= $\begin{bmatrix} 2 & -4 & 5 \\ 3 & 1 & -2 \\ -1 & 3 & 1 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

and B= $\begin{bmatrix} 15000 \\ 1000 \\ 4000 \end{bmatrix}$

$\Rightarrow$ X=A^{-1}B

|A|= $\begin{vmatrix} 2 & -4 & 5 \\ 3 & 1 & -2 \\ -1 & 3 & 1 \end{vmatrix}$

= $2 \begin{vmatrix} 1 & -2 \\ 3 & 1 \end{vmatrix} + 4 \begin{vmatrix} 3 & -2 \\ -1 & 1 \end{vmatrix} + 5 \begin{vmatrix} 3 & 1 \\ -1 & 3 \end{vmatrix}$

= 2 (1 + 6) + 4 (3 - 2) + 5 (9 + 1)

= 2 (7) + 4 (1) + 5(10) = 14 + 4 + 50 = 68.

adj A=

$$\begin{bmatrix} +\begin{vmatrix} 1 & -2 \\ 3 & 1 \end{vmatrix} & -\begin{vmatrix} 3 & -2 \\ -1 & 1 \end{vmatrix} & +\begin{vmatrix} 3 & 1 \\ -1 & 3 \end{vmatrix} \\ -\begin{vmatrix} -4 & 5 \\ 3 & 1 \end{vmatrix} & +\begin{vmatrix} 2 & 5 \\ -1 & 1 \end{vmatrix} & -\begin{vmatrix} 2 & -4 \\ -1 & 3 \end{vmatrix} \\ +\begin{vmatrix} -4 & 5 \\ 1 & -2 \end{vmatrix} & -\begin{vmatrix} 2 & 5 \\ 3 & -2 \end{vmatrix} & +\begin{vmatrix} 2 & -4 \\ 3 & 1 \end{vmatrix} \end{bmatrix}^T$$

= $\begin{bmatrix} +(1+6) & -(3-2) & +(9+1) \\ -(-4-15) & +(2+5) & -(6-4) \\ +(8-5) & -(4-15) & +(2+12) \end{bmatrix}$

= $\begin{bmatrix} 7 & -1 & 10 \\ 19 & 7 & -2 \\ 3 & 19 & 14 \end{bmatrix}^T = \begin{bmatrix} 7 & 19 & 3 \\ -1 & 7 & 19 \\ 10 & -2 & 14 \end{bmatrix}$

$\therefore A^{-1} = \frac{1}{|A|} adj = \frac{1}{68} \begin{bmatrix} 7 & 19 & 3 \\ -1 & 7 & 19 \\ 10 & -2 & 14 \end{bmatrix}$

$\therefore X = A^{-1}B = \frac{1}{68} \begin{bmatrix} 7 & 19 & 3 \\ -1 & 7 & 19 \\ 10 & -2 & 14 \end{bmatrix} \begin{bmatrix} 15000 \\ 1000 \\ 4000 \end{bmatrix}$

= $\frac{1}{68} \begin{bmatrix} 105000 + 19000 + 12000 \\ -15000 + 7000 + 76000 \\ 150000 - 2000 + 56000 \end{bmatrix}$

= $\frac{1}{68} \begin{bmatrix} 136000 \\ 68000 \\ 204000 \end{bmatrix} = \begin{bmatrix} 2000 \\ 1000 \\ 3000 \end{bmatrix}$

$\therefore$ x = 2000, y = 1000, z = 3000

Hence the prices per unit of the commodities A, B and C are Rs.2000, Rs.1000 and Rs.3000 respectively.

24) In a T20 match, Chennai Super Kings needed just 6 runs to win with 1 ball left to go in the last over. The last ball was bowled and the batsman at the crease hit it high up. The ball traversed along a path in a vertical plane and the equation of the path is $y = ax^2 + bx + c$ with respect to a xy-coordinate system in the vertical plane and the ball traversed through the points (10, 8), (20, 16) (30, 18) can you conclude that Chennai Super Kings won the match?

Justify your answer. (All distances are measured in metres and the meeting point of the plane of the path with the farthest boundary line is (70, 0).)

The path $y = ax^2 + bx + c$ passes through the points (10, 8), (20, 16), (40, 22). So, we get the system of equations 100a + 10b + c = 8, 400a + 20b + c = 16, 1600a + 40b + c = 22. To apply Cramer's rule, we find

$$\Delta = \begin{vmatrix} 100 & 10 & 1 \\ 400 & 20 & 1 \\ 1600 & 40 & 1 \end{vmatrix} = 1000 \begin{vmatrix} 1 & 1 & 1 \\ 4 & 2 & 1 \\ 16 & 4 & 1 \end{vmatrix} = 1000\,[-2 + 12 - 6] = -6000,$$

$$\Delta_1 = \begin{vmatrix} 8 & 10 & 1 \\ 16 & 20 & 1 \\ 22 & 40 & 1 \end{vmatrix} = 20 \begin{vmatrix} 4 & 1 & 1 \\ 8 & 2 & 1 \\ 11 & 4 & 1 \end{vmatrix} = 20[-8 + 3 + 10] = 100,$$

$$\Delta_2 = \begin{vmatrix} 100 & 8 & 1 \\ 400 & 16 & 1 \\ 1600 & 22 & 1 \end{vmatrix} = 200 \begin{vmatrix} 1 & 4 & 1 \\ 4 & 8 & 1 \\ 16 & 11 & 1 \end{vmatrix} = 200[-3 + 48 - 84] = -7800,$$

$$\Delta_3 = \begin{vmatrix} 100 & 10 & 8 \\ 400 & 20 & 16 \\ 1600 & 40 & 22 \end{vmatrix} = 2000 \begin{vmatrix} 1 & 1 & 4 \\ 4 & 2 & 8 \\ 16 & 4 & 11 \end{vmatrix} = 2000[-10 + 84 - 64] = 20000.$$

By Cramer's rule, we get $a = \frac{\Delta_1}{\Delta} = -\frac{1}{60}$, $b = \frac{\Delta_2}{\Delta} = \frac{7800}{6000} = \frac{78}{60} = \frac{13}{10}$, $c = \frac{\Delta_3}{\Delta} = \frac{20000}{6000} = -\frac{20}{6} = -\frac{10}{3}$.

So, the equation of the path is $y = \frac{1}{60}x^2 + \frac{13}{10}x - \frac{10}{3}$.

When x = 70, we get y = 6. So, the ball went by 6 metres high over the boundary line and it is impossible for a fielder standing even just before the boundary line to jump and catch the ball. Hence the ball went for a super six and the Chennai Super Kings won the match.

25) A family of 3 people went out for dinner in a restaurant. The cost of two dosai, three idlies and two vadais is Rs.150. The cost of the two dosai, two idlies and four vadais is Rs.200. The cost of five dosai, four idlies and two vadais is Rs.250. The family has Rs.350 in hand and they ate 3 dosai and six idlies and six vadais. Will they be able to manage to pay the bill within the amount they had ?

Let the cost of one dosa be Rs.x

The cost of one idli be Rs.y

and the cost of one vadai be Rs.z

By the given data,

2x+ 3y+ 2z =150

2x + 2y+ 4z=200

5x+4y+ 2z=250

$$\therefore \Delta = \begin{vmatrix} 2 & 3 & 2 \\ 2 & 2 & 4 \\ 5 & 4 & 2 \end{vmatrix}$$

$= 2\begin{vmatrix} 2 & 4 \\ 4 & 2 \end{vmatrix} - 3\begin{vmatrix} 2 & 4 \\ 5 & 2 \end{vmatrix} + 2\begin{vmatrix} 2 & 2 \\ 5 & 4 \end{vmatrix}$

=2(4 - 16) - 3(4 - 20) + 2(8 - 10)

= 2(- 12) - 3(- 16) + 2(- 2)

= - 24 + 48 - 4 = 20

$\Delta_1 = \begin{vmatrix} 150 & 3 & 2 \\ 200 & 2 & 4 \\ 250 & 4 & 2 \end{vmatrix}$

Taking 50 common from C3 we get,

$=100\begin{vmatrix} 3 & 3 & 1 \\ 4 & 2 & 2 \\ 5 & 4 & 1 \end{vmatrix}$

$= 100\left[3\begin{vmatrix} 2 & 2 \\ 4 & 1 \end{vmatrix} - 3\begin{vmatrix} 4 & 2 \\ 5 & 1 \end{vmatrix} + 1\begin{vmatrix} 4 & 2 \\ 5 & 4 \end{vmatrix}\right]$

= 100[3(2 - 8) - 3(4 - 10) + 1(16 - 10)]

= 100[3(-6) - 3(- 6) + 6]

= 100[- 18 + 18 + 6] = 600

$\Delta_2 = \begin{vmatrix} 2 & 150 & 2 \\ 2 & 200 & 4 \\ 5 & 250 & 2 \end{vmatrix} = 100\begin{vmatrix} 2 & 3 & 1 \\ 2 & 4 & 2 \\ 5 & 5 & 1 \end{vmatrix}$

$= 100\left[3\begin{vmatrix} 4 & 2 \\ 5 & 1 \end{vmatrix} - 3\begin{vmatrix} 2 & 2 \\ 5 & 1 \end{vmatrix} + 1\begin{vmatrix} 2 & 4 \\ 5 & 5 \end{vmatrix}\right]$

= 100[2(4 - 10) - 3(2 - 10) + 1(10 - 20)]

= 100[2(- 6) - 3(- 8) + 1(- 10)]

= 100[- 12 + 24 - 10] = 100 [2] = 200.

$\Delta_3 = \begin{vmatrix} 2 & 3 & 150 \\ 2 & 2 & 200 \\ 5 & 4 & 250 \end{vmatrix} = 50\begin{vmatrix} 2 & 3 & 3 \\ 2 & 2 & 4 \\ 5 & 4 & 5 \end{vmatrix}$

$= 50\left[2\begin{vmatrix} 2 & 4 \\ 4 & 5 \end{vmatrix} - 3\begin{vmatrix} 2 & 4 \\ 5 & 5 \end{vmatrix} + 3\begin{vmatrix} 2 & 2 \\ 4 & 4 \end{vmatrix}\right]$

= 50 [2(10 - 16) - 3(10 - 20) + 3(8 - 10)]

= 50[2(- 6) - 3(- 10) +3(- 2)]

= 50 [- 12 + 30 - 6] = 50 [12] = 600.

$\therefore x = \frac{\Delta_1}{\Delta} = \frac{600}{20} = 30$

$y = \frac{\Delta_2}{\Delta} = \frac{200}{20} = 10$

$z = \frac{\Delta_3}{\Delta} = \frac{600}{20} = 30$

Hence, the price of one dosa be Rs.30,one idli be Rs.10 and the price of 1 vadai be Rs.30.

Also the cost on dosa, six idlies and six vadai is

= 3x + 6y + 6z = 3(30) + 6(10) + 6(30)

= 90 + 60 + 180 = Rs.330

Since the family had Rs.350 in hand, they will be able to manage to pay the bill.

26) The upward speed v(t)of a rocket at time t is approximated by $v(t) = at^2 + bt + c \le t \le 100$ where a, b and c are constants. It has been found that the speed at times t = 3, t = 6, and t = 9 seconds are respectively, 64, 133, and 208 miles per second respectively. Find the speed at time t = 15 seconds. (Use Gaussian elimination method.)

Since v(3) = 64, v(6) = 133,and v(9) = 208 , we get the following system of linear equations

9a + 3b + c = 64 ,

36a + 6b + c = 133 ,

81a + 9b + c = 208 .

We solve the above system of linear equations by Gaussian elimination method.

Reducing the augmented matrix to an equivalent row-echelon form by using elementary row operations, we get

$$[A \mid B] = \left[\begin{array}{ccc|c} 9 & 3 & 1 & 64 \\ 36 & 6 & 1 & 133 \\ 81 & 9 & 1 & 208 \end{array}\right] \xrightarrow{R_2 \to R_2 - 4R_1, R_3 \to R_3 - 9R_1} \left[\begin{array}{ccc|c} 9 & 3 & 1 & 64 \\ 0 & -6 & -3 & -123 \\ 0 & -18 & -8 & -368 \end{array}\right] \xrightarrow{R_2 \to R_2 \div (-3), R_3 \div (-2)} \left[\begin{array}{ccc|c} 9 & 3 & 1 & 64 \\ 0 & 2 & 1 & 41 \\ 0 & 9 & 4 & 184 \end{array}\right]$$

$$\xrightarrow{R_3 \to 2R_3} \left[\begin{array}{ccc|c} 9 & 3 & 1 & 64 \\ 0 & 2 & 1 & 41 \\ 0 & 18 & 8 & 368 \end{array}\right] \xrightarrow{R_3 \to R_3 - 9R_2} \left[\begin{array}{ccc|c} 9 & 3 & 1 & 64 \\ 0 & 2 & 1 & 41 \\ 0 & 0 & -1 & -1 \end{array}\right] \xrightarrow{R_3 \to (-1)R_3} \left[\begin{array}{ccc|c} 9 & 3 & 1 & 64 \\ 0 & 2 & 1 & 41 \\ 0 & 0 & 1 & 1 \end{array}\right].$$

Writing the equivalent equations from the row-echelon matrix, we get

9a + 3b + c = 64, 2b + c = 41, c = 1.

By back substitution, we get c = 1, b = $\frac{(41-c)}{2} = \frac{(41-1)}{2} = 20$, a = $\frac{64-3b-c}{9} = \frac{64-60-1}{9} = \frac{1}{3}$.

So, we get $v(t) = \frac{1}{3}t^2 + 20t + 1$. Hence, $v(15) = \frac{1}{3}(225) + 20(15) + 1 = 75 + 300 + 1 = 376$.

27) If $ax^2 + bx + c$ is divided by x + 3, x − 5, and x − 1, the remainders are 21,61 and 9 respectively. Find a,b and c. (Use Gaussian elimination method.)

Let $P(x) = ax^2+bx+c$

Given P(-3) = 21

[∵ P(x) ÷x+3, the remainder is 21]

⇒ a(-3)2+b(-3)+c=21

⇒ 9a-3b+c=21

Also, P(5)=61

⇒ a(5)2+b(5)+c=61

[using remainder theorem]

⇒ 25x+5b+c=61 ....(2)

and P(1)=9

⇒ a(1)2+b(1)+c=9

⇒ a+b+c=9 ....(3)

Reducing the augment matrix to an equivalent row-echelon form using elementary row operations, we get

$$\left[\begin{array}{ccc|c} 9 & - & 1 & 21 \\ 25 & 5 & 1 & 61 \\ -1 & 1 & 1 & 9 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_3} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 9 \\ 25 & 5 & 1 & 61 \\ 9 & -3 & 1 & 21 \end{array}\right]$$

$$\xrightarrow{R_3 \to R_3 - 9R_1 R_2 \to R_2 - 25R_1} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 9 \\ 0 & -20 & -24 & -164 \\ 0 & -12 & -8 & -60 \end{array}\right]$$

$$R_2 \to R_2 \div R_3 \to R_3 \div 4 \underset{\longrightarrow}{} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 9 \\ 0 & -5 & -6 & -41 \\ 0 & -3 & -2 & -15 \end{array}\right]$$

$$R_3 \to R_3 - \tfrac{3}{5}R_2 \underset{\longrightarrow}{} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 9 \\ 0 & -5 & -6 & -41 \\ 0 & 0 & \frac{8}{5} & \frac{48}{5} \end{array}\right]$$

$$R_3 \to 5R_3 \underset{\longrightarrow}{} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 9 \\ 0 & -5 & -6 & -41 \\ 0 & 0 & 8 & 48 \end{array}\right]$$

Writing the equivalent equations from the row-echelon matrix we get,

a+b+c = 9 ....(1)

- 5b-6c = -41 .....(2)

8c=48

$\Rightarrow$ c=$\frac{48}{8}$=6

Substituting c=6 in (2) we get,

$\Rightarrow$ -5b-6(6)=-41

$\Rightarrow$ -5b=-41+36=-5

$\Rightarrow$ -5b=-41+36=-5

$\Rightarrow$ b=$\frac{-5}{-5}$=1

Substituting b = 1, c = 6 in (1) we get,

a+1+6=9

$\Rightarrow$ a+7=9

$\Rightarrow$ a=9-7

$\Rightarrow$ a=2

$\therefore$ a=2, b=1, and c=6

28) An amount of Rs.65,000 is invested in three bonds at the rates of 6%,8% and 9% per annum respectively. The total annual income is Rs.4,800. The income from the third bond is Rs.600 more than that from the second bond. Determine the price of each bond. (Use Gaussian elimination method.)

Let the price of bond invested in 6%, 8% and 9% rates be let Rs.x, Rs.y and Rs.z respectively

$\therefore$ By the given data, x + y + z = 65000 ....(1)

$\frac{6\times x\times 1}{100} + \frac{8\times y\times 1}{100} + \frac{9\times z\times 1}{100}$=4800

[$\because$ Intrest=$\frac{PNR}{100}$]

$\Rightarrow$ $\frac{6x}{100} + \frac{8y}{100} + \frac{9z}{100}$=4800

$\Rightarrow$ 6x+8y+9z=480000 ......(2)

Also, $\frac{9z}{100} = 600 + \frac{8y}{100}$

$\Rightarrow$ $\frac{-8y}{100} + \frac{9y}{100}$=600

$\Rightarrow$ -8y+9z=60000 ....(3)

Reducing the augmented matrix to an equivalent row-echelon form by using elementary row operation, we get

$$\begin{bmatrix} 1 & 1 & 1 & 65000 \\ 6 & 8 & 9 & 480000 \\ 0 & -8 & 9 & 60000 \end{bmatrix}$$

$$\xrightarrow{R_2 \to R_2 - 6R_1} \begin{bmatrix} 1 & 1 & 1 & 65000 \\ 0 & 2 & 3 & 90000 \\ 0 & -8 & 9 & 60000 \end{bmatrix}$$

$$\xrightarrow{R_3 \to R_3 + 4R_2} \begin{bmatrix} 1 & 1 & 1 & 65000 \\ 0 & 2 & 3 & 90000 \\ 0 & 0 & 21 & 420000 \end{bmatrix}$$

Writing the equivalent from the row echelon matrix we get,

x+y+z=65000 ....(1)

2y+z=90000 .....(2)

21z=42000

$\Rightarrow$ z=$\frac{420000}{21}$=20000

Substituting z = 20,000 in (2),

2y + 3(20,000) = -90000

$\Rightarrow$ 2y+60,000 =90,000

$\Rightarrow$ 2y=90,000-60,000

=30,000

$\Rightarrow$ y=$\frac{30,000}{2}$=15,000

Substitutingy = 15,000 and z = 20,000 in (1) we get,

x+15,000+20,000=65000

$\Rightarrow$ x+35,000=65000

$\Rightarrow$ x=65,000-35,000

$\Rightarrow$ 30,000

Thus the price of 6% bond is f 30,000 the price of 8% bond is f 15,000 and the price of 9% bond is f 20,000 is Rs.20,000.

29) A boy is walking along the path y = ax^2 + bx + c through the points (−6, 8),(−2, −12) , and (3, 8) . He wants to meet his friend at P(7,60) . Will he meet his friend? (Use Gaussian elimination method.)

