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11 th Mathematics

Unit 7 : Matrices and Determinants

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Topics :

1. Matrices
2. Algebraic Operations on Matrices
3. Transpose of a Matrix
4. Symmetric and Skew-symmetric Matrices
5. Properties of Determinants
6. Application of Factor Theorem to Determinants
7. Product of Determinants
8. Area of a Triangle
9. Singular and non-singular Matrices

Matrices

Definition:

A matrix is a set or group of numbers arranged in a square or rectangular array enclosed by two brackets

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Properties:

- A specified number of rows and a specified number of columns
- Two numbers (rows x columns) describe the dimensions or **order of the matrix.**

A matrix is denoted by a bold capital letter and the elements within the matrix are denoted by lower case letters

e.g. matrix [**A**] with elements a_{ij}

$$\mathbf{A}_{m \times n} = \begin{bmatrix} a_{11} & a_{12}\ldots & a_{ij} & a_{in} \\ a_{21} & a_{22}\ldots & a_{ij} & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & a_{ij} & a_{mn} \end{bmatrix}$$

TYPES OF MATRICES

1. Column matrix or vector:

A matrix is said to be a **column matrix** if it has only one column

$$\begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix}$$

2. Row matrix or vector

A matrix is said to be a **row matrix** if it has only one row.

$$\begin{bmatrix} 1 & 1 & 6 \end{bmatrix}$$

3. Rectangular matrix

A matrix is said to be rectangular if the number of rows is not equal to the number of columns

$$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 2 & 0 & 3 & 3 & 0 \end{bmatrix}$$

4. Square matrix

A matrix is said to be **square** if the number of rows is equal to the number of columns

(a square matrix **A** has an order of m)

$$\begin{bmatrix} 1 & 1 & 1 \\ 9 & 9 & 0 \\ 6 & 6 & 1 \end{bmatrix}$$

5. Diagonal matrix

A square matrix is said to be **diagonal** if at least one element of principal diagonal is non-zero and all the other elements are zero.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

6. Unit or Identity matrix - I

A diagonal matrix is said to be **identity** if all of its diagonal elements are equal to one, denoted by I

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

7. Null (zero) matrix - 0

A matrix is said to be **a null or zero matrix** if all of its elements are equal to zero. It is denoted by O

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

8. Triangular matrix

A square matrix is said to be triangular if all of its elements above the principal diagonal are zero **(lower triangular matrix)** or all of its elements below the principal diagonal are zero **(upper triangular matrix)**.

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 5 & 2 & 3 \end{bmatrix} \qquad \begin{bmatrix} 1 & 8 & 9 \\ 0 & 1 & 6 \\ 0 & 0 & 3 \end{bmatrix}$$

8a. Upper triangular matrix

A square matrix whose elements below the main diagonal are all zero

$$\begin{bmatrix} 1 & 8 & 9 \\ 0 & 1 & 6 \\ 0 & 0 & 3 \end{bmatrix}$$

8b. Lower triangular matrix

A square matrix whose elements above the main diagonal are all zero

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 5 & 2 & 3 \end{bmatrix}$$

9. Scalar matrix

A diagonal matrix is said to be scalar if all of its diagonal elements are the same

$$\begin{bmatrix} a_{ij} & 0 & 0 \\ 0 & a_{ij} & 0 \\ 0 & 0 & a_{ij} \end{bmatrix}$$

EQUALITY OF MATRICES

Two matrices are said to be equal only when all corresponding elements are equal

Therefore their size or dimensions are equal as well

Note :

- A Square matrix is said to be singular if

  $|A| = 0$

- A Square matrix is said to be skew symmetric if

  $|A| \neq 0$

Transpose of a Matrix :

The **transpose** of a matrix is obtained by interchanging rows and columns of *A* and is denoted by A^T

- $(A+B)^T = A^T + B^T$
- $(AB)^T = B^T A^T$
- $(kA)^T = kA^T$
- $(A^T)^T = A$

Symmetric and Skew-symmetric Matrices :

- A Square matrix is said to be symmetric if

  $\mathbf{A} = \mathbf{A}^T$

- A Square matrix is said to be skew symmetric if

  $\mathbf{A} = -\mathbf{A}^T$

Theorem :

For any square matrix A with real number entries, $\mathbf{A} + \mathbf{A}^T$ is a symmetric matrix and $\mathbf{A} - \mathbf{A}^T$ is a skew-symmetric matrix.

Theorem :

Any square matrix can be expressed as the sum of a symmetric matrix and a skew-symmetric matrix.

Note :

1.A matrix which is both symmetric and skew-symmetric is a zero matrix.

2.For any square matrix A with real entries, $\mathbf{A} + \mathbf{A}^T$ is symmetric and $\mathbf{A} - \mathbf{A}^T$ is skew-symmetric and further $\mathbf{A} = \frac{1}{2}(\mathbf{A} + \mathbf{A}^T) + \frac{1}{2}(\mathbf{A} - \mathbf{A}^T)$

Determinants :

Note :

- Determinants can be defined only for square matrices.
- For a square matrix A, $|A|$ is read as **determinant of *A*.**
- Matrix is only a representation whereas determinant is a value of a matrix.

Properties of Determinants :

Property 1

The determinant of a matrix remains unaltered if its rows are changed into columns and columns into rows. That is, $|A|^T = |A^T|$

Property 2

If any two rows / columns of a determinant are interchanged, then the determinant changes in sign but its absolute value remains unaltered

Property 3

If there are n interchanges of rows (columns) of a matrix A then the determinant of the resulting matrix is $(-1)^n)\,|A|$.

Property 4

If two rows (columns) of a matrix are identical, then its determinant is zero.

Property 5

If a row (column) of a matrix A is a scalar multiple of another row (or column) of A, then its determinant is zero.

Note 7.8

(i) If all entries of a row or a column are zero, then the determinant is zero.

(ii) The determinant of a triangular matrix is obtained by the product of the principal diagonal elements.

Property 6

If each element in a row (or column) of a matrix is multiplied by a scalar k, then the determinant is multiplied by the same scalar k.

Property 7

If each element of a row (or column) of a determinant is expressed as sum of two or more terms then the whole determinant is expressed as sum of two or more determinants.

Property 8

If, to each element of any row (column) of a determinant the equi-multiples of the corresponding entries of one or more rows (columns) are added or subtracted, then the value of the determinant remains unchanged

Application of Factor Theorem to Determinants.

Factor Theorem:

If each element of a matrix A is a polynomial in x and if $|A|$ vanishes for $x = a$, then $(x - a)$ is a factor of $|A|$.

Note

(i) This theorem is very much useful when we have to obtain the value of the determinant in 'factors' form.

(ii) If we substitute b for a in the determinant $|A|$, any two of its rows or columns become identical, then $|A| = 0$, and hence by factor theorem $(a - b)$ is a factor of $|A|$.

(iii) If r rows (columns) are identical in a determinant of order n $(n \geq r)$, when we put $x = a$, then $(x - a)^{r-1}$ is a factor of $|A|$.

(iv) A square matrix (or its determinant) is said to be in cyclic symmetric form if each row is obtained from the first row by changing the variables cyclically.

(v) If the determinant is in cyclic symmetric form and if m is the difference between the degree of the product of the factors (obtained by substitution) and the degree of the product of the leading diagonal elements and if

(1) m is zero, then the required factor is a constant k

(2) m is 1, then the required factor is $k(a + b + c)$ and

(3) m is 2, then the required factor is $k(a^2 + b^2 + c^2) + l\,(ab + bc + ca)$.

