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11 Th Mathematics study matrials

For Students

CHAPTER 8 – Vector Algebra-I

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Topics :

- Scalars and vectors
- Representation of a vector and types of vectors
- Algebra of vectors
- Position vectors
- Resolution of vectors
- Direction cosines and directon ratios
- Product of vectors

CHAPTER 8 – Vector Algebra-I

Scalar :

A scalar is a quantity that is determined by its magnitude.

Example : distance, length, speed, temperature

vector

A **vector** is a quantity that is determined by both its magnitude and its direction and hence it is a directed line segment

Example : force, displacement, and velocity

Representation of a vector and types of vectors

Initial point and terminal point

The tail point A is called the **initial point** and the tip point B is called the **terminal point** of the vector $\vec{a}$. The initial point of a vector is also taken as origin of the vector

Free vector and localized vector

If we have a liberty to choose the origin of the vector at any point then it is said to be **free vector,** whereas if it is restricted to a certain specified point then the vector is said to be **localized vector.**

Co-initial vectors and co-terminous *vector*

Co-initial vectors are having the same initial point. On the other hand, the **co-terminous vectors** are having the same terminal point

collinear or parallel *vector*

Two or more vectors are said to be **collinear** or **parallel** if they have same line of action or have the lines of action parallel to one another.
Two or more vectors are said to be **coplanar** if they lie on the same plane or parallel to the same plane.

Equal vector

Two vectors are said to be **equal** if they have same direction and same magnitude

Zero vector

Zero vector is a vector which has zero magnitude and an arbitrary direction and it is denoted by . $\vec{0}$.

unit vector

A vector of magnitude 1 is called a **unit vector**

like vectors and unlike vectors :

Two vectors are said to be **like vectors** if they have the same direction. Two vectors are said to be **unlike vectors** if they have opposite directions.

Triangle law of addition

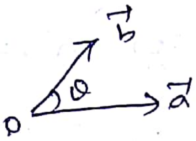
If two vectors are represented in magnitude and direction by the two sides of a triangle taken in order, then their sum is represented by the third side taken in the reverse order.

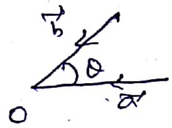
8.8 product of vectors:

1. sclar product (dot product)
2. vector product (cross product)

8.8.1 Angle between two vectors:

The angle between two vectors $\vec{a}$ & $\vec{b}$ is the angle between their directions when these directions both converges or both diverges from their point of intersection





Scalar product:

let $\vec{a}$ & $\vec{b}$ are any two non-zero vectors and θ is angle between $\vec{a}$ & $\vec{b}$. then sclar product of $\vec{a}$ and $\vec{b}$ is denoted by $\vec{a}\cdot\vec{b}$ and defined by

$$\vec{a}\cdot\vec{b} = |\vec{a}|\,|\vec{b}|\cos\theta$$

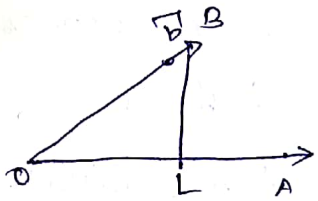
Geometric meaning of sclar (dot) product:

let $\vec{OA} = \vec{a}$

$\vec{OB} = \vec{b}$

θ be angle between $\vec{a}$ & $\vec{b}$

Draw $BL \perp^r OA$



OL is called projection of $\vec{B}$ on $\vec{a}$

$$\vec{a}\cdot\vec{b} = |\vec{a}|\,|\vec{b}|\cos\theta$$

$$= (OA)(OB)\left(\frac{OL}{OB}\right)$$

$$= OA\cdot(OL)$$

$$\vec{a}\cdot\vec{b} = |\vec{a}|\ (\text{Projection of } \vec{b} \text{ on } \vec{a})$$

$$\text{Projection of } \vec{b} \text{ on } \vec{a} = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|}$$

iii) projection of $\vec{a}$ on $\vec{b}$ $= \frac{\vec{b}\cdot\vec{a}}{|\vec{b}|}$.

Properties of sclar product:

(i) Sclar product of two vectors is commutative.

(ie) $\vec{a}\cdot\vec{b} = \vec{b}\cdot\vec{a}$

Proof $\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\theta$

$= |\vec{b}||\vec{a}|\cos\theta$

$\vec{a}\cdot\vec{b} = \vec{b}\cdot\vec{a}$

2) Nature of sclar product:

If $\theta = 0$ then $\vec{a}\cdot\vec{b} = ab$ (vectors parallel & same direction)

If $\theta = \pi$ then $\vec{a}\cdot\vec{b} = -ab$) " " & oposite direction

If $\theta = \pi/2$ then $\vec{a}\cdot\vec{b} = 0$ (vectors are perpendicular)

If $0 < \theta < \pi/2$ then $\vec{a}\cdot\vec{b} > 0$

If $\pi/2 < \theta < \pi$ then $\vec{a}\cdot\vec{b} < 0$.

3. $\vec{a}\cdot\vec{b} = 0 \Rightarrow |\vec{a}| = 0$ or $|\vec{b}| = 0$ or $\theta = \pi/2$

4. For any non zero vectors $\vec{a}$ & $\vec{b}$,

$\vec{a}\cdot\vec{b} = 0 \Leftrightarrow \vec{a} \perp \vec{b}$

5. $\vec{a}\cdot\vec{a} = |\vec{a}||\vec{a}| = |\vec{a}|^2 = a^2$

6.

$\cdot$	$\vec{i}$	$\vec{j}$	$\vec{k}$
$\vec{i}$	1	0	0
$\vec{j}$	0	1	0
$\vec{k}$	0	0	1

7. $\lambda\vec{a}\cdot\mu\vec{b} = \lambda\mu(\vec{a}\cdot\vec{b}) = (\lambda\mu\vec{a})\cdot\vec{b} = \vec{a}\cdot(\lambda\mu\vec{b})$

8. $\vec{a}\cdot(\vec{b}+\vec{c}) = \vec{a}\cdot\vec{b} + \vec{a}\cdot\vec{c}$

$(\vec{a}+\vec{b})\cdot\vec{c} = \vec{a}\cdot\vec{c} + \vec{b}\cdot\vec{c}$

$\vec{a}\cdot(\vec{b}-\vec{c}) = \vec{a}\cdot\vec{b} - \vec{a}\cdot\vec{c}$.

9) vector identities:

$$(\vec{a}+\vec{b})^2 = |\vec{a}|^2 + |\vec{b}|^2 + 2\vec{a}.\vec{b}$$

$$(\vec{a}-\vec{b})^2 = |\vec{a}|^2 + |\vec{b}|^2 - 2\vec{a}.\vec{b}$$

$$(\vec{a}+\vec{b})(\vec{a}-\vec{b}) = |\vec{a}|^2 - |\vec{b}|^2$$

10) If $\vec{a} = a_1\vec{i} + a_2\vec{j} + a_3\vec{j}$

$\vec{b} = b_1\vec{i} + b_2\vec{j} + b_3\vec{k}$ Then

$$\vec{a}.\vec{b} = a_1b_1 + a_2b_2 + a_3b_3$$

11) If θ is the angle between $\vec{a}$ & $\vec{b}$ then

$$\cos\theta = \frac{\vec{a}.\vec{b}}{|\vec{a}||\vec{b}|}$$

12) $|\vec{a}+\vec{b}| \leq |\vec{a}| + |\vec{b}|$

13 $|\vec{a}.\vec{b}| \leq |\vec{a}|\,|\vec{b}|$

Problems:

Result: If $\vec{a} = a_1\vec{i} + a_2\vec{j} + a_3\vec{k}$

$\vec{b} = b_1\vec{i} + b_2\vec{j} + b_3\vec{k}$

Then $\vec{a}.\vec{b} = a_1b_1 + a_2b_2 + a_3b_3$.

Problems:

1. Find $\vec{a}.\vec{b}$ when

(i) $\vec{a} = \vec{i} - \vec{j} + 5\vec{k}$ and $\vec{b} = 3\vec{i} - 2\vec{k}$

Soln $\vec{a} = \vec{i} - \vec{j} + 5\vec{k}$

$\vec{b} = 3\vec{i} + 0\vec{j} - 2\vec{k}$

$\therefore \vec{a}.\vec{b} = 3 - 0 - 10$

$= -7$

(ii) $\vec{a}$ & $\vec{b}$ represent the points $(2,3,-1)$, $(-1,2,3)$

Soln

$\vec{a} = 2\vec{i}+3\vec{j}-\vec{k}$

$\vec{b} = -\vec{i}+2\vec{j}+3\vec{k}$

$\therefore \vec{a}\cdot\vec{b} = -2+6-3$

$= 1$

2) Find $\vec{a}\cdot\vec{b}$ when

(i) $\vec{a} = \vec{i}-2\vec{j}+\vec{k}$ and $\vec{b} = 3\vec{i}-4\vec{j}-2\vec{k}$

Soln

$\vec{a} = \vec{i}-2\vec{j}+\vec{k}$

$\vec{b} = 3\vec{i}-4\vec{j}-2\vec{k}$

$\therefore \vec{a}\cdot\vec{b} = 3+8-2$

$= 9$

(ii) $\vec{a} = 2\vec{i}+2\vec{j}-\vec{k}$ and $\vec{b} = 6\vec{i}-3\vec{j}+2\vec{k}$

Soln

$\vec{a} = 2\vec{i}+2\vec{j}-\vec{k}$

$\vec{b} = 6\vec{i}-3\vec{j}+2\vec{k}$

$\therefore \vec{a}\cdot\vec{b} = 12-6-2$

$= 4.$

3. Find $(\vec{a}+3\vec{b})\cdot(2\vec{a}-\vec{b})$ if $\vec{a} = \vec{i}+\vec{j}+2\vec{k}$ and $\vec{b} = 3\vec{i}+2\vec{j}-\vec{k}$

Soln

$\vec{a} = \vec{i}+\vec{j}+2\vec{k}$

$3\vec{b} = 9\vec{i}+6\vec{j}-3\vec{k}$

$(\vec{a}+3\vec{b}) = 10\vec{i}+7\vec{j}-\vec{k}$

$2\vec{a} = 2\vec{i}+2\vec{j}+4\vec{k}$

$-\vec{b} = -3\vec{i}-2\vec{j}+\vec{k}$

$(2\vec{a}-\vec{b}) = -\vec{i}+0\vec{j}+5\vec{k}$

$(\vec{a}+3\vec{b})\cdot(2\vec{a}-\vec{b}) = (10)(-1)+(7)(0)+(-1)(5)$

$= -10+0-5$

$= -15$

3.3 (3)

Note $\vec{a} \perp \vec{b} \Rightarrow \vec{a}\cdot\vec{b} = 0.$

4. Find the value of λ for which the vectors $\vec{a}$ & $\vec{b}$ are $\perp$r where

(i) $\vec{a} = 2\vec{i} + \lambda\vec{j} + \vec{k}$ and $\vec{b} = \vec{i} - 2\vec{j} + 3\vec{k}$

Soln

$$\vec{a} \perp \vec{b}$$
$$\Rightarrow \vec{a}\cdot\vec{b} = 0$$
$$2(1) + \lambda(-2) + (1)(3) = 0$$
$$2 - 2\lambda + 3 = 0$$
$$-2\lambda = -5$$
$$\lambda = 5/2$$

(ii) $\vec{a} = 2\vec{i} + 4\vec{j} - \vec{k}$ and $\vec{b} = 3\vec{i} - 2\vec{j} + \lambda\vec{k}$

Soln

$$\vec{a} \perp \vec{b}$$
$$\Rightarrow \vec{a}\cdot\vec{b} = 0$$
$$(2)(3) + 4(-2) + (-1)(\lambda) = 0$$
$$6 - 8 - \lambda = 0$$
$$-2 - \lambda = 0$$
$$-\lambda = 2$$
$$\lambda = -2$$

5. If $\vec{a} = 2\vec{i} + 2\vec{j} + 3\vec{k}$, $\vec{b} = -\vec{i} + 2\vec{j} + \vec{k}$ & $\vec{c} = 3\vec{i} + \vec{j}$ S.T $\vec{a} + \lambda\vec{b}$ is $\perp$r to $\vec{c}$

Soln

$$(\vec{a} + \lambda\vec{b}) \perp \vec{c}$$
$$\Rightarrow (\vec{a} + \lambda\vec{b})\cdot\vec{c} = 0$$
$$\cancel{(2+\lambda(-1))\,0 + (2+2\lambda)(3) + (3+\lambda)}(1) = 0$$
$$6 + 6\lambda \cancel{+ \lambda + 3} = 0$$
$$(\vec{a}\cdot\vec{c}) + \lambda(\vec{b}\cdot\vec{c}) = 0$$
$$(6 + 2 + 0) + \lambda(-3 + 2) = 0$$
$$8 - \lambda = 0 \qquad \lambda = 8$$

6) If $|\vec{a}+\vec{b}| = |\vec{a}-\vec{b}|$ P.T $\vec{a} \perp \vec{b}$

Soln

$$|\vec{a}+\vec{b}| = |\vec{a}-\vec{b}|$$

$$|\vec{a}+\vec{b}|^2 = |\vec{a}-\vec{b}|^2$$

$$(\vec{a}+\vec{b})^2 = (\vec{a}-\vec{b})^2$$

$$|\vec{a}|^2+|\vec{b}|^2+2\vec{a}\cdot\vec{b} = |\vec{a}|^2+|\vec{b}|^2-2\vec{a}\cdot\vec{b}$$

$$2\vec{a}\cdot\vec{b} = -2\vec{a}\cdot\vec{b}$$

$$4\vec{a}\cdot\vec{b} = 0$$

$$\vec{a}\cdot\vec{b} = 0$$

$$\Rightarrow \vec{a} \perp \vec{b}$$

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7) Show that the vectors $\vec{a} = 2\vec{i}+3\vec{j}+6\vec{k}$, $\vec{b} = 6\vec{i}+2\vec{j}-3\vec{k}$, $\vec{c} = 3\vec{i}-6\vec{j}+2\vec{k}$ are mutually ~~perpen~~ orthogonal.

