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11 Th Mathematics study matrials

For Students

CHAPTER 11 – Integration

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Topics:

- Basic rules of integration
- Integrals of the form $f(ax+b)$
- Properties of integrals
- Simple applications
- Methods of integration

Integration

List of formulae

1. $\int x^n \, dx = \frac{x^{n+1}}{n+1} + c$

2. $\int e^x \, dx = e^x + c$

3. $\int k \, dx = kx + c$

3. $\int \frac{1}{x} dx = \log x + c$

5. $\int \sin x \, dx = -\cos x + c$

6. $\int \cos x \, dx = \sin x + c$

7. $\int \sec^2 x \, dx = \tan x + c$

8. $\int \text{cosec}^2 x \, dx = -\cot x + c$

9. $\int \sec x \tan x \, dx = \sec x + c$

10. $\int \text{cosec}\, x \cot x \, dx = -\text{cosec}\, x + c$

11. $\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x + c$

12. $\int \frac{dx}{1+x^2} = \tan^{-1} x + c$

13. $\int \frac{dx}{x\sqrt{x^2-1}} = \sec^{-1} x + c$

14. $\int \tan x \, dx = \log(\sec x) + c$

15. $\int \cot x \, dx = \log(\sin x) + c$

16. $\int \sec x \, dx = \log(\sec x + \tan x) + c$

17. $\int \text{cosec}\, x \, dx = -\log(\text{cosec}\, x + \cot x) + c$

18. $\int \log x \, dx = x \log x - x + c$

19. $\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\left(\frac{x}{a}\right) + c$

20. $\int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + c$

21. $\int \frac{dx}{\sqrt{x^2+a^2}} = \log\left|x+\sqrt{x^2+a^2}\right| + c$

22. $\int \frac{dx}{\sqrt{x^2-a^2}} = \log\left|x+\sqrt{x^2-a^2}\right| + c$

23. $\int \frac{dx}{a^2-x^2} = \frac{1}{2a}\log\left|\frac{a+x}{a-x}\right| + c$

24. $\int \frac{dx}{x^2-a^2} = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right| + c$

25. $\int \sqrt{a^2-x^2}\, dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right) + c$

26. $\int \sqrt{x^2-a^2}\, dx = \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}\log\left|x+\sqrt{x^2-a^2}\right| + c$

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27. $\int \sqrt{x^2+a^2}\,dx = \frac{x}{2}\sqrt{x^2+a^2} + \frac{a^2}{2}\log\left|x+\sqrt{x^2+a^2}\right| + c$

28. **Integration by parts**

$$\int u\,dv = uv - \int v\,du$$

Remark 1.

To integrate the second and third powers of sine and cosine of angles use the following formulae

1. $\sin^2 x = \frac{1-\cos 2x}{2}$ 2. $\cos^2 x = \frac{1+\cos 2x}{2}$

3. $\sin^3 x = \frac{3\sin x - \sin 3x}{4}$ 4. $\cos^3 x = \frac{3\cos x + \cos 3x}{4}$

Remark 2

To integrate the products of sine and cosine of angles use the following formulae

1. $\sin A \cos B = \frac{1}{2}[\sin(A+B) + \sin(A-B)]$
2. $\cos A \sin B = \frac{1}{2}[\sin(A+B) - \sin(A-B)]$
3. $\cos A \cos B = \frac{1}{2}[\cos(A+B) + \cos(A-B)]$
4. $\sin A \sin B = \frac{1}{2}[\cos(A-B) - \cos(A+B)]$

$$\boxed{\sin^{-1}\left(\frac{2x}{1+x^2}\right) = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right) = \tan^{-1}\left(\frac{2x}{1-x^2}\right) = 2\tan^{-1}x}$$

Types of Integrals

1. $\int \frac{f'(x)}{f(x)}dx = \log|f(x)| + c$

2. $\int \frac{f'(x)}{\sqrt{f(x)}}dx = 2\sqrt{f(x)} + c$

3. $\int [f(x)]^n f'(x)\, dx = \frac{[f(x)]^{n+1}}{n+1} + c$

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problems:

1. Integrate x^{10}

Soln $I = \int x^{10}\,dx$

$= \frac{x^{11}}{11} + c$

$\int x^n\,dx = \frac{x^{n+1}}{n+1} + c.$

2. Integrate x^{11}

Soln $I = \int x^{11}\,dx$

$= \frac{x^{12}}{12} + c.$

$\int x^n\,dx = \frac{x^{n+1}}{n+1} + c$

3. Integrate $\frac{1}{x^{10}}$

Soln

$I = \int \frac{1}{x^{10}}\,dx$

$= \int x^{-10}\,dx$

$= \frac{x^{-9}}{-9} + c$

$= -\frac{1}{9x^9} + c.$

$\int x^n\,dx = \frac{x^{n+1}}{n+1} + c$

4. Integrate $\frac{1}{x^7}$

Soln $I = \int x^{-7}\,dx$

$= \frac{x^{-6}}{-6} + c$

$= -\frac{1}{6x^6} + c$

$\int x^n\,dx = \frac{x^{n+1}}{n+1} + c$

5) Integrate $\sqrt{x}$

Soln $I = \int \sqrt{x}\,dx$

$= \int x^{1/2}\,dx = \frac{x^{1/2+1}}{1/2+1} + c = \frac{x^{3/2}}{3/2} + c = \frac{2}{3}x^{3/2} + c$

6. Integrate $1/\sqrt{x}$

Soln $I = \int \frac{1}{\sqrt{x}} dx$

$= \int x^{-1/2} dx$

$= \frac{x^{-1/2+1}}{-\frac{1}{2}+1} + c$

$= \frac{x^{1/2}}{1/2} + c$

$= 2\sqrt{x} + c$

$\int x^n dx = \frac{x^{n+1}}{n+1} + c$

7. Integrate $x^{5/8}$

Soln $I = \int x^{5/8} dx + c$

$= \frac{x^{5/8+1}}{5/8+1} + c$

$= 8/13 \; x^{13/8} + c.$

$\int x^n dx = \frac{x^{n+1}}{n+1} + c$

8. Integrate $\frac{1}{\sin^2 x} d.$

Soln $I = \int \frac{1}{\sin^2 x} dx$

$= \int \operatorname{cosec}^2 x \, dx$ $\quad \because \int \operatorname{cosec}^2 x \, dx = -\cot x$

$= -\cot x + c$

9) Integrate $\frac{1}{\cos^2 x}$.

Soln $I = \int \frac{1}{\cos^2 x} dx$

$= \int \sec^2 x \, dx$ $\quad \because \int \sec^2 x \, dx = \tan x + c$

$= \tan x + c$

10) Integrate $\frac{\tan x}{\cos x}$

Soln

$$I = \int \frac{\tan x}{\cos x}\, dx$$

$$= \int \sec x \tan x\, dx$$

$$= \sec x + c$$

$\int \sec x \tan x\, dx = \sec x + c.$

11) Integrate $\frac{\cot x}{\sin x}$

Soln

$$I = \int \frac{\cot x}{\sin x}\, dx$$

$$= \int \frac{1}{\sin x} \cot x\, dx$$

$$= \int \operatorname{cosec} x \cot x\, dx$$

$$= -\operatorname{cosec} x + c$$

$\int \operatorname{cosec} x \cot x\, dx = -\operatorname{cosec} x + c$

12) Integrate $\frac{\cos x}{\sin^2 x}$

Soln

$$I = \int \frac{\cos x}{\sin x} \cdot \frac{1}{\sin x}\, dx$$

$$= \int \cot x \operatorname{cosec} x\, dx$$

$$= -\operatorname{cosec} x + c.$$

13) Integrate $\frac{\sin x}{\cos^2 x}$

Soln

$$I = \int \frac{\sin x}{\cos x} \cdot \frac{1}{\cos x}\, dx$$

$$= \int \tan x \sec x\, dx$$

$$= \int \sec x \tan x\, dx$$

$$= \sec x + c$$

$\because \int \sec x \tan x\, dx = \sec x + c.$

14) Integrate $\frac{1}{\cos^2 x} dx$

Soln $I = \int \sec^2 x \, dx$ $\quad \because \frac{1}{\cos x} = \sec x$

$= \tan x + c$ $\quad \int \sec^2 x \, dx = \tan x$.

15. Integrate 12^3

Soln $I = \int 12^3 \, dx$

$= 12^3 \int dx$ $\quad \int dx = x + c$

$= 12^3 x + c$

16) Integrate $\frac{x^{24}}{x^{25}}$

Soln $I = \int \frac{x^{24}}{x^{25}} dx$

$= \int \frac{1}{x} dx$ $\quad \int \frac{1}{x} dx = \log x$.

$= \log|x| + c$

17) Integrate $\int \frac{x^2}{x^3} dx$

Soln $I = \int \frac{x^2}{x^3} dx$

$= \int \frac{1}{x} dx$

$= \log|x| + c$

18) Integrate e^x

Soln $I = \int e^x dx = e^x + c$ $\quad \int e^x dx = e^x$

19) Integrate $1/e^{-x}$

$I = \int \frac{1}{e^{-x}} dx$

$= \int e^x dx$

$= e^x + c$

20) $\int \frac{1}{\sqrt{1-x^2}} dx = \sin^{-1}x + c$

21) $\int \frac{1}{1+x^2} dx = \tan^{-1}x + c$

22) $\int (1-x^2)^{-1/2} dx$

$= \int \frac{1}{\sqrt{1-x^2}} dx$

$= \sin^{-1}x + c.$

11.4 Integrals of the form $\int f(ax+b)\,dx$

If any constant is multiplied with the independent variable x, then the same fundamental formula can be used after dividing its by the co.eff. of x.

(ie)

If $\int f(x)\,dx = g(x) + c$

Then $\int f(ax+b)\,dx = \frac{1}{a} g(ax+b) + c$

$\therefore$

1. $\int (ax+b)^n dx = \frac{1}{a}\frac{(ax+b)^{n+1}}{n+1} + c$

2. $\int \frac{1}{ax+b} dx = \frac{1}{a}\log(ax+b) + c$

3. $\int e^{ax+b} dx = \frac{1}{a} e^{ax+b} + c$

4. $\int \sin(ax+b)\,dx = \frac{1}{a}(-\cos(ax+b)) + c$

5. $\int \cos(ax+b)\,dx = \frac{1}{a}\sin(ax+b) + c$

6. $\int \sec^2(ax+b) = \frac{1}{a}\tan(ax+b) + c$

7. $\int \operatorname{cosec}^2(ax+b)\,dx = \frac{-1}{a}\cot(ax+b) + c$

8. $\int \sec(ax+b)\tan(ax+b)\,dx = \frac{1}{a}\sec(ax+b)+c$

9. $\int \operatorname{cosec}(ax+b)\cot(ax+b)\,dx = -\frac{1}{a}\operatorname{cosec}(ax+b)+c$

10. $\int \frac{1}{1+(ax+b)^2}\,dx = \frac{1}{a}\tan^{-1}(ax+b)+c$

11. $\int \frac{1}{\sqrt{1-(ax)^2}}\,dx = \frac{1}{a}\sin^{-1}(ax)+c$

12 $\int a^x\,dx = \frac{a^x}{\log a}+c.$

Problems!

1. Integrate $(x+5)^6$

Soln $I = \int (x+5)^6\,dx$

$= \frac{(x+5)^7}{7}+c$

2. Integrate $(4x+6)^6$

Soln $I = \int (4x+6)^6\,dx$

$= \frac{1}{4}\frac{(4x+6)^7}{7}+c$

3. Integrate $\frac{1}{(2-3x)^4}$

Soln $I = \int \frac{1}{(2-3x)^4}\,dx$

$= \int (2-3x)^{-4}\,dx$

$= \frac{1}{(-3)}\frac{(2-3x)^{-3}}{-3}$

$= \frac{1}{9}(2-3x)^{-3}+c$

4) Integrate $\frac{1}{(3x+7)^4}$

Soln $I = \int \frac{1}{(3x+7)^4} dx$

$= \int (3x+7)^{-4} dx$

$= \frac{1}{3} \frac{(3x+7)^{-3}}{-3} + c$

$= -\frac{1}{9} \frac{1}{(3x+7)^3} + c.$

5) Integrate $(3x+2)^{1/2}$

Soln $I = \int (3x+2)^{1/3} dx$

$= \frac{1}{3} \frac{(3x+2)^{1/3+1}}{1/3+1} + c$

$= 1/3 \frac{(3x+2)^{4/3}}{4/3} + c$

$= \frac{1}{4} (3x+2)^{4/3} + c.$

6. Integrate $\int \sqrt{15-2x}\, dx$

Soln $I = \int \sqrt{15-2x}\, dx$

$= \int (15-2x)^{1/2} dx$

$= \frac{1}{-2} \frac{(15-2x)^{1/2+1}}{1/2+1} + c$

$= -1/2 \frac{(15-2x)^{3/2}}{3/2}$

$= -1/3 (15-2x)^{3/2}$

7) Evaluate $\int \sin 3x\, dx$

$I = \int \sin 3x\, dx$

$= \frac{-1}{3} \cos 3x dx.$

$\int \sin(ax+b) dx$

$= \frac{-1}{a} \cos(ax+b) + c$

8) $\int \sin(2x+4)\,dx$

$I = \int \sin(2x+4)\,dx$

$= \frac{-1}{2}\cos(2x+4)+c$

9) Integrate $\cos(5-11x)\,dx$

$I = \int \cos(5-11x)\,dx$

$= \frac{1}{-11}\sin(5-11x)+c.$

10) Integrate $\operatorname{cosec}^2(5x-7)$

Soln $I = \int \operatorname{cosec}^2(5x-7)\,dx$

$= \frac{1}{5}(-\cot(5x-7))+c$

$= -\frac{1}{5}\cot(5x-7)+c.$

11) Integrate $\sec^2(3+4x)\,d.$

Soln $I = \int \sec^2(3+4x)\,dx$

$= \frac{1}{4}\tan(3+4x)+c.$

12) Integrate $\sec^2 x/5$

Soln $I = \int \sec^2 x/5\,dx$

$= \frac{1}{1/5}\int \tan x/5)+c$

$= 5\tan x/5+c.$

13) Integrate $\operatorname{cosec}(5x+3)\cot(5x+3)$

Soln $I = \int \operatorname{cosec}(5x+3)\cot(5x+3)\,dx$

$= \frac{1}{5}(-\operatorname{cosec}(5x+3))+c.$

14) Integrate $30\sec(2-15x)\tan(2-15x)$

Soln
$$I = \int 30\sec(2-15x)\tan(2-15x)\,dx$$
$$= 30\left(\frac{1}{-15}\right)\sec(2-15x)+c$$
$$= -2\sec(2-15x)+c$$