Giveny = ax^2 + bx + c ...(1)

(-6,8) lies on (1)

$\Rightarrow$ 8=a(-6)^2+b(-6)+c

$\Rightarrow$ 8=36z-6b+c ....(2)

(-2,12) lies on (1)

$\Rightarrow$ -12=a(-2)^2+b(-2)+c

$\Rightarrow$ -12=4a-2b+c ......(3)

Also (3,8) lies on (1)

$\Rightarrow$ 8=a(3)^2+b(3)+c

$\Rightarrow$ 8=9a+3b+c ....(4)

Reducing the augment matrix to an equivalent row-echelon form by using elementary. row operations, we get,

$$\begin{bmatrix} 36 & -6 & 1 & 8 \\ 4 & -2 & 1 & -12 \\ 0 & 3 & 1 & 8 \end{bmatrix} \xrightarrow{R_2 \to 9R_2 - R_1 R_3 \to 4R_3 - R_1} \begin{bmatrix} 36 & -6 & 1 & 8 \\ 0 & -12 & 8 & -116 \\ 0 & 18 & 3 & 24 \end{bmatrix}$$

$\xrightarrow{R_2 \to R_2 \div 4 R_3 \to R_3 \div 3}$ $\left[\begin{array}{ccc|c} 36 & -6 & 1 & -8 \\ 0 & -3 & 2 & -29 \\ 0 & 0 & 5 & -8 \end{array}\right]$

$\xrightarrow{R_3 \to R_3 + 2R_2}$ $\left[\begin{array}{ccc|c} 36 & -6 & 1 & -8 \\ 0 & -3 & 2 & -29 \\ 0 & 0 & 5 & -50 \end{array}\right]$

Writing the equivalent equation from the row echelon matrix, we get 36a - 6b + c = 8 ....(1)

-3b+2c=-29 ....(2)

5c=-50

$\Rightarrow$ c=$\frac{-50}{5}$=-10

Substituting c = -10 in (2) we get,

-3b+2(-10)=-29

$\Rightarrow$ -3b+2(-10)=-29

$\Rightarrow$ -3b-20=-29

$\Rightarrow$ -3b=-9

$\Rightarrow$ b=$\frac{-9}{-3}$=3

Substituting b = 3 and c = -10 in (1) we get,

36a-6(3)-10=8

$\Rightarrow$ 36a-18-10=8

$\Rightarrow$ 36a-28=8

$\Rightarrow$ 6a=8+28=36

$\Rightarrow$ a=$\frac{36}{36}$=1

$\therefore$ a=1, b=3, c=-10

Hence the path of the boy is

y=1(x^2)+3(x)-10

$\Rightarrow$ y=x^2+3x-10

Since his friend is at P(7, 60),

60=$(7)^2$+3(7)-10

$\Rightarrow$ 60=49+21-10

$\Rightarrow$ 60=70-10=60

$\Rightarrow$ 60=60

Since (7, 60) satisfies his path, he can meet his friend who is at P(7, 60)

30) Test for consistency of the following system of linear equations and if possible solve:

x + 2y - z = 3, 3x - y + 2z = 1, x - 2y + 3z = 3, x - y + z + 1 = 0

Here the number of unknowns is 3.

The matrix form of the system is AX = B, where

A = $\begin{bmatrix} 1 & 2 & -1 \\ 3 & -1 & 2 \\ 1 & -2 & 3 \\ 1 & -1 & 1 \end{bmatrix}$, X = $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$, B = $\begin{bmatrix} 3 \\ 1 \\ 3 \\ -1 \end{bmatrix}$.

The augmented matrix is [A | B] = $\left[\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 3 & -1 & 2 & 1 \\ 1 & -2 & 3 & 3 \\ 1 & -1 & 1 & -1 \end{array}\right]$.

Applying Gaussian elimination method on [A | B], we get

$$[A\,|\,B] \xrightarrow[\substack{R_3 \to R_3 - R_1,\\ R_4 \to R_4 - R_1}]{R_2 \to R_2 - 3R_1} \left[\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 0 & -7 & 5 & -8 \\ 0 & -4 & 4 & 0 \\ 0 & -3 & 2 & 4 \end{array}\right] \xrightarrow[\substack{R_3 \to (-1)R_3\\ R_4 \to (-1)R_4}]{R_2 \to (-1)R_2,} \left[\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 0 & 7 & -5 & 8 \\ 0 & 4 & -4 & 0 \\ 0 & 3 & -2 & 4 \end{array}\right]$$

$$\xrightarrow[R_4 \to 7R_4 - 3R_2]{R_3 \to 7R_3 - 4R_2} \left[\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 0 & 7 & -5 & 8 \\ 0 & 0 & -8 & -32 \\ 0 & 0 & 1 & 4 \end{array}\right] \xrightarrow{R_3 \to R_3 \div (-8)} \left[\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 0 & 7 & -5 & 8 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 1 & 4 \end{array}\right] \xrightarrow{R_4 \to R_4 - R_3} \left[\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 0 & 7 & -5 & 8 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

There are three non-zero rows in the row-echelon form of [A | B].So, ρ([A | B]) = 3.

So, the row-echelon form of A is $\begin{bmatrix} 1 & 2 & -1 \\ 0 & 7 & -5 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$. There are three non-zero rows in it. So ρ(A) = 3.

Hence, ρ(A) = ρ([A | B]) = 3.

From the echelon form, we write the equivalent system of equations

x + 2y − z = 3,7y − 5z = 8, z = 4, 0 = 0.

The last equation 0=0 is meaningful. By the method of back substitution, we get

z = 4

7y − 20 = 8 ⇒ y = 4 ,

x = 3 − 8 + 4 ⇒ x = −1.

So, the solution is (x = −1, y = 4, z = 4).(Note that A is not a square matrix.)

Here the given system is consistent and the solution is unique.

31) Investigate for what values of λ and μ the system of linear equations

x + 2y + z = 7, x + y + λz = μ, x + 3y − 5z = 5 has

(i) no solution

(ii) a unique solution

(iii) an infinite number of solutions

Here the number of unknowns is 3.

The matrix form of the system is AX = B, where A = $\begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & \lambda \\ 1 & 3 & -5 \end{bmatrix}$, X = $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$, B = $\begin{bmatrix} 7 \\ \mu \\ 5 \end{bmatrix}$.

Applying elementary row operations on the augmented matrix [A | B], we get

$$[A\,|\,B] = \left[\begin{array}{ccc|c} 1 & 2 & 1 & 7 \\ 1 & 1 & \lambda & \mu \\ 1 & 3 & -5 & 5 \end{array}\right] \xrightarrow{R_2 \leftrightarrow R_3} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 7 \\ 1 & 3 & -5 & 5 \\ 1 & 1 & \lambda & \mu \end{array}\right] \xrightarrow[R_3 \to R_3 - R_1]{R_2 \to R_2 - R_1,} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 7 \\ 0 & 1 & -6 & -2 \\ 0 & -1 & \lambda - 1 & \mu - 7 \end{array}\right] \xrightarrow{R_3 \to R_3 + R_2} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 7 \\ 0 & 1 & -6 & -2 \\ 0 & 0 & \lambda - 7 & \mu - 9 \end{array}\right]$$

.

(i) If λ =7 and μ = 9, then ρ(A) = 2 and ρ([A | B]) = 3. So ρ(A) ≠ ρ([A | B]) Hence the given system is inconsistent and has no solution.

(ii) If λ ≠ 7 and μ is any real number, then ρ(A) = 3 and ρ([A | B]) = 3.

So, ρ(A) = ρ([A | B]) = 3 = Number of unknown. Hence the given system is consistent and has a unique solution.

(iii) If λ = 7 and μ = 9, then ρ(A) = 2 and ρ([A | B]) = 2.

So, ρ(A) = ρ([A | B]) = 2. < Number of unknown. Hence the given system is consistent and has infinite number of solutions.

32) Find the value of k for which the equations kx - 2y + z = 1, x - 2ky + z = -2, x - 2y + kz = 1 have

(i) no solution

(ii) unique solution

(iii) infinitely many solution

kx-2y+z=1, -2ky+z=-2, x-2y+k=1

The matrix form of the system is AX = B where

$$\begin{bmatrix} k & -2 & 1 \\ 1 & -2k & 1 \\ 1 & -2 & k \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, B = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

Applying elementary row operation on the augment matrix [AIB] we get

$$[A/B] = \begin{bmatrix} k & -2 & 1 & 1 \\ 1 & -2k & 1 & | \; -2 \\ 1 & -2 & k & 1 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_3} \begin{bmatrix} 1 & -2 & k & 1 \\ 1 & -2k & 1 & | \; -2 \\ k & -2 & k & 1 \end{bmatrix}$$

$$\xrightarrow{R_2 \to R_2 + R_1 R_3 \to R_3 - kR_1} \begin{bmatrix} 1 & -2 & k & 1 \\ 1 & -2k+2 & k & | \; - \\ 0 & -2+2k & 1-k^2 & 1-k \end{bmatrix}$$

$$\xrightarrow{R_3 \to R_3 + R_2} \begin{bmatrix} 1 & -2 & k & 1 \\ 0 & -2k+2 & k & | \; -3 \\ 0 & 0 & 1-k^2 & 1-k \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & -2 & k & 1 \\ 0 & -2k+2 & 1-k & -3 \\ 0 & 0 & k^2-k+2 & -k-2 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & -2 & k & 1 \\ 0 & -2k+2 & 1-k & | \; -3 \\ 0 & 0 & (k+2)(1-k) & -k-2 \end{bmatrix} \quad ....(1)$$

Case (i) when k=1

$$[A \mid B] \to \begin{bmatrix} 1 & -2 & 1 & 1 \\ 0 & 0 & 0 & -3 \\ 0 & 0 & 0 & -3 \end{bmatrix} \xrightarrow{R_3 \to R_3 - R_2} \begin{bmatrix} 1 & -2 & 1 & 1 \\ 0 & 0 & 0 & | \; -3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Here ρ(A)=3 and ρ[A|B]=3

So, ρ(A)≠ρ[A|B] ⇒ The system has no solution

Case (ii) when k≠2, k≠--2

$$\begin{bmatrix} 1 & -2 & k & 1 \\ 0 & -2k+2 & 1-k & | \; -3 \\ 0 & 0 & \text{not zero} & \text{not zero} \end{bmatrix}$$

$\Rightarrow \rho$(A)=3 and ρ[A|B]=3

so, ρ(A)=ρ[A|B]=3 =the number of unknowns Hence, the system has unique solution. Case (iii) when k=-2

$$\rho[A \mid B] \rightarrow \begin{bmatrix} 1 & -2 & -2 & 1 \\ 1 & 6 & 3 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Here ρ(A)=2 and ρ[A|B]=2

$\therefore \rho$(A)=ρ[A|B]=2< 3 the number of unknowns so the system is consistent with infinitely many solutions.

33) Investigate the values of λ and m the system of linear equations 2x + 3y + 5z = 9, 7x + 3y - 5z = 8, 2x + 3y + λz = μ, have

(i) no solution

(ii) a unique solution

(iii) an infinite number of solutions.

2x+3y=9, 7x+3y-5z=8, 2x+3y+λz=μ

The matrix form of the system is AX = B where

A= $\begin{bmatrix} 2 & 3 & 5 \\ 7 & 3 & -5 \\ 2 & 3 & \lambda \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, B = \begin{bmatrix} 9 \\ 8 \\ \mu \end{bmatrix}$

Applying elementary row operations augmented matrix [AIB] we get

[A|B]= $\begin{bmatrix} 2 & 3 & 5 & | & 9 \\ 7 & 3 & -5 & | & 8 \\ 2 & 3 & \lambda & & \mu \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 7 & 3 & -5 & & 8 \\ 2 & 3 & 5 & | & 9 \\ 2 & 3 & \lambda & & \mu \end{bmatrix}$

$$\xrightarrow{R_2 \to R_2 - \frac{2}{7}R_1 R_3 \to R_3 - R_2} \begin{bmatrix} 7 & 3 & -5 & & -8 \\ 0 & \frac{15}{7} & \frac{45}{7} & | & \frac{45}{7} \\ 0 & 0 & \lambda - 5 & & 4 - 9 \end{bmatrix}$$

$$\xrightarrow{R_2 \to R_2 \times 7} \begin{bmatrix} 7 & 3 & -5 & & -8 \\ 0 & 15 & 45 & | & 47 \\ 0 & 0 & \lambda - 5 & & \mu - 9 \end{bmatrix}$$

Case (i) when λ=5

[A|B]= $\begin{bmatrix} 7 & 3 & -5 & & -8 \\ 0 & 15 & 45 & | & 47 \\ 0 & 0 & 0 & & -4 \end{bmatrix}$

Here ρ(A)=2 and ρ[A|B]=3

So, ρ(A) ≠ ρ[A|B]

Hence the system is inconsistent and has no solution

Case (ii) When λ≠5, μ≠9

[A|B]= $\begin{bmatrix} 7 & 3 & -5 & & -8 \\ 0 & 15 & 45 & | & 47 \\ 0 & 0 & not\ \ zero & & not\ \ zero \end{bmatrix}$

Here ρ(A)=3 and ρ[A|B]=3

$\therefore \rho$(A)=ρ[A|B]=3= number of unknowns

Hence, the system is consistent with solution

Case (ii When λ=5, μ=9

$$[A|B]=\left[\begin{array}{cccc|c} 7 & 3 & -5 & & -8 \\ 0 & 15 & 45 & & 47 \\ 0 & 0 & 0 & & 0 \end{array}\right]$$

Here $\rho(A)=2$ and $\rho[A|B]=2$

$\therefore \rho(A)=\rho[A|B]= 2<$ number of unknowns

∴ The system is consistent and has infinite number of solutions.

34) Determine the values of λ for which the following system of equations $(3\lambda - 8)x + 3y + 3z = 0$, $3x + (3\lambda - 8)y + 3z = 0$, $3x + 3y + (3\lambda - 8)z = 0$. has a non-trivial solution.

Here the number of unknowns is 3. So, if the system is consistent and has a non-trivial solution, then the rank of the coefficient matrix is equal to the rank of the augmented matrix and is less than 3.

So the determinant of the coefficient matrix should be 0.

Hence we get

$$\begin{vmatrix} 3\lambda-8 & 3 & 3 \\ 3 & 3\lambda-8 & 3 \\ 3 & 3 & 3\lambda-8 \end{vmatrix} = 0 \text{ or } \begin{vmatrix} 3\lambda-2 & 3\lambda-2 & 3\lambda-2 \\ 3 & 3\lambda-8 & 3 \\ 3 & 3 & 3\lambda-8 \end{vmatrix} = 0 \text{ (by applying } R_1 \to R_1+R_2+R_3)$$

$$\text{or } (3\lambda-2)\begin{vmatrix} 1 & 1 & 1 \\ 3 & 3\lambda-8 & 3 \\ 3 & 3 & 3\lambda-8 \end{vmatrix} = 0 \text{ (by taking out } (3\lambda-2) \text{ from } R_1)$$

$$\text{or } (3\lambda-2)\begin{vmatrix} 1 & 1 & 1 \\ 3 & 3\lambda-11 & 3 \\ 3 & 3 & 3\lambda-11 \end{vmatrix} = 0 \text{ (by applying } R_2 \to R_2-3R_1, R_3 \to R_3-3R_1)$$

or $(3\lambda - 2)(3\lambda - 11)^2$ 0. So $\lambda = \frac{2}{3}$ and $\lambda = \frac{11}{3}$.

We now give an application of system of linear homogeneous equations to chemistry. You are already aware of balancing chemical reaction equations by inspecting the number of atoms present on both sides. A direct method is explained in the following example.

35) By using Gaussian elimination method, balance the chemical reaction equation: $C_5H_8 + O_2 \longrightarrow CO_2 + H_2O$. (The above is the reaction that is taking place in the burning of organic compound called isoprene.)

We are searching for positive integers x_1, x_2, x_3 and x_4 such that

$x_1C_5H_8 + x_2O_2 \longrightarrow x_3CO_2 + x_4H_2O$. ......(1)

The number of carbon atoms on the left-hand side of (1) should be equal to the number of carbon atoms on the right-hand side of (1). So we get a linear homogenous equation

$5x_1 = x_3 \Rightarrow 5x_1 - x_3 = 0$. ....(2)

Similarly, considering hydrogen and oxygen atoms, we get respectively,

$8x_1 = 2x_4 \Rightarrow 4x_1 - x_4 = 0$. ....(3)

$2x_2 = 2x_3 + x_4 \Rightarrow 2x_2 - 2x_4 = 0$. ....(4)

Equations (2), (3), and (4) constitute a homogeneous system of linear equations in four unknowns.

The augmented matrix is $[A \mid B] = \left[\begin{array}{cccc|c} 5 & 0 & -1 & 0 & 0 \\ 4 & 0 & 0 & -1 & 0 \\ 0 & 2 & -2 & -1 & 0 \end{array}\right]$.

By Gaussian elimination method, we get

[A | B] $\xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{cccc|c} 4 & 0 & 0 & -1 & 0 \\ 5 & 0 & -1 & 0 & 0 \\ 0 & 2 & -2 & -1 & 0 \end{array}\right] \xrightarrow{R_2 \leftrightarrow R_3} \left[\begin{array}{cccc|c} 4 & 0 & 0 & -1 & 0 \\ 0 & 2 & -2 & -1 & 0 \\ 5 & 0 & -1 & 0 & 0 \end{array}\right] \xrightarrow{R_3 \to 4R_3 - 5R_1} \left[\begin{array}{cccc|c} 4 & 0 & 0 & -1 & 0 \\ 0 & 2 & -2 & -1 & 0 \\ 0 & 0 & -4 & 5 & 0 \end{array}\right]$.

Therefore, ρ(A) = ρ([A | B]) = 3 < 4 = Number of unknowns.

The system is consistent and has infinite number of solutions.

Writing the equations using the echelon form, we get $4x_1 - x_4 = 0$, $2x_2 - 2x_3 - x_4 = 0$, $-4x_3 + 5x_4 = 0$.

So, one of the unknowns should be chosen arbitrarily as a non-zero real number.

Let us choose $x_4 = t$, $t \neq 0$.Then, by back substitution, we get $x_3 = \frac{5t}{4}$, $x_2 = \frac{7t}{4}$, $x_1 = \frac{t}{4}$.

Since x_1, x_2, x_3 and x_4 are positive integers, let us choose t = 4.

Then, we get $x_1 = 1$, $x_2 = 7$, $x_3 = 5$ and $x_4 = 4$.

So, the balanced equation is $C_5H_8 + 7O_2 \longrightarrow 5CO_2 + 4H_2O$.

36) If the system of equations px + by + cz = 0, ax + qy + cz = 0, ax + by + rz = 0 has a non-trivial solution and p ≠ a,q ≠ b,r ≠ c, prove that $\frac{p}{p-a} + \frac{q}{q-b} + \frac{r}{r-c} = 2$.

Assume that the system px + by + cz = 0, ax + qy + cz = 0, ax + by + rz = 0 has a non-trivial solution.

So, we have $\begin{vmatrix} p & b & c \\ a & q & c \\ a & b & r \end{vmatrix} = 0$, Applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$ in the above equation,

we get $\begin{vmatrix} p & b & c \\ a-p & q-b & c \\ a-p & b & r-c \end{vmatrix} = 0$. That is, $\begin{vmatrix} p & b & c \\ -(p-a) & q-b & c \\ -(p-a) & b & r-c \end{vmatrix} = 0$.

Since p ≠ a, q ≠ b, r ≠ c, we get $(p - a)(q - b)(r - c)\begin{vmatrix} \frac{p}{p-a} & \frac{b}{q-b} & \frac{c}{r-c} \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{vmatrix} = 0$.

So, we have $\begin{vmatrix} \frac{p}{p-a} & \frac{b}{q-b} & \frac{c}{r-c} \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{vmatrix} = 0$.

Expanding the determinant, we get $\frac{p}{p-a} + \frac{b}{q-b} + \frac{c}{r-c} = 0$.

That is, $\frac{p}{p-a} + \frac{q-(q-b)}{q-b} + \frac{r-(r-c)}{r-c} = 0 \Rightarrow \frac{p}{p-a} + \frac{b}{q-b} + \frac{c}{r-c} = 2$.