Soln

$$\vec{a} = 2\vec{i}+3\vec{j}+6\vec{k}$$

$$\vec{b} = 6\vec{i}+2\vec{j}-3\vec{k}$$

$$\vec{c} = 3\vec{i}-6\vec{j}+2\vec{k}$$

$\vec{a}\cdot\vec{b} = 12+6-18 = 0 \Rightarrow \vec{a}\perp\vec{b}$

$\vec{b}\cdot\vec{c} = 18-12-6 = 0 \Rightarrow \vec{b}\perp\vec{c}$

$\vec{c}\cdot\vec{a} = 6-18+12 = 0. \Rightarrow \vec{c}\perp\vec{a}$

$\therefore \vec{a}, \vec{b}, \vec{c}$ are mutually orthogonal.

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8. Show that the vectors $-\vec{i}-2\vec{j}-6\vec{k}$, $2\vec{i}-\vec{j}+\vec{k}$ and $-\vec{i}+3\vec{j}+5\vec{k}$ form a right angle triangle.

Soln

Let $\vec{AB} = -\vec{i}-2\vec{j}-6\vec{k}$

$\vec{BC} = 2\vec{i}-\vec{j}+\vec{k}$

$\vec{CA} = -\vec{i}+3\vec{j}+5\vec{k}$

$AB = |\vec{AB}| = \sqrt{1+4+36} = \sqrt{41}$

$BC = |\vec{BC}| = \sqrt{4+1+1} = \sqrt{6}$

$CA = |\vec{CA}| = \sqrt{1+9+25} = \sqrt{35}$

$\vec{BC}\cdot\vec{CA} = -2-3+5$
$= 0$

$\vec{BC} \perp \vec{CA}$

$\Rightarrow ABC$ form a right angle triangle

$BC^2+CA^2 = 6+35$
$= 41$
$= (\sqrt{41})^2$
$= AB^2$

$\therefore$ Given vector form a right angled triangle.

9) Show that the points (2,-1,3) (4,3,1) (3,1,2) are collinear

Soln

Let $\vec{OA} = 2\vec{i} - \vec{j} + 3\vec{k}$

$\vec{OB} = 4\vec{i} + 3\vec{j} + \vec{k}$

$\vec{OC} = 3\vec{i} + \vec{j} + 2\vec{k}$

$\vec{AB} = \vec{OB} - \vec{OA}$

$= 2\vec{i} + 4\vec{j} - 2\vec{k}$

$AB = \sqrt{4+16+4}$

$= \sqrt{24}$

$\vec{BC} = \vec{OC} - \vec{OB}$

$= -\vec{i} - 2\vec{j} + \vec{k}$

$|\vec{BC}| = \sqrt{1+4+1}$

$= \sqrt{6}$

$AB \cdot BC = \sqrt{24} \cdot \sqrt{6}$

$= \sqrt{8 \times 3}\ \sqrt{3 \times 2}$

$= \sqrt{2 \times 2 \times 2 \times 2 \times 3 \times 3}$

$= 2 \times 2 \times 3$

$= 12 \longrightarrow ①$

$\vec{AB} \cdot \vec{BC} = -2 - 8 - 2$

$= -12$

$= -(AB)(BC)$ by ①

$\therefore$ AB, BC given are collinear

$\therefore$ A, B, C are collinear.

10) For any $\vec{r}$ S.T $\vec{r} = (\vec{r}\cdot\vec{i})\vec{i} + (\vec{r}\cdot\vec{j})\vec{j} + (\vec{r}\cdot\vec{k})\vec{k}$

Soln

Let $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$

$(\vec{r}\cdot\vec{i}) = (x\vec{i} + y\vec{j} + z\vec{k}) \cdot (\vec{i} + 0\vec{j} + 0\vec{k})$

$= x + 0 + 0$

$= x$

$(\vec{r}\cdot\vec{i})\vec{i} = x\vec{i}$

III'y $(\vec{r}\cdot\vec{j})\vec{j} = y\vec{j}$

$(\vec{r}\cdot\vec{k})\vec{k} = z\vec{k}$

$\therefore (\vec{r}\cdot\vec{i})\vec{i} + (\vec{r}\cdot\vec{j})\vec{j} + (\vec{r}\cdot\vec{k})\vec{k} = x\vec{i} + y\vec{j} + z\vec{k}$

$= \vec{r}$

(H.P)

11. If $\vec{a}, \vec{b}$ are unit vector and θ is the angle between then S.T

(i) $\sin \theta/2 = \frac{1}{2}|\vec{a}-\vec{b}|$ (ii) $\cos \theta/2 = \frac{1}{2}|\vec{a}+\vec{b}|$

(iii) $\tan \theta/2 = \frac{|\vec{a}-\vec{b}|}{|\vec{a}+\vec{b}|}$

Soln

Since $\vec{a}$ & $\vec{b}$ are unit vector.

$\therefore \left.\begin{array}{l}|\vec{a}|=1\\|\vec{b}|=1\end{array}\right\} \rightarrow (1)$

(i)

$|\vec{a}-\vec{b}|^2 = |\vec{a}|^2+|\vec{b}|^2-2\vec{a}\cdot\vec{b}$

$|\vec{a}-\vec{b}|^2 = 1+1-2|\vec{a}||\vec{b}|\cos\theta$

$|\vec{a}-\vec{b}|^2 = 2-2(1)(1)\cos\theta$

$|\vec{a}-\vec{b}|^2 = 2(1-\cos\theta)$ $1-\cos\theta = 2\sin^2\theta/2$

$|\vec{a}-\vec{b}|^2 = 2(2\sin^2\theta/2)$

$|\vec{a}-\vec{b}| = 2\sin\theta/2$

$\sin\theta/2 = \frac{1}{2}|\vec{a}-\vec{b}|. \longrightarrow (2)$

(ii) $|\vec{a}+\vec{b}|^2 = |\vec{a}|^2+|\vec{b}|^2+2\vec{a}\cdot\vec{b}$

$|\vec{a}+\vec{b}|^2 = 1+1+2|\vec{a}||\vec{b}|\cos\theta.$

$|\vec{a}+\vec{b}|^2 = 2+2(1)(1)\cos\theta$

$|\vec{a}+\vec{b}|^2 = 2+2\cos\theta$

$|\vec{a}+\vec{b}|^2 = 2(1+\cos\theta)$ $1+\cos\theta = 2\cos^2\theta/2$

$|\vec{a}+\vec{b}|^2 = 2(2\cos^2\theta/2)$

$|\vec{a}+\vec{b}| = 2\cos\theta/2$

$\cos\theta/2 = \frac{1}{2}|\vec{a}+\vec{b}| \longrightarrow (3)$

(iii) $\frac{(2)}{(3)} \Rightarrow \frac{\sin\theta/2}{\cos\theta/2} = \frac{|\vec{a}-\vec{b}|}{|\vec{a}-\vec{b}|}$

$\tan\theta/2 = \frac{|\vec{a}-\vec{b}|}{|\vec{a}+\vec{b}|}$

12) If $|\vec{a}| = 5$, $|\vec{b}| = 6$, $|\vec{c}| = 7$ and $\vec{a}+\vec{b}+\vec{c} = 0$ find $\vec{a}.\vec{b} + \vec{b}.\vec{c} + \vec{c}.\vec{a}$

Soln

$(\vec{a}+\vec{b}+\vec{c})^2 = 0$

$(\vec{a}+\vec{b}+\vec{c})^2 = 0$

$|\vec{a}|^2+|\vec{b}|^2+|\vec{c}|^2 + 2(\vec{a}.\vec{b}+\vec{b}.\vec{c}+\vec{c}.\vec{a}) = 0$

$25+36+49+2(\vec{a}.\vec{b}+\vec{b}.\vec{c}+\vec{c}.\vec{a}) = 0$

$2(\vec{a}.\vec{b}+\vec{b}.\vec{c}+\vec{c}.\vec{a}) = -110$

$\vec{a}.\vec{b}+\vec{b}.\vec{c}+\vec{c}.\vec{a} = \frac{-110}{2} = -55.$

13) Let $\vec{a}, \vec{b}, \vec{c}$ be three vectors such that $|\vec{a}| = 3$, $|\vec{b}| = 4$, $|\vec{c}| = 5$ and each one of them perpendicular to the sum of other two find $|\vec{a}+\vec{b}+\vec{c}|$

Soln

Given that

(i) $|\vec{a}| = 3$, $|\vec{b}| = 4$, $|\vec{c}| = 5 \longrightarrow (1)$

Let $\vec{a} \perp r$ to $(\vec{b}+\vec{c})$

$\therefore \vec{a}.(\vec{b}+\vec{c}) = 0$

$\vec{a}.\vec{b}+\vec{a}.\vec{c} = 0 \longrightarrow (2)$ llly

$|\vec{a}+\vec{b}+\vec{c}|^2 = |\vec{a}|^2+|\vec{b}|^2+|\vec{c}|^2+2($

Note:
If θ is the angle between $\vec{a}$ & $\vec{b}$ then

$$\cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|}$$

Problems:

1. If $|\vec{a}| = 10$, $|\vec{b}| = 15$ and $\vec{a}\cdot\vec{b} = 75\sqrt{2}$ find the angle between $\vec{a}$ & $\vec{b}$

Soln $|\vec{a}| = 10$ $|\vec{b}| = 15$, $\vec{a}\cdot\vec{b} = 75\sqrt{2}$

Let θ is the angle between $\vec{a}$ & $\vec{b}$

Then
$$\cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|}$$
$$= \frac{75\sqrt{2}}{10\times 15}$$
$$= \frac{\sqrt{2}}{2}$$
$$\cos\theta = \frac{1}{\sqrt{2}}$$
$$\theta = \cos^{-1}(1/\sqrt{2}) = \pi/4.$$

2. Find the angle between the vectors.

(i) $\vec{a} = 2\vec{i}+3\vec{j}-6\vec{k}$ & $\vec{b} = 6\vec{i}-3\vec{j}+2\vec{k}$

Soln $\vec{a}\cdot\vec{b} = (2)(6)+(3)(-3)+(-6)(2)$
$= 12-9-12$
$= -9$

$|\vec{a}| = \sqrt{4+9+36} = \sqrt{49} = 7$

$|\vec{b}| = \sqrt{36+9+4} = \sqrt{49} = 7$

Let θ is angle between $\vec{a}$ & $\vec{b}$

Then
$$\cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|}$$
$$= \frac{-9}{7\times 7}$$
$$\cos\theta = \frac{-9}{49} \qquad \theta = \cos^{-1}(-9/49)$$

(ii) $\vec{a} = \vec{i} - \vec{j}$ & $\vec{b} = \vec{j} - \vec{k}$

Soln

$\vec{a} \cdot \vec{b} = 0 - 1 + 0 = -1$

$|\vec{a}| = \sqrt{1+1} = \sqrt{2}$

$|\vec{b}| = \sqrt{1+1} = \sqrt{2}$

If θ is the angle between $\vec{a}$ & $\vec{b}$

Then $\cos\theta = \dfrac{\vec{a} \cdot \vec{b}}{|\vec{a}||\vec{b}|}$

$\cos\theta = \dfrac{-1}{\sqrt{2}\sqrt{2}}$

$\cos\theta = \dfrac{-1}{2}$

$\theta = \cos^{-1}(-1/2)$

$= 2\pi/3$

3) Find the angle between the vectors $5\vec{i} + 3\vec{j} + 4\vec{k}$ & $6\vec{i} - 8\vec{j} - \vec{k}$

Soln Let $\vec{a} = 5\vec{i} + 3\vec{j} + 4\vec{k}$

$\vec{b} = 6\vec{i} - 8\vec{j} - \vec{k}$

$\vec{a} \cdot \vec{b} = 30 - 24 - 4$

$\vec{a} \cdot \vec{b} = 2$

$|\vec{a}| = \sqrt{25+9+16} = \sqrt{50} = 5\sqrt{2}$

$|\vec{b}| = \sqrt{36+64+1} = \sqrt{101}$

If θ is the angle between $\vec{a}$ & $\vec{b}$ then

$\cos\theta = \dfrac{\vec{a} \cdot \vec{b}}{|\vec{a}||\vec{b}|}$

$= \dfrac{2}{(5\sqrt{2})\sqrt{101}}$

$\cos\theta = \dfrac{\sqrt{2}}{5\sqrt{101}}$

$\theta = \cos^{-1}\left(\dfrac{\sqrt{2}}{5\sqrt{101}}\right)$

4) If $\vec{a}, \vec{b}$ and $\vec{c}$ are 3 unit vectors satisfying $\vec{a} - \sqrt{3}\vec{b} + \vec{c} = 0$ find the angle between $\vec{a}$ & $\vec{c}$