$\int \sec(ax+b)\tan(ax+b)\,dx = \frac{1}{a}\sec(ax+b)+c$

15) Integrate e^{3x-6}

Soln
$$I = \int e^{3x-6}\,dx$$
$$= \frac{1}{3}e^{3x-6}+c$$

$\int e^{ax+b}\,dx = \frac{1}{a}e^{ax+b}+c$

16) Integrate e^{3x}

Soln
$$I = \int e^{3x}\,dx$$
$$= \frac{1}{3}e^{3x}+c.$$

$\int e^{ax+b}\,dx = \frac{1}{a}e^{ax+b}+c$

17) Integrate e^{8-7x}

Soln
$$I = \int e^{8-7x}\,dx$$
$$= \frac{1}{-7}e^{8-7x}+c.$$

18) Integrate e^{5-4x}

Soln
$$I = \int e^{5-4x}\,dx$$
$$= \frac{1}{-4}e^{5-4x}+c$$

19) Integrate $\frac{1}{3x-2}$

Soln
$$I = \int \frac{1}{3x-2}\,dx$$
$$= \frac{1}{3}\log|3x-2|+c$$

$\int \frac{1}{ax+b}\,dx = \frac{1}{a}\log|ax+b|+c$

20) $I = \int \frac{1}{6-4x} dx$

$\int \frac{1}{ax+b} dx = \frac{1}{a} \log|ax+b| + c$

$= \frac{1}{-4} \log|6-4x| + c$

21) $I = \int \frac{1}{5-4x} dx$

$= \frac{1}{-4} \log|5-4x| + c$

22) Integrate $\frac{1}{\sqrt{1-(4x)^2}}$

$\int \frac{1}{\sqrt{1-(ax)^2}} dx = \frac{1}{a} \sin^{-1}(ax) + c$

Soln $I = \int \frac{1}{\sqrt{1-(4x)^2}} dx$

$= \frac{1}{4} \sin^{-1} 4x + c$

23) Integrate $\frac{1}{\sqrt{1-(25x)^2}}$

Soln $I = \int \frac{1}{\sqrt{1-25x^2}} dx$

$= \int \frac{1}{\sqrt{1-(5x)^2}} dx$

$= \frac{1}{5} \sin^{-1}(5x) + c$

24) Integrate $\frac{1}{\sqrt{1-(9x)^2}}$

Soln $I = \int \frac{1}{\sqrt{1-(9x)^2}} dx$

$= \frac{1}{9} \sin^{-1}(9x) + c$

25) $I = \int \frac{1}{1+(6x)^2} dx = \frac{1}{6} \tan^{-1} 6x + c.$

26 $I = \int \frac{1}{1+(2x)^2} dx = \frac{1}{2} \tan^{-1} 2x + c$

11.5. properties of Integrals.

1. $\int k f(x) dx = k \int f(x) dx$ k - constant

2 $\int (f_1(x) + f_2(x)) dx = \int f_1(x) dx + \int f_2(x) dx$.

problems

Integrate the following w.r.t x.

1. $5x^4$

soln $I = \int 5x^4 dx$

$= 5 \int x^4 dx$

$= 5 \frac{x^5}{5} + c$

$= x^5 + c$.

2. $5x^2 - 4 + \frac{7}{x} + \frac{2}{\sqrt{x}}$

soln $I = \int (5x^2 - 4 + \frac{7}{x} + \frac{2}{\sqrt{x}}) dx$

$= 5 \int x^2 dx - 4 \int dx + 7 \int \frac{1}{x} dx + 2 \int \frac{1}{\sqrt{x}} dx$

$= \frac{5x^3}{3} - 4x + 7 \log |x| + 2 \frac{x^{-1/2+1}}{-1/2+1} + c$

$= \frac{5x^3}{3} - 4x + 7 \log |x| + 4\sqrt{x} + c$.

3) $2 \cos x - 4 \sin x + 5 \sec^2 x + \csc^2 x$

soln $I = \int (2 \cos x - 4 \sin x + 5 \sec^2 x + \csc^2 x) dx$

$= 2 \int \cos x \, dx - 4 \int \sin x \, dx + 5 \int \sec^2 x \, dx + \int \csc^2 x \, dx$

$= 2 \sin x + 4 \cos x + 5 \tan x - \cot x + c$.

4) $(x+4)^5 + \frac{5}{(2-5x)^4} - \operatorname{cosec}^2(3x-1)$

Soln $I = \int \left((x+4)^5 + \frac{5}{(2-5x)^4} - \operatorname{cosec}^2(3x-1) \right) dx$

$$= \frac{(x+4)^6}{6} + \frac{5}{(-3)(-5)} \frac{1}{(2-5x)^3} + \frac{1}{3}\cot^2(3x-1) + c$$

$$= \frac{(x+4)^6}{6} + \frac{1}{3}\frac{1}{(2-5x)^4} + \frac{1}{3}\cot^2(3x-1) + c.$$

5) $\frac{12}{(4x-5)^3} + \frac{6}{3x+2} + 16e^{4x+3}$

Soln $I = \int \left(\frac{12}{(4x-5)^3} + \frac{6}{3x+2} + 16e^{4x+3} \right) dx$

$$= 12\left(\frac{1}{4}\right)\left(\frac{-1}{2(4x-5)^2}\right) + 6\left(\frac{1}{3}\right)\log|3x+2| + \frac{16}{4}e^{4x+3} + c$$

$$= -\frac{3}{2(4x-5)^2} + 2\log|3x+2| + 4e^{4x+3} + c.$$

6. $\frac{15}{\sqrt{15x-4}} - 8\cot(4x+2)\operatorname{cosec}(4x+2)$

Soln $I = \int \left(\frac{15}{\sqrt{5x-4}} - 8\cot(4x+2)\operatorname{cosec}(4x+2) \right) dx$

$$= 15\int \frac{dx}{\sqrt{5x-4}} - 8\int \cot(4x+2)\operatorname{cosec}(4x+2)$$

$$= 15\left(\frac{1}{5}\right)(2\sqrt{5x-4}) - 8\left(\frac{1}{4}\right)(-\operatorname{cosec}(4x+2)) + c$$

$$= 6\sqrt{5x-4} + 2\operatorname{cosec}(4x+2) + c.$$

7. Integrate $\left(4\cos(5-2x) + 9e^{3x-6} + \frac{24}{6-4x}\right)$

Soln

$$I = \int\left(4\cos(5-2x) + 9e^{3x-6} + \frac{24}{6-4x}\right)dx$$

$$= \frac{4}{-2}\sin(5-2x) + \frac{9}{3}e^{3x-6} + \frac{24}{-4}\log|6-4x| + c$$

$$= -2\sin(5-2x) + 3e^{3x-6} - 6\log|6-4x| + c.$$

8. Integrate $\sec^2 x/5 + 18\cos 2x + 10\sec(5x+3)\tan(5x+3)$

Soln

$$I = \int\left(\sec^2 x/5 + 18\cos 2x + 10\sec(5x+3)\tan(5x+3)\right)dx$$

$$= \frac{1}{1/5}\tan x/5 + \frac{18}{2}\sin 2x + \frac{10}{5}\sec(5x+3) + c$$

$$= 5\tan x/5 + 9\sin 2x + 2\sec(5x+3) + c.$$

9. Integrate $\frac{8}{\sqrt{1-(4x)^2}} + \frac{27}{\sqrt{1-(3x)^2}} - \frac{15}{1+(5x)^2}$

Soln

$$I = \int\left(\frac{8}{\sqrt{1-(4x)^2}} + \frac{27}{\sqrt{1-(3x)^2}} - \frac{15}{1+(5x)^2}\right)dx$$

$$= \frac{8}{-4}\sin^{-1}4x + \frac{27}{3}\sin^{-1}3x - \frac{15}{5}\tan^{-1}5x + c$$

$$= 2\sin^{-1}4x + 9\sin^{-1}3x - 3\tan^{-1}5x + c$$

10. Integrate $\frac{6}{1+(3x+2)^2} - \frac{12}{\sqrt{1-(3-4x)^2}}$

Soln

$$I = \int\left(\frac{6}{1+(3x+2)^2} - \frac{12}{\sqrt{1-(3-4x)^2}}\right)dx$$

$$= \frac{6}{3}\tan^{-1}(3x+2) - \frac{12}{-4}\sin^{-1}(3-4x) + c$$

$$= 2\tan^{-1}(3x+2) + 3\sin^{-1}(3-4x) + c$$

11) Integrate $\frac{1}{3}\cos\left(\frac{x}{3}-4\right) + \frac{7}{7x+9} + e^{x/5+3}$

<u>Soln</u>

$$I = \int \left(\frac{1}{3}\cos\left(\frac{x}{3}-4\right) + \frac{7}{7x+9} + e^{x/5+3}\right)dx$$

$$= \frac{1}{3}\left(\frac{1}{1/3}\right)\sin\left(\frac{x}{3}-4\right) + \frac{7}{7}\log|7x+9| + \frac{1}{(1/5)}e^{x/5+3} + c$$

$$= \sin\left(\frac{x}{3}-4\right) + \log|7x+9| + 5e^{x/5+3} + c$$

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11.6 Simple applications.

1. If $f'(x) = 3x^2 - 4x + 5$ and $f(1) = 3$ find $f(x)$

Soln

$f'(x) = 3x^2 - 4x + 5$

Integrate on both sides

$\int f'(x)\,dx = \int (3x^2 - 4x + 5)\,dx$ $\qquad \int f'(x)\,dx = f(x)$

$f(x) = \frac{3x^3}{3} - \frac{4x^2}{2} + 5x + c$

$f(x) = x^3 - 2x^2 + 5x + c \longrightarrow (1)$

$f(1) = 3$

$1 - 2 + 5 + c = 3$

$4 + c = 3$

$c = 3 - 4$

$c = -1$

Using this in (1)

$f(x) = x^3 - 2x^2 + 5x - 1$

2) If $f'(x) = 4x - 5$ & $f(2) = 1$ find $f(x)$

Soln

$f'(x) = 4x - 5$

Integrate on both sides

$\int f'(x)\,dx = \int (4x - 5)\,dx$

$f(x) = \frac{4x^2}{2} - 5x + c$

$f(x) = 2x^2 - 5x + c \longrightarrow (1)$

$f(2) = 1$

$8 - 10 + c = 1$

$-2 + c = 1$

$c = 3$

(1) $\Rightarrow$ $f(x) = 2x^2 - 5x + 3$

3) If $f'(x) = 9x^2 - 6x$ & $f(0) = -3$ find $f(x)$

Soln

$$f'(x) = 9x^2 - 6x$$

$$\int f'(x)dx = \int (9x^2 - 6x)\,dx$$

$$f(x) = \frac{9x^3}{3} - \frac{6x^2}{2} + c$$

$$f(x) = 3x^3 - 3x^2 + c \longrightarrow (1)$$

$$f(0) = -3$$

$$0 - 0 + c = -3$$

$$c = -3$$

$$(1) \Rightarrow f(x) = 3x^3 - 3x^2 - 3.$$

---

(4) If $f''(x)$ ~~(x)~~ $12x - 6$ & $f(1) = 30$, $f'(1) = 5$ find $f(x)$

Soln

$$f''(x) = 12x - 6$$

$$\int f''(x)\,dx = \int (12x - 6)\,dx$$

$$f'(x) = \frac{12x^2}{2} - 6x + c$$

$$f'(x) = 6x^2 - 6x + c \longrightarrow (1)$$

$$f'(1) = 5$$

$$6 - 6 + c = 5$$

$$\cancel{f'(1)}\ c = 5$$

$$(1) \Rightarrow f'(x) = 6x^2 - 6x + 5$$

$$\int f'(x)\,dx = \int (6x^2 - 6x + 5)\,dx$$

$$f(x) = \frac{6x^3}{3} - \frac{6x^2}{2} + 5x + k$$

$$f(x) = 2x^3 - 3x^2 + 5x + k \longrightarrow (2)$$

$$f(1) = 30$$

$$2 - 3 + 5 + k = 30$$

$$4 + k = 30$$

$$k = 26$$

$$(2) \Rightarrow f(x) = 2x^3 - 3x^2 + 5x + 26$$

11.7 methods of integration

The following are four important methods of integration.

1. Integration by decomposition into sum or differences.
2. Integration by substitution.
3. Integration by parts.
4. Integration by successive reduction.