37) Determine the values of λ for which the following system of equations x + y + 3z = 0, 4x + 3y + λz = 0, 2x + y + 2z = 0 has

(i) a unique solution

(ii) a non-trivial solution

x+y+3z=0, 4x+3y+λz=0, 2x+y+2z=0

Reducing the augmented matrix to row - echelon form we get,

[A|0]= $\left[\begin{array}{ccc|c} 1 & 1 & 3 & 0 \\ 4 & 3 & \lambda & 0 \\ 2 & 1 & 2 & 0 \end{array}\right]$

$\xrightarrow{R_1 \leftrightarrow R_3} \left[\begin{array}{ccc|c} 1 & 1 & 3 & 0 \\ 0 & -1 & -4 & 0 \\ 0 & -1 & \lambda-2 & 0 \end{array}\right]$

$\xrightarrow{R_2 \to R_2 - 2R_1,\ R_3 \to R_3 - 4R_1} \left[\begin{array}{ccc|c} 1 & 1 & 3 & 0 \\ 0 & -1 & -4 & 0 \\ 0 & 0 & \lambda-8 & 0 \end{array}\right]$

$\xrightarrow{R_3 \to R_3 - R_1} \left[\begin{array}{ccc|c} 1 & 1 & 3 & 0 \\ 0 & -1 & -4 & 0 \\ 0 & 0 & not\ \ zero & 0 \end{array}\right]$

Case (i) when λ≠8

[A|0]= $\left[\begin{array}{ccc|c} 1 & 1 & 3 & 0 \\ 0 & -1 & -4 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

Here ρ(A)=3, ρ([A/0]=3

∴ ρ(A)=ρ([A/0])=3=the number of unknowns

∴ The given system is consistent and has unique solution.

Case (ii) when λ = 8

Here ρ(A)=2, ρ([A/0]=2

∴ ρ(A)=ρ([A/0])==2< 3 the number of unknowns,

∴ The system is consistent and has non-trivial solutions.

38) Solve the following system of linear equations by matrix inversion method:

x + y + z − 2 = 0, 6x − 4y + 5z − 31 = 0, 5x + 2y + 2z = 13.

x+y+z=2=0, 6x-4y+5z-31=0, 5x+2y+2z=13

The matrix form of the system is

$$\begin{bmatrix} 1 & 1 & 1 \\ 6 & -4 & 5 \\ 5 & 2 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 13 \end{bmatrix}$$

AX =B where A= $\begin{bmatrix} 1 & 1 & 1 \\ 6 & -4 & 5 \\ 5 & 2 & 2 \end{bmatrix}$

X= $\begin{bmatrix} x \\ y \\ z \end{bmatrix}, B = \begin{bmatrix} 2 \\ 31 \\ 13 \end{bmatrix}$

⇒ X=A^{-1}B

|A| = $\begin{vmatrix} 1 & 1 & 1 \\ 6 & -4 & 5 \\ 5 & 2 & 2 \end{vmatrix} = 1 \begin{vmatrix} -4 & 5 \\ 2 & 2 \end{vmatrix} - 1 \begin{vmatrix} 6 & 5 \\ 5 & 2 \end{vmatrix} + 1 \begin{vmatrix} 6 & -4 \\ 5 & 2 \end{vmatrix}$

adj A=

$$\begin{bmatrix} +\begin{vmatrix} -4 & 5 \\ 2 & 2 \end{vmatrix} & -\begin{vmatrix} 6 & 5 \\ 5 & 2 \end{vmatrix} & +\begin{vmatrix} 6 & -4 \\ 5 & 2 \end{vmatrix} \\ -\begin{vmatrix} 1 & 1 \\ 2 & 2 \end{vmatrix} & +\begin{vmatrix} 1 & 1 \\ 5 & 2 \end{vmatrix} & -\begin{vmatrix} 1 & 1 \\ 5 & 2 \end{vmatrix} \\ +\begin{vmatrix} 1 & 1 \\ -4 & 5 \end{vmatrix} & -\begin{vmatrix} 1 & 1 \\ 6 & 5 \end{vmatrix} & +\begin{vmatrix} 1 & 1 \\ 6 & -4 \end{vmatrix} \end{bmatrix}^T$$

$$= \begin{bmatrix} +(-8-10) & -(12-25) & +(12+20) \\ -(2-2) & +(2-5) & -(2-5) \\ +(5+4) & -(5-6) & +(-4-6) \end{bmatrix}^T$$

$$= \begin{bmatrix} -18 & 13 & 32 \\ 0 & -3 & 3 \\ 9 & 1 & -10 \end{bmatrix}^T$$

$$= \begin{bmatrix} -18 & 0 & 9 \\ 13 & -3 & 1 \\ 32 & 3 & -10 \end{bmatrix}$$

$$\therefore A^{-1} = \frac{1}{|A|} adjA = \frac{1}{27}\begin{bmatrix} -18 & 0 & 9 \\ 13 & -3 & 1 \\ 32 & 3 & -10 \end{bmatrix}$$

$\therefore X = A^{-1}B$

$$= \frac{1}{27}\begin{bmatrix} -18 & 0 & 9 \\ 13 & -3 & 1 \\ 32 & 3 & -10 \end{bmatrix}\begin{bmatrix} 2 \\ 31 \\ 13 \end{bmatrix}$$

$$= \frac{1}{27}\begin{bmatrix} -36 & +0 & +117 \\ 26 & -93 & +13 \\ 64 & +93 & -130 \end{bmatrix} = \frac{1}{27}\begin{bmatrix} 81 \\ -54 \\ 27 \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}$$

$\therefore$ x=3, y=-2, z=1

$\therefore$ Solution set is {3,-2,1}

39) Solve the following systems of linear equations by Cramer's rule:

$\frac{3}{x} - \frac{4}{y} - \frac{2}{z} - 0,\ \frac{1}{x} + \frac{2}{y} + \frac{1}{z} - 2 = 0,\ \frac{2}{x} - \frac{5}{y} - \frac{4}{z} + 1 = 0$

$\frac{3}{x} - \frac{4}{y} - \frac{2}{z} - 0,\ \frac{1}{x} + \frac{2}{y} + \frac{1}{z} - 2 = 0,\ \frac{2}{x} - \frac{5}{y} - \frac{4}{z} + 1 = 0$

Put $\frac{1}{x} = u, \frac{1}{y} = v, \frac{1}{z} = w$

We get 3u - 4v - 2w = 1, u + 2v + w = 2, 2u-'5v-4w = -1

$$\therefore \begin{vmatrix} 3 & -4 & -2 \\ 1 & 2 & 1 \\ 2 & -5 & -4 \end{vmatrix} = 3\begin{vmatrix} 2 & 1 \\ -5 & -4 \end{vmatrix} + 4\begin{vmatrix} 1 & 1 \\ 2 & -4 \end{vmatrix} - 2\begin{vmatrix} 1 & 2 \\ 2 & -5 \end{vmatrix}$$

= 3(- 8 + 5) + 4'(- 4 - 2) - 2(- 5 - 4)

= 3(- 3) + 4(- 6) - 2(- 9)

= - 9 - 24 + 18= -15

$\Delta_1 = \begin{vmatrix} 1 & -4 & -2 \\ 2 & 2 & 1 \\ -1 & -5 & -4 \end{vmatrix}$

$= 1\begin{vmatrix} 2 & 1 \\ -5 & -4 \end{vmatrix} + 4\begin{vmatrix} 2 & 1 \\ -1 & -4 \end{vmatrix} - 2\begin{vmatrix} 2 & 2 \\ -1 & -5 \end{vmatrix}$

= 1(- 8 + 5) + 4(- 8 + Il-2(- 10 + 2)

= 1(- 3) + 4(-7) - 2(- 8)

= - 3 - 28 + 16= -15

$\Delta_2 = \begin{vmatrix} 3 & 1 & -2 \\ 1 & 2 & 1 \\ 2 & -1 & -4 \end{vmatrix}$

$3\begin{vmatrix} 2 & 1 \\ -1 & -4 \end{vmatrix} - 1\begin{vmatrix} 1 & 1 \\ 2 & -4 \end{vmatrix} - 2\begin{vmatrix} 1 & 2 \\ 2 & -1 \end{vmatrix}$

= 3(- 8 + 1) - 1(- 4 - 2) - 2(- 1 - 4)

= 3(-7) - 1(- 6) - 2(- 5)

=-21+6+10=-5

$\Delta_3 = \begin{vmatrix} 3 & -4 & 1 \\ 1 & 2 & 2 \\ 2 & -5 & -1 \end{vmatrix}$

$3\begin{vmatrix} 2 & 2 \\ -5 & -1 \end{vmatrix} + 4\begin{vmatrix} 1 & 2 \\ 2 & -1 \end{vmatrix} - 2\begin{vmatrix} 1 & 2 \\ 2 & -5 \end{vmatrix}$

=3(-2+ 10)+4(-1-4)+ 1(-5-4)

= 3(8) + 4(- 5) + 1(- 9)

= 24 - 20 - 9 = - 5

$\therefore \frac{\Delta_1}{\Delta} = \frac{-15}{-15} = 1 \Rightarrow \frac{1}{x} = 1 \Rightarrow$ x=1

v= $\frac{\Delta_2}{\Delta} = \frac{-5}{-15} = \frac{1}{3} \Rightarrow \frac{1}{y} = \frac{1}{3} \Rightarrow$ y=3

w= $\frac{\Delta_3}{\Delta} = \frac{5}{-15} = \frac{1}{3} \Rightarrow \frac{1}{z} = \frac{1}{3} \Rightarrow$ z=3

∴ Solution set is {1,,3}

40) Using Gaussian Jordan method, find the values of λ and μ so that the system of equations 2x - 3y + 5z = 12, 3x + y + λz =μ, x - 7y + 8z = 17 has (i) unique solution (ii) infinite solutions and (iii) no solution.

The augmented matrix [AIB] is $\left[\begin{array}{ccc|c} 2 & -3 & 5 & 12 \\ 3 & 1 & \lambda & \mu \\ 1 & -7 & 8 & 17 \end{array}\right]$

$\xrightarrow{R_1 \leftrightarrow R_3} \left[\begin{array}{ccc|c} 1 & -7 & 8 & 17 \\ 3 & 1 & \lambda & \mu \\ 2 & -3 & 5 & 12 \end{array}\right]$

$\xrightarrow{R_2 \to R_2 - R_1 R_3 \to R_3 - 2R_1} \left[\begin{array}{ccc|c} 1 & -7 & 8 & 17 \\ 0 & 22 & \lambda - 51 & \mu - 51 \\ 0 & 11 & -11 & -22 \end{array}\right]$

$$R_2 \to R_2 - 2R_3 R_3 \to R_3 \div 11 \quad \longrightarrow \left[\begin{array}{ccc|c} 1 & -7 & 8 & 17 \\ 0 & 0 & \lambda - 2 & \mu - 7 \\ 0 & 1 & -1 & -2 \end{array}\right]$$

$$R_2 \leftrightarrow R_3 \quad \longrightarrow \left[\begin{array}{ccc|c} 1 & -7 & 8 & 17 \\ 0 & 1 & -1 & -2 \\ 0 & 0 & \lambda - 2 & \mu - 7 \end{array}\right]$$

Case (i) when $\lambda \neq 2$,

$\rho([AIB]) = 3$ and $\rho(A) = 3$

$\therefore \rho([AIB]) = \rho(A) = 3$ = the number of unknowns

$\therefore$ The system has unique solution

Case (ii) when $\lambda = 2$, $\mu = 7$

$$\left[\begin{array}{ccc|c} 1 & -7 & 8 & 17 \\ 0 & 1 & -1 & -2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Here $\rho(A) = 2$ and $\rho([AIB]) = 2$

$\therefore \rho(A) = \rho([AIB]) = 2 <$ number of unknowns

Thus the system is consistent with infinitely many solutions.

Case (iii) When $\lambda = 2$ and $\mu \neq 7$

$\rho(A) = 2$ and $\rho([AIB]) = 3$

$\therefore \rho(A) \neq \rho([AIB])$

Thus, the given system of equations is inconsistent.

41) Show that $\left(2 + i\sqrt{3}\right)^{10} + \left(2 - i\sqrt{3}\right)^{10}$ is real ii) $\left(\frac{19 + 9i}{5 - 3i}\right)^{15} - \left(\frac{8 + i}{I + 2i}\right)^{15}$ is purely imaginary.

i) Let = $\left(2 + i\sqrt{3}\right)^{10} + \left(2 - i\sqrt{3}\right)^{10}$

$$\bar{z} = \overline{\left(2 + i\sqrt{3}\right) + \left(2 - i\sqrt{3}\right)^{10}}$$

$$= \overline{\left(2 + i\sqrt{3}\right)^{10}} + \overline{\left(2 + i\sqrt{3}\right)^{10}}$$

$$= \left(\overline{2 + i\sqrt{3}}\right)^{10} + \left(\overline{2 - i\sqrt{3}}\right)^{10}$$

$$= \left(2 - i\sqrt{3}\right)^{10} + \left(\overline{2 + i\sqrt{3}}\right)^{10} = z$$

$\bar{z}$ z= $\Rightarrow$ z is real

ii) Let z= $\left(\frac{19 + 9i}{5 - 3i}\right)^{15} - \left(\frac{8 + i}{I + 2i}\right)^{15}$

Here, $\frac{19+9i}{5-3i} = \frac{(19+9i)(5+3i)}{(5-3i)(5+3i)}$

$= \frac{(95-27)+i(45+57)}{5^2+3^2} = \frac{68+102i}{34}$

=2+3i

and $\frac{8+i}{1+2i} = \frac{(8+i)(1-2i)}{(1+2i)(1-2i)}$

$= \frac{(8+2)+i(1-16)}{1^2+2^2} = \frac{10-15i}{5}$

=2-3i

Now z= $\left(\frac{19+9i}{5-3i}\right)^{15} - \left(\frac{8+i}{1+2i}\right)^{15}$

$\Rightarrow$ z=$(2+3i)^{15}$ - $(2-3i)^{15}$ (by (1) and (2))

Then by definition,

$$\bar{z} = \overline{\left((2+3i)^{15} - (2-3i)^{15}\right)}$$

$= \left(\overline{2+3i}\right)^{15} - \left(\overline{2-3i}\right)^{15}$ (using properties of conjugates)

$=(2-3i)^{15} - (2+3i)^{15} = -((2+3i)^{15} - (2-3i)^{15})$

$\Rightarrow \bar{z}$ =-z

Therefore, $\left(\frac{19+9i}{5-3i}\right)^{15} - \left(\frac{8+i}{1+2i}\right)^{15}$ is purely imaginary.

42) Let z_1, z_2, and z_3 be complex numbers such that $|z_1| = |z_2| = |z_3| = r > 0$ and $z_1+z_2+z_3 \neq 0$ prove that

$$\left|\frac{z_1z_2 + z_2z_3 + z_3z_1}{z_1 + z_2 + z_3}\right| = r$$

Given that $|z_1| = |z_2| = |z_3| = r \Rightarrow z_1\bar{z}_1 = z_2\bar{z}_2 = r^2$

$\Rightarrow z_1 = \frac{r^2}{\bar{z}_1}, z_2 = \frac{r^2}{\bar{z}_2}, z_3 = \frac{r^2}{\bar{z}_3}$

Therefore

$$z_1 + z_2 + z_3 = \frac{r^2}{\bar{z}_1} + \frac{r^2}{\bar{z}_2} + \frac{r^3}{\bar{z}_3}$$

$= r^2\left(\frac{\bar{z}_2\bar{z}_3 + \bar{z}_1\bar{z}_3 + \bar{z}_1\bar{z}_2}{\bar{z}_1\bar{z}_2\bar{z}_3}\right) \left(\because |z| = |\bar{z}| \text{ and } |z_1z_2z_3| = |z_1||z_2||z_3|\right)$

$$|z_1 + z_2 + z_3| = |r^2| \left|\frac{z_1z_3 + z_1\bar{z}_3 + z_1z_2}{z_1z_2\bar{z}_3}\right|$$

$= r^2\frac{|z_2z_3 + z_1z_3 + z_1z_2|}{|z_1||z_2||z_3|}$

$= |z_1 + z_2 + z_3| = r^2\frac{|z_2z_3 + z_1z_3 + z_1z_2|}{r^3} = \frac{|z_2z_3 + z_1z_3 + z_1z_2|}{r}$

$$\frac{|z_2z_3+z_1z_3+z_1z_2|}{|z_1+z_2+z_3|}=r$$

Thus, $\left|\frac{z_1z_3+z_1z_3+z_1z_2}{z_1+z_2+z_3}\right|=r$

43) If z_1, z_2, and z_3 are three complex numbers such that $|z_1|=1, |z_2|=2|z_3|=3$ and $|z_1+z_2+z_3|=1$, show that $|9z_1z_2+4z_1z_2+z_2z_3|=6$

Given $|z_1|=1$, $|z_2|=2$, $|z_3|=3$, $|z_1+z_2+z_3|=1$

$|z_1|^2=1_2 \Rightarrow z_1\bar{z}_1=1 \Rightarrow \bar{z}_1=\frac{1}{z_1}$

$|z_2|^2=4 \Rightarrow z_2\bar{z}_2=1 \Rightarrow \bar{z}_2=\frac{4}{z_2}$

$|z_3|^2=9 \Rightarrow z_3\bar{z}_3=1 \Rightarrow \bar{z}_3=\frac{9}{z_3}$

$$\therefore \left|9,\frac{1}{\bar{z}_1}.\frac{4}{\bar{z}_2}+4.\frac{1}{\bar{z}_1}.\frac{9}{\bar{z}_3}+\frac{4}{\bar{z}_2}.\frac{9}{\bar{z}_3}\right|$$

$$\left|\frac{36}{\bar{z}_1\bar{z}_2}+\frac{36}{\bar{z}_1\bar{z}_3}+\frac{36}{\bar{z}_2\bar{z}_3}\right| = \left|36\left(\frac{\bar{z}_3+\bar{z}_2+\bar{z}_1}{\bar{z}_1\bar{z}_2\bar{z}_3}\right)\right|$$

$$\left[\because |\bar{z}_1+\bar{z}_2+\bar{z}_3| = |\overline{z_1+z_2+z_3}|\right]$$

$$=\frac{36\,|\overline{z_1+z_2+z_3}|}{|\bar{z}_1|\,|\bar{z}_2|\,|\bar{z}_3|} = 36\frac{|\bar{z}_1+\bar{z}_2+\bar{z}_3|}{|\bar{z}_1|\,|\bar{z}_2|\,|\bar{z}_3|}$$

$$\left[\because |\bar{z}_1| = |z_1|, |\bar{z}_2 = |z_{21}|, |z_3| = |\bar{z}_3|\right]$$

$$=\frac{36(1)}{1(2)(3)}=\frac{\overset{6}{\cancel{36}}}{\cancel{6}}=6$$

$\therefore |9z_1+z_2+4z_1z_3+z_2z_3|=6$

44) If z=x+iy is a complex number such that $\text{Im}\left(\frac{2z+1}{iz+1}\right)=0$ show that the locus of z is $2x^2+2y^2+x-2y=0$

Given z=x+iy

$\text{Im}\left(\frac{2z+1}{iz+1}\right)=0$

$\Rightarrow \text{Im}\left(\frac{2(x+iy)+1}{i(x+iy)+1}\right)=0$

$\Rightarrow \text{Im}\left(\frac{(2x+1)+2iy}{ix+i^2y+1}\right)$

$\Rightarrow \text{Im}\left(\frac{(2x+1)+2iy}{ix-y+1}\right)$

$\left(\frac{(2x+1)+iy}{(1-y)+ix}\right)$

Multiply and divide by the conjugate of the denominator

We get Im $\left(\frac{(2x+1)+2iy}{(1-y)+ix} \times \frac{(1-y)-ix}{(1-y)-ix}\right)=0$

$\Rightarrow$ Im $\left(\frac{(2x+1)+2iy \times (1-y)-ix}{(1-y)^2+x^2}\right)$

Choosing the imaginably part we get,

$\frac{(2x+1)(-x)+2y(1-y)}{(1-y)^2+x^2}$

$\Rightarrow$ (2x+1)-x+2y(1-y)=0

$\Rightarrow$ $-2x^2-x+2y-2y^2=0$

$\Rightarrow$ $2x^2+2y^2+x-2y=0$

Hence, locus of z is $2x^2+2y^2+x-2y=0$

45) If z=x+iy and arg $\left(\frac{z-i}{z+2}\right) = \frac{\pi}{4}$, then show that $x^2+y^2+3x-3y+2=0$