Soln Let θ be the angle between $\vec{a}$ & $\vec{c}$

$$\vec{a} - \sqrt{3}\vec{b} + \vec{c} = 0$$
$$|\vec{a} + \vec{c}| = |\sqrt{3}\vec{b}|$$
$$|\vec{a} + \vec{c}| = \sqrt{3}|\vec{b}|$$
$$|\vec{a} + \vec{c}| = \sqrt{3}$$
$$|\vec{a} + \vec{c}|^2 = 3$$
$$|\vec{a}|^2 + |\vec{c}|^2 + 2\vec{a}\cdot\vec{c} = 3$$
$$1 + 1 + 2|\vec{a}||\vec{c}|\cos\theta = 3$$
$$2 + 2(1)(1)\cos\theta = 3$$
$$2\cos\theta = 1$$
$$\cos\theta = 1/2$$
$$\theta = \cos^{-1} 1/2$$
$$= \pi/3$$

5. Show that points $(4,3,-1)$ $(2,-4,5)$ & $(1,-1,0)$ form a right angle triangle.

Soln Trivially they form a triangle
it is enough show that one angle is $\pi/2$
so find the sides of the triangle.

let $\vec{OA} = 4\vec{i} + 3\vec{j} - \vec{k}$
$\vec{OB} = 2\vec{i} - 4\vec{j} - 5\vec{k}$
$\vec{OC} = \vec{i} - \vec{j} + 0\vec{k}$

$\vec{AB} = \vec{OB} - \vec{OA} = -2\vec{i} - \vec{j} + 4\vec{k}$
$\vec{BC} = \vec{OC} - \vec{OB} = -\vec{i} + 3\vec{j} - 5\vec{k}$
$\vec{CA} = \vec{OA} - \vec{OC} = 3\vec{i} - 2\vec{j} + \vec{k}$

$\vec{AB}\cdot\vec{CA} = -6 + 2 + 4 = 0$
$\therefore \vec{AB} \perp \vec{CA}$
$\Rightarrow \triangle ABC$ form a right angle triangle.

Note: The projection of $\vec{a}$ on $\vec{b} = \dfrac{\vec{a}\cdot\vec{b}}{|\vec{b}|}$

1. Find the projection of $\vec{i}+3\vec{j}+7\vec{k}$ on $2\vec{i}+6\vec{j}+3\vec{k}$

Soln

Let $\vec{a} = \vec{i}+3\vec{j}+7\vec{k}$

$\vec{b} = 2\vec{i}+6\vec{j}+3\vec{k}$

$\vec{a}\cdot\vec{b} = 2+18+21 = 41$

$|\vec{b}| = \sqrt{4+36+9} = 7$

projection of $\vec{a}$ on $\vec{b} = \dfrac{\vec{a}\cdot\vec{b}}{|\vec{b}|}$

$= \dfrac{41}{7}$ units

2. Find λ, when the projection of $\vec{a} = \lambda\vec{i}+\vec{j}+4\vec{k}$ on the vector $\vec{b} = 2\vec{i}+6\vec{j}+3\vec{k}$ is 4 units.

Soln

$\vec{a} = \lambda\vec{i}+\vec{j}+4\vec{k}$

$\vec{b} = 2\vec{i}+6\vec{j}+3\vec{k}$

$\vec{a}\cdot\vec{b} = 2\lambda+6+12 = 2\lambda+18$

$|\vec{b}| = \sqrt{4+36+9} = \sqrt{49} = 7$

now

projection of $\vec{a}$ on $\vec{b} = 4$

$\dfrac{\vec{a}\cdot\vec{b}}{|\vec{b}|} = 4$

$\dfrac{2\lambda+18}{7} = 4$

$2\lambda+18 = 28$

$2\lambda = 28-18$

$2\lambda = 10$

$\lambda = 10/2$

$\lambda = 5$.

9) Find the projection of $\vec{AB}$ on $\vec{CD}$ where A, B, C, D are the points (4, -3, 0) (7, -5, -1) (-2, 1, 3) (0, 2, 3)

Soln

$$\vec{OA} = 4\vec{i} - 3\vec{j} + 0\vec{k}$$
$$\vec{OB} = 7\vec{i} - 5\vec{j} - \vec{k}$$
$$\vec{OC} = -2\vec{i} + \vec{j} + 3\vec{k}$$
$$\vec{OD} = 0\vec{i} + 2\vec{j} + 3\vec{k}$$

$$\vec{AB} = \vec{OB} - \vec{OA} = 3\vec{i} - 2\vec{j} - \vec{k}$$
$$\vec{CD} = \vec{OD} - \vec{OC} = 2\vec{i} + \vec{j} + 2\vec{k}$$

$$\vec{AB} \cdot \vec{CD} = 6 - 2 - 2 = 2$$
$$|\vec{CD}| = \sqrt{4+1+4} = 3$$

$$\text{projection of } \vec{AB} \text{ on } \vec{CD} = \frac{\vec{AB} \cdot \vec{CD}}{|\vec{CD}|}$$
$$= 2/3.$$

8.8.4 Vector product

Vector product:

Let $\vec{a}$ & $\vec{b}$ be any two non zero vectors. Then the vector product of $\vec{a}$ & $\vec{b}$ is denoted by $\vec{a} \times \vec{b}$ and it is defined by

$$\vec{a} \times \vec{b} = |\vec{a}|\,|\vec{b}| \sin\theta \,\hat{n}$$

where θ is angle between $\vec{a}$ & $\vec{b}$ and $\hat{n}$ is unit $\perp r$ to $\vec{a}$ & $\vec{b}$

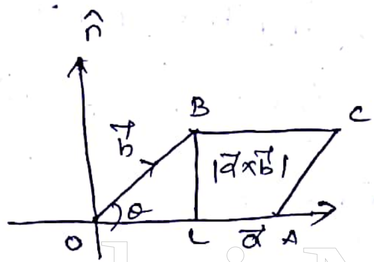
Geometrical meaning of vector product

Let $\vec{OA} = \vec{a}$

$\vec{OB} = \vec{b}$

$\angle AOB = \theta$.

Construct the parallogram OACB



from diagram

$$\sin\theta = \frac{BL}{OB}$$

$$BL = (OB)\sin\theta.$$

now

$$|\vec{a} \times \vec{b}| = |\vec{a}|\,|\vec{b}| \sin\theta$$
$$= |\vec{a}|\,(BL)$$
$$= \text{base} \times \text{hight}$$
$$= \text{area of the parallogram OACB}$$

Deduction:

1. The area of triangle whose sides are $\vec{a}$ & $\vec{b}$ $= \frac{1}{2}|\vec{a} \times \vec{b}|$

2. The area of parallogram whose sides are $\vec{a}$ & $\vec{b}$ $\} = |\vec{a} \times \vec{b}|$

Properties:

1. vector product is non commutative

proof $\vec{b} \times \vec{a} = |\vec{b}||\vec{a}| \sin\theta (-\hat{n})$ $\because$ $\vec{b}, \vec{a}, -\hat{n}$ form a right handed system.

$= -|\vec{a}||\vec{b}| \sin\theta \, \hat{n}$

$\vec{b} \times \vec{a} = -(\vec{a} \times \vec{b})$

2. If two vectors are parallel (collinear) Then $\vec{a} \times \vec{b} = 0$.

But $\vec{a} \times \vec{b} = 0 \Rightarrow \vec{a} = 0$ or $\vec{b} = 0$ or $\vec{a} \parallel \vec{b}$

3. For any two non zero vector $\vec{a}$ and $\vec{b}$,

$\vec{a} \times \vec{b} = 0 \Leftrightarrow \vec{a}$ parallel to $\vec{b}$

4. $\vec{a} \times \vec{a} = 0$

4 (a) $\vec{i}, \vec{j}, \vec{k}$ are right handed system.

$\therefore$ $\vec{i} \times \vec{i} = 0$ $\quad \vec{i} \times \vec{j} = \vec{k}$ $\quad \vec{j} \times \vec{i} = -\vec{k}$

$\vec{j} \times \vec{j} = 0$ $\quad \vec{j} \times \vec{k} = \vec{i}$ $\quad \vec{k} \times \vec{j} = -\vec{i}$

$\vec{k} \times \vec{k} = 0$ $\quad \vec{k} \times \vec{i} = \vec{j}$ $\quad \vec{i} \times \vec{k} = -\vec{j}$

$\times$	$\vec{i}$	$\vec{j}$	$\vec{k}$
$\vec{i}$	0	$\vec{k}$	$-\vec{j}$
$\vec{j}$	$-\vec{k}$	0	$\vec{i}$
$\vec{k}$	$\vec{j}$	$-\vec{i}$	0

5. $m\vec{a} \times n\vec{b} = mn(\vec{a} \times \vec{b}) = (mn\vec{a}) \times \vec{b} = \vec{a} \times (mn\vec{b})$

6. vector product is distributive over addition.

(ie) $\vec{a} \times (\vec{b} + \vec{c}) = (\vec{a} \times \vec{b}) + (\vec{a} \times \vec{c})$

$(\vec{a} + \vec{b}) \times \vec{c} = (\vec{a} \times \vec{c}) + (\vec{b} \times \vec{c})$

$\vec{a} \times (\vec{b} - \vec{c}) = (\vec{a} \times \vec{b}) - (\vec{a} \times \vec{c})$.

8) Working rule to find the cross product

$\vec{a} = a_1\vec{i} + a_2\vec{j} + a_3\vec{k}$

$\vec{b} = b_1\vec{i} + b_2\vec{j} + b_3\vec{k}$

$$\vec{a}\times\vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

9) If θ is angle between $\vec{a}$ & $\vec{b}$ then

$$\sin\theta = \frac{|\vec{a}\times\vec{b}|}{|\vec{a}||\vec{b}|}$$

10) The unit vector $\perp$r to both $\vec{a}$ & $\vec{b}$ are

$$\pm\hat{n} = \pm\frac{(\vec{a}\times\vec{b})}{|\vec{a}\times\vec{b}|}$$

problems:

1. Find $|\vec{a}\times\vec{b}|$ where $\vec{a} = 3\vec{i}+4\vec{j}$ & $\vec{b} = \vec{i}+\vec{j}+\vec{k}$

soln

$$\vec{a}\times\vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & 4 & 0 \\ 1 & 1 & 1 \end{vmatrix}$$

$$= \vec{i}(4-0) - \vec{j}(3-0) + \vec{k}(3-4)$$

$$\vec{a}\times\vec{b} = 4\vec{i} - 3\vec{j} - \vec{k}$$

$$|\vec{a}\times\vec{b}| = \sqrt{16+9+1}$$

$$= \sqrt{26}.$$

2. Find the magnitude of $\vec{a}\times\vec{b}$ if $\vec{a} = 2\vec{i}+\vec{j}+3\vec{k}$ and $\vec{b} = 3\vec{i}+5\vec{j}-2\vec{k}$

soln

$$\vec{a}\times\vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 1 & 3 \\ 3 & 5 & -2 \end{vmatrix}$$

$$= \vec{i}(-2-15) - \vec{j}(-4-9) + \vec{k}(10-3)$$

$$\vec{a}\times\vec{b} = -17\vec{i} + 13\vec{j} + 7\vec{k}$$

$$|\vec{a}\times\vec{b}| = \sqrt{289+169+49}$$

$$= \sqrt{507}$$

37) S.T $\vec{a}\times(\vec{b}+\vec{c})+\vec{b}\times(\vec{c}+\vec{a})+\vec{c}\times(\vec{a}+\vec{b})=\vec{0}$

Soln

$\vec{a}\times(\vec{b}+\vec{c})+\vec{b}\times(\vec{c}+\vec{a})+\vec{c}\times(\vec{a}+\vec{b})$

$=(\vec{a}\times\vec{b})+(\vec{a}\times\vec{c})+(\vec{b}\times\vec{c})+(\vec{b}\times\vec{a})+(\vec{c}\times\vec{a})+(\vec{c}\times\vec{b})$

$=(\vec{a}\times\vec{b})+(\vec{a}\times\vec{c})+(\vec{b}\times\vec{c})-(\vec{a}\times\vec{b})-(\vec{a}\times\vec{c})-(\vec{b}\times\vec{c})$

$=\vec{0}.$

Note:

Area of parallogram whose adjacent sides are $\vec{a}$ & $\vec{b}$ is

$|\vec{a}\times\vec{b}|$

1. Find the area of the parallogram whose adjacent sides are $\vec{a}=3\vec{i}+\vec{j}+4\vec{k}$ $\vec{b}=\vec{i}-\vec{j}+\vec{k}$

Soln

$$\vec{a}\times\vec{b}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ 3 & 1 & 4\\ 1 & -1 & 1\end{vmatrix}$$

$=\vec{i}(1+4)-\vec{j}(3-4)+\vec{k}(-3-1)$

$=5\vec{i}+\vec{j}-4\vec{k}$

$|\vec{a}\times\vec{b}|=\sqrt{25+1+16}$

$=\sqrt{42}$

$\therefore$ Area of the parallogram is $\sqrt{42}$ sq. units.