Decomposition methods:

problems:

1. Integrate $(1-x^3)^2$

Soln

$$I = \int (1-x^3)^2 dx \qquad (a-b)^2 = a^2+b^2-2ab$$

$$= \int (1+x^6-2x^3) dx$$

$$= x + \frac{x^7}{7} - \frac{2x^4}{4} + c$$

$$= x + \frac{x^7}{7} - \frac{x^4}{2} + c.$$

2. Integrate $\frac{x^2-x+1}{x^3}$

Soln

$$I = \int \frac{x^2-x+1}{x^3} dx$$

$$= \int \left(\frac{1}{x} - \frac{1}{x^2} + \frac{1}{x^3} \right) dx$$

$$= \log|x| + \frac{1}{x} - \frac{1}{2x^2} + c$$

3. integrate $\frac{x^3+4x^2-3x+2}{x^2}$

Soln

$$I = \int \frac{x^3+4x^2-3x+2}{x^2} dx$$

$$= \int \left(x+4 - \frac{3}{x} + \frac{2}{x} \right) dx$$

$$= \frac{x^2}{2} + 4x - 3\log|x| - \frac{2}{x} + c.$$

4. Integrate $\left(\sqrt{x} + \frac{1}{\sqrt{x}}\right)^2$

$$I = \int \left(\sqrt{x} + \frac{1}{\sqrt{x}}\right)^2 dx$$

$$= \int \left(x + \frac{1}{x} + 2\right) dx$$

$$= \frac{x^2}{2} + \log|x| + 2x + c.$$

5. Integrate $(2x-5)(36+4x)$

Soln

$$I = \int (2x-5)(36+4x)\, dx$$

$$= \int (72x + 8x^2 - 180 - 20x)\, dx$$

$$= \int (62x + 8x^2 - 180)\, dx$$

$$= 62\frac{x^2}{2} + \frac{8x^3}{3} - 180x + c$$

$$= 31x^2 + \frac{8}{3}x^3 - 180x + c$$

6. Integrate $\cos 5x \sin 3x$

$$I = \int \cos 5x \sin 3x\, dx$$

$$= \frac{1}{2}\int (\sin 8x - \sin 2x)\, dx$$

$$= \frac{1}{2}\left(\frac{-\cos 8x}{8} + \frac{\cos 2x}{2}\right) + c$$

$\cos A \sin B$

$= \frac{1}{2}(\sin(A+B) - \sin(A-B))$

7) Integrate $\cos^3 x$

Soln

$$I = \int \cos^3 x\, dx$$

$$= \frac{1}{4}\int (3\cos x + \cos 3x)\, dx$$

$$= \frac{1}{4}\left(3\sin x + \frac{\sin 3x}{3}\right) + c.$$

$\cos^3 x = \frac{1}{4}(3\cos x + \cos 3x)$

11.7.3 Change of variable:

Evaluate the following integrals:

1. $\int 2x\sqrt{1+x^2}\,dx$

Soln $I = \int 2x\sqrt{1+x^2}\,dx$

put
$1+x^2 = u$
$2x\,dx = du$.

$= \int \sqrt{u}\,du$

$= \int u^{1/2}\,du$

$= \frac{u^{3/2}}{3/2} + c$

$= \frac{2}{3}u^{3/2} + c$

$= \frac{2}{3}(1+x^2)^{3/2} + c.$

2. $\int \frac{x}{\sqrt{1+x^2}}\,dx$

Soln $I = \int \frac{x}{\sqrt{1+x^2}}\,dx$

$1+x^2 = u$
$2x\,dx = du$
$x\,dx = \frac{du}{2}$

$= \frac{1}{2}\int \frac{1}{\sqrt{u}}\,du.$

$= \frac{1}{2}\int u^{-1/2}\,du$

$= \frac{1}{2}\,\frac{u^{1/2}}{1/2} + c$

$= u^{1/2} + c$

$= \sqrt{1+x^2} + c.$

3. $\int \frac{1}{1+x^2}\,dx$

Soln $I = \int \frac{1}{1+x^2}\,dx$

$x = \tan u$
$dx = \sec^2 u\,du.$

$= \int \frac{\sec^2 x}{1+\tan^2 u}\,du.$

$= \int \frac{\sec^2 x}{\sec^2 x}\,du.$

$= \int du = u + c = \tan^{-1}x + c.$

4. $\int \frac{x^2}{1+x^6}\,dx$

Soln $I = \int \frac{x^2}{1+x^6}\,dx$

$x^3 = t$

$3x^2 dx = dt$

$x^2 dx = \frac{dt}{3}$

$= \frac{1}{3}\int \frac{dt}{1+t^2}$

$= \frac{1}{3}\tan^{-1}t + c$

$= \frac{1}{3}\tan^{-1}x^3 + c$

5. $\int x(a-x)^8\,dx$

Soln $I = \int x(a-x)^8\,dx$

$a-x = t$ | $a-t = x$

$-dx = dt$

$dx = -dt$

$= -\int (a-t)t^8\,dt$

$= \int (t-a)t^8\,dt$

$= \int (t^9 - at^8)\,dt$

$= \frac{t^{10}}{10} - \frac{at^9}{9} + c$

$= \frac{(a-x)^{10}}{10} - \frac{a(a-x)^9}{9} + c.$

6. $\int \frac{\sin\sqrt{x}}{\sqrt{x}}\,dx$

Soln $I = \int \frac{\sin\sqrt{x}}{\sqrt{x}}\,dx$

$t = \sqrt{x}$

$dt = \frac{1}{2\sqrt{x}}\,dx$

$2dt = \frac{1}{\sqrt{x}}\,dx$

$= 2\int \sin t\,dt$

$= 2(-\cos t) + c$

$= -2\cos\sqrt{x} + c.$

7) $\int \frac{\sin^{-1}x}{\sqrt{1-x^2}}\,dx$

$I = \int \frac{\sin^{-1}x}{\sqrt{1-x^2}}\,dx$

$t = \sin^{-1}x$

$dt = \frac{1}{\sqrt{1-x^2}}\,dx.$

$= \int t\,dt$

$= \int t\,dt$

$= \frac{t^2}{2} + c$

$= \frac{(\sin^{-1}x)^2}{2} + c$

8. $\int \frac{\sqrt{x}}{1+\sqrt{x}}\,dx$

soln $I = \int \frac{\sqrt{x}}{1+\sqrt{x}}\,dx$

$1+\sqrt{x} = t$

$\sqrt{x} = t-1$

$x = (t-1)^2$

$dx = 2t-2$

$= \int \frac{t-1}{1+t-1}(2t-2)\,dt$

$= 2\int \frac{t^2-1-2t}{t}\,dt$

$= 2\int \left(t-\frac{1}{t}-2\right)dt$

$= 2\left(\frac{t^2}{2} - \log t - 2t\right) + c$

$= t^2 - 2\log|t| - 4t + c$

$= (1+\sqrt{x})^2 - 2\log|1+\sqrt{x}| - 4(1+\sqrt{x}) + c.$

9) $\int x(1-x)^{17}\,dx$

soln $I = \int x(1-x)^{17}\,dx$

$1-x = t$

$-dx = dt$

$= \int (\cancel{t-1})\, t^{17}(-dt)$

$= \int (t-1)\,t^{17}(dt)$

$= \int (t^{18} - t^{17})\,dt$

$= \frac{t^{19}}{19} - \frac{t^{18}}{18} + c$

$= \frac{(1-x)^{19}}{19} - \frac{(1-x)^{18}}{18} + c.$

10 $\int \sin^5 x \cos^3 x\, dx$

Soln $I = \int \sin^5 x \cos^3 x\, dx$

$= \int \sin^5 x (1-\sin^2 x) \cos x\, dx$ $\quad \because 1-\sin^2 x = \cos^2 x$

$= \int (\sin^5 x - \sin^7 x) \cos x\, dx$ $\quad \sin x = t$, $\cos x\, dx = dt$

$= \int (t^5 - t^7)\, dt$

$= \frac{t^6}{6} - \frac{t^8}{8} + c$

$= \frac{\sin^6 x}{6} - \frac{\sin^8 x}{8} + c$

11) $\int \tan x \sqrt{\sec x}\, dx$

Soln $I = \int \tan \sqrt{\sec x}\, dx$ $\quad t = \sec x$, $dt = \sec x \tan x\, dx$, $\frac{dt}{\sec x} = \tan x\, dx$, $\frac{dt}{t} = \tan x\, dx$

$= \int \sqrt{t} \frac{dt}{t}$

$= \int \frac{1}{\sqrt{t}} dt$

$= 2\sqrt{t} + c$

$= 2\sqrt{\sec x} + c$

12) $\int \alpha \beta x^{\alpha-1} e^{-\beta x^\alpha} dx$

Soln $I = \int \alpha \beta x^{\alpha-1} e^{-\beta x^\alpha} dx$ $\quad t = x^\alpha$, $dt = \alpha x^{\alpha-1} dx$

$= \int \beta e^{-\beta t} dt$

$= \beta \frac{e^{-\beta t}}{-\beta} + c$

$= -e^{-\beta t} + c$

$= -e^{-\beta x^\alpha} + c.$

Important Results

1. $\int \frac{f'(x)}{f(x)} dx = \log|f(x)| + c$

2. $\int f'(x)[f(x)]^n dx = \frac{[f(x)]^{n+1}}{n+1} + c.$

problems:

1. $\int \tan x \, dx$

soln

$I = \int \tan x \, dx$

$= \int \frac{\sin x}{\cos x} dx$

$= -\int \frac{-\sin x}{\cos x} dx$

$= -\log|\cos x| + c$

$= \log|\sec x| + c.$

$f(x) = \cos x$

$f'(x) = -\sin x$

$\int \frac{f'(x)}{f(x)} dx = \log|f(x)| + c.$

2. $\int \cot x \, dx$

soln

$I = \int \cot x \, dx$

$= \int \frac{\cos x}{\sin x} dx$

$= \log \sin x + c.$

$f(x) = \sin x$

$f'(x) = \cos x$

$\int \frac{f'(x)}{f(x)} dx = \log|f(x)| + c$

3. $\int \operatorname{cosec} x \, dx$

soln

$I = \int \operatorname{cosec} x \, dx$

$= \int \frac{\operatorname{cosec} x . (\operatorname{cosec} x - \cot x)}{\operatorname{cosec} x - \cot x} dx$

$= \int \frac{\operatorname{cosec}^2 x - \cot x \operatorname{cosec} x}{\operatorname{cosec} x - \cot x}$

$f(x) = \operatorname{cosec} x - \cot x$

$f'(x) = \operatorname{cosec}^2 x - \operatorname{cosec} x \cot x$

$\int \frac{f'(x)}{f(x)} dx = \log|f(x)| + c$

$= \log|\operatorname{cosec}x - \cot x| + c.$

4) $\int \sec x \, dx$

Soln $I = \int \sec x \, dx$

$= \int \frac{\sec x(\sec x + \tan x)}{\sec x + \tan x} dx$

$= \int \frac{\sec^2 x + \sec x \tan x}{(\sec x + \tan x)} dx$

$f(x) = \tan x + \sec x$

$f'(x) = \sec^2 x + \sec x \tan x$

$= \log|\sec x + \tan x| + c.$

$\int \frac{f'(x)}{f(x)} dx = \log|f(x)| + c$

5) Important formula's

1. $\int \tan x \, dx = \log|\sec x| + c$
2. $\int \cot x \, dx = \log|\sin x| + c$
3. $\int \operatorname{cosec} x \, dx = \log|\operatorname{cosec}x - \cot x| + c$
4. $\int \sec x \, dx = \log|\sec x + \tan x| + c.$

Problems:

1. $\int \frac{2x+4}{x^2+4x+6} dx$

Soln $I = \int \frac{2x+4}{x^2+4x+6} dx$

$f(x) = x^2 + 4x + 6$

$f'(x) = 2x + 4$

$= \log|x^2+4x+6| + c.$

$\int \frac{f'(x)}{f(x)} dx = \log|f(x)| + c$

2. $\int \frac{e^x}{e^x - 1} dx$

$I = \int \frac{e^x}{e^x - 1} dx$

$= \log|e^x - 1| + c.$

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3. $\int \frac{10x^9 + 10^x \log_e 10}{10^x + x^{10}} dx$.

Soln

$$I = \int \frac{10x^9 + 10^x \log_e 10}{10^x + x^{10}} dx$$

$$= \log |10^x + x^{10}| + c.$$

$f(x) = x^{10} + 10^x$

$f'(x) = 10x^9 + 10^x \log_e 10$.