Given z=x+iy and arg $\left(\frac{z-i}{z+2}\right) = \frac{\pi}{4}$

$\Rightarrow$ arg(z-i)-arg(z+2)=$\frac{\pi}{4}$

$\Rightarrow$ arg(x+iy-i)-arg(x+iy+2)=$\frac{\pi}{4}$

$\Rightarrow$ arg(x+i(y-1)-arg((x+2)+iy)=$\frac{\pi}{4}$

$\Rightarrow$ $tan^{-1}\left(\frac{y-1}{x}\right) - tan^{-1}\left(\frac{y}{x+2}\right)=\frac{\pi}{4}$

$\Rightarrow$

$tan^{-1}\left(\frac{\frac{y-1}{x}-\frac{y}{x+2}}{1+\frac{y-1}{x}\cdot\frac{y}{x+2}}\right)$

$=\frac{\pi}{4}\left[\because tan^{-1}x - tan^{-1}y = tan^{-1}\left(\frac{x-y}{1+xy}\right)\right]$

$$\Rightarrow \frac{\left(\frac{(x+2)(y-1)-xy}{x(x+2)}\right)}{\left(\frac{x(x+2)+y(y-1)}{x(x+2)}\right)} = \tan\frac{\pi}{4} = 1$$

$\Rightarrow$ $\frac{(x+2)(y-1)-xy}{x(x+2)+y(y-1)}=1$

$\Rightarrow$ $-x+2y-2 = x^2+2x+y^2-y$

$\Rightarrow$ $x^2+2x+y^2-y+x-2y+2 =0$

$\Rightarrow$ $x^2+y^2+3x-3y+2=0$

46) Show that $\left(\frac{\sqrt{3}}{2}+\frac{i}{2}\right)^5 + \left(\frac{\sqrt{3}}{2}-\frac{i}{2}\right)^5 = -\sqrt{3}$

Let $\frac{\sqrt{3}}{2}+\frac{i}{2}$ =r(cosθ+i sinθ)

r= $\sqrt{\left(\frac{\sqrt{3}}{2}\right)^2+\left(\frac{1}{2}\right)^2}=\sqrt{\frac{3}{4}+\frac{1}{4}}=\sqrt{\frac{4}{4}}$ =1

α=

$$tan^{-1}\left|\frac{y}{x}\right| = tan^{-1}\left|\frac{\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right| = tan^{-1}\left(\frac{1}{\sqrt{3}}\right)=\frac{\pi}{6}$$

Since $\frac{\sqrt{3}}{2}+\frac{i}{2}$ lies is the I quadrant, θ=α

∴ $\frac{\sqrt{3}}{2}+\frac{i}{2}=1\left(cos\frac{\pi}{6}+isin\frac{\pi}{6}\right)$

∴ $\left(\frac{\sqrt{3}}{2}+\frac{i}{2}\right)^5=1^5\left(cos\frac{\pi}{6}+isin\frac{\pi}{6}\right)^5$

$=cos\frac{5\pi}{6}+isin\frac{5\pi}{6}$ ....(1) [De moivres theorem]

Similarly $\frac{\sqrt{3}}{2}+\frac{i}{2}=1\left(cos\frac{\pi}{6}-isin\frac{\pi}{6}\right)$

∴ $\left(\frac{\sqrt{3}}{2}-\frac{i}{2}\right)^5=1^5\left[cos\frac{\pi}{6}+isin\frac{\pi}{6}\right]^5$

$=cos\frac{5\pi}{6}-isin\frac{5\pi}{6}$ ....(2)

Adding (1) and (2) we get,

$\left(\frac{\sqrt{3}}{2}+\frac{i}{2}\right)^5+\left(\frac{\sqrt{3}}{2}-\frac{i}{2}\right)^5$

$= \cos\frac{5\pi}{6}+isin\frac{5\pi}{6}+\cos\frac{5\pi}{6}-isin\frac{5\pi}{6}$

$=2cos\frac{5\pi}{6}=2cos\left(\pi-\frac{\pi}{6}\right)$

$=-2cos\frac{\pi}{6}$ [∵ $\frac{5\pi}{6}$ lies in the II quard]

$=-2\left(\frac{\sqrt{3}}{2}\right)=-\sqrt{3}$

47) If $2cosa=x+\frac{1}{x}$ and $2cos\beta=y+\frac{1}{y}$, show that

i) $\frac{x}{y}+\frac{y}{x}=2cos(\alpha-\beta)$.

ii) $xy-\frac{1}{xy}=2isin(\alpha+\beta)$

iii) $\frac{x^m}{y^n} - \frac{y^n}{x^m} = 2isin(m\alpha - n\beta)$

iv) $x^m y^n + \frac{1}{x^m y^n} = 2cos(m\alpha + n\beta)$

i) Given 2cosα=x+$\frac{1}{x}$

⇒ 2cosα =$\frac{x^2+1}{x}$

⇒ x^2+1=2x cosα

⇒ x^2-2x cosα+1=0

⇒ $\frac{2cos\alpha \pm \sqrt{(-2cos\alpha)^2 - 4(1)(1)}}{2}$

$= \frac{2cos\alpha \pm \sqrt{4cos^2\alpha - 4}}{2}$ $\left[\because \frac{b \pm \sqrt{b^2 - 4ac}}{2a} \right]$

$= \frac{2cos\alpha \pm \sqrt{-sin^2\alpha}}{2}$

$= \frac{2cos\alpha \pm isin\alpha}{2}$ [∵ $sin^2\alpha + cos^2\alpha = 1$]

⇒ x^2=cosα±sinα

Also, 2cosβ=y+$\frac{1}{y}$

⇒ 2cosβ =$\frac{y^2+1}{y}$

⇒ y^2-2y cosβ+1=0

⇒ $\frac{2cos\beta \pm \sqrt{(-2cos^2\beta^2 - 4(1)(1)}}{2}$

$= \frac{2cos\beta \pm \sqrt{4cos^2\beta - 4}}{2} = \frac{2cos\beta \pm 2isin\beta}{2}$

⇒ y=cosβ±i sinβ

$\frac{x}{y} + \frac{y}{x} = 2cos(\alpha - \beta)$

$\frac{x}{y} = \frac{cos\alpha + isin\alpha}{cos\beta + isin\beta}$

=cos(α-β)+i sin(α-β) .....(1)

and $\frac{y}{x} = \frac{1}{x}$=cos(α-β)+i sin(α-β) ....(2)

(1)+(2) ⟶ =$\frac{x}{y} + \frac{y}{x}$

=cos(α-β)+i sin(α-β)+cos(α-β)-i sin(α-β)

=2cos(α-β)

ii) Given 2cosα=x+$\frac{1}{x}$

⇒ 2cosα =$\frac{x^2+1}{x}$

⇒ x^2+1=2x cosα

⇒ x^2-2x cosα+1=0

⇒ $\frac{2cos\alpha \pm \sqrt{(-2cos\alpha)^2 - 4(1)(1)}}{2}$

$= \frac{2cos\alpha \pm \sqrt{4cos^2\alpha - 4}}{2}$ $\left[\because \frac{b \pm \sqrt{b^2-4ac}}{2a}\right]$

$= \frac{2cos\alpha \pm \sqrt{-sin^2\alpha}}{2}$

$= \frac{2cos\alpha \pm isin\alpha}{2}$ [∵ $sin^2\alpha + cos^2\alpha = 1$]

$\Rightarrow x^2 = cos\alpha \pm sin\alpha$

Also, $2cos\beta = y + \frac{1}{y}$

$\Rightarrow 2cos\beta = \frac{y^2+1}{y}$

$\Rightarrow y^2 - 2y\,cos\beta + 1 = 0$

$\Rightarrow \frac{2cos\beta \pm \sqrt{(-2cos^2\beta^2 - 4(1)(1)}}{2}$

$= \frac{2cos\beta \pm \sqrt{4cos^2\beta - 4}}{2} = \frac{2cos\beta \pm 2isin\beta}{2}$

$\Rightarrow y = cos\beta \pm i\,sin\beta$

$$xy - \frac{1}{xy} = 2isin(\alpha + \beta)$$

$xy = (cos\alpha + i\,sin\alpha)(cos\beta + i\,sin\alpha\beta)$

$= cos(\alpha+\beta) + i\,isn(\alpha+\beta)$

$\frac{1}{xy} = cos(\alpha+\beta) - i\,isn(\alpha+\beta)$

$$\therefore xy - \frac{1}{xy} = cos(\alpha+\beta) + isin(\alpha+\beta) - cos(\alpha+\beta) + isin(\alpha+\beta)$$

$= 2i\,sin(\alpha+\beta)$

iii) Given $2cos\alpha = x + \frac{1}{x}$

$\Rightarrow 2cos\alpha = \frac{x^2+1}{x}$

$\Rightarrow x^2 + 1 = 2x\,cos\alpha$

$\Rightarrow x^2 - 2x\,cos\alpha + 1 = 0$

$\Rightarrow \frac{2cos\alpha \pm \sqrt{(-2cos\alpha)^2 - 4(1)(1)}}{2}$

$= \frac{2cos\alpha \pm \sqrt{4cos^2\alpha - 4}}{2}$ $\left[\because \frac{b \pm \sqrt{b^2-4ac}}{2a}\right]$

$= \frac{2cos\alpha \pm \sqrt{-sin^2\alpha}}{2}$

$= \frac{2cos\alpha \pm isin\alpha}{2}$ [∵ $sin^2\alpha + cos^2\alpha = 1$]

$\Rightarrow x^2 = cos\alpha \pm sin\alpha$

Also, $2cos\beta = y + \frac{1}{y}$

$\Rightarrow 2cos\beta = \frac{y^2+1}{y}$

$\Rightarrow y^2 - 2y\,cos\beta + 1 = 0$

$\Rightarrow \frac{2cos\beta \pm \sqrt{(-2cos^2\beta^2 - 4(1)(1)}}{2}$

$= \frac{2cos\beta \pm \sqrt{4cos^2\beta - 4}}{2} = \frac{2cos\beta \pm 2isin\beta}{2}$

$\Rightarrow$ y=cosβ±i sinβ

$$\frac{x^m}{y^n} - \frac{y^n}{x^m} = 2i\sin(m\alpha - n\beta)$$

$x^m=(\cos\alpha+\sin\alpha)^m = \cos m\alpha+i \sin m\alpha$ [By De moivre's theorem]

$y^n =(\cos\beta+i \sin\beta)^n =\cos n\beta+i \sin n\beta$

$\therefore \frac{x^m}{y^n} = \frac{\cos m\alpha + i\sin m\alpha}{\cos n\beta + i\sin n\beta}$

$=\cos(m\alpha-n\beta)+i \sin(m\alpha+n\beta)$

and $\frac{y^n}{x^m} = \frac{\frac{1}{x^m}}{y^n}$

$=\cos(m\alpha-n\beta)-i \sin(m\alpha-n\beta)$

$$\therefore \frac{x^m}{y^n} - \frac{y^n}{x^m} = \cos(m\alpha - n\beta) + i\sin(m\alpha - n\beta)$$

$$-\cos(m\alpha - n\beta) + i\sin(m\alpha - n\beta) = 2i\sin(m\alpha - n\beta)$$

iv) Given $2\cos\alpha=x+\frac{1}{x}$

$\Rightarrow 2\cos\alpha = \frac{x^2+1}{x}$

$\Rightarrow x^2+1=2x\cos\alpha$

$\Rightarrow x^2-2x\cos\alpha+1=0$

$\Rightarrow \frac{2\cos\alpha \pm \sqrt{(-2\cos\alpha)^2-4(1)(1)}}{2}$

$= \frac{2\cos\alpha \pm \sqrt{4\cos^2\alpha - 4}}{2} \quad \left[\because \frac{b \pm \sqrt{b^2-4ac}}{2a}\right]$

$= \frac{2\cos\alpha \pm \sqrt{-\sin^2\alpha}}{2}$

$= \frac{2\cos\alpha \pm i\sin\alpha}{2}$ [$\because \sin^2\alpha+\cos^2\alpha=1$]

$\Rightarrow x^2=\cos\alpha\pm\sin\alpha$

Also, $2\cos\beta=y+\frac{1}{y}$

$\Rightarrow 2\cos\beta = \frac{y^2+1}{y}$

$\Rightarrow y^2-2y\cos\beta+1=0$

$\Rightarrow \frac{2\cos\beta \pm \sqrt{(-2\cos^2\beta^2-4(1)(1)}}{2}$

$= \frac{2\cos\beta \pm \sqrt{4\cos^2\beta - 4}}{2} = \frac{2\cos\beta \pm 2i\sin\beta}{2}$

$\Rightarrow$ y=cosβ±i sinβ

$$x^m y^n + \frac{1}{x^m y^n} = 2\cos(m\alpha + n\beta)$$

$x^m y^n=(\cos\alpha+i \sin m\alpha) (\cos n\beta+i \sin n\beta)$

$\cos(m\alpha+n\beta)+i \sin(m\alpha+n\beta)$

$\frac{1}{x^m y^n}=\cos(m\alpha+n\beta)-i \sin(m\alpha+n\beta)$

$$\therefore x^m y^n + \frac{1}{x^m y^n} = \cos(m\alpha + n\beta) + i\sin(m\alpha + n\beta) + \cos(m\alpha + n\beta) - i\sin(m\alpha + n\beta)$$

=2cos(mα+nβ)

48) Solve the equation $z^3+27=0$.

$z^3=-27=(-1 \times 3)^3 = -1 \times 33$

$z=(-1)^{\frac{1}{3}} \times 3^{3 \times \frac{1}{3}} = (-1)^{\frac{1}{3}} \times 3$

$\therefore z=3[cos\pi + isin\pi]^{\frac{1}{3}}$

[∵ cosπ=-1 and sinπ=0]

$=3\left[cos\frac{1}{3}(2k\pi + \pi)isin\frac{1}{3}(2k\pi + \pi)\right]$

k=0,1,2

When k=0,

$z=3\left[cos\frac{1}{3}(\pi)isin\frac{1}{3}(\pi)\right] = 3cos\frac{\pi}{3}$

When k=1

$z=3\left[cos\frac{1}{3}(3\pi)isin\frac{1}{3}(3\pi)\right]$

=3[cosπ+i sinπ]=3(-1+0)=-

When k=2

$z=3\left[cos\frac{1}{3}(5\pi)isin\frac{1}{3}(5\pi)\right] = 3\left[cos5\frac{\pi}{3}\right]$

Hence, the roots are 3 cis$\frac{\pi}{3}$,-3,3 cis5$\frac{\pi}{3}$.

49) If z=x+iy and arg $\left(\frac{z-1}{z+1}\right) = \frac{\pi}{2}$,then show that $x^2+y^2=1$.

Now, $\frac{z-1}{z+1} = \frac{x+iy-1}{x+iy+1} = \frac{(x-1)+iy}{(x+1)+iy} = \frac{[(x-1)+iy][(x+1)-iy]}{[(x+1)+iy][(x+1)-iy]}$

$\Rightarrow \frac{z-1}{z+1} = \frac{\left(x^2+y^2-1\right)+i(2y)}{(x+1)^2+y^2}$

Since, arg $\left(\frac{z-1}{z+2}\right) = \frac{\pi}{2} \Rightarrow tan^{-1}\left(\frac{2y}{x^2+y^2-1}\right) = \frac{\pi}{2}$

$\Rightarrow \frac{2y}{x^2+y^2-1} = tan\frac{\pi}{2}$

$\Rightarrow x^2+y^2 = 1$

50) Find all cube roots of $\sqrt{3}+i$

Let $z^3 = \sqrt{3}+i = r(cos\theta + isin\theta)$

Then, $r = \sqrt{3+1} = 2$ and $\alpha = \theta = \frac{\pi}{6}$

Therefore,

$$z^3 = \sqrt{3} + i = 2\left(cos\frac{\pi}{6} + isin\frac{\pi}{6}\right)$$

$$\Rightarrow z = \sqrt[3]{2}\left(cos\left(\frac{\pi + 12k\pi}{18}\right) + isin\left(\frac{\pi + 12k\pi}{18}\right)\right), k=0,1,2.$$

Taking k = 0,1,2 , we get

k=0,

$$= 2^{\frac{1}{3}}\left(cos\frac{\pi}{18} + sin\frac{\pi}{18}\right)$$

k=1,

$$z = z^{\frac{1}{3}}\left(cos\frac{\pi}{18} + sin\frac{\pi}{18}\right)$$

k=2,

$$z = 2^{\frac{1}{3}}\left(cos\frac{25\pi}{18} + sin\frac{25\pi}{18}\right) = 2^{\frac{1}{3}}\left(-cos\frac{7\pi}{18} - sin\frac{7\pi}{18}\right)$$

51) Solve the equation $x^3-9x^2+14x+24=0$ if it is given that two of its roots are in the ratio 3:2.

Let ∝,β,ɤ be the roots of the equation

Given $\frac{\alpha}{\beta} = \frac{3}{2} \Rightarrow 2\alpha = 3\beta \Rightarrow \alpha = \frac{3}{2}\beta$

$\therefore \frac{3}{2}\beta, \beta, \gamma$ are the roots of the given equation

Then by vieta's formula,

$$\frac{3}{2}\beta + \beta + \gamma = \frac{-b}{a} = \frac{-(-9)}{1} = 9$$

$$\frac{5}{2}\beta + \gamma = 9 \Rightarrow \gamma = 9 - \frac{5}{2}\beta$$

$$\Rightarrow \gamma = \frac{18-5\beta}{2} \qquad ...(2)$$

Also $\frac{3}{2}\beta(\beta) + \beta\gamma + \left(\frac{3}{2}\beta\right)\gamma = \frac{c}{a} = \frac{14}{1} = 14$

$$\Rightarrow \frac{3}{2}\beta^2 + \frac{5}{2}\beta\left(\frac{18-5\beta}{2}\right) = 14 \quad [using \quad (2)]$$

$$\Rightarrow \frac{3}{2}\beta^2 + \frac{90\beta}{4} - \frac{25\beta^2}{4} = 14$$

Multiplying by 4, $6\beta^2 + 90\beta - 25\beta^2 = 56$

$19\beta^2 - 90\beta + 56 = 0$

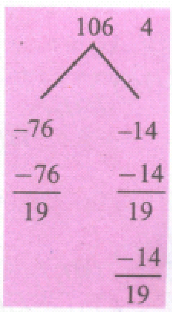
$\Rightarrow (\beta - 4)(19\beta - 14) = 0$

$\Rightarrow \beta = 4$

$\beta = \frac{14}{19}$

When β=4, the other roots are $\frac{3}{2}(4), 4, \frac{18-5}{2}(4)$

$\Rightarrow 6, 4, -1$



When $\beta = \frac{14}{19}$, the other roots are $\frac{3}{2}\beta, \beta\frac{18-5\beta}{2}[by(2)]$

$$\Rightarrow \frac{3}{2}\left(\frac{14}{19}\right), \frac{14}{19}, \frac{18-5\left(\frac{14}{19}\right)}{2} \Rightarrow \frac{21}{19}, \frac{14}{19}, \frac{136}{19}$$

52) Find a polynomial equation of minimum degree with rational coefficients, having $\sqrt{5}-\sqrt{3}$ as a root.