2. Find the area of the parallogram whose adjacent sides are $\vec{a}=1\vec{i}+2\vec{j}+3\vec{k}$ & $\vec{b}=3\vec{i}-2\vec{j}+\vec{k}$

Soln

$$\vec{a}\times\vec{b}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ 1 & 2 & 3\\ 3 & -2 & 1\end{vmatrix}$$

$=\vec{i}(2+6)-\vec{j}(1-9)+\vec{k}(-2-6)$

$=8\vec{i}+8\vec{j}-8\vec{k}$

$=8(\vec{i}+\vec{j}-\vec{k})$

$|\vec{a}\times\vec{b}|=8\sqrt{1+1+1}=8\sqrt{3}$

$\therefore$ area of parallogram $=8\sqrt{3}$ squ units.

8.4 (3)

Note:
The area of triangle whose sides are $\vec{a}, \vec{b} = \frac{1}{2}|\vec{a}\times\vec{b}|$

problem:

1. Find the area of a triangle having the points $A(1,0,0)$ $B(0,1,0)$ $C(0,0,1)$ as its vertices.

Soln: Let us find two sides of the triangle.

Let $\vec{OA} = \vec{i}$

$\vec{OB} = \vec{j}$

$\vec{OC} = \vec{k}$

$\therefore \vec{AB} = \vec{OB} - \vec{OA} = -\vec{i}+\vec{j}$

$\vec{AC} = \vec{OC} - \vec{OA} = -\vec{i}+\vec{k}$

$$\vec{AB}\times\vec{AC} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{vmatrix}$$

$$= \vec{i} - \vec{j}(-1) + \vec{k}(1)$$

$$= \vec{i}+\vec{j}+\vec{k}$$

$$|\vec{AB}\times\vec{AC}| = \sqrt{1+1+1} = \sqrt{3}$$

Area of triangle $= \frac{1}{2}|\vec{AB}\times\vec{AC}|$

$= \sqrt{3}/2$ sq. units.

2. Find the area of the triangle whose vertices are $A(3,-1,2)$ $B(1,-1,-3)$ $C(4,-3,1)$

Soln: Let us find two sides of the triangle.

$\vec{OA} = 3\vec{i}-\vec{j}+2\vec{k}$

$\vec{OB} = \vec{i}-\vec{j}-3\vec{k}$

$\vec{OC} = 4\vec{i}-3\vec{j}+\vec{k}$

$\vec{AB} = \vec{OB} - \vec{OA} = -2\vec{i}+0\vec{j}-5\vec{k}$

$\vec{AC} = \vec{OC} - \vec{OA} = \vec{i}-2\vec{j}-\vec{k}$

$$\vec{AB}\times\vec{AC} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -2 & 0 & -5 \\ 1 & -2 & -1 \end{vmatrix}$$

$$= -10\vec{i} - 3\vec{j} - 4\vec{k}$$

$$|\vec{AB}\times\vec{AC}| = \sqrt{100+9+16}$$

$$= \sqrt{125}$$

$$= 5\sqrt{5}$$

$\therefore$ Area of triangle $= \frac{1}{2}|\vec{AB}\times\vec{AC}|$

$$= \frac{5\sqrt{5}}{2} \text{ sq. units.}$$

3. If $\vec{a}, \vec{b}, \vec{c}$ are position vector of the vertices A, B, C of the triangle. Show that the area of triangle ABC is $\frac{1}{2}.|\vec{a}\times\vec{b} + \vec{b}\times\vec{c} + \vec{c}\times\vec{a}|$ Also deduce the condition for collinearity of the points A, B, C

soln

Note: If θ is the angle between $\vec{a}$ & $\vec{b}$ then

$$\sin\theta = \frac{|\vec{a}\times\vec{b}|}{|\vec{a}|.|\vec{b}|}$$

Problem:

1. Find the angle between the vectors $2\vec{i}+\vec{j}-\vec{k}$ & $\vec{i}+2\vec{j}+\vec{k}$ using vector product.

Soln Let $\vec{a} = 2\vec{i}+\vec{j}-\vec{k}$

$\vec{b} = \vec{i}+2\vec{j}+\vec{k}$

$$\vec{a}\times\vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 1 & -1 \\ 1 & 2 & 1 \end{vmatrix}$$

$= \vec{i}(1+2) - \vec{j}(2+1) + \vec{k}(4-1)$

$= 3\vec{i} - 3\vec{j} + 3\vec{k}$

$|\vec{a}\times\vec{b}| = 3\sqrt{1+1+1}$

$= 3\sqrt{3}$

$|\vec{a}| = \sqrt{4+1+1} = \sqrt{6}$

$|\vec{b}| = \sqrt{1+4+1} = \sqrt{6}$.

If θ is the angle between $\vec{a}$ & $\vec{b}$ then

$$\sin\theta = \frac{|\vec{a}\times\vec{b}|}{|\vec{a}||\vec{b}|}$$

$$= \frac{3\sqrt{3}}{\sqrt{6}\sqrt{6}}$$

$$= \frac{3\sqrt{3}}{6}$$

$$\sin\theta = \frac{\sqrt{3}}{2}$$

$$\theta = \sin^{-1}\left(\frac{\sqrt{3}}{2}\right) = \pi/2$$

2. Find cosine & sine angle between the vectors $\vec{a} = 2\vec{i}+\vec{j}+3\vec{k}$ $\vec{b} = 4\vec{i}-2\vec{j}+2\vec{k}$

Soln Let $\vec{a} = 2\vec{i}+\vec{j}+3\vec{k}$ $|\vec{a}| = \sqrt{4+1+9} = \sqrt{14}$

$\vec{b} = 4\vec{i}-2\vec{j}+2\vec{k}$ $|\vec{b}| = \sqrt{16+4+4} = \sqrt{24}$

$\vec{a}\cdot\vec{b} = 8-2+6 = 12$

$$\cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|} = \frac{12}{\sqrt{14}\sqrt{24}} = \sqrt{3/7}$$

$$\vec{a}\times\vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 1 & 3 \\ 4 & -2 & 2 \end{vmatrix}$$

$= \vec{i}(2+6) - \vec{j}(4-12) + \vec{k}(-4-4)$

$= 8\vec{i} + 8\vec{j} - 8\vec{k}$

$= 8(\vec{i}+\vec{j}-\vec{k})$

$|\vec{a}\times\vec{b}| = 8\sqrt{1+1+1} = 8\sqrt{3}$

now $\sin\theta = \dfrac{|\vec{a}\times\vec{b}|}{|\vec{a}||\vec{b}|}$

$= \dfrac{8\sqrt{3}}{\sqrt{14}\sqrt{24}}$

$= \dfrac{4\sqrt{3}}{\sqrt{7}\sqrt{12}}$

$= \dfrac{2}{\sqrt{7}}$

Note $\vec{a}\perp\vec{b} \Rightarrow \vec{a}\cdot\vec{b} = 0.$

3. If $\vec{a} = -3\vec{i}+4\vec{j}-7\vec{k}$ and $\vec{b} = 6\vec{i}+2\vec{j}-3\vec{k}$ verify

(i) $\vec{a} \perp \vec{a}\times\vec{b}$

(ii) $\vec{b}$ lr $\vec{a}\times\vec{b}$

soln

$$\vec{a}\times\vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -3 & 4 & -7 \\ 6 & 2 & -3 \end{vmatrix}$$

$= \vec{i}(-12+14) - \vec{j}(9+42) + \vec{k}(-6-24)$

$= 2\vec{i} - 51\vec{j} - 30\vec{k}$

$\vec{a}\cdot(\vec{a}\times\vec{b}) = -6+20+210$

$= 0$

$\therefore \vec{a}$ lr $\vec{a}\times\vec{b}$

(ii) $\vec{b} \cdot (\vec{a} \times \vec{b}) = 12 - 102 + 90$

$= 0$

$\therefore \vec{b} \perp r \ (\vec{a} \times \vec{b})$

Note :

1. The unit vector perpendicular to both $\vec{a}$ & $\vec{b}$ are

$$\pm \hat{n} = \pm \frac{(\vec{a} \times \vec{b})}{|\vec{a} \times \vec{b}|}$$

2. vector of magnitude λ, perpendicular to both $\vec{a}$ & $\vec{b}$ are

$$\pm \lambda \hat{n} = \pm \lambda \frac{(\vec{a} \times \vec{b})}{|\vec{a} \times \vec{b}|}$$

Problems:

1. Find the vector of magnitude '6' which are perpendicular to both $\vec{a}$ ~~and $\vec{b}$~~ $= 4\vec{i} - \vec{j} + 3\vec{k}$ $\vec{b} = -2\vec{i} + \vec{j} - 2\vec{k}$

soln

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 4 & -1 & 3 \\ -2 & 1 & -2 \end{vmatrix}$$

$= \vec{i}(2-3) - \vec{j}(-8+6) + \vec{k}(4-2)$

$= -\vec{i} + 2\vec{j} + 2\vec{k}$

$|\vec{a} \times \vec{b}| = \sqrt{1+4+4} = 3$

$\therefore$ vector of magnitude 6 perpendicular to both $\vec{a}$ & $\vec{b}$ $\square$

$$\pm 6 \hat{n} = \pm 6 \frac{(\vec{a} \times \vec{b})}{|\vec{a} \times \vec{b}|}$$

$$= \pm 6 \frac{(-\vec{i} + 2\vec{j} + 2\vec{k})}{3}$$

$$= \pm 2(-\vec{i} + 2\vec{j} + 2\vec{k})$$

2) Find the ~~unit~~ vector $10\sqrt{2}$ perpendicular magnitude that are to each of $\vec{a}$

$\vec{a} = \vec{i} + 2\vec{j} + \vec{k}$ & $\vec{b} = \vec{i} + 3\vec{j} + 4\vec{k}$

soln

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2 & 1 \\ 1 & 3 & 4 \end{vmatrix}$$

$$= 5\vec{i} - 3\vec{j} + \vec{k}$$

$$|\vec{a}\times\vec{b}| = \sqrt{25+9+1} = \sqrt{35}$$

$\therefore$ vector of magnitude $10\sqrt{3}$, $\perp r$ to both $\vec{a}$ and $\vec{b}$

$$\text{is} = \pm 10\sqrt{3}\,\frac{(\vec{a}\times\vec{b})}{|\vec{a}\times\vec{b}|}$$

$$= \pm \frac{10\sqrt{3}}{\sqrt{35}}(5\vec{i} - 3\vec{j} + \vec{k})$$

3. Find the unit vectors $\perp r$ to each vector $(\vec{a}+\vec{b})$ & $\vec{a}-\vec{b}$ where $\vec{a} = \vec{i}+\vec{j}+\vec{k}$, $\vec{b} = \vec{i}+2\vec{j}+3\vec{k}$

Soln

$$\vec{a} = \vec{i}+\vec{j}+\vec{k}$$

$$\vec{b} = \vec{i}+2\vec{j}+3\vec{k}$$

$$\vec{a}+\vec{b} = 2\vec{i}+3\vec{j}+4\vec{k}$$

$$\vec{a}-\vec{b} = 0\vec{i}-\vec{j}-2\vec{k}$$

$$(\vec{a}+\vec{b})\times(\vec{a}-\vec{b}) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 3 & 4 \\ 0 & -1 & -2 \end{vmatrix}$$

$$= \vec{i}(-6+4) - \vec{j}(-4) + \vec{k}(-2+0)$$

$$= -2\vec{i}+4\vec{j}-2\vec{k}$$

$$|(\vec{a}+\vec{b})\times(\vec{a}-\vec{b})| = \sqrt{4+16+4} = \sqrt{24}.$$

$\therefore$ unit vect $\perp r$ to $(\vec{a}+\vec{b})$ & $(\vec{a}-\vec{b})$ $= \pm\dfrac{-2\vec{i}+4\vec{j}-2\vec{k}}{\sqrt{24}}.$

4) S.T $|\vec{a}\times\vec{b}|^2 + (\vec{a}\cdot\vec{b})^2 = |\vec{a}|^2|\vec{b}|^2$

Soln

$$|\vec{a}\times\vec{b}| = |\vec{a}|\,|\vec{b}|\sin\theta$$

$$|\vec{a}\times\vec{b}|^2 = |\vec{a}|^2|\vec{b}|^2\sin^2\theta \longrightarrow (1)$$

$$\vec{a}\cdot\vec{b} = |\vec{a}|\,|\vec{b}|\cos\theta.$$

$$(\vec{a}\cdot\vec{b})^2 = |\vec{a}|^2|\vec{b}|^2\cos^2\theta \longrightarrow (2)$$

(1) + (2)

$$\Rightarrow |\vec{a}\times\vec{b}|^2 + (\vec{a}\cdot\vec{b})^2 = |\vec{a}|^2|\vec{b}|^2(\sin^2\theta + \cos^2\theta)$$

$$= |\vec{a}|^2|\vec{b}|^2(1) = |\vec{a}|^2|\vec{b}|^2$$

5) For any vector $\vec{a}$ S.T $|\vec{a}\times\vec{i}|^2+|\vec{a}\times\vec{j}|^2+|\vec{a}\times\vec{k}|^2=2|\vec{a}|^2$