4. $\int \frac{e^x - e^{-x}}{e^x + e^{-x}} dx$

Soln

$$I = \int \frac{e^x - e^{-x}}{e^x + e^{-x}} dx$$

$$= \log |e^x + e^{-x}| + c$$

$f(x) = e^x + e^{-x}$

$f'(x) = e^x - e^{-x}$

5. $\int \frac{\cot x}{\log \sin x} dx$

Soln

$$I = \int \frac{\cot x}{\log \sin x} dx$$

$$= \log |\log \sin x| + c$$

$f(x) = \cancel{\cot x}\ \log \sin x$

$f'(x) = \frac{1}{\sin x} \cdot \cos x$

$= \cot x$

6. $\int \frac{1}{x \log x} dx$

$$I = \int \frac{1}{x \log x}$$

$$= \int \frac{1/x}{\log x} dx$$

$$= \log |\log x| + c.$$

$f(x) = \log x$

$f'(x) = 1/x$

$\int \frac{f'(x)}{f(x)} dx = \log |f(x)| + c$

7) $\int \frac{\cos 2x}{(\sin x+\cos x)^2}\,dx$

Soln $I = \int \frac{\cos 2x}{(\cos x+\sin x)^2}$

$(\cos x+\sin x)^2$
$= \cos^2 x + \sin^2 x + 2\sin x\cos x$
$= 1+\sin 2x$

$= \int \frac{\cos 2x}{1+\sin 2x}\,dx$

$= \frac{1}{2}\int \frac{2\cos 2x}{1+\sin 2x}\,dx$

$= \frac{1}{2}\log|1+\sin 2x|+c$

8) $\int \frac{\sin 2x}{a^2+b^2\sin^2 x}\,dx$

$I = \int \frac{\sin 2x}{a^2+b^2\sin^2 x}\,dx$

$f(x) = a^2+b^2\sin^2 x$
$f'(x) = 0+b^2(2\sin x\cos x)$
$= b^2\sin 2x$

$= \frac{1}{b^2}\int \frac{b^2\sin 2x}{a^2+b^2\sin^2 x}\,dx$

$= \frac{1}{b}\log|a^2+b^2\sin^2 x|+c$

9) $\int \frac{1}{x\log x\log(\log x)}\,dx$

$= \int \frac{1}{x\log x\log(\log x)}\,dx$

$= \int \frac{\frac{1}{x\log x}}{\log\log x}\,dx$

$f(x) = \log\log x$
$f'(x) = \frac{1}{\log x}\cdot\frac{1}{x}$

$= \log|\log\log x|+c.$

10) $\int \frac{\cos x}{\cos(x-a)}\,dx$

$= \int \frac{\cos[x-a+a]}{\cos(x-a)}\,dx$

$= \int \frac{\cos(x-a)\cos a + \sin(x-a)\sin a}{\cos(x-a)}\,dx$

$= \int (\cos a + \tan(x-a)\sin a)\,dx$

$= x\cos a + \sin a \int \tan(x-a)\,dx$

$= x\cos a + \sin a \log|\sec(x-a)| + c.$

Integration by parts

$\int u\,dv = uv - \int v\,du.$

The success of this methods on the proper choice of u.

1. If integrand contains any non integrable functions directly from the formula like $\log x$, $\tan^{-1}x$ etc we have to take these non integrable function as u and other as dv.
2. If integrand contains both integrable function, and one of these is x^n (n·+ve integer) then take $u = x^n$.
3. For other case choice of u is ours.

Problems:

1. $\int x\cos x\,dx$

Soln

$I = \int x\cos x\,dx$ $\quad u = x \quad dv = \cos x$

$= uv - \int v\,du$ $\quad du = dx \quad v = \sin x$

$= x\sin x - \int \sin x\,dx$

$\int \sin x\,dx = -\cos x.$

$= x\sin x + \cos x + c$

2. $\int \log x\,dx$

$I = \int \log x\,dx$ $\quad u = \log x \quad dv = dx$

$= uv - \int v\,du$ $\quad du = \frac{1}{x}dx \quad v = x$

$= x\log x - \int x\frac{1}{x}dx$

$= x\log x - \int dx$

$= x\log x - x + c.$

3) $\int \sin^{-1}x \, dx$

$I = \int \sin^{-1}x \, dx$

$= uv - \int v\,du$

$u = \sin^{-1}x \qquad dv = dx$

$du = \frac{1}{\sqrt{1-x^2}}dx \qquad v = x$

$= x\sin^{-1}x - \int \frac{x}{\sqrt{1-x^2}}dx$

$= x\sin^{-1}x - \int \frac{-dt}{2\sqrt{t}}$

$1-x^2 = t$

$-2x\,dx = dt$

$x\,dx = -\frac{dt}{2}$

$= x\sin^{-1}x + \frac{1}{2}\int \frac{dt}{\sqrt{t}}$

$= x\sin^{-1}x + \frac{1}{2} 2\sqrt{t} + c$

$= x\sin^{-1}x + \sqrt{1-x^2} + c.$

4) $\int \tan^{-1}\left(\frac{2x}{1-x^2}\right)dx$

$I = \int \tan^{-1}\left(\frac{2x}{1-x^2}\right)dx$

$= uv - \int v\,du$

$x = \tan\theta$

$dx = \sec^2\theta \, d\theta.$

$= \int \tan^{-1}\left(\frac{2\tan\theta}{1-\tan^2\theta}\right) \sec^2\theta\, d\theta$

$= \int \tan^{-1}(\tan 2\theta)(\sec^2\theta)\, d\theta$

$= \int 2\theta \sec^2\theta\, d\theta$

$= 2\int \theta \sec^2\theta\, d\theta$

$u = \theta \qquad dv = \sec^2\theta$

$du = d\theta \qquad v = \tan\theta$

$= 2\left[\theta\tan\theta - \int \tan\theta\, d\theta\right]$

$= 2\left[\theta\tan\theta - \log|\sec\theta|\right] + c$

$= 2\left[\tan^{-1}x \tan^{-1}\tan x - \log|\sec\tan^{-1}x)\right] + c$

$= 2\left(x\tan^{-1}x - 2\log\left|\sqrt{1+x^2}\right|\right) + c.$

Bernoullis formula for integration by parts

$$\int u\,dv = uv - u'v_1 + u''v_2 - \dots$$

u', u'', u''' are successive derivatives of u.

v, v_1, v_2, v_3 are " integrals of dv.

problems:

1. $\int x e^x dx$

Soln

$dv = e^x$

$u = x \quad v = e^x$

$u' = 1 \quad v_1 = e^x$

$$I = \int x e^x dx$$

$$= uv - u'v_1 + u''v_2 - \dots$$

$$= xe^x - (1)e^x + c$$

$$= xe^x - e^x + c.$$

2. $\int x^2 e^{5x} dx$

$dv = e^{5x}$

$u = x^2 \quad v = e^{5x}/5$

$u' = 2x \quad v_1 = e^{5x}/25$

$u'' = 2 \quad v_2 = e^{5x}/125$

$u''' = 0$

$$I = \int x^2 e^{5x} dx$$

$$= uv - u'v_1 + u''v_2 - u'''v_3 + \dots$$

$$= x\frac{e^{5x}}{5} - 2x\frac{e^{5x}}{5^2} + 2\frac{e^{5x}}{5^3} + c$$

3. $\int x^3 \cos x\, dx$

$dv = \cos x$

$u = x^3 \quad v = \sin x$

$u' = 3x^2 \quad v_1 = -\cos x$

$u'' = 6x \quad v_2 = -\sin x$

$u''' = 6 \quad v_3 = \cos x$

$u^{iv} = 0$

$$I = \int x^3 \cancel{e^x} \cos x\, dx$$

$$= uv - u'v_1 + u''v_2 - u'''v_3 + \dots$$

$$= x^3 \sin x - 3x^2(\cos x) + 6x(-\sin x) - 6\cos x$$

$$= x^3 \sin x + 3x^2 \cos x - 6x \sin x - 6\cos x + c.$$

4) $\int x^3 e^{-x} dx$

$I = \int x^3 e^{-x} dx$

$= uv - u'v_1 + u''v_2 - u'''v_3 + \cdots$

$= x^3\left(\frac{e^{-x}}{-1}\right) - 3x^2 e^{-x} + 6x(-e^{-x}) - 6(e^{-x}) + c$

$= -x^3 e^{-x} - 3x^2 e^{-x} - 6x e^{-x} - 6e^{-x} + c$

5) $\int 9x e^{3x} dx$

Soln

$I = \int 9x e^{3x} dx$

$= 9\int x e^{3x} dx$

$= 9\left(x\frac{e^{3x}}{3} - (1)\frac{e^{3x}}{9}\right) + c$

$= 3x e^{3x} - e^{3x} + c.$

6) $\int x \sin 3x\, dx$

$I = \int x \sin 3x\, dx$

$= x\left(\frac{-\cos 3x}{3}\right) - (1)\left(\frac{-\sin 3x}{9}\right) + c$

$= \frac{-x\cos 3x}{3} + \frac{\sin 3x}{9} + c.$

7) $\int 25x e^{-5x} dx$

$I = \int 25x e^{-5x} dx$

$= 25\int x e^{-5x} dx$

$= 25\left(x\left(\frac{e^{-5x}}{-5}\right) - (1)\left(\frac{e^{-5x}}{25}\right)\right) + c$

$= -5x e^{-5x} - e^{-5x} + c.$

8) $\int 27x^2 e^{3x} dx$

$I = \int 27x^2 e^{3x} dx$

$= 27\int x^2 e^{3x} dx$

$= 27\left(x^2 \frac{e^{3x}}{3} - 2x\frac{e^{3x}}{9} + 2\frac{e^{3x}}{27}\right) + c$

$= 9x^2 e^{3x} - 6x e^{3x} + 2e^{3x} + c$

9) $\int x^2 \cos x \, dx$

Soln

$I = \int x^2 \cos x \, dx$

$= x^2(\sin x) - 2x(-\cos x) + 2(-\sin x) + c$

$= x^2 \sin x + 2x\cos x - 2\sin x + c$

10) $\int x^3 \sin x \, dx$

$I = \int x^3 \sin x \, dx$

$= x^3(-\cos x) - 3x^2(-\sin x) + 6x(\cos x) - 6\sin x + c$

$= -x^3\cos x + 3x^2 \sin x + 6x\cos x - 6\sin x + c$

11) $\int \frac{x\sin^{-1}x}{\sqrt{1-x^2}} dx$

Soln

$I = \int \frac{x\sin^{-1}x}{\sqrt{1-x^2}} dx$ $t = \sin^{-1}x$

$dt = \frac{1}{\sqrt{1-x^2}} dx$

$= \int t \sin t \, dt$

$= t(-\cos t) - (1)(-\sin t)$

$= -t\cos t + \sin t + c$

$= -\sin^{-1}x \cos t + \sin \sin^{-1}x + c$

$= -\sin^{-1}x\sqrt{1-\sin^2 t} + x + c = -\sin^{-1}x\sqrt{1-x^2} + x + c$

12) $\int x^5 e^{x^2} dx$

Soln $I = \int x^5 e^{x^2} dx$

$x^2 = t$

$2x\,dx = dt$

$x\,dx = \frac{dt}{2}$

$= \int t^2 e^t \frac{dt}{2}$

$= \frac{1}{2} \int t^2 e^t dt$

$= \frac{1}{2} (t^2 e^t - 2t e^t + 2e^t) + c$

$= \frac{1}{2} (x^4 e^{x^2} - 2x^2 e^{x^2} + 2e^{x^2}) + c.$

$= \frac{1}{2} e^{x^2} (x^4 - 2x^2 + 2) + c.$

13) $\int x \log x\, dx$

$I = \int x \log x\, dx$

$u = \log x$ $dv = x$

$v = \frac{x^2}{2}$

$= uv - \int v\,du$

$= \log x \left(\frac{x^2}{2}\right) - \int \frac{x^2}{2}\left(\frac{1}{x}\right) dx$

$= \frac{x^2}{2} \log x - \int \frac{x}{2} dx$

$= \frac{x^2}{2} \log x - \frac{x^2}{4} + c.$

11.7.8 Integrals of the form

(i) $\int e^{ax}\sin bx\,dx$

(ii) $\int e^{ax}\cos bx\,dx$

Result:

(i) $\int e^{ax}\sin bx\,dx = \frac{e^{ax}}{a^2+b^2}[a\sin bx - b\cos bx]+c$

(ii) $\int e^{ax}\cos bx\,dx = \frac{e^{ax}}{a^2+b^2}[a\cos bx + b\sin bx]+c$.

proof (i)

$I = \int e^{ax}\sin bx\,dx \longrightarrow (A)$

$u = e^{ax}$ $\quad dv = \sin bx$

$du = ae^{ax}dx$ $\quad \int dv = \int \sin bx$

$v = \frac{-\cos bx}{b}$

$= \int u\,dv$

$= uv - \int v\,du$

$= \frac{-e^{ax}\cos bx}{b} - \int \frac{ae^{ax}\cos bx}{-b}\,dx$

$I = \frac{-e^{ax}\cos bx}{b} + \frac{a}{b}\int e^{ax}\cos bx\,dx \longrightarrow ①$

$u = e^{ax}$ $\quad dv = \cos bx$

$du = ae^{ax}dx$ $\quad \int dv = \int \cos bx\,dx$

$v = \frac{\sin bx}{b}$

$= -\frac{e^{ax}\cos bx}{b} + \frac{a}{b}\left(uv - \int v\,du\right)$

$= -\frac{e^{-ax}\cos bx}{b} + \frac{a}{b}\left(\frac{e^{ax}\sin bx}{b} - \int \frac{ae^{ax}\sin bx}{b}\,dx\right)$

$= -\frac{e^{ax}\cos bx}{b} + \frac{a}{b^2}e^{ax}\sin bx - \frac{a^2}{b^2}\int e^{ax}\sin bx\,dx$

$I = -\frac{e^{2ax}\cos bx}{b} + \frac{a^2}{b^2}e^{ax}\sin bx - \frac{a^2}{b^2}I$