Product of Determinants :

(i) Row by column multiplication rule

(ii) Row by row multiplication rule

(iii) Column by column multiplication rule

(iv) Column by row multiplication rule

Note :

(i) If A and B are square matrices of the same order n, then $|AB| = |A||B|$ holds.

(ii) In matrices, although $AB \neq BA$ in general, we do have $|AB| = |BA|$ always.

Area of a Triangle :

$$\text{Area of the triangle} = \begin{bmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{bmatrix}$$

Singular and Non singular Matrices :

- A Square matrix is said to be **singular** if

  $|A| = 0$

- A Square matrix is said to be **non singular** if

  $|A| \neq 0$

Matrices and Determinants.

Problems:

1. Construct a 2×3 matrix whose $(i,j)^{th}$ element is given by
$a_{ij} = \frac{\sqrt{3}}{2}|2i-3j|$ $(1 \le i \le 2,\ 1 \le j \le 3)$

Soln

Let A be 2×3 matrix

$$\therefore A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$$

$a_{ij} = \frac{\sqrt{3}}{2}|2i-3j|$

$\therefore a_{11} = \frac{\sqrt{3}}{2}|2-3| = \frac{\sqrt{3}}{2}$

$$= \begin{bmatrix} \frac{\sqrt{3}}{2} & 2\sqrt{3} & \frac{7\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \sqrt{3} & \frac{5\sqrt{3}}{2} \end{bmatrix}$$

2. Construct an $m\times n$ matrix $A=[a_{ij}]$ where a_{ij} is given by

(i) $a_{ij} = \frac{(i-2j)^2}{2}$ with $m=2, n=3$

(ii) $a_{ij} = \frac{|3i-4j|}{4}$ with $m=3, n=4$

Soln

(i) From given data 'A' is 2×3 matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$$

$a_{ij} = \frac{(i-2j)^2}{2}$

$a_{11} = \frac{(-1)^2}{2} = \frac{1}{2}$

$$= \begin{bmatrix} 1/2 & 9/2 & 25/2 \\ 0 & 2 & 1/2 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 1 & 9 & 25 \\ 0 & 4 & 1 \end{bmatrix}$$

(ii) From given data 'A' is 3×4 matrix

$$\therefore A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix}$$

$a_{ij} = \frac{|3i-4j|}{4}$

$a_{11} = \frac{|3-4|}{4} = \frac{1}{4}$

$$= \begin{bmatrix} 1/4 & 5/4 & 9/4 & 13/4 \\ 2/4 & 2/4 & 6/4 & 10/4 \\ 5/4 & 1/4 & 3/4 & 7/4 \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} 1 & 5 & 9 & 13 \\ 2 & 2 & 6 & 10 \\ 5 & 1 & 3 & 7 \end{bmatrix}$$

Equality of matrices:

Two matrices $A = [a_{ij}]$ & $B = [b_{ij}]$ are equal if & only if

(i) Both A & B are same order

(ii) $a_{ij} = b_{ij} \quad \forall i, j$

For instance, if

$$\begin{bmatrix} x & y \\ u & v \end{bmatrix} = \begin{bmatrix} 2.5 & -1 \\ \frac{1}{\sqrt{2}} & 3/2 \end{bmatrix}$$

Then $x = 2.5 \quad y = -1$

$u = \frac{1}{\sqrt{2}} \quad v = 3/2$

Note: If either of condition (i) or (ii) does not hold then the matrix A & B are called unequal matrices.

problems:

1. Find x, y, a & b if $\begin{bmatrix} 3x+4y & b & x-2y \\ a+b & 2a-b & -3 \end{bmatrix} = \begin{bmatrix} 2 & 6 & 4 \\ 5 & -5 & -3 \end{bmatrix}$

soln

$3x + 4y = 2$

$x - 2y = 4$

$3x + 4y = 2$

$2x - 4y = 8$

$5x = 10$

$x = 2$

$\therefore 3(2) + 4y = 2$

$4y = 2 - 6$

$4y = -4$

$\boxed{y = -1}$

$a+b = 5$

$2a-b = -5$

$3a = 0$

$\boxed{a = 0}$

$0+b = 5$

$\boxed{b = 5}$

$\therefore x = 2, \; y = -1 \quad a = 0, \; b = 5$

2. Determine $x+y$ if $\begin{bmatrix} 2x+y & 4x \\ 5x-7 & 4x \end{bmatrix} = \begin{bmatrix} 7 & 7y-13 \\ y & x+6 \end{bmatrix}$

Soln

$$\begin{bmatrix} 2x+y & 4x \\ 5x-7 & 4x \end{bmatrix} = \begin{bmatrix} 7 & 7y-13 \\ y & x+6 \end{bmatrix}$$

$\therefore$ $4x = x+6$ $2x+y=7$

$3x = 6$ $2(2)+y=7$

$x=2$ $y = 7-4$

$y=3$

$\therefore$ $x+y = 2+3$

$=5$

3. Find the value of p, q, r if

$$\begin{bmatrix} p^2-1 & 0 & -31-q^3 \\ 7 & r+1 & 9 \\ -2 & 8 & s-1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -4 \\ 7 & 3/2 & 9 \\ -2 & 8 & -\pi \end{bmatrix}$$

Soln

$p^2-1=1$ $-31-q^3=-4$ $r+1=3/2$ $s-1=-\pi$

$p^2=2$ $-q^3=-27$ $r=3/2-1$ $s=1-\pi$.

$p=\pm\sqrt{2}$ $q=-3$ $r=1/2$

7.2.3 Algebric operations on matrices :

1. Scalar multiplication
2. addition & subtraction
3. multiplication of two matrices.

Sclar multiplication :

For a given matrix $A=[a_{ij}]$ and a scalar k, we define a scalar multiplication

$$KA = [b_{ij}] \quad \text{where } b_{ij} = Ka_{ij}$$

Ex:

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

$$KA = \begin{bmatrix} Ka_{11} & Ka_{12} \\ Ka_{21} & Ka_{22} \end{bmatrix}$$

Negative of a matrix:

Negative of a matrix $A = [a_{ij}]$ is denoted by $-A$ and it is defined by $-A = [-a_{ij}]$

Addition and subtraction of two matrices:

Let $A = [a_{ij}]$ and $B = [b_{ij}]$ are two matrices and addition & subtraction of A & B are defined by

$A+B = [c_{ij}]$ where $c_{ij} = a_{ij} + b_{ij}$

$A-B = [d_{ij}]$ where $d_{ij} = a_{ij} - b_{ij}$.

Note:

1. If A and B are not of the same order, then A+B and A−B are not defined.

Problems:

1. compute A+B and A−B if $A = \begin{bmatrix} 4 & \sqrt{5} & 7 \\ -1 & 0 & 0.5 \end{bmatrix}$ $B = \begin{bmatrix} \sqrt{3} & \sqrt{5} & 7.3 \\ 1 & 1/3 & 1/4 \end{bmatrix}$

Soln

$A = \begin{bmatrix} 4 & \sqrt{5} & 7 \\ -1 & 0 & 0.5 \end{bmatrix}$ $B = \begin{bmatrix} \sqrt{3} & \sqrt{5} & 7.3 \\ 1 & 1/3 & 1/4 \end{bmatrix}$

$A+B = \begin{bmatrix} 4+\sqrt{3} & 2\sqrt{3} & 14.3 \\ 0 & 1/3 & 3/4 \end{bmatrix}$

$A-B = \begin{bmatrix} 4-\sqrt{3} & 0 & -0.3 \\ -2 & -1/3 & 1/4 \end{bmatrix}$

2. Find A+B+C if A, B, C are given by

$A = \begin{bmatrix} \sin^2\theta & 1 \\ \cot^2\theta & 0 \end{bmatrix}$ $B = \begin{bmatrix} \cos^2\theta & 0 \\ -\operatorname{cosec}^2\theta & 1 \end{bmatrix}$ $C = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$