Given $(\sqrt{5}-\sqrt{3})$ is a root

$\Rightarrow \sqrt{5}+\sqrt{3}$

$\therefore$ Sum of the roots $= \sqrt{5}-\sqrt{3}+\sqrt{5}+\sqrt{3} = 2\sqrt{5}$

Product of the roots

$= (\sqrt{5}-\sqrt{3})(\sqrt{5}+\sqrt{3})$

$= (\sqrt{5})^2 - (\sqrt{3})^2 = 5 - 3 = 2$

$\therefore$ One of the factor is X^2 -x (sum of the roots) + product of the roots

$\Rightarrow x^2 - 2x\sqrt{5} + 2$

The other factor also will be $x^2 - 2x\sqrt{5} + 2$

$(x^2 - 2x\sqrt{5} + 2)(x^2 + 2x\sqrt{5} + 2) = 0$

$\Rightarrow (x^2 + 2 - 2\sqrt{5}x)(x^2 + 2 + 2\sqrt{5}x) = 0$

$\Rightarrow \left(x^2+2\right)^2 - (2\sqrt{5}x)^2 = 0$

$[\because (a+b)(a-b) = a^2 - b^2]$

$\Rightarrow x^4 + 4x^2 + 4(5)x^2 = 0$

$\Rightarrow x^4 + 4x^2 + 4 - 20x^2 = 0$

$\Rightarrow x^4 - 16x^2 + 4 = 0$

53) If 2+i and 3-$\sqrt{2}$ are roots of the equation $x^6-13x^5+62x^4-126x^3+65x^2+127x-140=0$, find all roots.

Since the coefficient of the equations are all rational numbers, 2+i and 3-$\sqrt{2}$ are roots, we get 2-i and 3+$\sqrt{2}$ are also roots of the given equation. Thus (x-(2+i)), (x-(2-i)), (x-(3-$\sqrt{2}$)) and (x-(3+$\sqrt{2}$)) are factors. Thus their product.

((x-(2+i))(x-(2-i))(x-(3-$\sqrt{2}$))(x-(3+$\sqrt{2}$)) is a factor of the given polynomial equation. That is, $(x^2-4x+5)(x^2-6x+7)$ is a factor. Dividing the given polynomial equation by this factor, we get the other factor as (x^2-3x-4) which implies that 4 and −1 are the other two roots. Thus

2+i,2-i,3+$\sqrt{2}$,-$\sqrt{2}$,-1, and 4 are the roots of the given polynomial equation.

54) Solve the equation (x-2)(x-7)(x-3)(x+2)+19=0

We can solve this fourth degree equation by rewriting it suitably and adopting a technique of substitution. Rewriting the equation as

(x−2)(x−3)(x−7)(x+2)+19 = 0

the given equation becomes

$(x^2-5x+6)(x^2-5x-14)+19 = 0$.

If we take $x^2 - 5x$ as y , then the equation becomes (y+6)(y−14)+19=0;

that is,

$y^2-8y-65 = 0$

Solving this we get solutions y=13 and y=−5. Substituting this we get two quadratic equations

$x^2-5x-13=0$ and $x^2-5x+5=0$

which can be solved by usual techniques. The solutions obtained for these two equations together give solutions as

$\frac{5\pm\sqrt{77}}{2}, \frac{5\pm\sqrt{5}}{2}$.

55) Determine k and solve the equation $2x^3-6x^2+3x+k=0$ if one of its roots is twice the sum of the other two roots.

Given cubic equation is $2x^3-6x^2+3x+k=0$

Here, a = 2, b = -6, c = 3, d= k

Let α,β,γ be the roots

Given $\alpha=2(\beta+\gamma) \Rightarrow \frac{\alpha}{2}=\beta+\gamma$...(1)

Now, $\alpha+\beta+\gamma=\frac{-b}{a}=-\frac{(-6)}{2}=3$

$\frac{\alpha}{2}+\alpha=3 \Rightarrow \frac{\alpha+2\alpha}{2}=3 \Rightarrow \frac{3\alpha}{2}=3$

$\Rightarrow \alpha=2$

$\alpha\beta\gamma=\frac{-d}{a}=\frac{-k}{2} \Rightarrow 2.\beta\gamma=\frac{-k}{2}$

$\beta\gamma=\frac{-k}{4}$ ...(2)

Also, $\alpha\beta+\beta\gamma+\gamma\alpha=\frac{c}{a}$

$2\beta+\beta\gamma+2\gamma=\frac{3}{2}$

$2(\beta+\gamma)+\beta\gamma=\frac{3}{2}$

$\alpha\frac{-k}{4}=\frac{3}{2}$ [from(1)&(2)]

Also, $2-\frac{k}{4}=\frac{3}{2}[\because \alpha=2]$

$2-\frac{3}{2}=\frac{k}{4} \Rightarrow \frac{1}{2}=\frac{k}{4}$

$k=\frac{4}{2} \Rightarrow k=2$

From(2), $\beta\gamma=\frac{-k}{4}=\frac{-2}{4}=\frac{-1}{2} \Rightarrow \gamma=\frac{-1}{2\beta}$

From (1), $\beta+\gamma=\frac{\alpha}{2}=\frac{2}{2}=1$

Substituting $\gamma=\frac{-1}{2\beta}$ We get

$\beta-\frac{1}{2\beta}=1 \Rightarrow 2\beta^2-1=2\beta \Rightarrow 2\beta-2\beta-1=0$

$\beta=\frac{2\pm\sqrt{4-4(2)(-1)}}{4}=\frac{2\pm\sqrt{4+8}}{4}$

$=\frac{2\pm\sqrt{12}}{4}=\frac{2\pm2\sqrt{3}}{4}$

$\beta=\frac{1\pm\sqrt{3}}{2}$

Hence the roots are $2, \frac{1+\sqrt{3}}{2}, \frac{1-\sqrt{3}}{2}$

56) Find all zeros of the polynomial $x^6-3x^5-5x^4+22x^3-39x^2-39x+135$, if it is known that 1+2i and $\sqrt{3}$ are two of its zeros.

Let f(x) $x^6-3x^5-5x^4+22x^3-39x^2-39x+135$

Given (1+2i) is a root $\Rightarrow$ (-2i) is also a root

Also $\sqrt{3}$ is a root $\Rightarrow$ $-\sqrt{3}$ is also a root.

Hence, the factors of f(x) are [x - (1 + 2i)]

$[x-(1-2i)][x\sqrt{3}][x+\sqrt{3}]$

$[(x-1)-2i][(x-1)+2i][x-\sqrt{3}][x+\sqrt{3}]$

$((x-1)^2+2^2)(x^2-3)=(x^2-2x+1+4)(x^2-3)$

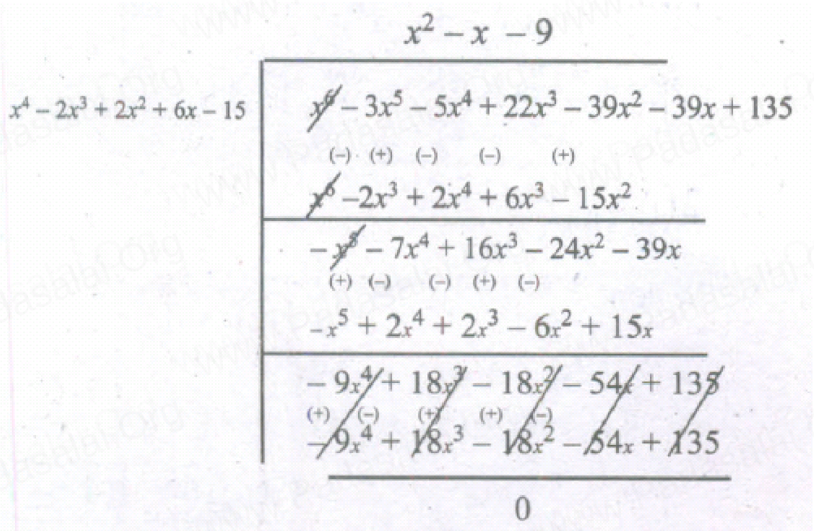
$\Rightarrow$ factor of f(x) is $(x^2-2x+5)(x^2-3)$

$\Rightarrow$ $x^4-3x^2-2x^3+6x+5x^2-15$

$\Rightarrow$ $(x^4-3x^2-2x^3+6x-15)$ is a factor of f(x)

To find the other factor, let us divide.f(x) by

$x^4-2x^3+2x^2+6x-15$



The other factor is x^2 - x - 9

$$\Rightarrow x = \frac{1 \pm \sqrt{(-1)^2 - 4(1)(-9)}}{2} \left[\because x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}\right]$$

$$\Rightarrow x = \frac{1 \pm \sqrt{37}}{2}$$

Hence the roots are

1-2i,1+2i, $\sqrt{3} - \sqrt{3}\frac{1+\sqrt{37}}{2}, \frac{1-\sqrt{37}}{2}$.

57) Solve the equation $6x^4$-$5x^3$-$38x^2$-5x+6=0 if it is known that $\frac{1}{3}$ is a solution.

This equation is Type- 2 even degree reciprocal equation. Hence, it can be rewritten as

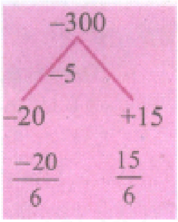
$$6(x^2 + \frac{1}{x^2}) - 5(x + \frac{1}{x}) - 38 = 0 \quad \ldots(1)$$

Put

$$x + \frac{1}{x} = y \Rightarrow \left(x + \frac{1}{x}\right)^2 = y^2$$

$$\Rightarrow x^2 + \frac{1}{x^2} + 2 = y^2 \Rightarrow x^2 + \frac{1}{x} = y^2 - 2$$

∴ (1)becomes as,



$$6(y^2 - 2) - 5y - 38 = 0$$

$$\Rightarrow 6y^2 - 12 - 5y - 38 = 0$$

$$\Rightarrow 6y^2 - 5y - 50 = 0$$

$$\Rightarrow (3y - 10)(2y + 5) = 0$$

$$\Rightarrow y = \frac{10}{3}, \frac{-5}{2}$$

Case (i) when $y = \frac{+10}{3}$

$$x + \frac{1}{x} = \frac{+10}{3}$$

$$\frac{x^2+1}{x} = \frac{+10}{3} \Rightarrow 3x^2 + 3 = 10x$$

$$3x^2 - 10x + 3 = 0$$

$(x-3)(x-\frac{1}{3})=0 \Rightarrow x=3, \frac{1}{3}$

case (ii) when $y=-\frac{5}{2} \Rightarrow x+\frac{1}{x}=-\frac{5}{2}$

$\Rightarrow \frac{x^2+1}{x}=\frac{-5}{2}$

$\Rightarrow 2x^2+2+5x=0$

$\Rightarrow 2x^2+5x+2=0$

$\Rightarrow (x+2)(2x+1)=0$

$\Rightarrow x=-2, \frac{-1}{2}$

$\therefore$ The roots are $3, \frac{1}{3}, -2, \frac{-1}{2}$.

58) Find (i) $\cos^{-1}(-\frac{1}{\sqrt{2}})$

ii) $\cos^{-1}(cos(-\frac{\pi}{3}))$

iii) $\cos^{-1}(cos(-\frac{7\pi}{6}))$

It is known that cos-1 x:[-1,1] $\rightarrow$ [0,π] is given by

$\cos^{-1}$x=y if and only if x=cos y for -1 $\le x \le 1 and 0 \le y \le \pi$

Thus, we have

i) $\cos^{-1}(-\frac{1}{\sqrt{2}})=\frac{3\pi}{4}$, since $\frac{3\pi}{4} \in [0,\pi]$ $\cos\frac{3\pi}{4}=\cos(\pi=\frac{\pi}{4})=-cos\frac{\pi}{4}=-\frac{1}{\sqrt{2}}$

ii) $\cos^{-1}(cos(-\frac{\pi}{3}))=\cos^{-1}(cos(-\frac{\pi}{3}))=\frac{\pi}{3}$, since $-\frac{\pi}{3} \notin [0,\pi]$, but $\frac{\pi}{3} \in [0,\pi]$

iii) $\cos^{-1}(cos(-\frac{7\pi}{6}))=\frac{5\pi}{6}$, since cos $(\frac{7x}{6})=cos(\pi+\frac{\pi}{6})=-\frac{\sqrt{3}}{2}=cos(\frac{5\pi}{6}) \in [0,\pi]$.

59) If a_1, a_2, a_3, ... an is an arithmetic progression with common difference d, prove that tan

$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$ $\left[tan^{-1}\left(\frac{d}{1+a_1a_2}\right)+tan^{-1}\left(\frac{d}{1+a_2a_3}\right)+....tan^{-1}\left(\frac{d}{1+a_na_{n-1}}\right)\right]=\frac{a_n-a_1}{1+a_1a_n}$

Now,

$$tan^{-1}\left(\frac{d}{1+a_1a_2}\right)=tan^{-1}\frac{a_n-a_1}{1+a_1a_n}=tan^{-1}a_2-tan^{-1}a_1$$

Similarly

$$,tan^{-1}\left(\frac{d}{1+a_1a_2}\right)=tan^{-1}\left(\frac{a_n-a_1}{1+a_1a_n}\right)=tan^{-1}a_3-tan^{-1}a_2$$

Continuing inductively, we get

$$tan^{-1}\left(\frac{d}{1+a_na_{n-1}}\right)=tan^{-1}\left(\frac{a_n-a_{n-1}}{1+a_na_{n-1}}\right)=tan^{-1}a_n-tan^{-1}a_{n-1}$$

Adding vertically, we get

$$tan^{-1}\left(\frac{d}{1+a_1a_2}\right)+tan^{-1}\left(\frac{d}{1+a_2a_3}\right)+....+tan^{-1}\left(\frac{d}{1+a_na_{n-1}}\right)tan[tan^{-1}a_n-tan^{-1}a_1]$$

$$tan\left[tan^{-1}\left(\frac{d}{1+a_1a_2}\right)+tan^{-1}\left(\frac{d}{1+a_2a_3}\right)+...+tan^{-1}\left(\frac{d}{1+a_na_{n-1}}\right)\right]=tan\left[tan^{-1}a_n-tan^{-1}a_1\right]$$

$$=\left[tan^{-1}\left(\frac{a_n-a_1}{1+a_1a_n}\right)\right]=\frac{a_n-a_1}{1+a_1a_n}$$

60) Solve $tan^{-1}\left(\frac{x-1}{x-2}\right) + tan^{-1}\left(\frac{x+1}{x+2}\right) = \frac{\pi}{4}$

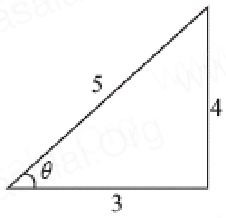
Now,

$$tan^{-1}\left(\frac{x-1}{x-2}\right) + tan^{-1}\left(\frac{x+1}{x+2}\right) = tan^{-1}\left[\frac{\frac{x-1}{x-2}+\frac{x+1}{x+2}}{1-\frac{x-1}{x-2}\left(\frac{x+1}{x+2}\right)}\right] = \frac{\pi}{4}$$

Thus, $\frac{\frac{x-1}{x-2}+\frac{x+1}{x+2}}{1-\frac{x-1}{x-2}\left(\frac{x+1}{x+2}\right)}$=1, which on simplification gives $2x^2-4=-3$

Thus, $x^2=\frac{1}{2}$ gives $x=\pm\frac{1}{\sqrt{2}}$

61) Solve $cos\left(sin^{-1}\left(\frac{x}{\sqrt{1+x^2}}\right)\right) = sin\left\{cot^{-1}\left(\frac{3}{4}\right)\right\}$



We know that

$$sin^{-1}\left(\frac{x}{\sqrt{1+x^2}}\right) = cos^{-1}\left(\frac{x}{\sqrt{1+x^2}}\right)$$

Thus,

$$cos\left(sin^{-1}\left(\frac{x}{\sqrt{1+x^2}}\right)\right) = \frac{1}{\sqrt{1+x^2}} \quad ...(1)$$

From the diagram, we have $cot^{-1}\left(\frac{3}{4}\right) = sin^{-1}\left(\frac{4}{5}\right)$

Hence, $sin\left\{cot^{-1}\left(\frac{3}{4}\right)\right\} = \frac{4}{5}$ ...(2)

Using (1) and (2) in the given equation, we $\frac{1}{\sqrt{1+x^2}} = \frac{4}{5}\ \sqrt{1+x^2} = \frac{5}{4}$

Thus, $x=\pm\frac{3}{4}$

62) Find the domain of the following functions

(i) $f(x) = sin^{-1}(2x - 3)$

(ii) $f(x) = sin^{-1}x + cos\ x$

The domain of sirr lx is [-1,1]

∴ $f(x) = sin^{-1}(2x - 3)$ is defined for all x,satisfying

$-1 \le 2x - 3 \le 1$

$\Rightarrow 3 - 1 \le 2x \le 1 + 3$

$\Rightarrow 2 \le 2x \le 4 \Rightarrow 1 \le x \le 2 \Rightarrow x\epsilon[1, 2]$

∴ Domain of $f(x) = sin^{-1}(2x - 3)$ is {I, 2].

(ii) The domain off(x) is [-1,1] and that ofcosx is R

∴ Domain of $f(x) = \sin^{-1}x + \cos x$ is

$[-1, 1] \cap R = [-1, 1]$

63)

Provethat $\tan^{-1}\left(\frac{1-x}{1+x}\right) - \tan^{-1}\left(\frac{1-y}{1+y}\right) = \sin^{-1}\left(\frac{y-x}{\sqrt{1+x^2}.\sqrt{1+y^2}}\right)$

LHS = $\tan^{-1}\left(\frac{1-x}{1+x}\right) - \tan^{-1}\left(\frac{1-y}{1+y}\right)$

$= \tan^{-1}(1) - \tan^{-1}(x) - (\tan^{-1}(1) - \tan^{-1}(y)$

$\left[\because \tan^{-1}\left(\frac{x-y}{1+xy}\right) = \tan^{-1}x - \tan^{-1}y\right]$

$= \tan^{-1}(1) - \tan^{-1}(x) - \tan^{-1}(1) + \tan^{-1}(y)$

$= \tan^{-1}(y) - \tan^{-1}(x)$

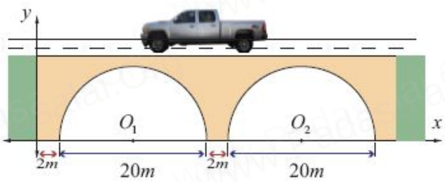
$= \tan^{-1}\left(\frac{y-x}{1+xy}\right)$

$= \sin^{-1}\left(\frac{y-x}{\sqrt{1+(yx)^2+(y-x)^2}}\right)$

$= \sin^{-1}\left(\frac{y-x}{\sqrt{(1+x^2)(1+x^2)}}\right)$

RHS

64) A road bridge over an irrigation canal have two semi circular vents each with a span of 20m and the supporting pillars of width 2m. Use Fig.5.16 to write the equations that model the arches.



Let O_1 O_2 be the centres of the two semi circular vents.

First vent with centre O1 (12,0) and radius r =10 yields equation to first semicircle as

$(x-12)^2+(y-0)^2 = 10^2$

$x^2+y^2-24x + 44 = 0$, $y > 0$

Second vent with centre O_2 (34,0) and radius r =10 yields equation to second vent as

$(x-34)^2+y^2 = 10^2$

$x^2+y^2-68x +1056 = 0$, $y > 0$.

65) Find the foci, vertices and length of major and minor axis of the conic

$4x^2+36y^2+40x-288y+532 = 0$.

Completing the square on x and y of $4x^2+36y^2+40x-288y+532 = 0$,

$4(x^2+10x+25-25)+36(y^2-8y+16-16)+532 = 0$, gives

$4(x^2+10x+25)+36(y^2-8y+16) = -532+100 + 576$

$4(x+5)^2+36(y-4)^2 = 144$.

Dividing both sides by 144 , the equation reduces to $\frac{(x+5)^2}{36}\frac{(y-4)^2}{4} = 1$

This is an ellipse with centre (-5,4) , major axis is parallel to x -axis, length of major axis is 12 and length of minor axis is 4. Vertices are (1,4) and (-11,4).

Now, $c^2 = a^2 - b^2 = 36 - 4 = 32$

and $c = \pm 4\sqrt{2}$

Then the foci are $(-5-4\sqrt{2}, 4)$ and $(-5+4\sqrt{2}, 4)$.

Length of the major axis = 2a =12 units and

the length of the minor axis = 2b =4 units.