$I + \frac{a^2}{b^2}I = \frac{a}{b^2}e^{ax}\sin bx - \frac{e^{ax}\cos bx}{b}$

$\left(1+\frac{a^2}{b^2}\right)I = \frac{e^{ax}}{b^2}(a\sin bx - b\cos bx)$

$$\frac{(a^2+b^2)}{b^2} I = \frac{e^{ax}}{b^2}(a\sin bx - b\cos bx) + c$$

$$I = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx) + c.$$

Proof : 2

$$I = \int e^{ax}\cos bx\,dx \longrightarrow (1)$$

$$= \int u\,dv \qquad u = e^{ax} \quad du = a e^{ax}\,dx \qquad dv = \cos bx \quad v = \frac{\sin bx}{b}$$

$$= uv - \int v\,du.$$

$$= \frac{e^{ax}\sin bx}{b} - \int \frac{a e^{ax}\sin bx}{b}\,dx$$

$$= \frac{e^{ax}\sin bx}{b} - \frac{a}{b}\int e^{ax}\sin bx\,dx$$

$$= \frac{e^{ax}\sin bx}{b} - \frac{a}{b}\left(uv - \int v\,du\right) \qquad u = e^{ax} \quad du = a e^{ax}\,dx \qquad dv = \sin bx \quad v = \frac{-\cos bx}{b}$$

$$= \frac{e^{ax}\sin bx}{b} - \frac{a}{b}\left(\frac{e^{ax}\cos bx}{-b} + \int \frac{\cos bx}{b}\left(a e^{ax}\,dx\right)\right)$$

$$= \frac{e^{ax}\sin bx}{b} + \frac{a}{b^2}e^{ax}\cos bx - \frac{a^2}{b^2}\int e^{ax}\cos bx\,dx$$

$$= \frac{1}{b^2}\left(b e^{ax}\sin bx + a\cos bx\right) - \frac{a^2}{b^2} I \qquad \text{by } (1)$$

$$I = \frac{e^{ax}}{b^2}\left(b\sin bx + a\cos bx\right) - \frac{a^2}{b^2} I$$

$$I + \frac{a^2}{b^2} I = \frac{e^{ax}}{b^2}\left(a\cos bx + b\sin bx\right)$$

$$\left(\frac{a^2+b^2}{b^2}\right) I = \frac{e^{ax}}{b^2}\left(a\cos bx + b\sin bx\right)$$

$$I = \frac{e^{ax}}{a^2+b^2}\left(a\cos bx + b\sin bx\right)$$

1. $\int e^{2x}\sin x\,dx$

Soln

$I = \int e^{2x}\sin x\,dx$

$\int e^{ax}\sin bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx)+c$

$= \frac{e^{2x}}{(2)^2+1^2}(2\sin x-\cos x)+c$

$= \frac{e^{2x}}{4+1}(2\sin x-\cos x)+c$

$= \frac{e^{2x}}{5}(2\sin x-\cos x)+c$

2) $\int e^{-3x}\sin 2x\,dx$

$I = \int e^{-3x}\sin 2x\,dx$

$= \frac{e^{-3x}}{(-3)^2+(2)^2}(-3\sin 2x - 2\cos 2x)+c$

$= \frac{-e^{-3x}}{13}(3\sin 3x+2\cos 2x)+c.$

3. $I = \int e^{-4x}\sin 2x\,dx$

Soln

$I = \int e^{-4x}\sin 2x\,dx$

$= \frac{e^{-4x}}{(-4)^2+(2)^2}((-4)\sin 2x - 2\cos 2x)+c$

$= \frac{e^{-4x}}{20}(-4\sin 2x-2\cos 2x)+c$

$= \frac{-e^{-4x}}{10}(2\sin 2x+\cos 2x)+c.$

4) $\int e^{-5x} \sin 3x\,dx$

Soln

$$I = \int e^{-5x} \sin 3x\,dx$$

$$= \frac{e^{-5x}}{(-5)^2+(3)^2}(-5\sin 3x - 3\cos 3x) + c$$

$$= -\frac{e^{-5x}}{34}(5\sin 3x + 3\cos 3x) + c$$

5) $\int e^{3x} \cos 2x\,dx$

$$\int e^{ax} \sin bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx) + c$$

$$I = \int e^{3x} \cos 2x\,dx$$

$$= \frac{e^{3x}}{3^2+2^2}(3\cos 2x + 2\sin 2x) + c$$

$$= \frac{e^{3x}}{13}(3\cos 2x + 2\sin 2x) + c$$

6) $\int e^{-x} \cos x\,dx$

$$I = \int e^{-x} \cos x\,dx$$

$$= \frac{e^{-x}}{(-1)^2+1^2}(-\cos x + \sin bx)$$

$$= \frac{e^{-x}}{2}(-\cos x + \sin bx)$$

7) $\int e^{-3x} \cos x\,dx$

$$= \frac{e^{-3x}}{9+1}(-3\cos x + \sin x) + c$$

$$= \frac{e^{-3x}}{10}(-3\cos x + \sin x) + c.$$

17.9 ①

Integrals of the form $\int e^{x}[f(x)+f'(x)]dx$

soln

$$\int e^{x}[f(x)+f'(x)]\,dx = e^{x}f(x)+c.$$

Problems:

1. Evaluate $\int e^{x}(\tan x+\log\sec x)\,dx$

soln

$$I = \int e^{x}(\tan x+\log\sec x)\,dx$$

$$= e^{x}f(x)+c$$

$$= e^{x}\log\sec x+c$$

~~$f(x) = \tan x$~~

$f(x) = \log\sec x$

$f'(x) = \frac{1}{\sec x}\cdot\sec x\tan x$

2. $\int e^{x}\sec x(1+\tan x)\,dx$

soln

$$I = \int e^{x}\sec x(1+\tan x)\,dx$$

$$= \int e^{x}(\sec x+\sec x\tan x)\,dx$$

$$= e^{x}f(x)+c$$

$$= e^{x}\sec x+c.$$

$f(x) = \sec x$

$f'(x) = \sec x\tan x$

3. $\int e^{x}(\sin x+\cos x)\,dx$

soln

$$I = \int e^{x}(\sin x+\cos x)\,dx$$

$$= e^{x}f(x)+c$$

$$= e^{x}\sin x+c.$$

$f(x) = \sin x$

$f'(x) = \cos x$

4) $\int e^x\left(\frac{1}{x}-\frac{1}{x^2}\right)dx$

Soln

$I=\int e^x\left(\frac{1}{x}-\frac{1}{x^2}\right)dx$

$f(x)=1/x$

$f'(x)=-1/x^2$

$=\int e^x\left(f(x)+f'(x)\right)dx$

$=e^x f(x)+c$

$=\frac{e^x}{x}+c$

5) $\int e^x\left(\frac{x-1}{2x^2}\right)dx$

$I=\int e^x\left(\frac{x-1}{2x^2}\right)dx$

$=\frac{1}{2}\int e^x\left(\frac{x-1}{x^2}\right)dx$

$=\frac{1}{2}\int e^x\left(\frac{1}{x}-\frac{1}{x^2}\right)dx$

$=\frac{1}{2}e^x f(x)+c$

$=\frac{1}{2}e^x\left(\frac{1}{x}\right)+c$

$=\frac{e^x}{2x}+c.$

6) $\int e^x\left(\frac{2+\sin 2x}{1+\cos 2x}\right)dx$

$I=\int e^x\left(\frac{2+\sin 2x}{1+\cos 2x}\right)dx$

$\sin 2x=2\sin x\cos x$

$1+\cos 2x=2\cos^2 x$

$=\int e^x\left[\frac{2+2\sin x\cos x}{2\cos^2 x}\right]dx$

$=\int e^x\left[\frac{1+\sin x\cos x}{\cos^2 x}\right]dx$

$=\int e^x\left[\sec^2 x+\tan x\right]dx$

11.9 (2)

$= \int e^{x}(\tan x + \sec^2 x)\,dx$

$= e^{x} f(x) + c$

$= e^{x} \tan x + c$

7) $\int e^{x}\left(\frac{1-x}{1+x^2}\right)^2 dx$

$I = \int e^{x}\left(\frac{1-x}{1+x^2}\right)^2 dx$

$= \int e^{x} \frac{(1-x)^2}{(1+x^2)^2}\,dx$

$= \int e^{x} \frac{1+x^2-2x}{(1+x^2)^2}\,dx$

$= \int e^{x}\left[\frac{1}{1+x^2} - \frac{2x}{(1+x^2)^2}\right]$

$f(x) = \frac{1}{1+x^2}$

$f'(x) = \frac{-2x}{(1+x^2)^2}$

$= e^{x} f(x) + c$

$= e^{x} \frac{1}{1+x^2} + c.$

8)

Integration of Rational algebric functions.

Type 1:

Integrals of the form

$$\int \frac{dx}{a^2 \pm x^2}, \int \frac{dx}{x^2 \pm a^2}, \int \frac{dx}{\sqrt{a^2 \pm x^2}}, \int \frac{dx}{\sqrt{x^2-a^2}}$$

1. $\int \frac{dx}{a^2-x^2} = \frac{1}{2a} \log\left|\frac{a+x}{a-x}\right| + c$

2. $\int \frac{dx}{x^2-a^2} = \frac{1}{2a} \log\left|\frac{x-a}{x+a}\right| + c$

3. $\int \frac{dx}{a^2+x^2} = \frac{1}{a} \tan^{-1}\left(\frac{x}{a}\right) + c$

4. $\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\left(\frac{x}{a}\right) + c$

5. $\int \frac{dx}{\sqrt{x^2-a^2}} = \log\left|x + \sqrt{x^2-a^2}\right| + c$

6. $\int \frac{dx}{\sqrt{x^2+a^2}} = \log\left|x + \sqrt{x^2+a^2}\right| + c$

problems:

1. Integrate $\frac{1}{4-x^2}$

Soln

$$I = \int \frac{dx}{4-x^2}$$

$$= \int \frac{dx}{2^2-x^2} \qquad \int \frac{dx}{a^2-x^2} = \frac{1}{2a} \log\left|\frac{a+x}{a-x}\right| + c.$$

$$= \frac{1}{4} \left|\frac{2+x}{2-x}\right| + c$$

2. Integrate $\frac{1}{25-4x^2}$

Soln $I = \int \frac{dx}{25-4x^2}$

$$= \int \frac{dx}{5^2-(2x)^2}$$

$$= \frac{1}{2}\left(\frac{1}{2(5)} \log\left|\frac{5+2x}{5-2x}\right|\right)+c$$

$$= \frac{1}{20} \log\left|\frac{5+2x}{5-2x}\right|+c$$

3) $\int \frac{1}{9x^2-4}\,dx$

Soln

$$I = \int \frac{1}{9x^2-4}\,dx$$

$$= \int \frac{1}{(3x)^2-(2)^2}\,dx \qquad \int \frac{dx}{x^2-a^2} = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right|+c$$

$$= \frac{1}{3}\left(\frac{1}{2(2)} \log\left(\frac{3x-2}{3x+2}\right)\right)+c$$

$$= \frac{1}{12} \log\left(\frac{3x-2}{3x+2}\right|+c$$

4) $\int \frac{1}{(x+1)^2-25}\,dx$

Soln

$$I = \int \frac{1}{(x+1)^2-25}\,dx$$

$$= \int \frac{1}{(x+1)^2-(5)^2} \qquad \int \frac{dx}{x^2-a^2} = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right|+c$$

$$= \frac{1}{2(5)} \log\left|\frac{x+1-5}{x+1+5}\right|+c$$

$$= \frac{1}{10} \log\left|\frac{x+4}{x+6}\right|+c.$$

10.10 (2)

5) $\int \frac{x^2}{x^2+5}\,dx$

Soln $I = \int \frac{x^2}{x^2+5}\,dx$

$= \int \frac{x^2+5-5}{x^2+5}\,dx$

$= \int \left(1 - \frac{5}{x^2+5}\right) dx$

$= \int dx - 5\int \frac{dx}{x^2+5}$

$= x - 5\int \frac{dx}{x^2+(\sqrt{5})^2}$

$= x - 5\,\frac{1}{\sqrt{5}}\tan^{-1}\left(\frac{x}{\sqrt{5}}\right) + c$

$= x - \sqrt{5}\tan^{-1}\left(\frac{x}{\sqrt{5}}\right) + c.$

$\int \frac{dx}{a^2+x^2}$

$= \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + c$

6. $\int \frac{dx}{\sqrt{1+4x^2}}$

Soln $I = \int \frac{dx}{\sqrt{1+4x^2}}$

$= \int \frac{dx}{\sqrt{1+(2x)^2}}$

$= \frac{1}{2}\log\left|2x+\sqrt{1+(2x)^2}\right| + c$

$= \frac{1}{2}\log\left|2x+\sqrt{1+4x^2}\right| + c$

$\int \frac{dx}{\sqrt{a^2+x^2}}$

$= \log\left|x+\sqrt{a^2+x^2}\right| + c$

7) $\int \frac{dx}{\sqrt{4x^2-25}}$

$I = \int \frac{dx}{\sqrt{4x^2-25}}$

$= \int \frac{dx}{\sqrt{(2x)^2-5^2}}$

$\int \frac{dx}{\sqrt{x^2-a^2}}$

$= \log\left|x+\sqrt{x^2-a^2}\right| + c$

$$= \frac{1}{2}\log\left|2x+\sqrt{(2x)^2-5^2}\right|+c$$

$$= \frac{1}{2}\log\left|2x+\sqrt{4x^2-25}\right|+c.$$

Type 2:

Integrals of the form $\int \frac{dx}{ax^2+bx+c}$, $\int \frac{dx}{\sqrt{ax^2+bx+c}}$

1. First we express ax^2+bx+c as sum or difference of two square terms.
   (ie) any one of Type 1 form.
2. Use the formulas of Type 1.

problems:

1. $\int \frac{1}{x^2-2x+5}\,dx$

soln $I = \int \frac{dx}{x^2-2x+5}$

$$= \int \frac{dx}{x^2-2x+1-1+5}$$

$$= \int \frac{dx}{(x-1)^2+4}$$

$$= \int \frac{dx}{(x-1)^2+2^2}$$

$$\int \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)+c$$

$$= \frac{1}{2}\tan^{-1}\left(\frac{x-1}{2}\right)+c.$$

2 $\int \frac{1}{6x-7-x^2}\,dx$

soln $I = \int \frac{dx}{6x-7-x^2}$

$$= \int \frac{-dx}{x^2-6x+7}$$

$$= \int \frac{-dx}{x^2-6x+9-9+7}$$

10.10 (3)

$$= \int \frac{-dx}{(x-3)^2-2}$$

$$= \int \frac{dx}{(\sqrt{2})^2-(x-3)^2}$$

$$= \frac{1}{2\sqrt{2}} \log \left| \frac{\sqrt{2}+(x-3)}{\sqrt{2}+(x-3)} \right| + c.$$