Soln

$A+B+C = \begin{bmatrix} \sin^2\theta+\cos^2\theta+0 & 1+0-1 \\ \cot^2\theta-\operatorname{cosec}^2\theta-1 & 0+1+0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -2 & 0 \end{bmatrix}$

3) Find $3B+4C-D$ if B, C, D are given by

$$B = \begin{bmatrix} 2 & 3 & 0 \\ 1 & -1 & 5 \end{bmatrix} \quad C = \begin{bmatrix} -1 & -2 & 3 \\ -1 & 0 & 2 \end{bmatrix} \quad D = \begin{bmatrix} 0 & 4 & -1 \\ 5 & 6 & -5 \end{bmatrix}$$

Soln

$$3B+4C-D = 3\begin{bmatrix} 2 & 3 & 0 \\ 1 & -1 & 5 \end{bmatrix} + 4\begin{bmatrix} -1 & -2 & 3 \\ -1 & 0 & 2 \end{bmatrix} - \begin{bmatrix} 0 & 4 & -1 \\ 5 & 6 & -5 \end{bmatrix}$$

$$= \begin{bmatrix} 6 & 9 & 0 \\ 3 & -3 & 15 \end{bmatrix} + \begin{bmatrix} -4 & -8 & 12 \\ -4 & 0 & 8 \end{bmatrix} + \begin{bmatrix} 0 & -4 & 1 \\ -5 & -6 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & -3 & 13 \\ -6 & -9 & 28 \end{bmatrix}$$

4. Simplify

$$\sec\theta\begin{bmatrix} \sec\theta & \tan\theta \\ \tan\theta & \sec\theta \end{bmatrix} - \tan\theta\begin{bmatrix} \tan\theta & \sec\theta \\ \sec\theta & \tan\theta \end{bmatrix}$$

Soln

$$A = \begin{bmatrix} \sec^2\theta & \sec\theta\tan\theta \\ \sec\theta\tan\theta & \sec^2\theta \end{bmatrix} - \begin{bmatrix} \tan^2\theta & \tan\theta\sec\theta \\ \sec\theta\tan\theta & \tan^2\theta \end{bmatrix}$$

$$= \begin{bmatrix} \sec^2\theta - \tan^2\theta & 0 \\ 0 & \sec^2\theta - \tan^2\theta \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

5. Determine the matrices A & B if they satisfy

$$2A - B + \begin{bmatrix} 6 & -6 & 0 \\ -4 & 2 & 1 \end{bmatrix} = 0 \quad \& \quad A - 2B = \begin{bmatrix} 3 & 2 & 8 \\ -2 & 1 & -7 \end{bmatrix}$$

Soln

$$2A - B = \begin{bmatrix} -6 & 6 & 0 \\ 4 & -2 & -1 \end{bmatrix} \longrightarrow (1)$$

$$2A - 4B = \begin{bmatrix} 6 & 4 & 16 \\ -4 & 2 & -14 \end{bmatrix}$$

$$-2A + 4B = \begin{bmatrix} -6 & -4 & -16 \\ 4 & -2 & 14 \end{bmatrix} \longrightarrow (2)$$

$(1)+(2) \Rightarrow$

$$3B = \begin{bmatrix} -12 & 2 & -16 \\ 8 & -4 & 13 \end{bmatrix}$$

$$B = \frac{1}{3}\begin{bmatrix} -12 & 2 & -16 \\ 8 & -4 & 13 \end{bmatrix}$$

From ①

$$A - 2B = \begin{bmatrix} 3 & 2 & 8 \\ -2 & 1 & -7 \end{bmatrix}$$

$$A = \begin{bmatrix} 3 & 2 & 8 \\ -2 & 1 & -7 \end{bmatrix} + 2B$$

$$= \begin{bmatrix} 3 & 2 & 8 \\ -2 & 1 & -7 \end{bmatrix} + \frac{2}{3}\begin{bmatrix} -12 & 2 & -16 \\ 8 & -4 & 13 \end{bmatrix}$$

$$= \frac{1}{3}\begin{bmatrix} 9 & 6 & 24 \\ -6 & 3 & -21 \end{bmatrix} + \frac{1}{3}\begin{bmatrix} -24 & 4 & -32 \\ 16 & -8 & 26 \end{bmatrix}$$

$$= \frac{1}{3}\begin{bmatrix} -15 & 10 & -8 \\ 10 & -5 & 5 \end{bmatrix}$$

Multiplication of matrices:

A matrix 'A' is said to be conformable for multiplication with a matrix B if the number of columns of A is equal to the number of rows of B.

Result:

(i) $\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} = \begin{bmatrix} aa_1 + bc_1 & ab_1 + bd_1 \\ ca_1 + dc_1 & cb_1 + dd_1 \end{bmatrix}$

problem:

1. If $A = \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}$, then find A^4

soln

$$A^2 = A \cdot A$$

$$= \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1+0 & a+a \\ 0+0 & 0+1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2a \\ 0 & 1 \end{bmatrix}$$

$$A^4 = A^2 \cdot A^2$$

$$= \begin{bmatrix} 1 & 2a \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2a \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1+0 & 2a+2a \\ 0+0 & 0+1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 4a \\ 0 & 1 \end{bmatrix}$$

2. Find A^2 if $A = \begin{bmatrix} 0 & c & b \\ c & 0 & a \\ b & a & 0 \end{bmatrix}$

Soln

$$A^2 = A \cdot A$$

$$= \begin{bmatrix} 0 & c & b \\ c & 0 & a \\ b & a & 0 \end{bmatrix} \begin{bmatrix} 0 & c & b \\ c & 0 & a \\ b & a & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0+c^2+b^2 & 0+0+ab & 0+ac+0 \\ 0+0+ab & c^2+0+a^2 & bc+0+0 \\ 0+ac+0 & bc+0+0 & b^2+a^2+0 \end{bmatrix}$$

$$= \begin{bmatrix} b^2+c^2 & ab & ac \\ ab & c^2+a^2 & bc \\ ac & bc & a^2+b^2 \end{bmatrix}$$

3. Solve x if $\begin{bmatrix} x & 2 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 2 \\ -1 & -4 & 1 \\ -1 & -1 & -2 \end{bmatrix} \begin{bmatrix} x \\ 2 \\ 1 \end{bmatrix} = 0$

Soln

$$\begin{bmatrix} x & 2 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 2 \\ -1 & -4 & 1 \\ -1 & -1 & -2 \end{bmatrix} \begin{bmatrix} x \\ 2 \\ 1 \end{bmatrix} = 0$$

$$\begin{bmatrix} x-2+1 & x-8+1 & 2x+2+2 \end{bmatrix} \begin{bmatrix} x \\ 2 \\ 1 \end{bmatrix} = 0$$

$$\begin{bmatrix} x-1 & x-7 & 2x+4 \end{bmatrix} \begin{bmatrix} x \\ 2 \\ 1 \end{bmatrix} = 0$$

$$x(x-1) + 2(x-7) + 1(2x+4) = 0$$

$$x^2+3x-10 = 0$$

$$(x-2)(x+5) = 0$$

$$x = 2, -5$$

$-10 < \begin{matrix} 5 \\ -2 \end{matrix}$ $\quad \begin{matrix} -10 \\ -3 \end{matrix}$

4) If $A = \begin{bmatrix} 4 & 2 \\ -1 & x \end{bmatrix}$ and s.t $(A-2I)(A-3I) = 0$ find the value of x.