Soln Let $\vec{a}=a_1\vec{i}+a_2\vec{j}+a_3\vec{k}$

$$|\vec{a}|^2=a_1^2+a_2^2+a_3^2 \longrightarrow (1)$$

$$\vec{a}\times\vec{i}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ a_1 & a_2 & a_3\\ 01 & 0 & 0\end{vmatrix}$$

$$=0\vec{i}-\vec{j}(a_3)+\vec{k}(a_2)$$

$$=0\vec{i}-a_3\vec{j}-a_2\vec{k}$$

$$|\vec{a}\times\vec{i}|^2=a_3^2+a_2^2$$

III'y $|\vec{a}\times\vec{j}|^2=a_1^2+a_3^2$

$$|\vec{a}\times\vec{k}|^2=a_1^2+a_2^2$$

$$\therefore |\vec{a}\times\vec{i}|^2+|\vec{a}\times\vec{j}|^2+|\vec{a}\times\vec{k}|^2=2(a_1^2+a_2^2+a_3^2)$$

$$=2|\vec{a}|^2 \quad \text{by } (1)$$

8.6. Resolution of vectors

Result:

① $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$

(i) $|\vec{r}| = \sqrt{x^2+y^2+z^2}$

(ii) $r = |\vec{r}| = \sqrt{x^2+y^2+z^2}$

(iii) drs of $\vec{r}$ are x, y, z

(iv) dcs of $\vec{r}$ are $x/r, y/r, z/r$

(v) l, m, n are dcs of a vector $\Leftrightarrow l^2+m^2+n^2$

② If A and B are points (x_1, y_1, z_1) & (x_2, y_2, z_2) then the components of $\vec{AB}$ in the direction of x, y & z axis are $(x_2-x_1)\vec{i}$, $(y_2-y_1)\vec{j}$, $(z_2-z_1)\vec{k}$

③ unit vector $= \dfrac{\vec{a}}{|\vec{a}|}$

4) collinear vector:

parallel vector:

Two vectors are parallel if they are sclar multiple of one another

(ie) $\vec{a} \parallel \vec{b} \Leftrightarrow \vec{a} = \lambda\vec{b}$

5) **coplanar vector:**

~~If~~ $\vec{a}, \vec{b}, \vec{c}$ are called coplanar if one vector is linear ~~catte~~. combination of other two vectors.

8.6 Resolution of vectors

Result:

$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$

1. drs are x, y, z
2. dcs are $\frac{x}{\sqrt{x^2+y^2+z^2}}, \frac{y}{\sqrt{x^2+y^2+z^2}}, \frac{z}{\sqrt{x^2+y^2+z^2}}$
3. sum of squres of dcs of $\vec{r}$ is 1.
4. $\sin^2\alpha + \sin^2\beta + \sin^2\gamma = 2$
5. l, m, n are dcs of a vector $\Leftrightarrow$ $l^2+m^2+n^2 = 1$
6. Any unit vector can be written as $\cos\alpha\vec{i} + \cos\beta\vec{j} + \cos\gamma\vec{k}$

Problem:

1. verify the following ratio's are dcs of some vector or not.

(i) $\frac{1}{5}, \frac{3}{5}, \frac{4}{5}$

soln

Let $(l, m, n) = \left(\frac{1}{5}, \frac{3}{5}, \frac{4}{5}\right)$

$$l^2+m^2+n^2 = \frac{1}{25} + \frac{9}{25} + \frac{16}{25}$$

$$= \frac{26}{25}$$

$$l^2+m^2+n^2 \neq 1$$

$\therefore$ $(1/5, 3/5, 4/5)$ not a dcs of any vector.

(ii) $\frac{1}{\sqrt{2}}, \frac{1}{2}, 1/2$

soln Let $(l, m, n) = (1/\sqrt{2}, 1/2, 1/2)$

$$l^2+m^2+n^2 = \frac{1}{2} + \frac{1}{4} + \frac{1}{4}$$

$$= 1$$

$\therefore$ $\left(\frac{1}{\sqrt{2}}, \frac{1}{2}, 1/2\right)$ is a dcs of some vector.

(ii) $\left(\frac{4}{3}, 0, \frac{3}{4}\right)$

soln Let $(l, m, n) = (4/3, 0, 3/4)$

$$l^2 + m^2 + n^2 = \frac{16}{9} + 0 + \frac{9}{16}$$

$$\neq 1$$

$\therefore (4/3, 0, 3/4)$ not a dcs of some vector.

(iv) can a vector direction angles $30^\circ, \cancel{45}, 60$

Soln
$$l = \cos 30^\circ = \sqrt{3}/2$$
$$m = \cos 45^\circ = 1/\sqrt{2}$$
$$n = \cos 60^\circ = 1/2$$

$$l^2 + m^2 + n^2 = \frac{3}{4} + \frac{1}{2} + \frac{1}{4}$$

$$\neq 1$$

$\therefore$ The angles $30, 45, 60$ not direction angles of any vector.

Note:

$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$

(i) drs of $\vec{r}$ are x, y, z $\quad r = \sqrt{x^2 + y^2 + z^2}$

(ii) dcs of $\vec{r}$ are $\frac{x}{r}, \frac{y}{r}, \frac{z}{r}$

Problems:

1. Find the drs & dcs of following vectors

(i) $3\vec{i} + 4\vec{j} - 6\vec{k}$

soln Let $\vec{r} = 3\vec{i} + 4\vec{j} - 6\vec{k}$

drs of $\vec{r}$ are $3, 4, -6$ $\quad r = \sqrt{9 + 16 + 36} = \sqrt{61}$

dcs of $\vec{r}$ are $3/r, 4/r, -6/r$

$$= \frac{3}{\sqrt{61}}, \frac{4}{\sqrt{61}}, \frac{-6}{\sqrt{61}}$$

(ii) $3\vec{i} - 4\vec{k}$

Soln Let $\vec{r} = 3\vec{i} + 0\vec{j} - 4\vec{k}$

$\therefore$ drs of $\vec{r}$ are $3, 0, -4$

dcs of $\vec{r}$ are $3/r, 0/r, -4/r$ $\qquad r = \sqrt{9+0+16} = \sqrt{25} = 5$

$= 3/5, 0, -4/5$

2) Find the dcs of a vector whose drs are.

(i) $(1, 2, 3)$

dcs $= \left(\frac{x}{r}, \frac{y}{r}, \frac{z}{r}\right)$ $\qquad r = \sqrt{1+4+9} = \sqrt{14}$

$= \left(\frac{1}{\sqrt{14}}, \frac{2}{\sqrt{14}}, \frac{3}{\sqrt{14}}\right)$

(ii) $(3, -1, 3)$

dcs $= \left(\frac{x}{r}, \frac{y}{r}, \frac{z}{r}\right)$ $\qquad r = \sqrt{9+1+9} = \sqrt{19}$

$= \left(\frac{3}{\sqrt{19}}, \frac{-1}{\sqrt{19}}, \frac{3}{\sqrt{19}}\right)$

(iii) $(0, 0, 7)$

dcs $= \left(\frac{x}{r}, \frac{y}{r}, \frac{z}{r}\right)$ $\qquad r = \sqrt{0+0+49} = 7$

$= (0, 0, 7/7)$

$= (0, 0, 1)$

3) Find the dcs & drs of the following vector.

Soln

(i) $3\vec{i} - 4\vec{j} + 8\vec{k}$

Let $\vec{r} = 3\vec{i} - 4\vec{j} + 8\vec{k}$

drs $= (3, -4, 8)$

dcs $= \left(\frac{x}{r}, \frac{y}{r}, \frac{z}{r}\right)$ $\qquad r = \sqrt{9+16+64} = \sqrt{89}$

$= \left(\frac{3}{\sqrt{89}}, \frac{-4}{\sqrt{89}}, \frac{8}{\sqrt{89}}\right)$

(ii) $3\vec{i}+\vec{j}+\vec{k}$

Soln

$dr_s = (3,1,1)$ $\quad r=\sqrt{9+1+1}=\sqrt{11}$

$dc_s = \left(\frac{3}{\sqrt{11}}, \frac{1}{\sqrt{11}}, \frac{1}{\sqrt{11}}\right)$

(iii) $\vec{j}$

Soln

$dr_s = (0,1,0)$ $\quad r=\sqrt{0+1+0}=1$

$dc_s = (0,1,0)$

(iv) $5\vec{i}-3\vec{j}-48\vec{k}$

$dr_s = (5,-3,-48)$ $\quad r=\sqrt{25+9+(48)^2}=\sqrt{2338}$

$dc_s = \left(\frac{5}{\sqrt{2338}}, \frac{-3}{\sqrt{2338}}, \frac{-48}{\sqrt{2338}}\right)$

(v) $3\vec{i}-3\vec{k}+4\vec{k}$

$dr_s = (3,-3,4)$ $\quad r=\sqrt{9+9+16}=\sqrt{34}$

$dc_s = \left(\frac{3}{\sqrt{34}}, \frac{-3}{\sqrt{34}}, \frac{4}{\sqrt{34}}\right)$

(vi) $\vec{i}-\vec{k}$

$dr_s = (1,0,-1)$ $\quad r=\sqrt{1+0+1}=\sqrt{2}$

$dc_s = \left(\frac{1}{\sqrt{2}}, 0, \frac{-1}{\sqrt{2}}\right)$

4) If $\left(\frac{1}{2}, \frac{1}{\sqrt{2}}, a\right)$ are dcs of some vector, then find a.

Soln

Sum of squares of dcs $=1$

$\left(\frac{1}{2}\right)^2+\left(\frac{1}{\sqrt{2}}\right)^2+a^2 = 1$

$\frac{1}{4}+\frac{1}{2}+a^2 = 1$

$\frac{3}{4}+a^2 = 1$

$a^2 = 1-\frac{3}{4}$

$a^2 = \frac{1}{4}$

$a = \pm\frac{1}{2}$

Note:

If A and B are the points (x_1,y_1,z_1) and (x_2,y_2,z_2) then the components of $\vec{AB}$ in the directions of x, y and z axis are $(x_2-x_1)\vec{i}$, $(y_2-y_1)\vec{j}$, $(z_2-z_1)\vec{k}$

Problem:

1. Find dcs of $\vec{AB}$ where A is $(2,3,1)$ & B is $(3,-1,2)$

Soln

$$\vec{OA} = 2\vec{i}+3\vec{j}+\vec{k}$$
$$\vec{OB} = 3\vec{i}-\vec{j}+2\vec{k}$$
$$\vec{AB} = \vec{OB}-\vec{OA}$$
$$= \vec{i}-4\vec{j}+\vec{k}$$

dcs $= \left(\frac{x}{r}, \frac{y}{r}, \frac{z}{r}\right)$

drs of $\vec{AB} = (1,-4,1)$ $\therefore r = \sqrt{1+16+1} = \sqrt{18}$

dcs of $\vec{AB} = \left(\frac{1}{\sqrt{18}}, \frac{-4}{\sqrt{18}}, \frac{1}{\sqrt{18}}\right)$

2. Find the dcs of the line joining $(2,3,1)$ & $(3,-1,2)$

Let $\vec{OA} = 2\vec{i}+3\vec{j}+\vec{k}$

$$\vec{OB} = 3\vec{i}-\vec{j}+2\vec{k}$$
$$\vec{AB} = \vec{OB}-\vec{OA}$$
$$= \vec{i}-4\vec{j}+\vec{k}$$

drs of $\vec{AB} = (1,-4,1)$

dcs of $\vec{AB} = \left(\frac{1}{\sqrt{18}}, \frac{-4}{\sqrt{18}}, \frac{1}{\sqrt{18}}\right)$.

III^{ly} dcs of $\vec{BA} = \left(\frac{-1}{\sqrt{18}}, \frac{4}{\sqrt{18}}, \frac{-1}{\sqrt{18}}\right)$.

Note:

The unit vector of $\vec{a}$ is $= \frac{\vec{a}}{|\vec{a}|}$

Problem: 1. Find the unit vector along the direction of the vector $5\vec{i}-2\vec{j}+4\vec{k}$

Soln Let $\vec{a} = 5\vec{i}-3\vec{j}+4\vec{k}$

$$|\vec{a}| = \sqrt{25+9+16} = \sqrt{50}.$$

$\therefore$ The unit vector along the direction of $\vec{a} = \frac{\vec{a}}{|\vec{a}|} = \frac{1}{\sqrt{50}}(5\vec{i}-3\vec{j}+4\vec{k})$

(2) Find the point whose position vector has magnitude 5 and parallel to the vector $4\vec{i}-3\vec{j}+10\vec{k}$.

Soln

Let $\vec{a} = 4\vec{i}-3\vec{j}+10\vec{k}$

The vector whose magnitude 5 and ~~patlel~~ parallel to $\vec{a}$ $\Big\} = 5\dfrac{\vec{a}}{|\vec{a}|}$

$$= \frac{5(4\vec{i}-3\vec{j}+10\vec{k})}{\sqrt{16+9+100}}$$

$$= \frac{5(4\vec{i}-3\vec{j}+10\vec{k})}{\sqrt{25}}$$

$$= \frac{1}{\sqrt{5}}(4\vec{i}-3\vec{j}+10\vec{k})$$

$\therefore$ So required point is $\left(\dfrac{4}{\sqrt{5}}, \dfrac{-3}{\sqrt{5}}, \dfrac{10}{\sqrt{5}}\right)$

(3) Find the value of m for which $m(\vec{i}+\vec{j}+\vec{k})$ is unit vector.