3. Integrate $\int \frac{1}{\sqrt{x^2+4x+2}}\, dx$

$$I = \int \frac{dx}{\sqrt{x^2+4x+2}}$$

$$\int \frac{dx}{\sqrt{x^2-a^2}} = \log\left|x+\sqrt{x^2-a^2}\right| + c$$

$$= \int \frac{dx}{\sqrt{x^2+4x+4-4+2}}$$

$$= \int \frac{dx}{\sqrt{(x+2)^2-2}}$$

$$= \int \frac{dx}{\sqrt{(x+2)^2-(\sqrt{2})^2}}$$

$$= \cancel{\frac{1}{2\sqrt{2}} \log \left| \frac{x+2-\sqrt{2}}{x+2+\sqrt{2}} \right| + c.} \quad = \log\left|(x+2)+\sqrt{x^2+4x+2}\right| + c$$

4. $\int \frac{1}{\sqrt{(2+x)^2-1}}\, dx$

$$I = \int \frac{dx}{\sqrt{(2+x)^2-1}}$$

$$\int \frac{dx}{\sqrt{x^2-a^2}}$$

$$= \log\left|(2+x)+\sqrt{(2+x)^2-1}\right| + c. \quad = \log\left|x+\sqrt{x^2-a^2}\right| + c.$$

5. $\int \frac{dx}{\sqrt{x^2-4x+5}}\, dx$

Soln

$$I = \int \frac{dx}{\sqrt{x^2-4x+5}}$$

$$= \int \frac{dx}{\sqrt{x^2-4x+4-4+5}}$$

$$\int \frac{dx}{\sqrt{x^2+a^2}} = \log|x+\sqrt{x^2+a^2}|+c$$

$$= \int \frac{dx}{\sqrt{(x-2)^2+1}}$$

$$= \log|(x-2)+\sqrt{(x-2)^2+1}|+c$$

$$= \log|(x-2)+\sqrt{x^2-4x+5}|+c.$$

6) $\int \frac{1}{\sqrt{x^2+12x+11}}\,dx$

$$I = \int \frac{dx}{\sqrt{x^2+12x+11}}$$

$$= \int \frac{dx}{\sqrt{x^2+12x+36-36+11}}$$

$$= \int \frac{dx}{\sqrt{(x+6)^2-25}}$$

$$\int \frac{dx}{\sqrt{x^2-a^2}} = \log|x+\sqrt{x^2-a^2}|+c$$

$$= \int \frac{dx}{\sqrt{(x+6)^2-5^2}}$$

$$= \log|(x+6)+\sqrt{(x+6)^2-5^2}|+c$$

$$= \log|x+6+\sqrt{x^2+12x+1}|+c$$

7) $\int \frac{1}{\sqrt{12+4x-x^2}}\,dx$

$$I = \int \frac{1}{\sqrt{12+4x-x^2}}\,dx$$

$$= \int \frac{1}{\sqrt{-(x^2-4x-12)}}\,dx$$

$$= \int \frac{1}{\sqrt{-(x^2-4x+4-4-12)}}\,dx$$

$$= \int \frac{dx}{\sqrt{-((x-2)^2-4^2)}}\,dx$$

$$= \int \frac{dx}{\sqrt{4^2-(x-2)^2}}$$

$$= \sin^{-1}\left(\frac{x-2}{4}\right)+c$$

$$\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\left(\frac{x}{a}\right)+c.$$

8) $\int \frac{1}{\sqrt{9+8x-x^2}}\,dx$

<u>Soln</u> $I = \int \frac{dx}{\sqrt{-(x^2-8x-9)}}$

$$= \int \frac{dx}{\sqrt{-(x^2-8x+16-16-9)}}$$

$$= \int \frac{dx}{\sqrt{-((x-4)^2-5^2)}}$$

$$= \int \frac{dx}{\sqrt{5^2-(x-4)^2}}$$

$$= \sin^{-1}\left(\frac{x-4}{5}\right)+c.$$

Integrals of the form $\int \frac{px+q}{ax^2+bx+c}$, $\int \frac{px+q}{\sqrt{ax^2+bx+c}}$

$$px+q = A\frac{d}{dx}(ax^2+bx+c) + B$$

$$px+q = A(2ax+b) + B$$

Equ. the co.eff of x & constant term

We get A & B

$$\therefore \int \frac{px+q}{ax^2+bx+c}\,dx = \int \frac{A(2ax+b)+B}{ax^2+bx+c}\,dx$$

$$= A\int \frac{2ax+b}{ax^2+bx+c}\,dx + B\int \frac{dx}{ax^2+bx+c}$$

$$= A\log|ax^2+bx+c| + c + B\int \frac{dx}{ax^2+bx+c}$$

Note:

1. $\int \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)+c$

2. $\int \frac{dx}{x^2-a^2} = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right|+c$

3. $\int \frac{dx}{a^2-x^2} = \frac{1}{2a}\log\left|\frac{a+x}{a-x}\right|+c.$

Problems:

1. Evaluate $\int \frac{3x+5}{x^2+4x+7}\,dx$

Soln

$$I = \int \frac{3x+5}{x^2+4x+7}\,dx \longrightarrow (1)$$

$$3x+5 = A\frac{d}{dx}(x^2+4x+7) + B$$

$$3x+5 = A(2x+4) + B$$

Eq. co.eff of x

$3 = 2A$

$A = 3/2$

eq. constant term

$5 = 4A + B$

$5 = 4(3/2) + B$

$5 = 6 + B$ $\boxed{B = -1}$

$$\therefore I = \int \frac{\frac{3}{2}(2x+4) - 1}{x^2+4x+7}\,dx$$

$$= \frac{3}{2}\int \frac{2x-1}{x^2+4x+7}\,dx - \int \frac{dx}{x^2+4x+7}$$

$$= \frac{3}{2}\log|x^2+4x+7| - \int \frac{dx}{x^2+4x+4-4+7}$$

$$= \frac{3}{2}\log|x^2+4x+7| - \int \frac{dx}{(x+2)^2+3}$$

$$= \frac{3}{2}\log|x^2+4x+7| - \int \frac{dx}{(x+2)^2+(\sqrt{3})^2}$$

$$= \frac{3}{2}\log|x^2+4x+7| - \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{x+2}{\sqrt{3}}\right) + c.$$

---

2) Evaluate $\int \frac{x+1}{x^2-3x+1}\,dx$

Soln

$$I = \int \frac{x+1}{x^2-3x+1}\,dx \longrightarrow (1)$$

$$x+1 = A\frac{d}{dx}(x^2-3x+1) + B$$

$$x+1 = A(2x-3) + B$$

Equ. the Co.eff of x & constant term.

$1 = 2A$

$A = \frac{1}{2}$

$1 = -3A + B$

$1 = -\frac{3}{2} + B$

$B = 1 + \frac{3}{2}$

$B = \frac{5}{2}$

(1) $\Rightarrow$

$$I = \int \frac{\frac{1}{2}(2x-3) + \frac{5}{2}}{x^2-3x+1}\,dx$$

$$= \frac{1}{2}\int \frac{2x-3}{x^2-3x+1}\,dx + \frac{5}{2}\int \frac{dx}{x^2-3x+1}$$

$$= \frac{1}{2}\log|x^2-3x+1| + \frac{5}{2}\int \frac{dx}{x^2-3x+\frac{9}{4}-\frac{9}{4}+1}$$

11.11 (2)

$$= \frac{1}{2}\log|x^2-3x+1| + \frac{5}{2}\int\frac{dx}{(x-3/2)^2-5/4}$$

$$= \frac{1}{2}\log|x^2-3x+1| + \frac{5}{2}\int\frac{dx}{(x-3/2)^2-(\sqrt{5}/2)^2}$$

$$= \frac{1}{2}\log|x^2-3x+1| + \frac{5}{2}\frac{1}{2(\sqrt{5}/2)}\log\left|\frac{x-3/2-\sqrt{5}/2}{x-3/2+\frac{\sqrt{5}}{2}}\right| + c$$

$$= \frac{1}{2}\log|x^2-3x+1| + \frac{\sqrt{5}}{2}\log\left|\frac{2x-3-\sqrt{5}}{2x-3+\sqrt{5}}\right| + c.$$

3) Evaluate $\int\frac{2x-3}{x^2+4x-12}dx$

Soln

$$I = \int\frac{2x-3}{x^2+4x-12}dx \longrightarrow (1)$$

$$2x-3 = A\frac{d}{dx}(x^2+4x-12)+B$$

$$2x-3 = A(2x+4)+B$$

equ. like coeff of x & constant term

$2 = 2A$ | $4A+B = -3$

$A = 1$ | $B = -3-4$

$B = -7$

$$I = 1\int\frac{2x+4}{x^2+4x-12}dx - 7\int\frac{dx}{x^2+4x-12}$$

$$= \log|x^2+4x-12| - 7\int\frac{dx}{x^2+4x+4-4-12}$$

$$= \log|x^2+4x-12| - 7\int\frac{dx}{(x+2)^2-16}$$

$$= \log|x^2+4x-12| - 7\int\frac{dx}{(x+2)^2-4^2}$$

$$= \log|x^2+4x-12| - 7\left[\frac{1}{8}\log\left|\frac{x+2-4}{x+2+4}\right|\right] + c$$

$$= \log|x^2+4x-12| - \frac{7}{8}\log\left|\frac{x-2}{x+6}\right| + c.$$

4. Evaluate $\int \frac{5x-2}{x^2+2x+2}\,dx$

$$I = \int \frac{5x-2}{x^2+2x+2}\,dx$$

$$5x-2 = A\frac{d}{dx}(x^2+2x+2) + B$$

$$5x-2 = A(2x+2) + B$$

equ. the co.eff of x & constant term.

$5 = 2A$ $\boxed{A = 5/2}$

$2A + B = -2$

$B = -2 - 2A$

$B = -2 - 2(5/2)$

$\boxed{B = -7}$

$$I = \frac{5}{2}\int \frac{2x+2}{x^2+2x+2}\,dx - 7\int \frac{dx}{x^2+2x+2}$$

$$= \frac{5}{2}\log|x^2+2x+2| - 7\int \frac{dx}{x^2+2x+1-1+2}$$

$$= \frac{5}{2}\log|x^2+2x+2| - 7\int \frac{dx}{(x+1)^2+1}$$

$$= \frac{5}{2}\log|x^2+2x+2| - 7\tan^{-1}(x+1) + c.$$

5. Evaluate $\int \frac{3x+1}{2x^2-2x+3}\,dx.$

Soln

$$I = \int \frac{3x+1}{2x^2-2x+3}\,dx$$

$$3x+1 = A\frac{d}{dx}(2x^2-2x+3) + B$$

$$3x+1 = A(4x-2) + B$$

Equ. the coff. of x & constant term.

$3 = 4A$ $\boxed{A = 3/4}$

$-2A + B = 1$

$B = 1 + 2A$

$= 1 + 2(3/4)$

$= 1 + 3/2 = 5/2$

$$I = \frac{3}{4}\log(2x^2-2x+3) + \frac{5}{3}\int \frac{1}{2x^2-2x+3}\,dx$$

$$= \frac{3}{4}\log(2x^2-2x+3) + \frac{5}{6}\int \frac{1}{x^2-x+\frac{1}{4}-\frac{1}{4}+\frac{3}{2}}\,dx$$

$$= \frac{3}{4}\log|2x^2-2x+3| + \frac{5}{6}\int \frac{1}{(x-\frac{1}{2})^2+(\frac{\sqrt{5}}{2})^2}\,dx$$

$$= \frac{3}{4}\log|2x^2-2x+3| + \frac{5}{6}\tan^{-1}\left(\frac{x-\frac{1}{2}}{\sqrt{5}/2}\right)+c.$$

Integrals of the form $\int \frac{px+q}{\sqrt{ax^2+bx+c}}\,dx$

$$px+q = A\frac{d}{dx}(ax^2+bx+c)+B$$

$$px+q = A(2ax+b)+B$$

Equ. the co.eff. of x & constant terms.

We get A & B.

$$\int \frac{px+q}{\sqrt{ax^2+bx+c}} = \int \frac{A(2ax+b)}{\sqrt{ax^2+bx+c}}\,dx + \int \frac{B}{\sqrt{ax^2+bx+c}}\,dx$$

$$= A(2\sqrt{ax^2+bx+c}) + B\int \frac{dx}{\sqrt{ax^2+bx+c}}$$

$$=$$

$$\therefore \int \frac{px+q}{\sqrt{ax^2+bx+c}}\,dx = A\left(2\sqrt{ax^2+bx+c}\right) + B\int \frac{dx}{\sqrt{ax^2+bx+c}}.$$

---

problems:

1. Evaluate $\int \frac{x+2}{\sqrt{x^2-1}}\,dx$

$$I = \int \frac{x+2}{\sqrt{x^2-1}}\,dx \longrightarrow (1)$$

$$x+2 = A\frac{d}{dx}(x^2-1)+B$$

$$x+2 = A(2x)+B$$

equ. the co.eff of x & constant term

$2A = 1$ $B = 2$

$A = \frac{1}{2}$

Now

$$\int \frac{x+2}{\sqrt{x^2-1}}\,dx = A\left(2\sqrt{x^2-1}\right) + B\int \frac{dx}{\sqrt{x^2-1}}\,dx$$

$$= \frac{1}{2}\left(2\sqrt{x^2-1}\right) + 2\log\left|x+\sqrt{x^2-1}\right| + c$$

$$= \sqrt{x^2-1} + 2\log\left|x+\sqrt{x^2-1}\right| + c.$$

---

2) Evaluate $\int \frac{2x+3}{\sqrt{x^2+4x+1}}\,dx$

$$I = \int \frac{2x+3}{\sqrt{x^2+4x+1}}\,dx \longrightarrow (1)$$

$$2x+3 = A\frac{d}{dx}(x^2+4x+1) + B$$

$$2x+3 = A(2x+4) + B$$

Equ. the co.eff of x & constant term.