Soln

$$A - 2I = \begin{bmatrix} 4 & 2 \\ -1 & x \end{bmatrix} + \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 2 \\ -1 & x-2 \end{bmatrix}$$

$$A - 3I = \begin{bmatrix} 4 & 2 \\ -1 & x \end{bmatrix} + \begin{bmatrix} -3 & 0 \\ 0 & -3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 \\ -1 & x-3 \end{bmatrix}$$

$$(A-2I)(A-3I) = 0$$

$$\begin{bmatrix} 2 & 2 \\ -1 & x-2 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ -1 & x-3 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 2-2 & 4+2x-6 \\ -1+2-x & -2+(x-2)(x-3) \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\therefore \quad 4+2x-6 = 0$$

$$2x-2 = 0$$

$$2x = 2$$

$$x = 1$$

5. If $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ a & b & -1 \end{bmatrix}$ show that A^2 is unit matrix.

Soln

$$A^2 = A \cdot A$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ a & b & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ a & b & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 1+0+0 & 0+0+0 & 0+0+0 \\ 0+0+0 & 0+1+0 & 0+0+0 \\ a+0-a & 0+b-b & 0+0+1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= I_3$$

6. If $A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix}$ and $A^3 - 6A^2 + 7A + kI = 0$ find k.

Soln

$$A^2 = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 1+0+4 & 0+0+0 & \cancel{20}+6 \\ 0+0+2 & 0+4+0 & 0+2+3 \\ 2+0+6 & 0+0+0 & 4+0+9 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 0 & 8 \\ 2 & 4 & 5 \\ 8 & 0 & 13 \end{bmatrix}$$

$$A^3 = A^2 \cdot A$$

$$= \begin{bmatrix} 5 & 0 & 8 \\ 2 & 4 & 5 \\ 8 & 0 & 13 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 5+0+16 & 0+0+0 & 10+0+24 \\ 2+0+10 & 0+8+0 & 4+4+15 \\ 8+0+26 & 0+0+0 & 16+0+39 \end{bmatrix}$$

$$= \begin{bmatrix} 21 & 0 & 34 \\ 12 & 8 & 23 \\ 34 & 0 & 55 \end{bmatrix}$$

now

$$A^3 - 6A^2 + 7A + kI = 0 .$$

Taking all element in $A, A^2, A^3, I, 0$. & using the given equ.

$$21 - 6(5) + 7(1) + k(1) = 0$$

$$28 - 30 + k = 0$$

$$\boxed{k = 2}$$

7.2.4 Properties of matrix adition, scalar multiplication & product of matrices.

1. $A+B = B+A$
2. $A+(B+C) = A+(B+C)$
3. $A+O = O+A = A.$
4. $A+(-A) = (-A)+A = O$
5. $(a+b)A = aA + bA$
6. $\alpha(A+B) = \alpha A + \alpha B$
7. $\alpha(bA) = (\alpha b)A.$
8. $1A = A$
9. $OA = O.$

properties of matrix multiplication

1. $A(BC) = (AB)C$
2. $A(B+C) = AB + AC$
3. $(A+B)C = AC + BC$
4. $\alpha(AB) = (\alpha A)B$
5. $AI = IA = A$

Problems:

1. Give your own examples of matrices satisfying the following conditions in each case:

(i) A and B such that $AB \neq BA$

(ii) A and B s.t $AB = 0 = BA$, $A \neq 0$ & $B \neq 0$

(iii) A and B s.t $AB = 0$ and $BA \neq 0$.

Soln

(i) A and B s.t $AB \neq BA$.

Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ $B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$

$$AB = \begin{bmatrix} 0+0 & 0+0 \\ 0+0 & 0+0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$BA = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0+0 & 0+0 \\ 1+0 & 0+0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

$\therefore AB \neq BA$.

(ii) A and B s.t. $AB = BA = 0$, $A \neq 0$, $B \neq 0$

Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ $B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

$$AB = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$BA = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$AB = BA = 0$, $A \neq 0$, $B \neq 0$.

(iii) A and B s.t $AB = 0$ and $BA \neq 0$.

Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

$B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$

$AB = 0$ & $BA \neq 0$ from (i).

2. If A is a square matrix s.t $A^2 = A$, find the value of $7A - (I+A)^3$

Soln

$$7A - (I+A)^3 = 7A - (I + 3A + 3A^2 + A^3)$$

$$= 7A - [I + 3A + 3A + A]$$

$$= 7A - [I + 7A]$$

$$= -I$$

$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$

$A^2 = A$

$A^3 = A^2 \cdot A$

$= A \cdot A$

$= A^2$

$= A$

3. Show that $f(x)f(y) = f(x+y)$ where $f(x) = \begin{bmatrix} \cos x & -\sin x & 0 \\ \sin x & \cos x & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Soln

$$f(x)f(y) = \begin{bmatrix} \cos x & -\sin x & 0 \\ \sin x & \cos x & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} \cos x & -\sin x & 0 \\ \sin x & \cos x & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \cos x\cos y - \sin x \sin y & -\cos x \sin y - \sin x \cos y & 0+0+0 \\ \sin x \cos y + \cos x \sin y & -\sin x \sin y + \cos x \cos y & 0+0+0 \\ 0+0+0 & 0+0+0 & 0+0+1 \end{bmatrix}$$

$$= \begin{bmatrix} \cos(x+y) & -\sin(x+y) & 0 \\ \sin(x+y) & \cos(x+y) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$f(x)f(y) = f(x+y)$$

4. verify the property $A(B+C) = AB + AC$ when the matrices A, B, C are given by

$$A = \begin{bmatrix} 2 & 0 & -3 \\ 1 & 4 & 5 \end{bmatrix} \quad B = \begin{bmatrix} 3 & 1 \\ -1 & 0 \\ 4 & 2 \end{bmatrix} \quad C = \begin{bmatrix} 4 & 7 \\ 2 & 1 \\ 1 & -1 \end{bmatrix}$$

Soln

(i) $A(B+C)$

$$B+C = \begin{bmatrix} 7 & 8 \\ 1 & 1 \\ 5 & 1 \end{bmatrix}$$

$$A(B+C) = \begin{bmatrix} 2 & 0 & -3 \\ 1 & 4 & 5 \end{bmatrix}\begin{bmatrix} 7 & 8 \\ 1 & 1 \\ 5 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 14+0-15 & 16+0-3 \\ 7+4+25 & 8+4+5 \end{bmatrix} = \begin{bmatrix} -1 & 13 \\ 36 & 17 \end{bmatrix} \rightarrow ①$$

(ii) AB+AC :

$$AB = \begin{bmatrix} 6+0+12 & 2+0-16 \\ 3-4+20 & 1+0+10 \end{bmatrix} = \begin{bmatrix} -6 & -14 \\ 19 & 11 \end{bmatrix}$$

$$AC = \begin{bmatrix} 8+0-3 & 14+0+3 \\ 4+8+5 & 7+4-5 \end{bmatrix} = \begin{bmatrix} 5 & 17 \\ 17 & 6 \end{bmatrix}$$

$$AB+AC = \begin{bmatrix} -1 & 13 \\ 36 & 17 \end{bmatrix} \longrightarrow (2)$$

$\therefore$ From (1) & (2)

$$A(B+C) = AB+AC.$$

5. Find the matrix A which satisfies the relation

$$A\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} = \begin{bmatrix} -7 & -8 & -9 \\ 2 & 4 & 6 \end{bmatrix}$$

Soln

From the given data A is 2x2 matrix.