66) Find the centre, foci, and eccentricity of the hyperbola $11x^2 - 25y^2 - 44x + 50y - 256 = 0$

Rearranging terms in the equation of hyperbola to bring it to standard form,

we have, $11(x^2-4x)-25(y^2-2y)-256 = 0$

$11(x-2)^2 - 25(y-1)^2 = 256-44+25$

$11(x-2)^2 - 25(y-1)^2 = 275$

$$\frac{(x-2)^2}{25} - \frac{(y-1)^2}{11} = 1$$

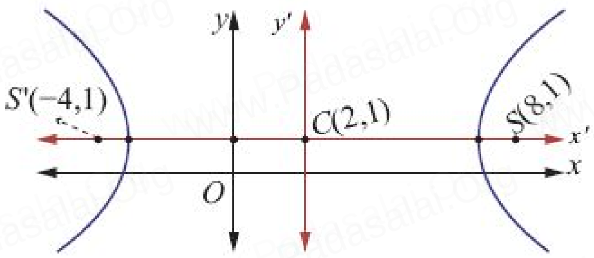
Centre (2,1). $a^2 = 25, b^2 = 11$

$c^2 = a^2 + b^2$

$= 25+11 = 36$

Therefore, $c = \pm 6$

and $e = \frac{c}{a} = \frac{6}{5}$ and the coordinates of foci are(8,1) and(-4,1)



67) Show that the line $x - y + 4 = 0$ is a tangent to the ellipse $x^2 + 3y^2 = 12$. Also find the coordinates of the point of contact.

$x^2 + 3y^2 = 12$

$\div 12$ we get, $\frac{x^2}{12} + \frac{y^2}{4} = 1$

$\therefore a^2 = 12, b^2 = 4$

The line x-y+ 4 = 0 can be rewritten as y=x+4.

$\therefore m=1, c=4$

The condition for y = mx + 4 to be a tangent to the ellipse is $c^2 = a^2m^2 + b^2$

$\therefore (4)^2 = 12(1)^2 + 4$

$\Rightarrow 16 = 12 + 4$

$\Rightarrow 16 = 16$

Since the condition is satisfied, the line x - y + 4 = 0 is a tangent to the ellipse $x^2 + 3y^2 = 12$.

Also, the point of contact is $\left(-\frac{a^2m}{c}, \frac{b^2}{c}\right)$

$$\Rightarrow \left(-\frac{12(1)}{4}, \frac{4}{4}\right) \Rightarrow (-3, 1)$$

$\therefore$The point of contact is (-3, 1).

68) A concrete bridge is designed as a parabolic arch. The road over bridge is 40m long and the maximum height of the arch is 15m. Write the equation of the parabolic arch.

From the graph the vertex is at (0,0) and the parabola is open down

Equation of the parabola is x2 = -4ay

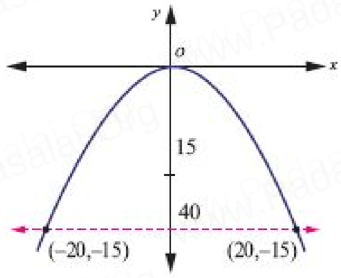
(-20,-15) and (20,-15) lie on the parabola

$20^2 = -4a(-15)$

$4a = \frac{400}{15}$

x2 = $\frac{-80}{3}$ x y

Therefore equation is $3x^2 = -80y$



69) Two coast guard stations are located 600 km apart at points A(0,0) and B(0,600). A distress signal from a ship at P is received at slightly different times by two stations. It is determined that the ship is 200 km farther from station A than it is from station B. Determine the equation of hyperbola that passes through the location of the ship.

Since the centre is located at (0,300) , midway between the two foci, which are the coast guard stations, the equation is

$\frac{(y-300)^2}{a^2} - \frac{(x-0)^2}{b^2} = 1$.... (1)

To determine the values of a and b , select two points known to be on the hyperbola and substitute each point in the above equation. The point(0,400) lies on the hyperbola, since it is 200 km further from Station A than from station B.

$\frac{(400-300)^2}{a^2} - \frac{O}{b^2} = 1 \frac{100^2}{a^2} = 1, a^2 = 10000.$ There is also a point (x,600) on the hyperbola such that $600_2+x_2=(x+200)_2$

$360000 + x^2 = x^2+400x+40000$

x = 800

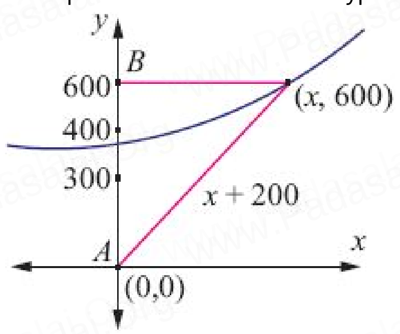
Substituting in (1), we have $\frac{(600-300)^2}{10000} - \frac{(800-0)^2}{b^2} = 1$

$9 - \frac{640000}{b^2} = 1$

$b^2=80000$

Thus the required equation of the hyperbola is $\frac{(y-300)^2}{10000} - \frac{x^2}{80000} = 1$

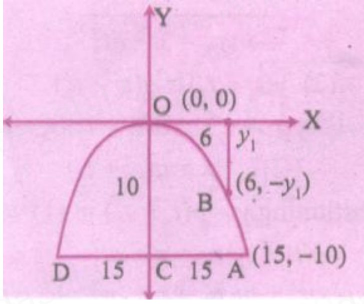
The ship lies somewhere on this hyperbola. The exact location can be determined using data from a third station.



70) A bridge has a parabolic arch that is 10m high in the centre and 30m wide at the bottom. Find the height of the arch 6m from the centre, on either sides.

Let the parabolic bridge to a open downward parabola. Then its equation is $x^2 = -4ay$ ......(1)

Since 10 m is the height in the centre (0, 0) and AD = 30m.



∴ A is the point (15, -10) which is in the IV quadrant A(1 5, -10) lies in (1)

$15^2 = -4a(-10)$

$\Rightarrow \frac{225}{10} = 4a \Rightarrow \frac{45}{2}$

(1) becomes $x2 = \frac{-45}{2}y$ .....(2)

To find the height ofthe arch at x = 6 m, the point B(6, $-y_1$) lies in (2)

$62 = \frac{-45}{2}(-y_1)$

$\Rightarrow 36 = \frac{-45y_1}{2} \Rightarrow \frac{36 \times 2}{45} = y_1$

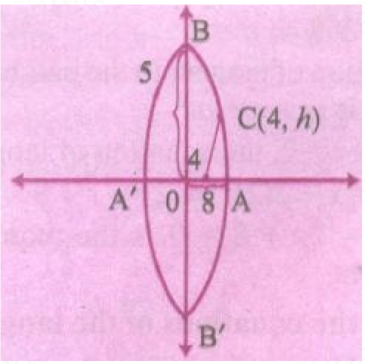
$\Rightarrow y_1 = \frac{8}{5} = 1.6$

∴ Height of the arch on either sides

= 10-1.6 = 8.4m

71) A tunnel through a mountain for a four lane highway is to have a elliptical opening. The total width of the highway (not the opening) is to be 16m, and the height at the edge of the road must be sufficient for a truck 4m high to clear if the highest point of the opening is to be 5m approximately . How wide must the opening be?

Let the cross section ofthe tunnel be in elliptical form.



Given AA' 16 m $\Rightarrow$ OA = 8 m and OB = 5m

∴ Equation of the ellipse is of the form $\frac{x^2}{5^2} + \frac{y^2}{8^2} = 1$ ......(1)

Let the width of the opening be 2h. At a distance of 4 m high, C(4. h) is a point on th ellipse

∴ (1) becomes,

$\frac{x^2}{5^2} + \frac{y^2}{8^2} = 1$

$\Rightarrow \frac{16}{25} + \frac{y^2}{64} = 1 \Rightarrow \frac{y^2}{64} = 1 - \frac{16}{25}$

$\Rightarrow \frac{y^2}{64} = \frac{25-16}{25} = \frac{9}{25}$

$\Rightarrow y_2 = 64 \times \frac{9}{25}$

$\Rightarrow y = 8 \times \frac{3}{5}$

$\Rightarrow y = \frac{24}{5}$

$\Rightarrow$ y = 4.8

$\therefore$ Width of the highway for the opening is 2y = 2(4.8) = 9.6 m

72) At a water fountain, water attains a maximum height of 4m at horizontal distance of 0 5 . m from its origin. If the path of water is a parabola, find the height of water at a horizontal distance of 0.75m from the point of origin.

Let the equation of the parabola be

$(x - h)^2 = -4a(y - k)$.

Here the vertex is (0, 5,4)

Equation of the parabola is $(x - 0.5)^2$

$= -4a(y-4)$ ...(1)

O(0, 0) is a point on the parabola

$(0 - 0.5)^2 = -4a\,(0 - 4)$

$\Rightarrow \left(\frac{-1}{2}\right)^2 = -4a(-4)$

$\Rightarrow \frac{1}{4} = 16a \Rightarrow a = \frac{1}{64}$

$\therefore$ (1) becomes as $(x - 0.5)^2 = -4 \times \frac{1}{64}(y - 4)$

Also $D(0.75, y_1)$ is a point on the parabola

$\therefore (0.75 - 0.5)^2 = \frac{-1}{16}(y_1 - 4)$

$\Rightarrow \left(\frac{1}{4}\right)^2 = \frac{-1}{6}(y_1 - 4)$

$\Rightarrow \frac{1}{16} = \frac{1}{16}(y_1 - 4)$

$\Rightarrow 1 = -y_1 + 4$

$\Rightarrow y_1 = -1 + 4 = 3$m

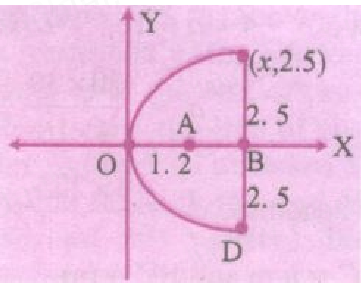
Height of the water at a horizontal distance of 0.75m is 3m

73) An engineer designs a satellite dish with a parabolic cross section. The dish is 5m wide at the opening, and the focus is placed 1 2 . m from the vertex

(a) Position a coordinate system with the origin at the vertex and the x -axis on the parabola's axis of symmetry and find an equation of the parabola.

(b) Find the depth of the satellite dish at the vertex.

Let the cross section of the satellite dish be an right open parabola.



Its equation is $y^2 = 4ax$

Since focus is placed 1.2 m from the vertex OA = 1.2 m and BC = 2.5 m since the width of the dish is 5m.

From the diagram, a = 1.2m

$\therefore y^2 = 4(1.2)x$

(a) $\Rightarrow y^2 = 4.8x$ ...(1)

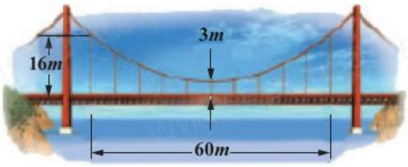
(b) Since $(x_1, 2.5)$ lines on (1)$(2.5)^2 = 4.8(x_1)$

$x_1 = \frac{2.5 \times 2.5}{4.8}$

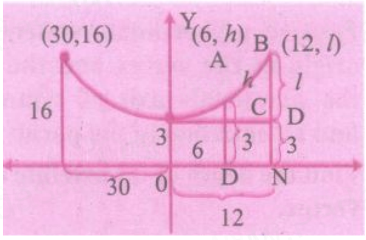
$x_1 = 1.3$ m

∴ Depth of the satellite dish at the vertex is 1.3 m.

74) Parabolic cable of a 60m portion of the roadbed of a suspension bridge are positioned as shown below. Vertical Cables are to be spaced every 6m along this portion of the roadbed. Calculate the lengths of first two of these vertical cables from the vertex.



Let the of the parbola be $x^2 = 4ay$ (1)



Since (30, 16) is a point on (1),

we get $30^2 = 4 \times a \times 16$

$\Rightarrow a = \frac{30 \times 30}{4 \times 16} = \frac{225}{16}$

∴ becomes, $x^2 = x^2 = \frac{4 \times 225}{16} y = \frac{225}{4} y$

Let AC = h m and BD = lm

∴ A(6, h) is a point on the parabola [∵ OD = 6]

$\therefore 6^2 = \frac{225}{4} \times h$

$\Rightarrow h = \frac{36 \times 4}{225} \Rightarrow h = 0.64$

∴ AD = 3 + h = 3 + 0.64 = 3.64 m

Also (12, l) is a point on the parabola

[∵ ON =6 + 6 = 12]

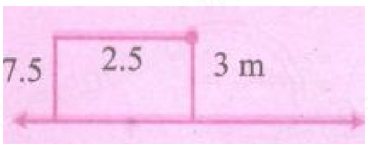
$\therefore 12^2 = \frac{225}{4} \times l$

$\Rightarrow l = \frac{12 \times 12 \times 4}{225} = \frac{576}{225} = 2.56 = 5.56$m

Hence the length of first two vertical cables are 3.64 m and 5.56 m.

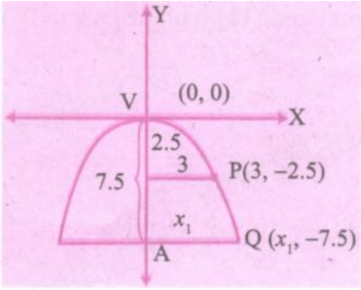
75) Assume that water issuing from the end of a horizontal pipe, 7 5 . m above the ground, describes a parabolic path. The vertex of the parabolic path is at the end of the pipe. At a position 2 5 . m below the line of the pipe, the flow of water has curved outward 3m beyond the vertical line through the end of the pipe. How far beyond this vertical line will the water strike the ground?

As per the given information, we can take the parabola as open downward.



∴ Its equation is $x^2 = -4ay$ ...(1)

Let P be a point on the flow paths, 2.5 m below the line of the pipe and 3m beyond the vertical line through the end of the pipe.



∴ Pis (3, -2.5)

∴ (1) becomes $3^2 = -4a(-2.5)$

⇒ $\frac{-9}{2.5} = 4a$

∴ (1) becomes, $x_1{}^2 = \frac{-9}{2.5}y$ .....(2)

Let x_1 be the distance between the bottom of the verticallineon the ground from the pipeendand the point on which this water touches the ground. But the height of the pipe from the ground is 7.5m.

∴ (x_1 -7.5) lies on ....(2)

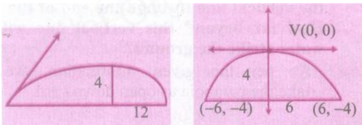
⇒ $x_1{}^2 = \frac{-9}{2.5}(-7.5)$

⇒ $x_1 = \sqrt{9 \times 3} = 3\sqrt{3}$ m

The water strikes the ground $3\sqrt{3}$ m beyond the vertical line.

76) On lighting a rocket cracker it gets projected in a parabolic path and reaches a maximum height of 4m when it is 6m away from the point of projection. Finally it reaches the ground 12m away from the starting point. Find the angle of projection.

By taking the vertex; at the origin, the parabola is open downward.



Its equation is $x^2 = -4ay$

It passes through (6, -4)

∴ 36 =-4a(-4) ⇒ 4a = - $\frac{36}{4}$ = 9

∴ (1) becomes, $x^2 = -9y$

To find the slope at (-6,-4)

Differentiating (1) with respect to 'x' we get,

$2x = -9\frac{dy}{dx}$

⇒ $\frac{dy}{dx} = \frac{-2x}{9}$

At (-6, -4), $\frac{dy}{dx} = -2\frac{(-6)}{9} = \frac{12}{9} = \frac{4}{3}$

∴ $tan\theta = \frac{4}{3} \Rightarrow \theta = tan^{-1}\left(\frac{4}{3}\right)$

∴ The angle of projection is $tan^{-1}\left(\frac{4}{3}\right)$

77) Find the vertex, focus, equation of directrix and length of the latus rectum of the following:

$x^2 - 2x + 8y + 17 = 0$

$x^2 - 2x + 8y + 17 = 0$

$x^2 - 2x = -8y - 17$

Adding 1 both sides, we get

$x^2 - 2x + 1 = -8y - 17 + 1$

$\Rightarrow$ $(x - 1)^2 = -8y - 16 = -8(y + 2)$

$\Rightarrow$ $(x - 1)^2 = -8(y + 2)$

This is a open downward parabola, latus rectum $4a = 8$ $\Rightarrow$ $a = 2$.

(a) Vertex is (1, -2)

$\Rightarrow$ h = 1, k=-2

(b) focus is (0 + h, - a + k)

$\Rightarrow$ (0+ 1,-2-2)

$\Rightarrow$ (1,-4)

(c) Equation of directrix is y = k + a

$\Rightarrow$ y = -2 + 2 $\Rightarrow$ y = 0

(d) Length of latus rectum is 4a = 8 units.

78) Find the vertex, focus, equation of directrix and length of the latus rectum of the following: $y^2-4y-8x+12=0$

$y^2 - 4y - 8x + 12 = 0$

$y^2-4y=8x-12$

Adding 4 both sides, we get,

y - 4y + 4 = 8x - 12 + 4 = 8x - 8

$\Rightarrow$ $(y - 2)^2 = 8(x - 1)$

This is a right open parabola and latus rectum is $4a = 8$ $\Rightarrow$ $a = 2$.

(a) Vertex is (1, 2) $\Rightarrow$ h = 1, k= 2

(b) focus is (h + a, 0 + k)

$\Rightarrow$ (1 + 2, 0 + 2)

$\Rightarrow$ (3, 2)

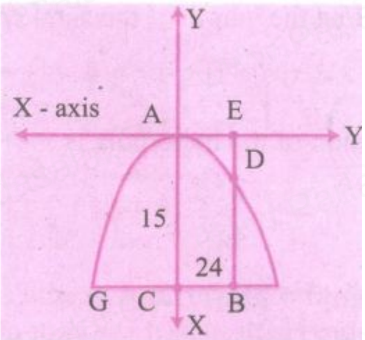
(c) Equation of directrix is x = h - a

$\Rightarrow$ x= 1-2

$\Rightarrow$ x= -1

(d) Length of latus rectum is 4a = 8 units.

79) The guides of a railway bridge is a parabola with its vertex at the highest point 15 m above the ends. If the span is 120 m, find the height of the bridge at 24 m from the middle point.



Let us take the axis AX as the and the tangent AY at A as y-axis.

Equation of the parabola is $y^2 = 4ax$

CA 15, FG = 120

CF = CG= 60

F is (15, 60)

Since F lies on (1), 602 =Aa(15) $\therefore$ $\Rightarrow$ a = 60

$y^2 = 240x$

When y = 24, $24^2 = 240(x)$

$\Rightarrow x = \frac{24 \times 24}{240}$

$x = \frac{24}{10} = \frac{12}{5} = 2.4$

From the diagram

BD=BE-ED= 15-2.4 = 12.6m.

Hence the required height is 12.6 m.

80) A kho-kho player In a practice Ion while running realises that the sum of tne distances from the two kho-kho poles from him is always 8m. Find the equation of the path traced by him of the distance between the poles is 6m.