Sol Since $m(\vec{i}+\vec{j}+\vec{k})$ is unit vector.

$$\therefore\ m|\vec{i}+\vec{j}+\vec{k}| = 1$$

$$m\sqrt{1+1+1} = 1$$

$$m\sqrt{3} = 1$$

$$m = 1/\sqrt{3}.$$

(4) Find the unit vector parallel to ~~3a~~ $3\vec{a}-2\vec{b}+4\vec{c}$ if $\vec{a} = 3\vec{i}-\vec{j}-4\vec{k}$, $\vec{b} = -2\vec{i}+4\vec{j}-3\vec{k}$ & $\vec{c} = \vec{i}+2\vec{j}-\vec{k}$

Soln

$$3\vec{a} = 9\vec{i}-3\vec{j}-12\vec{k}$$

$$-2\vec{b} = 4\vec{i}-8\vec{j}+6\vec{k}$$

$$4\vec{c} = 4\vec{i}+8\vec{j}-4\vec{k}$$

$\therefore\ 3\vec{a}-2\vec{b}+4\vec{c} = 17\vec{i}-3\vec{j}-10\vec{k}$

$\therefore$ unit vector parallel to $3\vec{a}-2\vec{b}+4\vec{c} = \dfrac{17\vec{i}-3\vec{j}-10\vec{k}}{\sqrt{(17)^2+(-3)^2+(-10)^2}}$

$$= \frac{17\vec{i}-3\vec{j}-10\vec{k}}{\sqrt{398}}.$$

5) If $\vec{a} = 2\vec{i}+3\vec{j}-4\vec{k}$ & $\vec{b} = 3\vec{i}-4\vec{j}-5\vec{k}$ & $\vec{c} = -3\vec{i}+2\vec{j}+3\vec{k}$ Find the magnitude and dcs of
(i) $\vec{a}+\vec{b}+\vec{c}$ (ii) $3\vec{a}-2\vec{b}+5\vec{c}$

Soln

$$\vec{a}+\vec{b}+\vec{c} = 2\vec{i}+\vec{j}-6\vec{k}$$

$$|\vec{a}+\vec{b}+\vec{c}| = \sqrt{4+1+36} = \sqrt{41}$$

$$\text{dcs} = \left(\frac{x}{r}, \frac{y}{r}, \frac{z}{r}\right)$$

$$= \left(\frac{2}{\sqrt{41}}, \frac{1}{\sqrt{41}}, \frac{-6}{\sqrt{41}}\right)$$

(ii)

$$3\vec{a} = 6\vec{i}+9\vec{j}-12\vec{k}$$

$$-2\vec{b} = -6\vec{i}+8\vec{j}+10\vec{k}$$

$$5\vec{c} = -15\vec{i}+10\vec{j}+15\vec{k}$$

$$3\vec{a}-2\vec{b}+5\vec{c} = -15\vec{i}+27\vec{j}+13\vec{k}$$

$$\text{magnitude} = \sqrt{(15)^2+(27)^2+(13)^2} = \sqrt{1123}.$$

$$\therefore \text{dcs} = \left(\frac{x}{r}, \frac{y}{r}, \frac{z}{r}\right)$$

$$= \left(\frac{-15}{\sqrt{1123}}, \frac{27}{\sqrt{1123}}, \frac{13}{\sqrt{1123}}\right)$$

6) Show that the vectors $2\vec{i}-\vec{j}+\vec{k}$, $3\vec{i}-4\vec{j}-4\vec{k}$, $\vec{i}-3\vec{j}-5\vec{k}$ form a right angle triangle.

Soln Let

$$\vec{OA} = 2\vec{i}-\vec{j}+\vec{k}$$

$$\vec{OB} = 3\vec{i}-4\vec{j}-4\vec{k}$$

$$\vec{OC} = \vec{i}-3\vec{j}-5\vec{k}$$

$$\vec{AB} = \vec{OB}-\vec{OA} = \vec{i}-3\vec{j}-5\vec{k} \qquad |\vec{AB}| = \sqrt{1+9+25} = \sqrt{35}$$

$$\vec{BC} = \vec{OC}-\vec{OB} = -2\vec{i}+\vec{j}-\vec{k} \qquad BC = \sqrt{4+1+1} = \sqrt{6}$$

$$\vec{CA} = \vec{OA}-\vec{OC} = \vec{i}+2\vec{j}+6\vec{k} \qquad CA = \sqrt{1+4+36} = \sqrt{41}$$

$$AB^2+BC^2 = 35+6$$

$$= 41$$

$$= (\sqrt{41})^2 = (CA)^2$$

$\therefore$ ABC form a right angle triangle

7) S.T the points whose position vectors $2\vec{i}+4\vec{j}+3\vec{k}$, $4\vec{i}+3\vec{j}+9\vec{k}$ & $10\vec{i}-\vec{j}+6\vec{k}$ form a right-angle triangle.

Soln

Let A, B, C be given points & O is origin.

$\therefore \vec{OA} = 2\vec{i}+4\vec{j}+3\vec{k}$

$\vec{OB} = 4\vec{i}+3\vec{j}+9\vec{k}$

$\vec{OC} = 10\vec{i}-\vec{j}+6\vec{k}$

$\vec{AB} = \vec{OB}-\vec{OA} = 2\vec{i}-3\vec{j}+6\vec{k}$ $\therefore AB = \sqrt{4+9+36} = \sqrt{49} = 7$

$\vec{BC} = \vec{OC}-\vec{OB} = 6\vec{i}-2\vec{j}+3\vec{k}$ $BC = \sqrt{36+4+9} = \sqrt{49} = 7$

$\vec{CA} = \vec{OA}-\vec{OC} = -8\vec{i}+5\vec{j}-3\vec{k}$ $CA = \sqrt{64+25+9} = \sqrt{98}$.

$AB^2+BC^2 = 49+49$

$= 98$

$= (\sqrt{98})^2$

$AB^2+BC^2 = CA^2$

$\therefore$ ABC form a right-angle triangle.

8) Show that the point A(1,1,1) B(1,2,3) & C(2,-1,1) are vertices of an isosceles triangle.

Soln

Let $\vec{OA} = \vec{i}+\vec{j}+\vec{k}$

$\vec{OB} = \vec{i}+2\vec{j}+3\vec{k}$

$\vec{OC} = 2\vec{i}-\vec{j}+\vec{k}$

$\vec{AB} = \vec{OB}-\vec{OA} = \vec{i}-\vec{j}+2\vec{k}$ $AB = \sqrt{0+1+4} = \sqrt{5}$

$\vec{CA} = \vec{OA}-\vec{OC} = -\vec{i}+2\vec{j}+0\vec{k}$ $CA = \sqrt{1+4+0} = \sqrt{5}$.

$\vec{BC} = \vec{OC}-\vec{OA} = \vec{i}-3\vec{j}-2\vec{k}$ $BC = \sqrt{1+9+4} = \sqrt{14}$.

$AB = CA$.

$\therefore \Delta ABC$ is isosceles triangle.

9. The position vector of vertices of a triangle are $\vec{i}+2\vec{j}+3\vec{k}$, $3\vec{i}-4\vec{j}+5\vec{k}$, $-2\vec{i}+3\vec{j}-7\vec{k}$. Find the parameter of the triangle.

Soln
let $\vec{OA} = \vec{i}+2\vec{j}+3\vec{k}$

$\vec{OB} = 3\vec{i}-4\vec{j}+5\vec{k}$

$\vec{OC} = -2\vec{i}+3\vec{j}-7\vec{k}$

$\vec{AB} = \vec{OB}-\vec{OA} = 2\vec{i}-6\vec{j}+2\vec{k} \quad \therefore AB = \sqrt{4+4+36} = \sqrt{44}$

$\vec{BC} = \vec{OC}-\vec{OB} = -5\vec{i}+7\vec{j}-12\vec{k} \quad \therefore BC = \sqrt{25+49+144} = \sqrt{218}$

$\vec{CA} = \vec{OA}-\vec{OC} = 3\vec{i}-\vec{j}+10\vec{k} \quad CA = \sqrt{9+1+100} = \sqrt{110}$.

$\therefore$ parameter of the triangle $\} = AB+BC+CA$

$= \sqrt{44}+\sqrt{218}+\sqrt{110}$.

Parallel vector :

Two vectors are parallel if they are sclar multiple of one another

(i) $\vec{a} \parallel \vec{b} \Leftrightarrow \vec{a} = \lambda\vec{b}$.

problems :

1. Find the value of λ for which the vector $\vec{a} = 3\vec{i}+2\vec{j}+9\vec{k}$ and $\vec{b} = \vec{i}+\lambda\vec{j}+3\vec{k}$ are parallel.

Soln $\vec{a} \parallel^{le} \vec{b}$

$\therefore \vec{a} = 3\vec{i}+2\vec{j}+9\vec{k}$

$\vec{a} = 3(\vec{i}+\frac{2}{3}\vec{j}+3\vec{k})$

$\therefore \vec{a} \parallel \vec{b}$

$\therefore \vec{b} = 3\vec{a}$

$\vec{i}+\lambda\vec{j}+3\vec{k} = 3(\vec{i}+\frac{2}{3}\vec{j}+3\vec{k})$

$\therefore \lambda = 2/3$

2) The position vector of the points P, Q, R, S are $\vec{i}+\vec{j}+\vec{k}$, $2\vec{i}+5\vec{j}$, $3\vec{i}+2\vec{j}-3\vec{k}$, $\vec{i}-6\vec{j}-\vec{k}$ respect. P.T $PQ \parallel RS$.

Soln

Let $\vec{OP} = \vec{i}+\vec{j}+\vec{k}$

$\vec{OQ} = 2\vec{i}+5\vec{j}+0\vec{k}$

$\vec{PQ} = \vec{OQ}-\vec{OP}$

$= \vec{i}+4\vec{j}-\vec{k}$

$\vec{OR} = 3\vec{i}+2\vec{j}-3\vec{k}$

$\vec{OS} = \vec{i}-6\vec{j}-\vec{k}$

$\vec{RS} = \vec{OS}-\vec{OR}$

$= -2\vec{i}-8\vec{j}+2\vec{k}$

$= -2(\vec{i}+4\vec{j}-\vec{k})$

$\therefore \vec{PQ} = \vec{i}+4\vec{j}-\vec{k}$

$\vec{RS} = -2(\vec{i}+4\vec{j}-\vec{k})$

$\vec{RS} = -2(\vec{PQ})$

$\therefore \vec{RS} \parallel \vec{PQ}$

Coplanar vector:

If $\vec{a}, \vec{b}, \vec{c}$ are are coplanar if one vector is a linear combination of other two vectors.

Problems:

1. Show that the vectors $5\vec{i}+6\vec{j}+7\vec{k}$, $7\vec{i}-8\vec{j}+9\vec{k}$, $3\vec{i}+20\vec{j}+5\vec{k}$ are coplanar vector.

Soln

Let $5\vec{i}+6\vec{j}+7\vec{k} = s(7\vec{i}-8\vec{j}+9\vec{k})+t(3\vec{i}+20\vec{j}+5\vec{k})$

$\therefore$
$$7s+3t=5 \longrightarrow (1)$$
$$-8s+20t=6 \longrightarrow (2)$$
$$9s+5t=7$$

$8\times(1) \Rightarrow \quad 56s+24s = 40$

$7\times(2) \Rightarrow \quad -56s+140s = 42$

$164s = 82$

$s = \frac{82}{164}$

$s = 1/2$

$(1) \Rightarrow 7/2+3t=5$

$3t = 5-7/2$

$3t = 3/2$

$t = 1/2$

$\therefore$ one vector is linear combination of other two.

$\therefore$ given vectors are coplanar.

2. S.T the vectors are coplanar

(i) $\vec{i}-2\vec{j}+3\vec{k}$, $-2\vec{i}+3\vec{j}-4\vec{k}$, $-\vec{j}+2\vec{k}$

(ii) $5\vec{i}+6\vec{j}+7\vec{k}$, $7\vec{i}-8\vec{j}+9\vec{k}$, $3\vec{i}+20\vec{j}+5\vec{k}$

(i) Let $-2\vec{i}+3\vec{j}-4\vec{k} = s(\vec{i}-2\vec{j}+3\vec{k}) + t(0\vec{i}-\vec{j}+2\vec{k})$

equ. the co.eff of $\vec{i}, \vec{j}, \vec{k}$

$s+0t = -2 \Rightarrow s = -2$

$-2s-t = 3$

$-2(-2)-t = 3$

$4-t = 3$

$t = 4-3$

$t = 1$

$\therefore t=1, s=-2$

one vector is linear combination of other two vectors.

$\therefore$ given 3 vectors are coplanar.