$$\text{Let } A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

now

$$A\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} = \begin{bmatrix} -7 & -8 & -9 \\ 2 & 4 & 6 \end{bmatrix}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} = \begin{bmatrix} -7 & -8 & -9 \\ 2 & 4 & 6 \end{bmatrix}$$

$$\begin{bmatrix} a+4b & 2a+5b & 3a+6b \\ c+4d & 2c+5d & 3c+6d \end{bmatrix} = \begin{bmatrix} -7 & -8 & -9 \\ 2 & 4 & 6 \end{bmatrix}$$

$a+4b = -7 \rightarrow (1)$

$2a+5b = -8 \rightarrow (2)$

$c+4d = 2 \rightarrow (3)$

$2c+5d = 4 \rightarrow (4)$

$(2) \Rightarrow \quad 2a+5b = -8$

$-2\times(1) \Rightarrow \quad -2a-8b = 14$

$-3b = 6$

$b = -2$

$(1) \Rightarrow \quad a = 1$

$(4) \Rightarrow \quad 2c+5d = 4$

$-2\times(3) \Rightarrow \quad -2c-8d = -4$

$-3d = 0$

$d = 0$

$(3) \Rightarrow \quad \therefore c+0 = 2$

$c = 2$

$$\therefore A = \begin{bmatrix} 1 & -2 \\ 2 & 0 \end{bmatrix}$$

7.2.5 Transpose of a matrix:

The Transpose of a matrix $A = [a_{ij}]$ is denoted by A^T and it is defined by

$$A^T = [a_{ji}]$$

Ex:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \text{ Then } A^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$$

Results:

1. $(A^T)^T = A$
2. $(KA)^T = KA^T$
3. $(A+B)^T = A^T + B^T$
4. $(AB)^T = B^T A^T$

problems:

1. If $A = \begin{bmatrix} 4 & 6 & 2 \\ 0 & 1 & 5 \\ 0 & 3 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 1 & -1 \\ 3 & -1 & 4 \\ -1 & 2 & 1 \end{bmatrix}$ verify

(i) $(AB)^T = B^T A^T$ (ii) $(A+B)^T = A^T + B^T$ (iii) $(A-B)^T = A^T - B^T$

(iv) $(3A)^T = 3A^T$.

Soln

(i) $(AB)^T = B^T A^T$

$$AB = \begin{bmatrix} 4 & 6 & 2 \\ 0 & 1 & 5 \\ 0 & 3 & 2 \end{bmatrix} \begin{bmatrix} 0 & 1 & -1 \\ 3 & -1 & 4 \\ -1 & 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 16 & 2 & 22 \\ -2 & 9 & 9 \\ 7 & 1 & 14 \end{bmatrix}$$

$$(AB)^T = \begin{bmatrix} 16 & -2 & 7 \\ 2 & 9 & 1 \\ 22 & 9 & 14 \end{bmatrix} \rightarrow (1)$$

$$B^T A^T = \begin{bmatrix} 0 & 3 & -1 \\ 1 & -1 & 2 \\ -1 & 4 & 1 \end{bmatrix} \begin{bmatrix} 4 & 0 & 0 \\ 6 & 1 & 3 \\ 2 & 5 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 16 & -2 & 7 \\ 2 & 9 & 1 \\ 22 & 9 & 14 \end{bmatrix} \longrightarrow ②$$

①, ② $\Rightarrow$

$$(AB)^T = B^T A^T$$

(ii) $(A+B)^T = A^T + B^T$:

$$A+B = \begin{bmatrix} 4 & 6 & 2 \\ 0 & 1 & 5 \\ 0 & 3 & 2 \end{bmatrix} + \begin{bmatrix} 0 & 1 & -1 \\ 3 & -1 & 4 \\ -1 & 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 7 & 1 \\ 3 & 0 & 9 \\ -1 & 5 & 3 \end{bmatrix}$$

$$(A+B)^T = \begin{bmatrix} 4 & 3 & -1 \\ 7 & 0 & 5 \\ 1 & 9 & 3 \end{bmatrix} \longrightarrow ①$$

$$A^T + B^T = \begin{bmatrix} 4 & 0 & 0 \\ 6 & 1 & 3 \\ 2 & 5 & 2 \end{bmatrix} + \begin{bmatrix} 0 & 3 & -1 \\ 1 & -1 & 2 \\ -1 & 4 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 3 & -1 \\ 7 & 0 & 5 \\ 1 & 9 & 3 \end{bmatrix} \longrightarrow ②$$

①, ② $\Rightarrow$

$$(A+B)^T = A^T + B^T$$

(iii) $(A-B)^T = A^T - B^T$:

$$A - B = \begin{bmatrix} 4 & 6 & 2 \\ 0 & 1 & 5 \\ 0 & 3 & 2 \end{bmatrix} - \begin{bmatrix} 0 & 1 & -1 \\ 3 & -1 & 4 \\ -1 & 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 5 & 3 \\ -3 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

$$(A-B)^T = \begin{bmatrix} 4 & -3 & 1 \\ 5 & 2 & 1 \\ 3 & 1 & 1 \end{bmatrix} \longrightarrow ①$$

$$A^T - B^T = \begin{bmatrix} 4 & 0 & 0 \\ 6 & 1 & 3 \\ 2 & 5 & 2 \end{bmatrix} - \begin{bmatrix} 0 & -3 & -1 \\ 1 & -1 & 2 \\ -1 & 4 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & -3 & 1 \\ -3 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix} \longrightarrow ②$$

①, ② $\Rightarrow (A-B)^T = A^T - B^T$

(iv) $(3A)^T = 3A^T$

$$3A = \begin{bmatrix} 12 & 18 & 6 \\ 0 & 3 & 15 \\ 0 & 9 & 6 \end{bmatrix}$$

$$(3A)^T = \begin{bmatrix} 12 & 0 & 0 \\ 18 & 3 & 9 \\ 6 & 15 & 6 \end{bmatrix}$$

$$= 3\begin{bmatrix} 4 & 0 & 0 \\ 6 & 1 & 3 \\ 2 & 5 & 2 \end{bmatrix}$$

$$(3A)^T = 3(A^T)$$

2. If $A^T = \begin{bmatrix} 4 & 5 \\ -1 & 0 \\ 2 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & -1 & 1 \\ 7 & 5 & -2 \end{bmatrix}$ verify the following

(i) $(A+B)^T = A^T + B^T$ (ii) $(A-B)^T = A^T - B^T$ (iii) $(B^T)^T = B$

Soln

$$A^T = \begin{bmatrix} 4 & 5 \\ -1 & 0 \\ 2 & 3 \end{bmatrix} \qquad A = \begin{bmatrix} 4 & -1 & 2 \\ 5 & 0 & 3 \end{bmatrix}$$

$$B^T = \begin{bmatrix} 2 & 7 \\ -1 & 5 \\ 1 & -2 \end{bmatrix} \qquad B = \begin{bmatrix} 2 & -1 & 1 \\ 7 & 5 & -2 \end{bmatrix}$$

$$A^T + B^T = \begin{bmatrix} 6 & 12 \\ -2 & 5 \\ 3 & -1 \end{bmatrix} \qquad A+B = \begin{bmatrix} 6 & -2 & 3 \\ 12 & 5 & -1 \end{bmatrix}$$

$$A^T + B^T = (A+B)^T$$

(ii) $(A-B)^T = A^T - B^T$

$$A-B = \begin{bmatrix} 2 & 0 & 1 \\ -2 & -5 & 5 \end{bmatrix}$$

$$(A-B)^T = \begin{bmatrix} 2 & -2 \\ 0 & -5 \\ 1 & 5 \end{bmatrix} \longrightarrow (1)$$

$$A^T - B^T = \begin{bmatrix} 2 & -2 \\ 0 & -5 \\ 1 & 5 \end{bmatrix}$$

$$= (A-B)^T \quad \text{From (1)}$$

$$B^T = \begin{bmatrix} 2 & 7 \\ -1 & 5 \\ 1 & -5 \end{bmatrix}$$

$$(B^T)^T = \begin{bmatrix} 2 & -1 & 1 \\ 7 & 5 & -5 \end{bmatrix} = B$$

3. If A is 3x4 matrix and B is a matrix such that both A^TB and BA^T defined, what is the order of B?

Soln

A is 3×4 matrix.