Given $F_1P + F_2P = 8$

By the focal property of ellipse

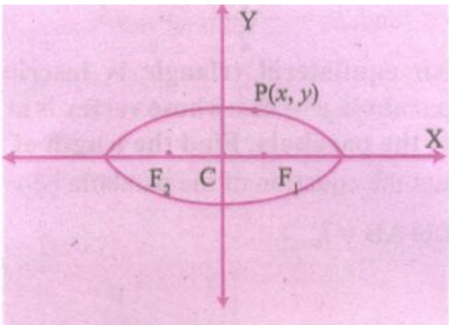
$F_1P + F_2P = 2a$

$\therefore 2a = 8 \Rightarrow a = 4$

and distance between the foci = $F_1F_2 = 6$

$2ae = 6 \Rightarrow ae = 3$

$\therefore 4(e) = 3 \Rightarrow e \frac{3}{4}$



$\therefore b^2 = a^2(1- e^2)$

$= 16\left(1 - \left(\frac{3}{4}\right)^2\right) = 16\left(1 - \frac{9}{10}\right) = 16\left(\frac{7}{16}\right) = 7$

$\therefore$ The path traced by him is an ellipse and its equation is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$\Rightarrow \frac{x^2}{16} - \frac{y^2}{7} = 1$

81) Identify the type of conic and find centre, foci, vertices, and directrices of each of the following :

$18x^2+12y^2-144x+48y+120 = 0$

$18x^2 + 12y^2 - 144x + 48y + 120 = 0$

Given equation is

$18x^2 + 12y^2 - 144x + 48y + 120 = 0$

$18x^2 - 144x + 12y^2 + 48y = -120$

$\Rightarrow 18(x^2 - 8x) + 12(y^2 + 4y) = -120$

$\Rightarrow 18(x^2-8x+ 16-16)+ 12(y^2 +4y+4-4)=-120$

$18(x - 4)^2 - 288 + 12 (y + 2)^2 - 48 = -120$

$\Rightarrow 18(x - 4)^2 + 12(y + 2)^2 = -120 + 288 + 48$

$\Rightarrow 18(x - 4)^2 + 12(y + 2)^2 = 216$

Dividing by 216 we get,

$\frac{18(x-4)^2}{216} + \frac{12(y+2)^2}{216} = 1$

$\Rightarrow \frac{(x-4)^2}{12} + \frac{(y+2)^2}{18} = 1$

This is an equation of the ellipse with major axis parallel to y-axis,

$\therefore a^2 = 18, b^2 = 12$

$\therefore c^2 = a^2 - b^2 = 18 - 12 = 6 \Rightarrow c = \sqrt{6}$

$$e = \sqrt{1 - \frac{b^2}{a^2}} = \sqrt{1 - \frac{12}{18}} = \sqrt{\frac{18-12}{18}}$$

$$= \sqrt{\frac{6}{18}} = \sqrt{\frac{1}{3}}$$

(a) Center is (4, -2)

$\Rightarrow$ h =4, k= -2

(b) Vertices are (h, k-a), (h, k + a)

$\Rightarrow (4, -2 - 3\sqrt{2}), (4, -2 + 3\sqrt{2})$

$[\because a^2 = 18 \Rightarrow a = \sqrt{18} = 3\sqrt{2}]$

(c) Foci are (h, k - c), (h, k + c)

$\Rightarrow (4, -2 - \sqrt{6}), (4, -2 + \sqrt{6})$

(d) Equation of directrices are $y + 2 = \pm \frac{a}{e}$

$\Rightarrow y + 2 = \pm \frac{a}{e}$

$\Rightarrow y - 2 = \pm \frac{3\sqrt{2}}{\frac{1}{\sqrt{3}}} = \pm 3\sqrt{2} \times \sqrt{3} = \pm 3\sqrt{6}$

$\Rightarrow y + 2 = \pm 3\sqrt{6}, y + 2 = -3\sqrt{6}$

$\Rightarrow y = -2 + 3\sqrt{6}$ and $y = -2 - 3\sqrt{6}$

82) Identify the type of conic and find centre, foci, vertices, and directrices of each of the following :

$9x^2 - y^2 - 36x - 6y + 18 = 0$

$9x^2 - y^2 - 36x - 6y + 18 = 0$

Given equation is $9x^2 - y^2 - 36x - 6y + 18 = 0$

$\Rightarrow 9x^2 - 36x - (y^2 + 6y) = -18$

$\Rightarrow 9(x^2 - 4x) - (y^2 + 6y) = -18$

$\Rightarrow 9(x^2 - 4x + 4 - 4) - (y^2 + 6y + 9 - 9) = -18$

$\Rightarrow 9(x-2)^2 - 36 - (y+3)^2 + 9 = -18$

$\Rightarrow 9(x-2)^2 - (y+3)^2 = -18 + 36 - 9$

$\Rightarrow 9(x-2)^2 - (y+3)^2 = 9$

Dividing by 9 we get, $\frac{(x-2)^2}{1} - \frac{(y+3)^2}{9} = 1$

This is an equation of the hyperbola whose transverse axis is parallel to x-axis.

$a^2 = 1, b^2 = 9$

$\therefore c^2 = a^2 + b^2 = 1 + 9 = 10 \Rightarrow c = \sqrt{10}$

$$e = \sqrt{1 - \frac{b^2}{a^2}} = \sqrt{1 - \frac{9}{1}} = \sqrt{10}$$

a) Center is (2, -3)

$\Rightarrow$ h =2, k = -3

(b) Foci are (h + c, k), (17 - c, k)

$\Rightarrow (2+\sqrt{10}, -3), (2-\sqrt{10}, -3)$

(c) Vertic ar (h + a, k) (h - a, k)

$\Rightarrow$ (2+ 1,-3), (2-1,-3)

$\Rightarrow$ (3, -3) (1, -3)

(d) Equation of directrices are x - 2 = $\pm \frac{a}{e}$

$\Rightarrow x - 2 = \pm \frac{1}{\sqrt{10}}$

$x = 2 \pm \frac{1}{\sqrt{10}}$

$\Rightarrow x = 2 + \frac{1}{\sqrt{10}}$ and $x = 2 - \frac{1}{\sqrt{10}}$

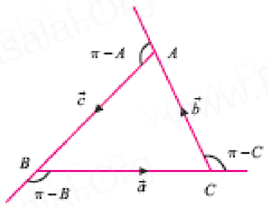
83) With usual notations, in any triangle ABC, prove by vector method that $\frac{a}{sinA} = \frac{b}{sinB} = \frac{c}{sinc}$

With usual notations in triangle, ABC let $\vec{BC} = \vec{a}$, $\vec{CA} = \vec{b}$ and $\vec{AB} = \vec{c}$. Then $\vec{BC} = \vec{a}$, $\vec{CA} = \vec{b}$ and $\vec{AB} = \vec{c}$

Since in ΔABC, $\vec{BC} + \vec{CA} + \vec{AB} = 0$ we have $\vec{BC} \times (\vec{BC} + \vec{CA} + \vec{AB}) = \vec{0}$

Simplyfying, we get,

$\vec{BC} \times \vec{CA} = \vec{AB} \times \vec{BC}$ ....(1)



Similarly, since $\vec{BC} + \vec{CA} + \vec{AB} = 0$, we have

Simplifying, we get $\vec{BC} \times \vec{CA} = \vec{CA} \times \vec{AB}$

From equations (1) and (2), we get ....(2)

$\vec{AB} \times \vec{BC} = \vec{CA} \times \vec{AB} = \vec{BC} \times \vec{CA}$

So, $\left| \vec{AB} \times \vec{BC} \right| = \left| \vec{CA} \times \vec{AB} \right| = \left| \vec{BC} \times \vec{CA} \right|$

ca sin(π − B) = bc sin(π - A) = ab sin (π - C)

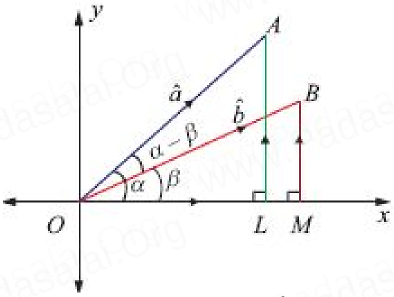
That is, ca sin B = bc sin A = absinC . Dividing by abc , we get

$\frac{sinA}{A} = \frac{sinB}{b} = \frac{sinC}{c}$ or $\frac{a}{sinA} = \frac{b}{sinB} = \frac{c}{sinC}$

84) Prove by vector method that sin(α −β)=sinα cosβ −cosα sinβ

Let $\hat{a} = \vec{OA}$ and $\hat{b} = \vec{OB}$ be the unit vectors making angles α and β respectively, with positive x -axis, where A and B are as shown in the diagram. Then, we get

$\hat{a}$ = cosα $\hat{i}$ + sinα$\hat{j}$ and $\hat{b}$ ˆ = cosβ $\hat{i}$ + sin β $\hat{j}$,



The angle between $\hat{a}$ and $\hat{b}$ is α −β and, the vectors $\hat{b}, \hat{a}, \hat{k}$ form a right-handed system.

Hence, we get $\hat{b} \times \hat{a} = \left|\hat{b}\right| \left|\hat{a}\right| \sin(\alpha-\beta)\hat{k} = \sin(\alpha-\beta)\hat{k}$

On the other hand,

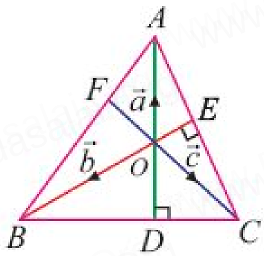
$$\hat{b} \times \hat{a} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ cos\beta & sin\beta & 0 \\ sin\alpha & sin\alpha & 0 \end{vmatrix} = (\sin\alpha\cos\beta - \cos\alpha\sin\beta)\hat{k}$$

Hence, by equations (1) and (2), we get

sin(α −β) = sin α cos β − cos α sin β

85) Prove by vector method that the perpendiculars (attitudes) from the vertices to the opposite sides of a triangle are concurrent.

Consider a triangle ABC in which the two altitudes AD and BE intersect at O. Let CO be produced to meet AB at F. We take O as the origin and let $\vec{OA} = \vec{a}, \vec{OB} = \vec{b}$ and $\vec{OC} = \vec{c}$



Since $\vec{AD}$ is perpendicular to $\vec{BC}$, we have $\vec{OA}$ is perpendicular to $\vec{BC}$, and hence we get $\vec{OA} \cdot \vec{BC} = 0$. Thnat is, $\vec{a}.(\vec{c} - \vec{b}) = 0$, which means

$\vec{a}.\hat{c} - \hat{a}.\hat{b} = 0$...............(1)

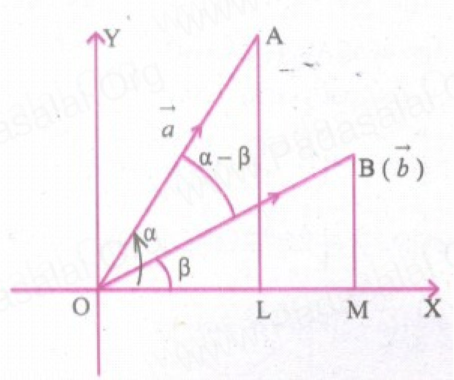
Similarly, since $\vec{BE}$ is perpendicular to $\vec{CA}$, we have $\vec{OB}$ is perpendicular to $\vec{CA}$, and hence we get $\vec{OB}.\vec{CA} = 0$. That is,

$\vec{a}.\hat{c} - \hat{b}.\hat{c} = 0$............(2)

Adding equations (1) and (2), gives $\vec{a}.\hat{c} - \hat{b}.\hat{c} = 0$. That is, $\hat{c}(\hat{a} - \hat{b}) = 0$

That is $\vec{OC} \cdot \vec{BA} = 0$. Therefore, $\vec{BA}$ is perpendicular to $\vec{OC}$. Which implies that $\vec{CF}$ is perpendicular to $\vec{AB}$. Hence, the perpendicular drawn from C to the side AB passes through O. Therefore, the altitudes are concurrent

86) Using vector method, prove that cos(α − β)=cos α cos β +sin α sin β



Let $\hat{a} = \vec{OA}$ and $\hat{b} = \vec{OB}$ be the unit vectors and which make angles α,β respectively with positive x-axis Draw AL and BM ⊥ to axis

Then $|\vec{OL}| = |\vec{OA}|$ =cosα=cosα

$|\vec{LA}| = |\vec{OB}|$ sinα=sinα

[∵ ΔOAL, sinα=$\frac{opp}{hyp}$ $cos\alpha = \frac{adj}{hyp}$]

∴ $\vec{OL} = |\vec{OL}|\hat{i} = cos\alpha\hat{i}$

$\vec{LA} = |\vec{LA}|\hat{j} = sin\alpha\hat{j}$

∴ $\hat{a} = \vec{OA} = \vec{OL} + \vec{LA}$ [Using Δ law of addition]

$\hat{a} = cos\alpha\hat{i} + sin\alpha\hat{j}$

Similarly $\hat{b} = cos\alpha\hat{i} + sin\alpha\hat{j}$ ......(2)

The angle between $\hat{a}$ and $\hat{b}$ is (α-β)

∵ $\hat{a}.\hat{b} = |a||\hat{b}|$ cos(α-β)=cos(α-β) ....(3)

[∵ $|\hat{a}| = |b|$ =1]

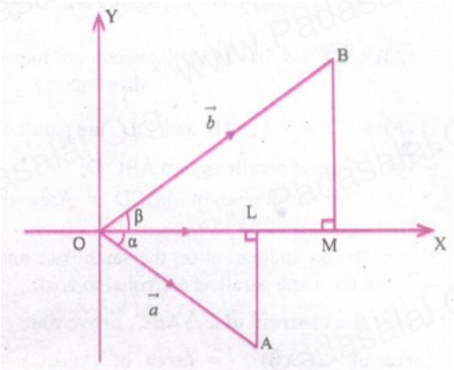
Also $\hat{a}.\hat{b} = (cos\alpha\hat{i} + sin\alpha\hat{j})(cos\beta\hat{i} + sin\beta\hat{j})$

$\hat{a}\hat{b}$=cosα cosβ + sinα sinβ .....(4)

From (3) and (4), we get

cos(α-β)=cosα cosβ + sinα sinβ

87) Prove by vector method that sin(α + β)=sin α cos β + cos α sin β



Let $\hat{a} = \vec{OA}$ and $\hat{b} = \vec{OB}$ be the unit vectors and which make angles α,β respectively with positive x-axis

Draw AL and BM ⊥ to x-axis

Then $|\vec{OL}| = |\vec{OA}|cos\alpha \Rightarrow \vec{OL} = |\vec{OL}|\hat{i} = cos\alpha\hat{i}$

$|\vec{LA}| = |\vec{OB}|$ sinα

$\Rightarrow \vec{LA} = |\vec{OB}|\hat{j} = sin\alpha(-\hat{j}) = -sin\alpha\hat{j}$

[$\vec{LA}$ is in the opp direction of y axis]

$\hat{a} = \vec{OA} = \vec{OL} + \vec{LA} = cos\alpha\hat{i} - sin\alpha\check{j}$ ...(1)

Similarly $\hat{b} = \vec{OB} = \vec{OM} + \vec{MB} = cos\beta\hat{i} + sin\beta\hat{j}$ ....(2)

Now $\hat{a} \times \hat{b} = |\hat{a}||\hat{b}|sin(\alpha+\beta)\hat{k} = sin(\alpha+\beta)\hat{k}$ ....(3)

[$|\hat{a}| = |\hat{b}| = 1$]

Also

$$\hat{a} \times \hat{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ cos\alpha & -sin\alpha & 0 \\ cos\beta & cos\beta & 0 \end{vmatrix}$$

$= \hat{i}(0) - \hat{j}(0) + \hat{k}(\cos\alpha \sin\beta + \sin\alpha \cos\beta)$

$= (\sin\alpha \cos\beta + \cos\alpha \sin\beta)\hat{k}$ ....(4)

using (3) and (4), $\sin(\alpha+\beta) = \sin\alpha \cos\beta + \cos\alpha \sin\beta$

88) If $\vec{a} = \hat{i} - \vec{j}, \vec{b} = \hat{i} - \hat{j} - 4\hat{k}, \vec{c} = 3\hat{j} - \hat{k}$ and $\vec{d} = 2\hat{i} + 5\hat{j} + \hat{k}$

(i) $(\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d}) = [\vec{a}, \vec{b}, \vec{d}]\vec{c} - [\vec{a}, \vec{b}, \vec{c}]\vec{d}$

(ii) $(\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d}) = [\vec{a}, \vec{c}, \vec{d}]\vec{b} - [\vec{b}, \vec{c}, \vec{d}]\vec{a}$

By definition,

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -1 & 0 \\ 1 & -1 & -4 \end{vmatrix} = 4\hat{i} + 4\hat{j}, \vec{c} \times \vec{d} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 3 & -1 \\ 2 & 5 & 1 \end{vmatrix} = 8\hat{i} - 2\hat{j} - 6\hat{k}$$

..........(1)

$$(\vec{a} \times \vec{b})(\vec{c} \times \vec{d}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 4 & 4 & 0 \\ 8 & -2 & -6 \end{vmatrix} = 24\hat{i} + 24\hat{j} - 40\hat{k}$$

On the other hand, we have

$[\vec{a}\vec{b}\vec{d}]c - [a, \vec{b}, c]\vec{d} = 28(3\vec{j} - \vec{k}) - 12(2\hat{i} + 5\hat{j} + \hat{k}) = -24\hat{i} + 24\hat{j} - 40\hat{k}$ ....(2)

Therefore, from equations (1) and (2), identity (i) is verified.

The verification of identity (ii) is left as an exercise to the reader

89) Find the point of intersection of the lines $\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4}$ and $\frac{x-4}{5} = \frac{y-1}{2} = z$

Every point on the line $\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4}$ = s (say) is of the form (2s + 1, 3s + 2, 4s + 3) and every point on the line $\frac{x-4}{5} = \frac{y-1}{2} = z = t$ (say) is of the form (5t + 4, 2t + 1, t). So, at the point of intersection, for some values of s and t, we have

(2s + 1, 3s + 2, 4s + 3) = (5t + 4, 2t + 1, t)

Therefore, 2s - 5t = 3, 3s - 2t = -1 and 4s - t = -3. Solving the first two equations we get t = -1, s = -1. These values of s and t satisfy the third equation. Therefore, the given lines intersect. Substituting, these values of t or s in the respective points, the point of intersection is (-1,- 1, -1)

90) Determine whether the pair of straight lines $\vec{r}(2\hat{i} + 3\hat{j} - \hat{k}) + t(2\hat{i} + 3\hat{j} + 2\hat{k}), \vec{r} = (2\hat{j} - 3\hat{k}) + s(\hat{i} + 2\hat{j} + 3\hat{k})$ are parallel. Find the shortest distance between them.

Comparing the given two equations with

$\vec{r} = \vec{a} + s\vec{b}$ and $\vec{r} = \vec{c} + s\vec{d}$

We have $\vec{a} = 2\hat{i} + 3\hat{j} - 3\hat{k}, \vec{b} = 2\hat{i} + 3\hat{j} + 4\hat{k}, \vec{c} = \hat{i} + 2\hat{j}, \vec{d} = \hat{i} + 2\hat{j} + 3\hat{k}$

Clearly, $\vec{b}$ is not a scalar multiple of $\vec{d}$. So, the two vectors are not parallel and hence the two lines are not parallel.

The shortest distance between the two straight lines is given by

$$\delta = \frac{\left|(\vec{c}\times\vec{a}).(\vec{b}\times\vec{d})\right|}{\left|\vec{b}\times\vec{d}\right|}$$

$$\vec{b}\times\vec{d} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 3 & 4 \\ 1 & 2 & 3 \end{vmatrix} = \hat{i} - 2\hat{j} + \hat{k}$$

$(\vec{c} - \vec{a}).(\vec{b}\times\vec{d}) = (-2\hat{i} - 4\hat{j} - 6\hat{k}).(\hat{i} - 2\hat{j} + \hat{k}) = 0$

Therefore, the distance between the two given straight lines is zero.Thus, the given lines intersect each other.

91) Show that the lines $\frac{x-3}{3} = \frac{y-3}{-1} = z - 1$ and $\frac{x-6}{2} = \frac{z-1}{3}, y - 2 = 0$ intersect. Also find the point of intersection

Given lines are $\frac{x-3}{3} = \frac{y-3}{-1}$ ....(1)

and z-1=0

$\Rightarrow$ z=1

and $\frac{x-6}{2} = \frac{z-1}{3}$ ....(2)

and y-2=0 $\Rightarrow$ y=2

Substituting y = 2 and z = 1 in (1) we get

$\frac{x-3}{3} = \frac{2-3}{-1} = \frac{-1}{-1} = 1 \Rightarrow x - 3 = 3 \Rightarrow x = 6$

The point of intersection is (6, 2, 1)

Let us check whether (6, 2, 1) satisfies (1) and (2)

(1) $\rightarrow \frac{6-6}{2} = \frac{1-1}{3} \Rightarrow 0 = 0$

(2) $\rightarrow \frac{6-3}{3} = \frac{1-3}{-1} \Rightarrow -1 = -1$

Hence, the given two lines intersect and the point of intersection is (6, 2, 1).