Coplanarity of Four Points : The necessary and sufficient condition that four points with P.V.'s $\vec{a}, \vec{b}, \vec{c}, \vec{d}$ are coplanar is that there exist scalar x, y, z, t not all zero such that $x\vec{a} + y\vec{b} + z\vec{c} + t\vec{d} = 0 \Rightarrow x + y + z + t = 0$

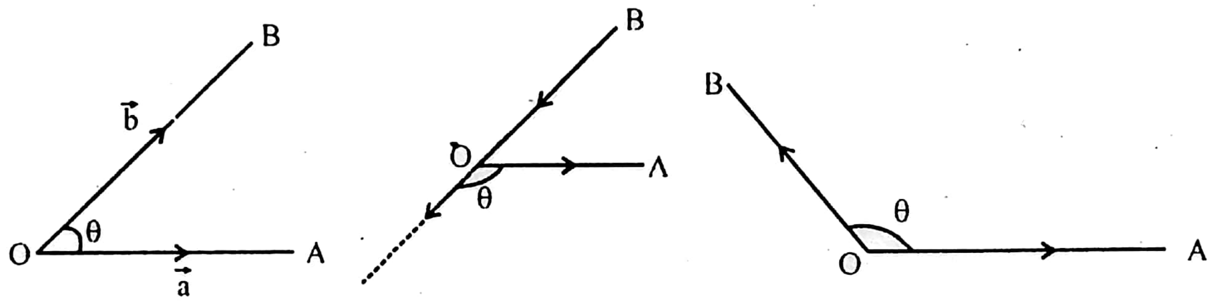
SCALAR PRODUCT OF TWO VECTORS

If $\vec{a}$ and $\vec{b}$ are two non-zero vectors then the scalar or dot product of $\vec{a}$ and $\vec{b}$ is denoted by $\vec{a} \cdot \vec{b}$ and is defined as $\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}|\cos\theta$

where θ is the angle between the two vectors and $0 \le \theta \le \pi$.

Note :

1. The angle between two vectors $\vec{a}$ and $\vec{b}$ is defined as the smaller angle θ between them when they are drawn with the same initial point.



Usually, we take $0 \le \theta \le \pi$. Angle between two like vectors is 0 and angle between two unlike vectors is π.

2. The dot product is a scalar quantity, since it is the product of three scalars $|\vec{a}|, |\vec{b}|$ and $\cos\theta$.
3. If either $\vec{a}$ and $\vec{b}$ is the null vector, the scalar product is the scalar zero.
4. If $\hat{a}$ and $\hat{b}$ are two unit vectors, than $\hat{a} \cdot \hat{b} = \cos\theta$. Thus, the scalar product of two vectors is equal to the cosine of the angle between their directions.
5. Since the scalar product of two vectors is only a number, we see that (i) $(\vec{a} \cdot \vec{b})(\vec{c} \cdot \vec{d})$ is only a number being the product of two numbers $\vec{a} \cdot \vec{b}$ and $\vec{c} \cdot \vec{d}$, (ii) $(\vec{a} \cdot \vec{b})\vec{c}$ is a vector whose modulus is $\vec{a} \cdot \vec{b}$ times that of $\vec{c}$.

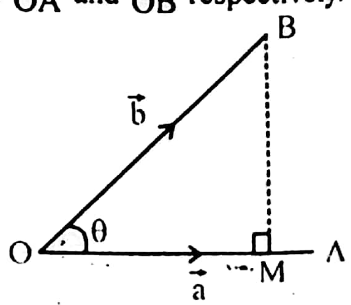
Geometrical Interpretation of Scalar Product :

Suppose the vectors $\vec{a}$ and $\vec{b}$ are represented by the directed line segments $\overrightarrow{OA}$ and $\overrightarrow{OB}$ respectively. Then

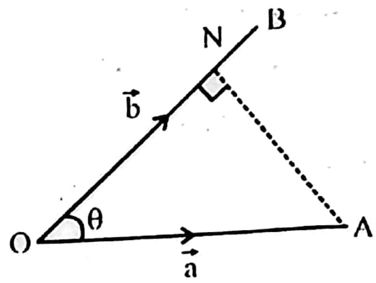
$$\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}|\cos\theta = OA(OB\cos\theta) = OA(OM)$$

$$= |\vec{a}| \times (\text{Scalar component of } \vec{b} \text{ along } \vec{a})$$

$$= |\vec{a}| \times (\text{Projection of } \vec{b} \text{ on } \vec{a})$$



Also,

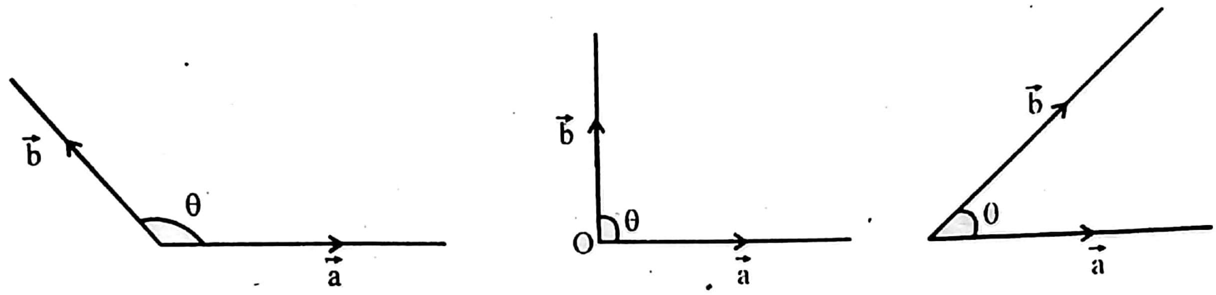


$\vec{a} \cdot \vec{b} = |\vec{b}||\vec{a}|\cos\theta = OB(OA\cos\theta) = OB(ON)$

$= |\vec{b}| \times$(Scalar component of $\vec{a}$ along $\vec{b}$)

$= |\vec{b}| \times$(Projection of $\vec{a}$ on $\vec{b}$)

Sign of the scalar product : If $\vec{a}, \vec{b}$ be the two non-zero vectors, then the scalar product $\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}|\cos\theta$,



is positive, negative, or zero, according as the angle θ, between the vectors is acute, obtuse, or right.

θ is acute $\Rightarrow \cos\theta > 0 \Rightarrow \vec{a} \cdot \vec{b} > 0$

θ is right $\Rightarrow \cos\theta = 0 \Rightarrow \vec{a} \cdot \vec{b} = 0$

θ is obtuse $\Rightarrow \cos\theta < 0 \Rightarrow \vec{a} \cdot \vec{b} < 0$

Length of a vector as a scalar product : If $\vec{a}$ be any vector, then the scalar product

$\vec{a} \cdot \vec{a} = |\vec{a}||\vec{a}|\cos 0 = |\vec{a}|^2 = a^2 \Leftrightarrow a = |\vec{a}| = +\sqrt{\vec{a} \cdot \vec{a}}$

Condition of perpendicularity : We notice that if $\vec{a} \cdot \vec{b}$ be any two vectors, then their scalar product $\vec{a} \cdot \vec{b}$ will vanish, if and only if, either at least one of the two vectors is the zero vector or the two vectors are at right angles to each other. Thus, the scalar product of two non-zero vectors is zero if they are at right angles to each other, and conversely.

$\boxed{\vec{a} \cdot \vec{b} = 0 \Leftrightarrow \vec{a} \perp \vec{b},\ \vec{a} \text{ and } \vec{b} \text{ being non-zero vectors.}}$

Scalar Product in terms of components :

In the view of definition of scalar products of vectors, we have

(i) $\hat{i} \cdot \hat{i} = \hat{j} \cdot \hat{j} = \hat{k} \cdot \hat{k} = 1$

(ii) $\hat{i} \cdot \hat{j} = \hat{j} \cdot \hat{k} = \hat{k} \cdot \hat{i} = 0$

Hence, if two vectors $\vec{a}$ and $\vec{b}$ are given by $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$, we have

$\vec{a} \cdot \vec{b} = a_1b_1 + a_2b_2 + a_3b_3$.

VI. Properties of scalar products :

(i) $\vec{a}\cdot\vec{b}=\vec{b}\cdot\vec{a}$ (Commutative Law)

(ii) $\vec{a}\cdot(\vec{b}+\vec{c})=\vec{a}\cdot\vec{b}+\vec{a}\cdot\vec{c}$ (Distributive Law)

(iii) There is no associative law for the scalar product since $\vec{a}\cdot(\vec{b}\cdot\vec{c})$ is not defined as $\vec{b}\cdot\vec{c}$ is scalar and the scalar product is an operation on two vectors.

(iv) If the scalar product of a vector $\vec{r}$ with each of three non-coplanar vectors is zero then $\vec{r}$ must be the zero vector, because no non-zero vector can be perpendicular to three non-coplanar vectors.

In particular, if a vector $\vec{r}$ is perpendicular to every vector then $\vec{r}$ be a zero vector.

(v) $\vec{a}\cdot(-\vec{b})=-(\vec{a}\cdot\vec{b})$; $(-\vec{a})\cdot(-\vec{b})=(\vec{a}\cdot\vec{b})$;

(vi) $m\vec{a}\cdot\vec{b}=m(\vec{a}\cdot\vec{b})=\vec{a}\cdot m\vec{b}$

(vii) $m\vec{a}\cdot n\vec{b}=mn(\vec{a}\cdot\vec{b})=\vec{a}\cdot mn\vec{b}$

(viii) $|\vec{a}+\vec{b}|=|\vec{a}-\vec{b}|\Leftrightarrow\vec{a}\perp\vec{b}$

(ix) Cancellation law does not hold necessarily.

If $\vec{a}\cdot\vec{b}=\vec{a}\cdot\vec{c}$, then $\vec{b}$ is not necessarily equal to $\vec{c}$ since $\vec{a}\cdot(\vec{b}-\vec{c})=0$ is also true when $\vec{a}$ is perpendicular to $\vec{b}-\vec{c}$. Thus $\vec{a}\,(\vec{b}-\vec{c})=0$ if

(i) $\vec{a}=0$ or (ii) $\vec{b}=\vec{c}$ or (iii) $\vec{a}$ is perpendicular to $\vec{b}-\vec{c}$.

(x) If $\vec{a}$ and $\vec{b}$ be parallel vectors, then $\vec{a}\cdot\vec{b}=\begin{cases}ab, \text{ if } \vec{a} \text{ and } \vec{b} \text{ are like}\\ -ab, \text{ if } \vec{a} \text{ and } \vec{b} \text{ are unlike}\end{cases}$

SOME USEFUL VECTOR IDENTITIES

(i) $(\vec{a}+\vec{b})\cdot(\vec{a}-\vec{b})=\vec{a}^2-\vec{b}^2=|\vec{a}|^2-|\vec{b}|^2=a^2-b^2$

(ii) $(\vec{a}+\vec{b})^2=\vec{a}^2+2\vec{a}\cdot\vec{b}+\vec{b}^2=a^2+2\vec{a}\cdot\vec{b}+b^2$

$\Rightarrow|\vec{a}+\vec{b}|^2=|\vec{a}|^2+|\vec{b}|^2+2(\vec{a}\cdot\vec{b})$ [$\because$ Square of a vector = square of its modulus]

(iii) Similarly $(\vec{a}-\vec{b})^2=\vec{a}^2-2\vec{a}\cdot\vec{b}+\vec{b}^2=a^2-2\vec{a}\cdot\vec{b}+b^2$

$|\vec{a}-\vec{b}|^2=|\vec{a}|^2+|\vec{b}|^2-2(\vec{a}\cdot\vec{b})$

(iv) $|\vec{a}+\vec{b}|^2+|\vec{a}-\vec{b}|^2=2(|\vec{a}|^2+|\vec{b}|^2)$ and $|\vec{a}+\vec{b}|^2-|\vec{a}-\vec{b}|^2=4(\vec{a}\cdot\vec{b})$

$\therefore\vec{a}\cdot\vec{b}=\frac{1}{4}[(\vec{a}+\vec{b})^2-(\vec{a}-\vec{b})^2]$

(v) If $\vec{a}=a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$, then $|\vec{a}|^2=\vec{a}\cdot\vec{a}=a_1^2+a_2^2+a_3^2$ $\quad\therefore|\vec{a}|=\sqrt{a_1^2+a_2^2+a_3^2}$

APPLICATION OF SCALAR PRODUCT OF VECTORS

1. **To find angle between two vectors :** If θ be the angle between two non-zero vectors, $\vec{a}$, $\vec{b}$ then we have $\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\theta = ab\cos\theta$

$$\Rightarrow \cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|} = \frac{\vec{a}\cdot\vec{b}}{ab} = \frac{\vec{a}}{a}\cdot\frac{\vec{b}}{b} = \hat{a}\cdot\hat{b}, \text{ or } \theta = \cos^{-1}(\hat{a}\cdot\hat{b})$$

where $\hat{a}, \hat{b}$ are the unit vectors in the directions of $\vec{a}$ and $\vec{b}$ respectively.