A^T is 4×3 matrix.

A^TB is defined. $\Rightarrow$ B has 3 rows.

BA^T is defined $\Rightarrow$ B has 4 colums.

$\therefore$ B is 3×4 matrix

7.2.6. Symetric and Skew symmetric matrix:

Symmetric matrix:

A square matrix 'A' is called symmetric if

$A^T = A$.

Example:

$$A = \begin{bmatrix} 2 & 3 \\ 3 & 1 \end{bmatrix}$$

$$A^T = \begin{bmatrix} 2 & 3 \\ 3 & 1 \end{bmatrix} = A. \quad A \text{ is symmetric.}$$

Skew symmetric matrix:

A square matrix 'A' is said to be skew symmetric if $A^T = -A$.

Ex:

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

$$A^T = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$$= -A \quad A \text{ is skew symmetric.}$$

Results:

1. zero matrix is both symmetric & skew symmetric.
2. In skew symmetric matrix main diagonal elements are zero.

Theorem:

For any square matrix A with real entries, $A+A^T$ is a symmetric matrix and $A-A^T$ is skew symmetric matrix.

Proof:

Let A be a matrix. (square)

We shall show that

(i) $A+A^T$ is symmetric

(ii) $A-A^T$ is skew symmetric.

Now

$$(A+A^T)^T = A^T + (A^T)^T$$
$$= A^T + A$$
$$= (A+A^T)$$

$\therefore$ $A+A^T$ is symmetric matrix.

Also

$$(A-A^T)^T = A^T - (A^T)^T$$
$$= A^T - A$$
$$= -(A-A^T)$$

$\therefore$ $A-A^T$ is skew symmetric

Theorem:

A any square matrix can be expressed as sum of symmetric matrix and skew symmetric matrix

Proof: Let A be a square matrix

now $A = \frac{1}{2}(A+A^T) + \frac{1}{2}(A-A^T)$

W.K.T $A+A^T$, $A-A^T$ are symmetric & skew symmetric resp.

$\Rightarrow \frac{1}{2}(A+A^T), \frac{1}{2}(A-A^T)$ are symmetric & "

$\therefore$ Hence the result

problem:

1. Let A & B two symmetric matrices. prove that $AB = BA$ $\Leftrightarrow$ AB is a symmetric matrix

Soln Let A & B are symmetric matrix

$\therefore A^T = A$

$B^T = B$.

Let $AB = BA$ we shall show that AB is symmetric

now $(AB)^T = B^T A^T$

$= BA$

$(AB)^T = AB$

$\therefore$ AB is symmetric

Conversely
Let AB is symmetric. We shall S.T. $AB = BA$

$\therefore (AB)^T = (AB)^T$

$= B^T A^T$

$AB. = BA$

Hence the proof

2. If A & B are symmetric matrices of same order P.T
(i) $AB + BA$ is a symmetric matrix
(ii) $AB - BA$ is a skew symmetric matrix

Soln Let A & B are symmetric matrix

$A^T = A$, $B^T = B$

(i) $(AB+BA)^T = (AB)^T + (BA)^T$

$= B^T A^T + A^T B^T$

$= BA + AB$

$(AB+BA)^T = AB + BA.$

$\therefore$ AB+BA is symmetric matrix

(ii) $(AB-BA)^T = (AB)^T - (BA)^T$

$= B^T A^T - A^T B^T$

$= BA - AB$

$(AB-BA)^T = -(AB-BA)$

$\therefore$ AB-BA is skew symmetric matrix.

3. Construct the matrix $A = [a_{ij}]_{3\times3}$ where $a_{ij} = i-j$ State A is symmetric or skew symmetric.

soln

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -1 & -2 \\ 1 & 0 & -1 \\ 2 & 1 & 0 \end{bmatrix} \qquad a_{ij} = i-j$$

$A^T = -A$

$\therefore$ A is skew symmetric

4. Express the following matrices as the sum of symmetric and skew symmetric matrix:

(i) $\begin{bmatrix} 4 & -2 \\ 3 & -5 \end{bmatrix}$ (ii) $\begin{bmatrix} 3 & 3 & -1 \\ -2 & -2 & 1 \\ -4 & -5 & 2 \end{bmatrix}$

soln

(i) $A = \begin{bmatrix} 4 & -2 \\ 3 & -5 \end{bmatrix} \therefore A^T = \begin{bmatrix} 4 & 3 \\ -2 & -5 \end{bmatrix}$

$A+A^T = \begin{bmatrix} 8 & 1 \\ 1 & -10 \end{bmatrix} \quad A - A^T = \begin{bmatrix} 0 & -5 \\ 5 & 0 \end{bmatrix}$

$$A = \frac{1}{2}(A+A^T) + \frac{1}{2}(A-A^T)$$

$$= \frac{1}{2}\begin{bmatrix} 8 & 1 \\ 1 & -10 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 0 & -5 \\ 5 & 0 \end{bmatrix}$$

(ii) $A = \begin{bmatrix} 3 & 3 & -1 \\ -2 & -2 & 1 \\ -4 & -5 & 2 \end{bmatrix}$ $\therefore$ $A^T = \begin{bmatrix} 3 & -2 & -4 \\ 3 & -2 & -5 \\ -1 & 1 & 2 \end{bmatrix}$

$A+A^T = \begin{bmatrix} 6 & 1 & -5 \\ 1 & -4 & -4 \\ -5 & -4 & 4 \end{bmatrix}$ $A-A^T = \begin{bmatrix} 0 & 5 & 3 \\ -5 & 0 & 6 \\ -3 & -6 & 0 \end{bmatrix}$

$$\therefore A = \frac{1}{2}(A+A^T) + \frac{1}{2}(A-A^T)$$

$$= \frac{1}{2}\begin{bmatrix} 6 & 1 & -5 \\ 1 & -4 & -4 \\ -5 & -4 & 4 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 0 & 5 & 3 \\ -5 & 0 & 6 \\ -3 & -6 & 0 \end{bmatrix}$$

5) Express the matrix $A = \begin{bmatrix} 1 & 3 & 5 \\ -6 & 8 & 3 \\ -4 & 6 & 5 \end{bmatrix}$ as the sum of symmetric & skew symmetric matrices.

Soln

$A = \begin{bmatrix} 1 & 3 & 5 \\ -6 & 8 & 3 \\ -4 & 6 & 5 \end{bmatrix}$ $A^T = \begin{bmatrix} 1 & -6 & -4 \\ 3 & 8 & 6 \\ 5 & 3 & 5 \end{bmatrix}$

$A+A^T = \begin{bmatrix} 2 & -3 & 1 \\ -3 & 16 & 9 \\ 1 & 9 & 10 \end{bmatrix}$ $A-A^T = \begin{bmatrix} 0 & 9 & 9 \\ -9 & 0 & -3 \\ -9 & 3 & 0 \end{bmatrix}$

$$\therefore A = \frac{1}{2}(A+A^T) + \frac{1}{2}(A-A^T)$$

$$= \frac{1}{2}\begin{bmatrix} 2 & -3 & 1 \\ -3 & 16 & 9 \\ 1 & 9 & 10 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 0 & 9 & 9 \\ -9 & 0 & -3 \\ -9 & 3 & 0 \end{bmatrix}$$

A is sum symmetric & skew symmetric matrices.