92) Find the vector parametric, vector non-parametric and Cartesian form of the equation of the plane passing through the points (-1, 2, 0), (2, 2, -1)and parallel to the straight line $\frac{x-1}{1} = \frac{2y+1}{2} = \frac{z+1}{-1}$

The required plane is parallel to the given line and so it is parallel to the vector $\vec{c} = \hat{i} + \hat{j} - \hat{k}$ and the plane passes through the points $\vec{a} = -\hat{i} + 2\hat{j}, \vec{b} = 2\hat{i} + 2\hat{j} - \hat{k}$

(i) vector equation of the plane in parametric form is $\vec{r} = \vec{a} + s(\vec{b} - \vec{a}) + t\vec{c}$, where s, t ∈ R

which implies that $\vec{r} = (-\hat{i} + 2\hat{j}) + s(3\hat{i} - \hat{k}) + t(\hat{i} + \hat{j} - \hat{k})$, where s, t ∈ R

(ii) vector equation of the plane in non-parametric form is $(\vec{r} - \vec{a}).(\vec{b} - \vec{a})\times\vec{c}) = 0$

Now,

$$(\vec{b} - \vec{a})\times\vec{c} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 0 & -1 \\ 1 & -1 & -1 \end{vmatrix} = \hat{i} + 2\hat{j} + 3\hat{k}$$

we have $(\vec{r} - (-\hat{i} + 2\hat{j})).(\hat{i} + 2\hat{j} + 3\hat{k}) = 0 \Rightarrow \vec{r}.(\hat{i} + 2\hat{j} + 3\hat{k}) = 3$

If $\vec{r}.(\hat{i} + 2\hat{j} + 3\hat{k})$ is the position vector of an arbitrary point on the plane, then from the above equation, we get the Cartesian equation of the plane as x + 2y + 3z = 3

93) Find the non-parametric form of vector equation, and Cartesian equation of the plane passing through the point (2,3,6) and parallel to the straight lines $\frac{x-1}{2} = \frac{y+1}{3} = \frac{z-3}{1}$ and $\frac{x+3}{2} = \frac{y-3}{-5} = \frac{z+1}{-3}$

The plane passes through the point.

$a = 2\hat{i} + 3\hat{j} + 6\hat{k}$ and parallel to the lines $\frac{x-1}{2}$

$= \frac{y+1}{3} = \frac{z-3}{1} and \frac{x+3}{2} = \frac{y-3}{-5} = \frac{z+1}{-3}$

$\Rightarrow \vec{b} = 2\hat{i} + 3\hat{j} + \hat{k} and \quad c = 2\hat{i} - 5\hat{j} - 3\hat{k}$

$$\vec{b} \times \vec{c} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 3 & 4 \\ 2 & -5 & -3 \end{vmatrix}$$

$= \hat{i}(-9+5) - \hat{j}(-6-2) + \hat{k}(-10-6)$

$= -4\hat{i} + 8\hat{j} - 16\hat{k}$

The non-parametric vector equation of the plane is

$(r.a).(\vec{b} \times c) = 0,$

$\Rightarrow [\vec{r}(2\hat{i} + 3\hat{j} + 16\hat{k}).(-4\hat{i} + 8\hat{j} - 16\hat{k})] = 0$

$\Rightarrow [r.(-4\hat{i} + 8\hat{j} - 16\hat{k})] - (-8 + 24 - 96) = 0$

$\Rightarrow \vec{r}.(-4\hat{i} + 8\hat{j} - 16\hat{k}) = -80$

$\div -4,$ We get

$r.(\hat{i} - 2\hat{j} + 4\hat{k}) = 20$

$Let \quad \vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$\Rightarrow (x\hat{i} + y\hat{j} + z\hat{k}).(\hat{i} - 2\hat{j} + 4\hat{k}) = 20$

$\Rightarrow x = 2y + 4z = 20$

$\Rightarrow x - 2y + 4z - 20 = 0$

94) Find the parametric form of vector equation, and Cartesian equations of the plane passing through the points (2, 2,1), (9,3,6) and perpendicular to the plane 2x + 6y + 6z = 9

Given plane is passing through the points

$a = 2\hat{i} + 2\hat{j} + 2\hat{k}\vec{b} = 9\hat{i} + 3\hat{j} + 6\hat{k}$

Equation of the given plane is 2x + 6y + 6z = 9. It can be written as $\vec{r}.(2\hat{i} + 6\hat{j} + 6\hat{k}) = 9$

Since the given plane is perpendicular to $2\hat{i} + 6\hat{j} + 6\hat{k}$, the required plane is parallel to $c = 2\hat{i} + 6\hat{j} + 6\hat{k}$. Hence, parametric form of vector equation of plane passing through two points and parallel to a vector is

$r = a + s(\vec{b} - a) + tc, s, t \in R$

$\vec{r} = 2\hat{i} + 2\hat{j} + \hat{k} + s(7\hat{i} + \hat{j} + 5\hat{k}) + t(2\hat{i} + 6\hat{j} + 6\hat{k}), s, t \in R$

Cartesian equation of the plane is

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ c_1 & c_2 & c_3 \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} x-2 & y-2 & z-1 \\ 7 & 1 & 5 \\ 2 & 6 & 6 \end{vmatrix} = 0$$

$\Rightarrow$ (x-2)(6-30) - (y-2)(42-10) + (z-1)(42-2) = 0

$\Rightarrow$ (x - 2)(-24) - (y - 2)(32) + (z - 1)(40) = 0

⇒ 24x + 48 - 32y + 64 + 40z - 40 = 0

⇒ -24x - 32y + 40z + 72 = 0

÷ + - 8 we get

3'x+ 4y - 5z - 9 = 0 is the Cartesian form.

∴ The parametric form of vector equation is

$\vec{r} = \vec{r}(3\vec{i} + 4\vec{j} - 5\vec{k}) = 9$

95) Find parametric form of vector equation and Cartesian equations of the plane passing through the points (2, 2,1), (1, −2,3) and parallel to the straight line passing through the points (2,1,−3) and (−1,5,−8)

The plane passes through two points

$a = 2\hat{i} + 2\hat{j} + \hat{k} and \vec{b} = \hat{i} - 2\hat{j} + 3\hat{k}$

The straight line passing through the points

(2, 1, -3) and (-1,5, -8) is

$\frac{x-x_1}{x_2-x_1} = \frac{y-y_1}{y_2-y_1} = \frac{z-z_1}{z_2-z_1}.$

$\Rightarrow \frac{x-2}{-1-2} = \frac{y-1}{5-1} = \frac{z+3}{-8+3}$

$\Rightarrow \frac{x-2}{-3} = \frac{y-1}{4} = \frac{z+3}{-5}$

Hence the required plane is parallel to the vector

$c = -3\hat{i} + 4\hat{j} - 5\hat{k}$

The parametric form of vector equation of the plane passing through two points $a, \vec{b}$ parallel to a vector

$c is r = a + s(\vec{b} - a) + tc, s, t \in R,$

$\vec{r}.2\hat{i} + 2\hat{j} + \hat{k} + s(-\hat{i} - 4\hat{j} + 2\hat{k}) + t(3\hat{i} - 4\hat{j} + 5\hat{k})$ s, t ∈ R

Cartesian form of the plane passing through two points and parallel to a vector is

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ x_2-x_1 & y_2-y_1 & z_2-z_1 \\ c_1 & c_2 & c_3 \end{vmatrix} = 0$$

[∵(x_1, Y_1, Z_1) is (2, 2,1), (X_2, Y_2, Z_2) is (-1, -2,3) & c_1, c_2, c_3 is -3, 4 -5]

$$\Rightarrow \begin{vmatrix} x-2 & y-2 & z-1 \\ -1 & -4 & 2 \\ -3 & 4 & -5 \end{vmatrix} = 0$$

⇒ (x - 2)(20 - 8) - (y- 2)(5+ 6) + (z-1)(-4 -12) = 0

⇒ (x - 2)(12) - (y - 2)(11) + (z - 1)(-16) = 0

⇒ 12x- 24 -11y + 22 -16z + 16 = 0

⇒ 12x-11y-16z+ 14 =0

96) Find the non-parametric form of vector equation of the plane passing through the point (1, −2, 4) and perpendicular to the plane x + 2y −3z = 11 and parallel to the line $\frac{x+7}{3} = \frac{y+3}{-1} = \frac{z}{1}$

Equation of the plane passing through the point

$= a = \hat{i} - 2\hat{j} + 4$ (1)

Equation of the given plane is X + 2y - 3z = 11

$\Rightarrow \vec{r}.(\hat{i} + 2\hat{j} - 3\hat{k}) = 11$

The given plane is perpendicular to the vector $\hat{i} + 2\hat{j} - 3\hat{k}$

∴ The required plane is parallel to the vector

$\vec{b} = \hat{i} + 2\hat{j} - 3\hat{k}$ (2)

The given plane is parallel to the line

$\frac{x+7}{3} = \frac{y+3}{-1} = \frac{z}{1}$ Whose direction ratios are 3,-1,1

The required plane is parallel to the vector

$\vec{c} = 3\hat{i} - \hat{j} + \hat{k}$ (3)

∴ The non-parametric vector equation of the plane passing through a point ($\vec{a}$) and parallel to two vectors $\vec{b}$ and $\vec{c}$ is

$(\vec{r} - \vec{a}).(\vec{b} \times \vec{c}) = 0$

$$\vec{b} \times c = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & -3 \\ 3 & -1 & 1 \end{vmatrix}$$

$= \hat{i}(2-3) - \hat{j}(1+9) + \hat{k}(-1-6)$

$= -\hat{i} - 10\hat{j} - 7\hat{k}$

$\therefore (r - (\hat{i} - 2\hat{j} + 4\hat{k})).(-\hat{i} - 10\hat{j} - 7\hat{k}) = 0$

$[\vec{r} - (-\hat{i} - 10\hat{j} - 7\hat{k})] - [(\hat{i} - 2\hat{j} + 4\hat{k}).(-\hat{i} - 10\hat{j} - 7\hat{k})] = 0$

$\Rightarrow r.(-\hat{i} - 10\hat{j} - 7\hat{k}) - [-1 + 20 - 28] = 0$

$\Rightarrow \vec{r}.(-\hat{i} - 10\hat{j} - 7\hat{k}) \times 9 = 0$

$\Rightarrow r.(\hat{i} + 10\hat{j} + 7\hat{k}) - 9 = 0$

$\Rightarrow \vec{r}.(\hat{i} + 10\hat{j} + 7\hat{k}) = 9$

Let $\Rightarrow r = x\hat{i} + y\hat{j} + z\hat{k}$

$\therefore (x\hat{i} + y\hat{j} + z\hat{k}).(\hat{i} + 10\hat{j} + 7\hat{k}) = 9$

$\Rightarrow$ + 10y+ 7z= 9 which is the required Cartesian equation of the plane.

97) Find the parametric vector, non-parametric vector and Cartesian form of the equations of the plane passing through the points (3,6,−2), (−1,−2,6) , and (6,−4,−2).

The plane passing through three points namely

$a = 3\hat{i} + 2\hat{k} + 6\hat{k},$

$\vec{b} = 3\hat{i} - 2\hat{j} + 6\hat{k}$ and

$c = 6\hat{i} - 4\hat{j} - 2\hat{k}$

Now, $\vec{b} - a = (-1-3)\hat{i} + (-2-6)\hat{j} + (-2+2)\hat{k}$

$= -4\hat{i} - 8\hat{j} + 8\hat{k}$

$\vec{c} - \vec{a} = (6-3)\hat{i} + (-4-6)\hat{j} + (-2+2)\hat{k}$

$= 3\hat{i} - 10\hat{j}$

The parametric form of vector equation of the plane passing through three points is

$\vec{r} = \vec{a} + s(\vec{b} - \vec{a}) + t(\vec{c} - \vec{a}), s, t \in R$

$\Rightarrow r = (3i + 6i - 2k) + s(4\hat{i} - 8\hat{j} + 8\hat{k}) + (3\hat{i} - 10\hat{j})s, t \in R$

The parametric form of vector equation of the plane passing through three points is

$[r - a, \vec{b} - a, c - a]^{=0}$

$\Rightarrow \vec{r} - (3i + 6i - 2k).[(4\hat{i} - 8\hat{j} 8\hat{k}) + (3\hat{i} - 10\hat{j})] = 0$

Cartesian equation is

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ x_2-x_1 & y_2-y_1 & z_2-z_1 \\ x_3-x_1 & y_3-y_1 & z_3-z_1 \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} x-3 & y-6 & x+2 \\ -4 & -8 & 8 \\ 3 & -10 & 0 \end{vmatrix} = 0$$

$\Rightarrow$ (x-3)(0 + 80) - (y-6)(0- 24) + (z + 2)(40 + 24) = 0

$\Rightarrow$ 80x - 240 +24y- 144 + +64z+ 128 = 0

$\Rightarrow$ 80x + 24y + 64z - 256 = 0

÷ 8 we get

10x + 3y + 8z,- 32 = 0

98) If the straight lines $\frac{x-1}{2} = \frac{y+1}{\lambda} = \frac{z}{2}$ and $\frac{x-1}{2} = \frac{y+1}{\lambda} = \frac{z}{\lambda}$ are coplanar, find λ and equations of the planes containing these two lines.

$\frac{x-1}{2} = \frac{y+1}{\lambda} = \frac{z}{2}$ and $\frac{x-1}{2} = \frac{y+1}{\lambda} = \frac{z}{\lambda}$

$\therefore a = \hat{i} - \hat{j}, \vec{b} = 2\hat{i} + \lambda\hat{j} + 2\hat{k}$

$c = -\hat{i} - \hat{j}, \vec{d} = 5\hat{i} + 2\hat{j} + \lambda\vec{k}$

$\left(\vec{c} - \vec{a}\right) = -2\hat{i},$

and

$$\left(\vec{b} \times \vec{d}\right) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & \lambda & 2 \\ 5 & 2 & \lambda \end{vmatrix}$$

$= \hat{i}\left(\lambda^2 - 4\right) - \hat{j}(2\lambda - 10) + \hat{k}(4 - 5\lambda)$

Since the given lines are co-planar,

$\left(\vec{c} - \vec{a}\right).\left(\vec{b} \times \vec{d}\right) = 0$

$\Rightarrow \left(-2\hat{i}\right).\left[\left(\lambda^2 - 4\right)\hat{i} - \hat{j}(2\lambda - 10) + \hat{k}(4 - 5\lambda)\right] = 0$

$\Rightarrow -2\left(\lambda^2 - 4\right) = 0$

$\Rightarrow \lambda^2 = 4 \qquad [\because -2 \neq 0]$

$\Rightarrow \lambda = \pm\sqrt{4} = \pm 2$

The Cartesian equation of the plane containing the given lines is

$$\begin{vmatrix} x-x_2 & y-y_2 & z-z_2 \\ b_1 & b_2 & b_3 \\ d_1 & d_2 & d_3 \end{vmatrix}$$

$$\Rightarrow \begin{vmatrix} x+1 & y+1 & z \\ 2 & 2 & 2 \\ 5 & 2 & 2 \end{vmatrix} = 0[\because \lambda = 2]$$

$\Rightarrow (x+1)(4-4) - (y+1)(4-10) + z(4-10) = 0$

$\Rightarrow (x+1)(0) - (y+1)(-6) + z(-6) = 0$

$\Rightarrow 6(y+1) - 6z = 0$

$\Rightarrow y + 1 - z = 0$

$\Rightarrow y - z + 1 = 0$ which is the required equation of the plane containing the given lines

99) Find the coordinates of the foot of the perpendicular and length of the perpendicular from the point (4,3,2) to the plane x + 2y + 3z = 2

Given equation of plane is x + 2y + 3z = 2

Length of perpendicular from (4, 3, 2) to the plane is

$$d = \frac{4 + 2(3) + 3(2)}{\sqrt{1^2 + 2^2 + 3^2}} = \frac{4 + 6 + 6}{\sqrt{14}}$$

$$= \frac{14}{\sqrt{14}} = \frac{\sqrt{14}.\sqrt{14}}{\sqrt{14}} = \sqrt{14} \text{ units}$$

Let us find the image of the point (4,3,2) to the plane x + 2y + 3z = 2

Here $u = 4\hat{i} + 3\hat{j} + 2\hat{k}, n = \hat{i} + 2\hat{j} + 3\hat{k}$

Then the image $\vec{v} = \vec{u} + \frac{2[p - (u.n)]}{|\vec{n}|^2}$

$$v = \left(4\hat{i} + 3\hat{j} + 2\hat{k}\right) + \frac{2[2 - (4 + 6 + 6)]}{\left(\sqrt{1^2 + 2^2 + 3^2}\right)}\left(\hat{i} + 2\hat{j} + 3\hat{k}\right)$$

$$= \left(4\hat{i} + 3\hat{j} + 2\hat{k}\right) + \frac{2(2-16)}{14}\left(\hat{i} + 2\hat{j} + 3\hat{k}\right)$$

$$= \left(4\hat{i} + 3\hat{j} + 2\hat{k}\right) + \frac{2(-14)}{14}\left(\hat{i} + 2\hat{j} + 3\hat{k}\right)$$

$$= \left(4\hat{i} + 3\hat{j} + 2\hat{k}\right) - 2\left(\hat{i} + 2\hat{j} + 3\hat{k}\right)$$

$$= \left(4\hat{i} + 3\hat{j} + 2\hat{k}\right) - 2\left(\hat{i} + 4\hat{j} - 6\hat{k}\right)$$

$$= 2\hat{i} - \hat{j} - 4\hat{k}$$

$\therefore$ The foot of the $\perp$ from (4,3,2) to the plane is

$$\frac{\left(4\hat{i} + 3\hat{j} + 2\hat{k}\right) + \left(2\hat{i} - \hat{j} - 4\hat{k}\right)}{2}$$

$$= \frac{6\hat{i} + 2\hat{j} - 2\hat{k}}{2} = 3\hat{i} + \hat{j} - \hat{k}$$

Hence, the co-ordinates of' the foot of the perpendicular is (3, 1, -1)

100) Find the vector and Cartesian equation of the plane passing through the point (1,1, -1) and perpendicular to the planes x + 2y + 3z - 7 = 0 and 2x - 3y + 4z = 0

The normal vector to the planes

x + 2y + 3z - 7 = 0, 2x - 3y + 4z = 0 are

$\vec{b} = \hat{i} + 2\hat{j} + 3\hat{k}$ and $\vec{c} = 2\hat{i} - 3\hat{j} + 4\hat{k}$

$\therefore$ The required planes passes through the point $\vec{a} = \hat{i} + \hat{j} + \hat{k}$ and parallel to two vector 5 namely $\vec{b}$ and $\vec{c}$

$\therefore$ The Parametric form of vectors equation of the plans is $\vec{r} = \vec{a} + s\vec{b} + t\vec{c}$ s, t $\in$ R

$$\vec{r} = \left(\vec{i} + \vec{j} - \vec{k}\right) + s\left(\vec{i} + 2\vec{j} + 3\vec{k}\right) + t\left(2\vec{i} - 3\vec{j} + 4\vec{k}\right),$$

Cartesian equation is

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} x-1 & y-1 & z+1 \\ 1 & 2 & 3 \\ 2 & -3 & 4 \end{vmatrix} = 0$$

⇒ (x - 1) (8 + 9) - (y - 1)(4 - 6) + (z + 1)(-3 -4) = 0

⇒ 17 (x - 1) +2 (y - 1) -7 (z + 1) = 0

⇒ 17x - 17 + 2y - 2 - 7z - 7 = 0

⇒ 17x + 2y - 7z - 26 = 0