$2A = 2$ $4A + B = 3$

$A = 1$ $4 + B = 3$

$\boxed{B = -1}$

(1) becomes.

$$\int \frac{2x+3}{\sqrt{x^2+4x+1}}\,dx = A\left(2\sqrt{x^2+4x+1}\right) + B\int \frac{dx}{\sqrt{x^2+4x+1}}$$

$$= 2\sqrt{x^2+4x+1} - \int \frac{dx}{\sqrt{x^2+4x+4-4+1}}$$

$$= 2\sqrt{x^2+4x+1} - \int \frac{dx}{\sqrt{(x+2)^2-(\sqrt{3})^2}}$$

$$= 2\sqrt{x^2+4x+1} - \log\left|x+2+\sqrt{(x+2)^2-(\sqrt{3})^2}\right| + c$$

$$= 2\sqrt{x^2+4x+1} - \log\left|x+2+\sqrt{x^2+4x+1}\right| + c$$

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3. Evaluate $\int \frac{2x+1}{\sqrt{9+4x-x^2}} dx$

Soln

$$I = \int \frac{2x+1}{\sqrt{9+4x-x^2}} dx \longrightarrow (1)$$

$$2x+1 = A\frac{d}{dx}(9+4x-x^2) + B$$

$$2x+1 = A(4-2x) + B$$

Equ. the co.eff of x & constant term

$-2A = 2$ $\qquad$ $4A + B = 1$

$A = -1$ $\qquad$ $-4 + B = 1$

$B = 5$

$\therefore$ (1) becomes

$$\int \frac{2x+1}{\sqrt{9+4x-x^2}} dx = A(2\sqrt{9+4x-x^2}) + B\int \frac{dx}{\sqrt{9+4x-x^2}}$$

$$= -2\sqrt{9+4x-x^2} + B\int \frac{-dx}{\sqrt{x^2-4x-9}}$$

$$= -2\sqrt{9+4x-x^2} + 5\int \frac{-dx}{\sqrt{x^2-4x+4-4-9}}$$

$$= -2\sqrt{9+4x-x^2} + 5\int \frac{-dx}{\sqrt{(x-2)^2-(\sqrt{13})^2}}$$

$$= -2\sqrt{9+4x-x^2} + 5\int \frac{dx}{\sqrt{(\sqrt{13})^2-(x-2)^2}}$$

$$= -2\sqrt{9+4x-x^2} + 5\sin^{-1}\left(\frac{x-2}{\sqrt{13}}\right) + c.$$

4) Evaluate: $\int \frac{2x+3}{\sqrt{x^2+x+1}} dx$

Soln

$$I = \int \frac{2x+3}{\sqrt{x^2+x+1}} dx \longrightarrow (1)$$

$$2x+3 = A\frac{d}{dx}(x^2+x+1) + B$$

$$2x+3 = A(2x+1)+B$$

Equ. the co.off of x & constant term.

$2A = 2$ $\qquad$ $A+B=3$

$A = 1$ $\qquad$ $B = 3-A$

$B = 3-1 = 2$

$\therefore$ (1) becomes

$$\int \frac{2x+3}{\sqrt{x^2+x+1}}\,dx = A\left(2\sqrt{x^2+x+1}\right) + B\int \frac{dx}{\sqrt{x^2+x+1}}$$

$$= 2\sqrt{x^2+x+1} + 2\int \frac{dx}{\sqrt{x^2+x+1/4-1/4+1}}$$

$$= 2\sqrt{x^2+x+1} + 2\int \frac{dx}{(x+1/2)^2+3/4}$$

$$= 2\sqrt{x^2+x+1} + 2\int \frac{dx}{(x+1/2)^2+(\sqrt{3}/2)^2}$$

$$= 2\sqrt{x^2+x+1} + 2\log\left|(x+1/2) + \sqrt{(x+1/2)^2+(\sqrt{3}/2)^2}\right| + c$$

$$= 2\sqrt{x^2+x+1} + 2\log\left|x+\tfrac{1}{2}+\sqrt{x^2+x+1}\right| + c.$$

5. Evaluate $\int \frac{5x-7}{\sqrt{3x-x^2-2}}\,dx$

Hints

$A = -5/2$

$B = 1/2$

$$\therefore\ I = -5\sqrt{3x-x^2+2} + \tfrac{1}{2}\sin^{-1}\left(\frac{2x-3}{\sqrt{17}}\right)+c.$$

Type IV

Integrals of the form $\int\sqrt{a^2\pm x^2}\,dx$, $\int\sqrt{x^2-a^2}\,dx$.

Results:

1. $\int\sqrt{a^2-x^2}\,dx = \frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)+c$

2. $\int\sqrt{x^2-a^2}\,dx = \frac{x}{2}\sqrt{x^2-a^2}-\frac{a^2}{2}\log\left|x+\sqrt{x^2-a^2}\right|+c$

3. $\int\sqrt{x^2+a^2}\,dx = \frac{x}{2}\sqrt{x^2+a^2}+\frac{a^2}{2}\log\left|x+\sqrt{x^2+a^2}\right|+c$.

Problems:

1. Evaluate $\int\sqrt{4-x^2}\,dx$

Soln

$I = \int\sqrt{4-x^2}\,dx$

$= \int\sqrt{2^2-x^2}\,dx$

$= \frac{x}{2}\sqrt{2^2-x^2}+\frac{4}{2}\sin^{-1}\left(\frac{x}{2}\right)+c$

$= \frac{x}{2}\sqrt{4-x^2}+2\sin\left(\frac{x}{2}+c\right)$

$\int\sqrt{a^2-x^2}\,dx$

$=\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)+c$

2. Evaluate $\int\sqrt{9-(2x+5)^2}\,dx$

Soln

$I = \int\sqrt{9-(2x+5)^2}\,dx$

$= \int\sqrt{3^2-(2x+5)^2}\,dx$

$= \frac{2x+5}{\cancel{4}}\left(\sqrt{\cancel{3^2}-(2x+5)}\right.$

$= \frac{1}{2}\left(\frac{2x+5}{2}\sqrt{9-(2x+5)^2}+\frac{9}{2}\sin^{-1}\left(\frac{2x+5}{3}\right)\right)+c$

3) $\int \sqrt{(6-x)(x-4)}\, dx$

$\int \sqrt{a^2-x^2}\, dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)+c.$

Soln

$I = \int \sqrt{(6-x)(x-4)}\, dx$

$= \int \sqrt{6x-24-x^2+4x}\, dx$

$= \int \sqrt{-x^2+10x-24}\, dx$

$= \int \sqrt{-(x^2-10x+24)}\, dx$

$= \int \sqrt{-((x-5)^2-25+24)}\, dx$

$= \int \sqrt{-((x-5)^2-1)}\, dx$

$= \int \sqrt{1-(x-5)^2}\, dx$

$= \frac{(x-5)}{2}\sqrt{1-(x-5)^2} + \frac{1}{2}\sin^{-1}\left(\frac{x-5}{1}\right)+c.$

4) $\int \sqrt{(x-3)(5-x)}\, dx$

$\int \sqrt{a^2-x^2}\, dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)+c$

Soln

$I = \int \sqrt{(x-3)(5-x)}\, dx$

$= \int \sqrt{-x^2+8x-15}\, dx$

$= \int \sqrt{-(x^2-8x+15)}\, dx$

$= \int \sqrt{-(x^2-8x+16-16+15)}\, dx$

$= \int \sqrt{-((x-4)^2-1)}\, dx$

$= \int \sqrt{1-(x-4)^2}\, dx$

$= \frac{x-4}{2}\sqrt{1-(x-4)^2} + \frac{1}{2}\sin^{-1}(x-4) + c.$

5) Evaluate $\int \sqrt{81+(2x+1)^2}\, dx$

Soln

$$I = \int \sqrt{81+(2x+1)^2}\, dx$$

$$= \int \sqrt{9^2+(2x+1)^2}\, dx$$

$$= \frac{1}{2}\left(\frac{2x+1}{2}\sqrt{81+(2x+1)^2} + \frac{81}{2}\log\left|(2x+1)+\sqrt{81+(2x+1)^2}\right| \right) + c.$$

6. Evaluate $\int \sqrt{x^2+2x+10}\, dx$

Soln

$$I = \int \sqrt{x^2+2x+10}\, dx$$

$$= \int \sqrt{x^2+2x+1-1+10}\, dx$$

$$= \int \sqrt{(x+1)^2+9}\, dx$$

$$= \int \sqrt{(x+1)^2+3^2}\, dx$$

$$= \frac{x+1}{2}\sqrt{(x+1)^2+3^2} + \frac{9}{2}\log\left|(x+1)+\sqrt{(x+1)^2+9}\right| + c$$

7. Evaluate $\int \sqrt{(x+1)^2-4}\, dx$

Soln

$$I = \int \sqrt{(x+1)^2-4}\, dx$$

$$= \int \sqrt{(x+1)^2-2^2}\, dx$$

$$= \frac{x+1}{2}\sqrt{(x+1)^2-4} - \frac{4}{2}\log\left|(x+1)+\sqrt{(x+1)^2-4}\right| + c$$

$$= \frac{x+1}{2}\sqrt{(x+1)^2-4} - 2\log\left|(x+1)+\sqrt{(x+1)^2-4}\right| + c.$$

8) Evaluate $\int \sqrt{x^2-2x-3}\,dx$

Soln $I = \int \sqrt{x^2-2x-3}\,dx$

$= \int \sqrt{x^2-2x+1-1-3}\,dx$

$= \int \sqrt{(x-1)^2-4}\,dx$

$= \int \sqrt{(x-1)^2-2^2}\,dx$

$= \frac{(x-1)}{2}\sqrt{(x-1)^2-4} - \frac{4}{2}\log\left|(x-1)+\sqrt{x^2-2x-3}\right| + c$

$= \frac{(x-1)^2}{2}\sqrt{x^2-2x-3} - 2\log\left|(x-1)+\sqrt{x^2-2x-3}\right| + c.$

9) Evaluate $\int \sqrt{25x^2-9}\,dx$

Soln

$I = \int \sqrt{25x^2-9}\,dx$

$= \int \sqrt{(5x)^2-3^2}$

$= \frac{1}{5}\left(\frac{5x}{2}\sqrt{(5x)^2-3^2} - \frac{3^2}{2}\log\left|5x+\sqrt{(5x)^2-3^2}\right|\right) + c$

$= \frac{1}{5}\left(\frac{5x}{2}\sqrt{25x^2-9} - \frac{9}{2}\log\left|5x+\sqrt{25x^2-9}\right|\right) + c.$

10. Evaluate $\int \sqrt{x^2+x+1}\,dx$

$I = \int \sqrt{x^2+x+1}\,dx$

$= \int \sqrt{x^2+x+\frac{1}{4}-\frac{1}{4}+1}$

$= \int \sqrt{(x+1/2)^2+\frac{3}{4}}$

$= \int \sqrt{(x+1/2)^2+(\sqrt{3}/2)^2}$

$= \frac{(x+1/2)}{2}\sqrt{(x+1/2)^2+(\sqrt{3}/2)^2} + \frac{(\sqrt{3}/2)^2}{2}\log\left|(x+1/2)+\sqrt{(x+1/2)^2+(\sqrt{3}/2)^2}\right| + c$

Integration Formulas

1. Common Integrals

Indefinite Integral

Method of substitution

$$\int f(g(x))g'(x)dx = \int f(u)du$$

Integration by parts

$$\int f(x)g'(x)dx = f(x)g(x) - \int g(x)f'(x)dx$$

Integrals of Rational and Irrational Functions

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C$$

$$\int \frac{1}{x}dx = \ln|x| + C$$

$$\int c\,dx = cx + C$$

$$\int xdx = \frac{x^2}{2} + C$$

$$\int x^2 dx = \frac{x^3}{3} + C$$

$$\int \frac{1}{x^2}dx = -\frac{1}{x} + C$$

$$\int \sqrt{x}dx = \frac{2x\sqrt{x}}{3} + C$$

$$\int \frac{1}{1+x^2}dx = \arctan x + C$$

$$\int \frac{1}{\sqrt{1-x^2}}dx = \arcsin x + C$$

Integrals of Trigonometric Functions

$$\int \sin x\,dx = -\cos x + C$$

$$\int \cos x\,dx = \sin x + C$$

$$\int \tan x\,dx = \ln|\sec x| + C$$

$$\int \sec x\,dx = \ln|\tan x + \sec x| + C$$

$$\int \sin^2 x\,dx = \frac{1}{2}(x - \sin x \cos x) + C$$

$$\int \cos^2 x\,dx = \frac{1}{2}(x + \sin x \cos x) + C$$

$$\int \tan^2 x\,dx = \tan x - x + C$$

$$\int \sec^2 x\,dx = \tan x + C$$

Integrals of Exponential and Logarithmic Functions

$$\int \ln x\,dx = x\ln x - x + C$$

$$\int x^n \ln x\,dx = \frac{x^{n+1}}{n+1}\ln x - \frac{x^{n+1}}{(n+1)^2} + C$$

$$\int e^x\,dx = e^x + C$$

$$\int b^x\,dx = \frac{b^x}{\ln b} + C$$

$$\int \sinh x\,dx = \cosh x + C$$

$$\int \cosh x\,dx = \sinh x + C$$

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2. Integrals of Rational Functions