6) (i) For what value of x, the matrix $A = \begin{bmatrix} 0 & 1 & -2 \\ -1 & 0 & x^3 \\ 2 & -3 & 0 \end{bmatrix}$ is skew symmetric

(ii) If $\begin{bmatrix} 0 & p & 3 \\ 2 & q^2 & -1 \\ r & 1 & 0 \end{bmatrix}$ is skew symmetric find p, q, r

Soln

(i) If A is skew symmetric then

$$x^3 = 3$$
$$x = 3^{1/3}$$

(ii) A is skew symmetric

$q^2 = 0$ $\quad p = -2 \quad r = -3$

$q = 0$

7. If $A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ x & 2 & y \end{bmatrix}$ is a matrix s.t $AA^T = 9I$ find x, y

Soln

$$AA^T = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ x & 2 & y \end{bmatrix} \begin{bmatrix} 1 & 2 & x \\ 2 & 1 & 2 \\ 2 & -2 & y \end{bmatrix} = 9 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1+4+4 & 2+2-4 & x+4-2y \\ 2+2-4 & 4+1+4 & 2x-2-2y \\ x+4+2y & 2x+2-2y & x^2+4+y \end{bmatrix} = \begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix}$$

$\therefore$ $x - 2y = -4 \longrightarrow$ ①

$2x - 2y = -2 \longrightarrow$ ②

① ⇒ $\therefore$ $x - 2y = -4$

② ⇒ $\quad\; x + y = -7$

$\quad\; - \quad - \quad\; +$

$$-3y = -3$$
$$y = 1$$

① ⇒ $x - 6 = -4$

$x = -4 + 6$

$\boxed{x = 2}$

8. Find the matrix A s.t $\begin{bmatrix} 2 & -1 \\ 1 & 0 \\ -3 & 4 \end{bmatrix} A^T = \begin{bmatrix} -1 & -8 & -10 \\ 1 & 2 & -5 \\ 9 & 22 & 15 \end{bmatrix}$

Determinants:

To every square matrix $A = [a_{ij}]$ of order n, we can associate a number called determinant of 'A'

If $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$ then $|A| = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$

Note:

i) Determinants can be defined only for square matrix

ii. |A| is read as determinant of A.

(iii) Matrix is only representation whereas determinant is the value of matrix

Determinant of matrices:

(i) Matrix of order 1:

$A = [a]$ then $|A| = a$.

(ii) Matrix of order 2:

$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ then $|A| = a_{11}a_{22} - a_{21}a_{12}$.

(iii) Matrix of order 3:

$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ then $|A| = a_{11}[(a_{22} a_{33}) - (a_{32}a_{23})]$

$- a_{12}[(a_{21}a_{33} - a_{31}a_{23}]$

$+ a_{13}[a_{21}a_{32} - a_{31}a_{22}]$

Problems:

(i) Evaluate $\begin{vmatrix} 2 & 4 \\ -1 & 2 \end{vmatrix}$

Soln let $A = \begin{vmatrix} 2 & 4 \\ -1 & 2 \end{vmatrix}$

$|A| = 4+4$

$= 8$.

2) Evaluate $\begin{vmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{vmatrix}$

Soln

Let $A = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$

$|A| = \cos^2\theta + \sin^2\theta$

$= 1.$

3. Find $|A|$, if $A = \begin{bmatrix} 0 & \sin\alpha & \cos\alpha \\ \sin\alpha & 0 & \sin\beta \\ \cos\alpha & -\sin\alpha & 0 \end{bmatrix}$

Soln

$|A| = 0 - \sin\alpha(-\cos\alpha\sin\alpha) + \cos\alpha(-\sin\alpha\sin\beta)$

$= 0.$

Properties of determinants:

Property 1:

The determinant of a matrix remains unaltered if its rows changed into columns and colums into rows

(or) $|A| = |A^T|$

Property 2:

If any two rows / colums of a determinants are interchanged, then the determinant change in sign but its absolute value remains unaltred.

Proof:

Let $|A| = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

$= a_1(b_2c_3 - b_3c_2) - b_1(a_3c_2 - a_2c_3) + c_1(a_3b_2 - a_2b_3)$ → (1)

Let $|A_1| = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \\ a_2 & b_2 & c_2 \end{vmatrix}$ $(R_2 \leftrightarrow R_3)$

$= a_1(b_3c_2 - b_2c_3) - b_1(a_3c_2 - a_2c_3) + c_1(a_3b_2 - a_2b_3)$

$= -a_1(b_2c_3 - b_3c_2) + b_1(a_2c_3 - a_3c_2) - c_1(a_2b_3 - a_3b_2)$

$= -[a_1(b_2c_3 - b_3c_2) - b_1(a_2c_3 - a_3c_2) + c_1(a_2b_3 - a_3b_2)]$

$= -|A|$

$\therefore |A_1| = -|A_1|$

Property is verified.

Property 3:

If there are n interchangs of rows (colums) of a matrix A then the determinants of the resulting matrix is $(-1)^n |A|$

Property: 4:

If two rows (colums) of a matrix are identical then its determinant is zero.

verification:

Let $|A| = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

$= -\begin{vmatrix} a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \\ a_2 & b_2 & c_2 \end{vmatrix}$ $(R_1 \to R_2)$

$|A| = -|A|$

$2|A| = 0$

$|A| = 0.$

Property 5: If a rows (colums) of a matrix is a scalar multiple of another row (or column) of A, then its determinant zero.

Note:

1. If all entries of a row or colums are zero, then the determinant is zero.
2. The determinant of a triangular matrix is product of principal diagonal elements.

Property: 6 If each element in a row (or column) of a matrix is multiplied by a scalar k, then the determinant is multiplied by the same scalar k.

verification:

Let $|A| = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

$= a_1(b_2c_3 - b_3c_2) - b_1(a_2b_3 - a_3c_2) + c_1(a_2b_3 - a_3b_2) \longrightarrow (1)$

Let

$|A_1| = \begin{vmatrix} ka_1 & kb_1 & kc_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

$= ka_1(b_2c_3 - b_3c_2) - kb_1(a_2c_3 - a_3c_2) + kc_1(a_2b_3 - a_3b_2)$

$= k[a_1(b_2c_3 - b_3c_2) - b_1(a_2c_3 - a_3c_2) + c_1(a_2b_3 - a_3b_2)]$

$= k|A|$

Note:

1. $|AB| = |A|\,|B|$
2. If $AB = 0$ then either $|A| = 0$ or $|B| = 0$
3. $|A^n| = (|A|)^n$.

property 7: If each element of a row (or column) of a determinant is expressed as sum of two or more terms then the whole determinant is expressed as sum of two are more determinants.