The angle between two no-zero vectors $\vec{a}$ and $\vec{b}$ is given by

$$\cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|} = \frac{\text{Dot product of the vectors}}{\text{Product of their moduli}}$$

Further, if $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$. Then the angle θ between $\vec{a}$ and $\vec{b}$ is given by $\cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|} = \frac{a_1b_1 + a_2b_2 + a_3b_3}{\sqrt{a_1^2 + a_2^2 + a_3^2}\sqrt{b_1^2 + b_2^2 + b_3^2}}$

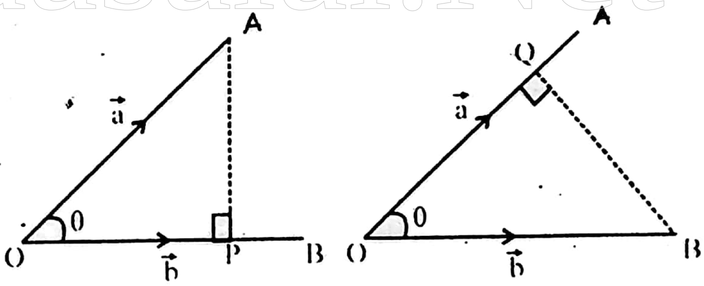
2. **To find Projection and Component of a Vector :** We know that the scalar product of two vectors is the product of the magnitude of either vector and the projection of the other in that direction.

$\vec{a}\cdot\vec{b} = |\vec{a}|$. Projection of $\vec{b}$ on $\vec{a}$ or

$\vec{a}\cdot\vec{b} = |\vec{b}|$, projection of $\vec{a}$ on $\vec{b}$.

It follows from the above that projection of

$$\vec{a} \text{ on } \vec{b} = \frac{\vec{a}\cdot\vec{b}}{|\vec{b}|} = \vec{a}\cdot\hat{b}$$



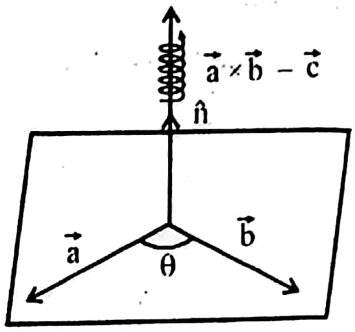
$$\text{Projection of } \vec{b} \text{ on } \vec{a} = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|} = \vec{b}\cdot\hat{a}.$$

Vector component of a vector $\vec{a}$ on $\vec{b}$ is the product of scalar projection of $\vec{a}$ on $\vec{b}$ and a units vector along $\vec{b}$. Thus, vector component of $\vec{a}$ on $\vec{b}$

$$= \frac{\vec{a}\cdot\vec{b}}{|\vec{b}|}\cdot\hat{b} = \frac{\vec{a}\cdot\vec{b}}{|\vec{b}|}\cdot\frac{\vec{b}}{|\vec{b}|} = \frac{(\vec{a}\cdot\vec{b})}{|\vec{b}|^2}\vec{b}.$$

Similarly, the vector projection of $\vec{b}$ on $\vec{a} = \frac{(\vec{a}\cdot\vec{b})}{|\vec{a}|^2}\vec{a}$.

VECTOR OR CROSS PRODUCT OF TWO VECTORS



The vector product of two vectors $\vec{a}$ and $\vec{b}$ is a vector $\vec{c}$ whose magnitude is $ab\sin\theta$, where $a=|\vec{a}|$, $b=|\vec{b}|$ and θ is the angle between the vectors $\vec{a}$ and $\vec{b}$ and the direction of $\vec{c}$ is perpendicular to both the vectors $\vec{a}$ and $\vec{b}$, such that $\vec{a}$, $\vec{b}$ and $\vec{c}$ form a right-handed screw.

The vector product of the vectors $\vec{a}$ and $\vec{b}$ is denoted by $\vec{a}\times\vec{b}$ and from the above definition, we get

$$\vec{a}\times\vec{b}=(|\vec{a}||\vec{b}|\sin\theta)\hat{n} \qquad ....(1)$$

or $$\vec{a}\times\vec{b}=(ab\sin\theta)\hat{n} \qquad .....(2)$$

where $a=|\vec{a}|$, $b=|\vec{b}|$, θ is the angle between the vectors $\vec{a}$ and $\vec{b}$, and $\hat{n}$ is a unit vector perpendicular to both $\vec{a}$ and $\vec{b}$ such that $\vec{a},\vec{b},\hat{n}$ from a right-handed triad of vectors.

Note :

1. If $\vec{a}=\vec{b}$ or if $\vec{a}$ is parallel to $\vec{b}$, then $\sin\theta=0$ and so $\vec{a}\times\vec{b}=0$.
2. We define that the vector product of any vector with a null vector is a null vector is a null vector i.e., $\vec{a}\times\vec{b}=\vec{0}$ if either $\vec{a}=\vec{0}$ or $\vec{b}=\vec{0}$.
3. The direction of $\vec{a}\times\vec{b}$ is regarded positive if the rotation from $\vec{a}$ to $\vec{b}$ appears to be anticlockwise.
4. $\vec{a}\times\vec{b}$ is perpendicular to the plane which contains both $\vec{a}$ and $\vec{b}$.

   Thus, the unit vector perpendicular to both $\vec{a}$ and $\vec{b}$ or to the plane containing $\vec{a}$ and $\vec{b}$ is given by

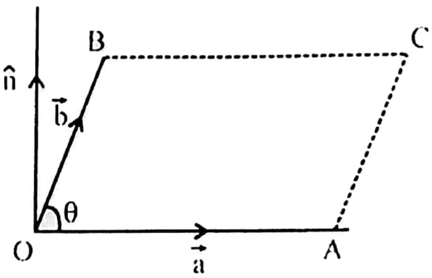
   $$\hat{n}=\frac{\vec{a}\times\vec{b}}{|\vec{a}\times\vec{b}|}=\frac{\vec{a}\times\vec{b}}{ab\sin\theta}.$$

5. Vector product of two parallel or collinear vectors is zero. If the vectors $\vec{a}$ and $\vec{b}$ are parallel or collinear, then $\theta = 0$ or $180°$, so in each case $\sin\theta = 0$. In particular $\vec{a}\times\vec{a}=\vec{0}$.
6. If $\vec{a}\times\vec{b}=0$, then $\vec{a}=\vec{0}$ or $\vec{b}=\vec{0}$ or $\vec{a}$ and $\vec{b}$ are parallel or collinear.
7. Vector product of two perpendicular vectors

   If $\theta=90°$, then $\sin\theta=1$, i.e., $\vec{a}\times\vec{b}=(ab)\hat{n}$ or $|\vec{a}\times\vec{b}|=|ab\hat{n}|=ab|\hat{n}|=ab$.
8. Vector product of two unit vectors. If $\hat{a}$ and $\hat{b}$, are unit vectors, then

   $a=|\hat{a}|=1;\ b=|\hat{b}|=1.\ \therefore \hat{a}\times\hat{b}=ab\sin\theta.\hat{n}=(\sin\theta).\hat{n}$

Geometrical Interpretation of Cross Product



Let $\overrightarrow{OA} = \vec{a}$ and $\overrightarrow{OB} = \vec{b}$ be two non-parallel and non-null vectors. Let θ be the angle between OA and OB and $a = |\vec{a}|$, $b = |\vec{b}|$. Then by definition.

$\vec{a} \times \vec{b} = (ab\sin\theta)\hat{n}$ or $|\vec{a} \times \vec{b}| = |(ab\sin\theta)\hat{n}| = (ab\sin\theta)|\hat{n}| = ab\sin\theta = |a||b|\sin\theta$ .....(1)

Now complete the parallelogram OACB. Then area of parallelogram OACB

$$= 2\,(\text{are}\ \Delta\ OAB) = 2\left[\frac{1}{2}OA.OB\sin\theta\right] = ab\sin\theta = |\vec{a} \times \vec{b}| \qquad \text{.......From (1)}$$

Hence $\vec{a} \times \vec{b}$ is a vector perpendicular to both $\vec{a}$ and $\vec{b}$ whose magnitude is equal to the area of the parallelogram whose adjacent sides are the vectors $\vec{a}$ and $\vec{b}$.

Note : Vector area of Δ OAB $= \frac{1}{2}$[area of parallelogram OACB] $= \frac{1}{2}(\vec{a} \times \vec{b})$

Area of Δ OAB $= \frac{1}{2}|\vec{a} \times \vec{b}|$

PROPERTIES OF VECTOR PRODUCT

1. **Vector product is not commulative :** The two vector products $\vec{a} \times \vec{b}$ and $\vec{b} \times \vec{a}$ are equal in magnitude and opposite in direction.

   $\therefore \vec{b} \times \vec{a} = -\vec{a} \times \vec{b}$ ....(i)

   Hence, we conclude that $\vec{a} \times \vec{b} \neq \vec{b} \times \vec{a}$.

   Due to the result (i), vector product is said to be anti-commulative.

2. **The vector product of a vector $\vec{a}$ with itself is a null vector, i.e.,** $\vec{a} \times \vec{a} = \vec{0}$.

3. **Vector product is associative with respect to a scalar.** Thus if $\vec{a}$ and $\vec{b}$ are any two vectors and m a scalar, then

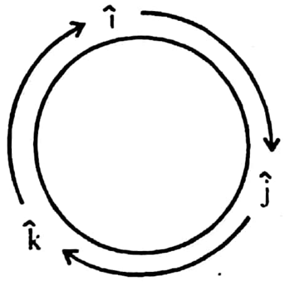
   $m(\vec{a} \times \vec{b}) = (m\vec{a}) \times \vec{b} = \vec{a} \times (m\vec{b})$

4. **Distributive Law :** For any two vectors $\vec{a}$, $\vec{b}$, $\vec{c}$

   $\vec{a} \times (\vec{b} + \vec{c}) = (\vec{a} \times \vec{b}) + (\vec{a} \times \vec{c})$.

VECTOR PRODUCT IN TERMS OF COMPONENTS

Let $\hat{i}, \hat{j}, \hat{k}$ be the orthonormal triad of unit vectors then we have $\hat{i} \times \hat{j} = \{(1)\,(1) \sin 90°\}\hat{n}$ where $\hat{n}$ is a unit vector perp. to both $\hat{i}$ and $\hat{j}$, i.e. $\hat{n} = \hat{k}$.



$\therefore \hat{i} \times \hat{j} = (1)\hat{k} = \hat{k} \Rightarrow \hat{i} \times \hat{j} = \hat{k}$. Similarly, $\therefore \hat{j} \times \hat{k} = \hat{i},\ \hat{k} \times \hat{i} = \hat{j}$.

Also, $\hat{i} \times \hat{i} = \hat{j} \times \hat{j} = \hat{k} \times \hat{k} = \vec{0}$

Further, as $\vec{a} \times \vec{b} = -(\vec{b} \times \vec{a})$, so we have $\hat{i} \times \hat{j} = -\hat{j} \times \hat{i}$, etc.

Hence $\hat{i} \times \hat{j} = \hat{k} = -\hat{j} \times \hat{i},\ \hat{j} \times \hat{k} = \hat{i} = -\hat{k} \times \hat{j},\ \hat{k} \times \hat{i} = \hat{j} = -\hat{i} \times \hat{k}$

Let $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k},\ \vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$. Then

$$\vec{a} \times \vec{b} = (a_1\hat{i} + a_2\hat{j} + a_3\hat{k}) \times (b_1\hat{i} + b_2\hat{j} + b_3\hat{k})$$

$$= a_1b_1(\hat{i} \times \hat{i}) + a_1b_2(\hat{i} \times \hat{j}) + a_1b_3(\hat{i} \times \hat{k}) + a_2b_1(\hat{j} \times \hat{i}) + a_2b_2(\hat{j} \times \hat{j}) + a_2b_3(\hat{j} \times \hat{k})$$
$$+ a_3b_1(\hat{k} \times \hat{i}) + a_3b_2(\hat{k} \times \hat{j}) + a_3b_3(\hat{k} \times \hat{k})$$

$$= (a_2b_3 - a_3b_2)\hat{i} + (a_3b_1 - a_1b_3)\hat{j} + (a_1b_2 - a_2b_1)\hat{k}$$

[Using results on cross products of orthonormal triads obtained above]

i.e., $$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

APPLICATION OF VECTOR PRODUCT :

1. Angle between two non-zero, non-parallel vectors

$\vec{a} \times \vec{b} = (ab \sin\theta)\hat{n}$ or $|\vec{a} \times \vec{b}| = |(ab \sin\theta)\hat{n}|$

$\Rightarrow |\vec{a} \times \vec{b}| = |(ab \sin\theta\hat{n}| = ab \sin\theta = |\vec{a}||\vec{b}| \sin\theta$

$$\therefore \sin\theta = \frac{|\vec{a} \times \vec{b}|}{|\vec{a}||\vec{b}|} = \frac{\text{modulus of } (\vec{a} \times \vec{b})}{(\text{modulus of } \vec{a})(\text{modulus of } \vec{b})}$$

$$\therefore \sin\theta = \sqrt{\left\{\frac{(a_1b_2 - a_2b_1)^2 + (a_2b_3 - a_3b_2)^2 + (a_3b_1 - a_1b_3)^2}{(a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2)}\right\}}$$ [In terms of components]

Note :

If $0 < \sin\theta < 1$, then θ will have two value θ_1 and θ_2 such that $0 < \theta_1 < 90°$ and $90° < \theta_2 < 180°$. In this case angle between two vectors cannot be determined by the above formula. Hence, angle between two vectors should be evaluated using dot product.