Integrals involving ax + b

$$\int (ax+b)^n\,dx = \frac{(ax+b)^{n+1}}{a(n+1)} \qquad (\mathit{for}\ n \neq -1)$$

$$\int \frac{1}{ax+b}\,dx = \frac{1}{a}\ln|ax+b|$$

$$\int x(ax+b)^n\,dx = \frac{a(n+1)x-b}{a^2(n+1)(n+2)}(ax+b)^{n+1} \qquad (\mathit{for}\ n \neq -1, n \neq -2)$$

$$\int \frac{x}{ax+b}\,dx = \frac{x}{a} - \frac{b}{a^2}\ln|ax+b|$$

$$\int \frac{x}{(ax+b)^2}\,dx = \frac{b}{a^2(ax+b)} + \frac{1}{a^2}\ln|ax+b|$$

$$\int \frac{x}{(ax+b)^n}\,dx = \frac{a(1-n)x-b}{a^2(n-1)(n-2)(ax+b)^{n-1}} \qquad (\mathit{for}\ n \neq -1, n \neq -2)$$

$$\int \frac{x^2}{ax+b}\,dx = \frac{1}{a^3}\left(\frac{(ax+b)^2}{2} - 2b(ax+b) + b^2\ln|ax+b|\right)$$

$$\int \frac{x^2}{(ax+b)^2}\,dx = \frac{1}{a^3}\left(ax+b-2b\ln|ax+b| - \frac{b^2}{ax+b}\right)$$

$$\int \frac{x^2}{(ax+b)^3}\,dx = \frac{1}{a^3}\left(\ln|ax+b| + \frac{2b}{ax+b} - \frac{b^2}{2(ax+b)^2}\right)$$

$$\int \frac{x^2}{(ax+b)^n}\,dx = \frac{1}{a^3}\left(-\frac{(ax+b)^{3-n}}{n-3} + \frac{2b(a+b)^{2-n}}{n-2} - \frac{b^2(ax+b)^{1-n}}{n-1}\right) \qquad (\mathit{for}\ n \neq 1,2,3)$$

$$\int \frac{1}{x(ax+b)}\,dx = -\frac{1}{b}\ln\left|\frac{ax+b}{x}\right|$$

$$\int \frac{1}{x^2(ax+b)}\,dx = -\frac{1}{bx} + \frac{a}{b^2}\ln\left|\frac{ax+b}{x}\right|$$

$$\int \frac{1}{x^2(ax+b)^2}\,dx = -a\left(\frac{1}{b^2(a+xb)} + \frac{1}{ab^2x} - \frac{2}{b^3}\ln\left|\frac{ax+b}{x}\right|\right)$$

Integrals involving ax^2 + bx + c

$$\int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\,arctg\,\frac{x}{a}$$

$$\int \frac{1}{x^2-a^2}\,dx = \begin{cases} \frac{1}{2a}\ln\frac{a-x}{a+x} & \mathit{for}\ |x|<|a| \\ \frac{1}{2a}\ln\frac{x-a}{x+a} & \mathit{for}\ |x|>|a| \end{cases}$$

$$\int \frac{1}{ax^2+bx+c}dx = \begin{cases} \frac{2}{\sqrt{4ac-b^2}}\arctan\frac{2ax+b}{\sqrt{4ac-b^2}} & for\, 4ac-b^2>0 \\ \frac{2}{\sqrt{b^2-4ac}}\ln\left|\frac{2ax+b-\sqrt{b^2-4ac}}{2ax+b+\sqrt{b^2-4ac}}\right| & for\, 4ac-b^2<0 \\ -\frac{2}{2ax+b} & for\, 4ac-b^2=0 \end{cases}$$

$$\int \frac{x}{ax^2+bx+c}dx = \frac{1}{2a}\ln\left|ax^2+bx+c\right| - \frac{b}{2a}\int\frac{dx}{ax^2+bx+c}$$

$$\int \frac{mx+n}{ax^2+bx+c}dx = \begin{cases} \frac{m}{2a}\ln\left|ax^2+bx+c\right| + \frac{2an-bm}{a\sqrt{4ac-b^2}}\arctan\frac{2ax+b}{\sqrt{4ac-b^2}} & for\ 4ac-b^2>0 \\ \frac{m}{2a}\ln\left|ax^2+bx+c\right| + \frac{2an-bm}{a\sqrt{b^2-4ac}}\operatorname{arctanh}\frac{2ax+b}{\sqrt{b^2-4ac}} & for\, 4ac-b^2<0 \\ \frac{m}{2a}\ln\left|ax^2+bx+c\right| - \frac{2an-bm}{a(2ax+b)} & for\ 4ac-b^2=0 \end{cases}$$

$$\int \frac{1}{\left(ax^2+bx+c\right)^n}dx = \frac{2ax+b}{(n-1)\left(4ac-b^2\right)\left(ax^2+bx+c\right)^{n-1}} + \frac{(2n-3)2a}{(n-1)\left(4ac-b^2\right)}\int\frac{1}{\left(ax^2+bx+c\right)^{n-1}}dx$$

$$\int \frac{1}{x\left(ax^2+bx+c\right)}dx = \frac{1}{2c}\ln\left|\frac{x^2}{ax^2+bx+c}\right| - \frac{b}{2c}\int\frac{1}{ax^2+bx+c}dx$$

3. Integrals of Exponential Functions

$$\int xe^{cx}dx = \frac{e^{cx}}{c^2}(cx-1)$$

$$\int x^2e^{cx}dx = e^{cx}\left(\frac{x^2}{c}-\frac{2x}{c^2}+\frac{2}{c^3}\right)$$

$$\int x^n e^{cx}dx = \frac{1}{c}x^n e^{cx} - \frac{n}{c}\int x^{n-1}e^{cx}dx$$

$$\int \frac{e^{cx}}{x}dx = \ln|x| + \sum_{i=1}^{\infty}\frac{(cx)^i}{i\cdot i!}$$

$$\int e^{cx}\ln x dx = \frac{1}{c}e^{cx}\ln|x| + E_i(cx)$$

$$\int e^{cx}\sin bx dx = \frac{e^{cx}}{c^2+b^2}(c\sin bx - b\cos bx)$$

$$\int e^{cx}\cos bx dx = \frac{e^{cx}}{c^2+b^2}(c\cos bx + b\sin bx)$$

$$\int e^{cx}\sin^n x dx = \frac{e^{cx}\sin^{n-1}x}{c^2+n^2}(c\sin x - n\cos bx) + \frac{n(n-1)}{c^2+n^2}\int e^{cx}\sin^{n-2}dx$$

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4. Integrals of Logarithmic Functions

$$\int \ln cx dx = x\ln cx - x$$

$$\int \ln(ax+b)dx = x\ln(ax+b) - x + \frac{b}{a}\ln(ax+b)$$

$$\int (\ln x)^2 dx = x(\ln x)^2 - 2x\ln x + 2x$$

$$\int (\ln cx)^n dx = x(\ln cx)^n - n\int (\ln cx)^{n-1} dx$$

$$\int \frac{dx}{\ln x} = \ln|\ln x| + \ln x + \sum_{n=2}^{\infty} \frac{(\ln x)^i}{i \cdot i!}$$

$$\int \frac{dx}{(\ln x)^n} = -\frac{x}{(n-1)(\ln x)^{n-1}} + \frac{1}{n-1}\int \frac{dx}{(\ln x)^{n-1}} \qquad (for\, n \neq 1)$$

$$\int x^m \ln x dx = x^{m+1}\left(\frac{\ln x}{m+1} - \frac{1}{(m+1)^2}\right) \qquad (for\ m \neq 1)$$

$$\int x^m (\ln x)^n dx = \frac{x^{m+1}(\ln x)^n}{m+1} - \frac{n}{m+1}\int x^m (\ln x)^{n-1} dx \qquad (for\ m \neq 1)$$

$$\int \frac{(\ln x)^n}{x} dx = \frac{(\ln x)^{n+1}}{n+1} \qquad (for\, n \neq 1)$$

$$\int \frac{\ln x^n}{x} dx = \frac{(\ln x^n)^2}{2n} \qquad (for\, n \neq 0)$$

$$\int \frac{\ln x}{x^m} dx = -\frac{\ln x}{(m-1)x^{m-1}} - \frac{1}{(m-1)^2 x^{m-1}} \qquad (for\, m \neq 1)$$

$$\int \frac{(\ln x)^n}{x^m} dx = -\frac{(\ln x)^n}{(m-1)x^{m-1}} + \frac{n}{m-1}\int \frac{(\ln x)^{n-1}}{x^m} dx \qquad (for\, m \neq 1)$$

$$\int \frac{dx}{x\ln x} = \ln|\ln x|$$

$$\int \frac{dx}{x^n \ln x} = \ln|\ln x| + \sum_{i=1}^{\infty} (-1)^i \frac{(n-1)^i (\ln x)^i}{i \cdot i!}$$

$$\int \frac{dx}{x(\ln x)^n} = -\frac{1}{(n-1)(\ln x)^{n-1}} \qquad (for\, n \neq 1)$$

$$\int \ln(x^2+a^2)dx = x\ln(x^2+a^2) - 2x + 2a\tan^{-1}\frac{x}{a}$$

$$\int \sin(\ln x)dx = \frac{x}{2}(\sin(\ln x) - \cos(\ln x))$$

$$\int \cos(\ln x)dx = \frac{x}{2}(\sin(\ln x) + \cos(\ln x))$$

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5. Integrals of Trig. Functions

$$\int \sin x dx = -\cos x$$

$$\int \cos x dx = -\sin x$$

$$\int \sin^2 x dx = \frac{x}{2} - \frac{1}{4}\sin 2x$$

$$\int \cos^2 x dx = \frac{x}{2} + \frac{1}{4}\sin 2x$$

$$\int \sin^3 x dx = \frac{1}{3}\cos^3 x - \cos x$$

$$\int \cos^3 x dx = \sin x - \frac{1}{3}\sin^3 x$$

$$\int \frac{dx}{\sin x} xdx = \ln\left|\tan\frac{x}{2}\right|$$

$$\int \frac{dx}{\cos x} xdx = \ln\left|\tan\left(\frac{x}{2}+\frac{\pi}{4}\right)\right|$$

$$\int \frac{dx}{\sin^2 x} xdx = -\cot x$$

$$\int \frac{dx}{\cos^2 x} xdx = \tan x$$

$$\int \frac{dx}{\sin^3 x} = -\frac{\cos x}{2\sin^2 x} + \frac{1}{2}\ln\left|\tan\frac{x}{2}\right|$$

$$\int \frac{dx}{\cos^3 x} = \frac{\sin x}{2\cos^2 x} + \frac{1}{2}\ln\left|\tan\left(\frac{x}{2}+\frac{\pi}{4}\right)\right|$$

$$\int \sin x \cos x dx = -\frac{1}{4}\cos 2x$$

$$\int \sin^2 x \cos x dx = \frac{1}{3}\sin^3 x$$

$$\int \sin x \cos^2 x dx = -\frac{1}{3}\cos^3 x$$

$$\int \sin^2 x \cos^2 x dx = \frac{x}{8} - \frac{1}{32}\sin 4x$$

$$\int \tan x dx = -\ln|\cos x|$$

$$\int \frac{\sin x}{\cos^2 x} dx = \frac{1}{\cos x}$$

$$\int \frac{\sin^2 x}{\cos x} dx = \ln\left|\tan\left(\frac{x}{2}+\frac{\pi}{4}\right)\right| - \sin x$$

$$\int \tan^2 x dx = \tan x - x$$

$$\int \cot x dx = \ln|\sin x|$$

$$\int \frac{\cos x}{\sin^2 x} dx = -\frac{1}{\sin x}$$

$$\int \frac{\cos^2 x}{\sin x} dx = \ln\left|\tan\frac{x}{2}\right| + \cos x$$

$$\int \cot^2 x dx = -\cot x - x$$

$$\int \frac{dx}{\sin x \cos x} = \ln|\tan x|$$

$$\int \frac{dx}{\sin^2 x \cos x} = -\frac{1}{\sin x} + \ln\left|\tan\left(\frac{x}{2}+\frac{\pi}{4}\right)\right|$$

$$\int \frac{dx}{\sin x \cos^2 x} = \frac{1}{\cos x} + \ln\left|\tan\frac{x}{2}\right|$$

$$\int \frac{dx}{\sin^2 x \cos^2 x} = \tan x - \cot x$$

$$\int \sin mx \sin nx dx = -\frac{\sin(m+n)x}{2(m+n)} + \frac{\sin(m-n)x}{2(m-n)} \qquad m^2 \neq n^2$$

$$\int \sin mx \cos nx dx = -\frac{\cos(m+n)x}{2(m+n)} - \frac{\cos(m-n)x}{2(m-n)} \qquad m^2 \neq n^2$$

$$\int \cos mx \cos nx dx = \frac{\sin(m+n)x}{2(m+n)} + \frac{\sin(m-n)x}{2(m-n)} \qquad m^2 \neq n^2$$

$$\int \sin x \cos^n x dx = -\frac{\cos^{n+1} x}{n+1}$$

$$\int \sin^n x \cos x dx = \frac{\sin^{n+1} x}{n+1}$$

$$\int \arcsin x dx = x \arcsin x + \sqrt{1-x^2}$$

$$\int \arccos x dx = x \arccos x - \sqrt{1-x^2}$$

$$\int \arctan x dx = x \arctan x - \frac{1}{2}\ln\left(x^2+1\right)$$

$$\int \operatorname{arc}\cot x dx = x \operatorname{arc}\cot x + \frac{1}{2}\ln\left(x^2+1\right)$$