~~verifi~~ (i.e)

$$\begin{vmatrix} a_1+m_1 & b_1 & c_1 \\ a_2+m_2 & b_2 & c_2 \\ a_3+m_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} m_1 & b_1 & c_1 \\ m_2 & b_2 & c_2 \\ m_3 & b_3 & c_3 \end{vmatrix}$$

Problems:

1. If A is a square matrix, $|A| = 2$, find the value of $|AA^T|$

soln

$|AA^T| = |A||A^T|$

$= |A|\,|A| \qquad |A| = |A^T|$

$= 2 \times 2$

$= 4$

2. If A & B are square matrix of order 3 s.T $|A| = -1$ & $|B| = 3$ find the value of $|3AB|$

soln

$|3AB| = 9|AB|$

$= 9|A||B|$

$= 9 \times (-1) \times 3$

$= -27$

3. verifiy $\det(AB) = \det(A)\det(B)$ for $A = \begin{bmatrix} 4 & 3 & -2 \\ 1 & 0 & 7 \\ 2 & 3 & -5 \end{bmatrix}$

$B = \begin{bmatrix} 1 & 3 & 3 \\ -2 & 4 & 0 \\ 9 & 7 & 5 \end{bmatrix}$

soln

$AB = \begin{bmatrix} 4 & 3 & -2 \\ 1 & 0 & 7 \\ 2 & 3 & -5 \end{bmatrix} \begin{bmatrix} 1 & 3 & 3 \\ -2 & 4 & 0 \\ 9 & 7 & 5 \end{bmatrix}$

$= \begin{bmatrix} 4-6-18 & 12+12-14 & 12+0-10 \\ 1+0+63 & 3+0+49 & 3+0+35 \\ 2-6-45 & 6+12-35 & 6+0-25 \end{bmatrix}$

$= \begin{bmatrix} -20 & 10 & 2 \\ 64 & 52 & 38 \\ -49 & -17 & -19 \end{bmatrix}$

$|AB| = -20(-988+646) - 10(-1216+1862) + 2(-1088+2548)$

$= 6840 - 6460 + 2920 = 3300 \longrightarrow (1)$

$|A| = 4(-21) - 3(-5-14) - 2(3)$

$= -84 + 57 - 6$

$= -33$

$|B| = 1(20) - 3(-10) + 3(-14-36)$

$= 20 + 30 - 150$

$= -100$

$|A||B| = 3300$

$|A||B| = |AB|$ ——— by (1)

H.P.

4) If $\lambda = -2$ find the value of $\begin{vmatrix} 0 & 2\lambda & 1 \\ \lambda^2 & 0 & 3\lambda^2+1 \\ -1 & 6\lambda-1 & 0 \end{vmatrix}$

Soln

$$\begin{vmatrix} 0 & 2\lambda & 1 \\ \lambda^2 & 0 & 3\lambda^2+1 \\ -1 & 6\lambda-1 & 0 \end{vmatrix} = \begin{vmatrix} 0 & -4 & 1 \\ 4 & 0 & 3(4)+1 \\ -1 & 6(-2)-1 & 0 \end{vmatrix}$$

$$= \begin{vmatrix} 0 & -4 & 1 \\ 4 & 0 & 13 \\ -1 & -13 & 0 \end{vmatrix}$$

$= 0 + 4(0+13) + 1(-52)$

$= 52 - 52$

$= 0$

5) Find the roots of $\begin{vmatrix} 1 & 4 & 20 \\ 1 & -2 & 5 \\ 1 & 2x & 5x^2 \end{vmatrix} = 0$

Soln

$$\begin{vmatrix} 1 & 4 & 20 \\ 1 & -2 & 5 \\ 1 & 2x & 5x^2 \end{vmatrix} = 0$$

7.3. (4)

$1(-10x^2-10x) + 4(5x^2-5) + 20(2x+2) = 0$

$-10x^2-10x - 20x^2+20+40x+40 = 0$

$-30x^2+30x+60 = 0$

$-x^2+x+2 = 0$

$x^2-x-2 = 0$

$(x-2)(x+1) = 0$

$-2 < \begin{matrix} -2 \\ 1 \end{matrix}$, -1

$x = -1, 2$

6) write the general form of 3×3 skew symmetric matrix and prove that its determinant is zero.

Soln

$$|A| = \begin{vmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{vmatrix}$$

General form of 3×3 skew symmetric.

$= 0 - a(bc) + b(ac)$

$= -abc + abc$

$= 0.$

7) without expanding, evaluate

(i) $\begin{vmatrix} 2 & 3 & 4 \\ 5 & 6 & 8 \\ 6x & 9x & 12x \end{vmatrix}$ (ii) $\begin{vmatrix} x+y & y+z & z+x \\ z & x & y \\ 1 & 1 & 1 \end{vmatrix}$

Soln

(i) $\begin{vmatrix} 2 & 3 & 4 \\ 5 & 6 & 8 \\ 6x & 9x & 12x \end{vmatrix}$

$= 3x \begin{vmatrix} 2 & 3 & 4 \\ 5 & 6 & 8 \\ 2 & 3 & 4 \end{vmatrix}$ Taking out 3x from R_3

$= 3x(0)$ $R_1 \equiv R_3$

$= 0.$

(ii) $\begin{vmatrix} x+y & y+z & z+x \\ z & x & y \\ 1 & 1 & 1 \end{vmatrix}$

$= \begin{vmatrix} x+y+z & x+y+z & x+y+z \\ z & x & y \\ 1 & 1 & 1 \end{vmatrix}$ $R_1 \to R_1 + R_3$

$= (x+y+z) \begin{vmatrix} 1 & 1 & 1 \\ z & x & y \\ 1 & 1 & 1 \end{vmatrix}$

$= (x+y+z)(0) \quad \because R_1 \equiv R_3$

$= 0.$

8. S.T $\begin{vmatrix} x+2a & y+2b & z+2c \\ x & y & z \\ a & b & c \end{vmatrix} = 0$

Soln

$\begin{vmatrix} x+2a & y+2b & z+2c \\ x & y & z \\ a & b & c \end{vmatrix} = \begin{vmatrix} x & y & z \\ x & y & z \\ a & b & c \end{vmatrix} + \begin{vmatrix} 2a & 2b & 2c \\ x & y & z \\ a & b & c \end{vmatrix}$

$= 0 + 2 \begin{vmatrix} a & b & c \\ x & y & z \\ a & b & c \end{vmatrix}$

$= 0 + 2(0)$

$= 0.$

9) S.T $\begin{vmatrix} \sec^2\theta & \tan^2\theta & 1 \\ \tan^2\theta & \sec^2\theta & -1 \\ 38 & 36 & 2 \end{vmatrix} = 0.$

Soln

$\begin{vmatrix} \sec^2\theta & \tan^2\theta & 1 \\ \tan^2\theta & \sec^2\theta & -1 \\ 38 & 36 & 2 \end{vmatrix}$

$= \begin{vmatrix} \sec^2\theta - \tan^2\theta & \tan^2\theta & 1 \\ \tan^2\theta - \sec^2\theta & \sec^2\theta & 1 \\ 2 & 36 & -2 \end{vmatrix}$ $R_1 \to R_1 \mp C_2$

$$= \begin{vmatrix} 1 & \tan^2\theta & 1 \\ -1 & \sec^2\theta & -1 \\ 2 & 36 & 2 \end{vmatrix}$$

$= 0 \qquad C_1 \equiv C_3$

10) S.T $\begin{vmatrix} s & a^2 & b^2+c^2 \\ s & b^2 & c^2+a^2 \\ s & c^2 & a^2+b^2 \end{vmatrix} = 0$

Soln

$$\begin{vmatrix} s & a^2 & b^2+c^2 \\ s & b^2 & c^2+a^2 \\ s & c^2 & a^2+b^2 \end{vmatrix}$$

$$= s \begin{vmatrix} 1 & a^2 & a^2+b^2+c^2 \\ 1 & b^2 & a^2+b^2+c^2 \\ 1 & c^2 & a^2+b^2+c^2 \end{vmatrix} \qquad c_3 \to c_2 + c_3$$

$$= s(a^2+b^2+c^2) \begin{vmatrix} 1 & a^2 & 1 \\ 1 & b^2 & 1 \\ 1 & c^2 & 1 \end{vmatrix}$$

$= s(a^2+b^2+c^2)\ (0) \qquad (R_1 \equiv C_2)$

$= 0$

11) S.T $\begin{vmatrix} b+c & bc & b^2c^2 \\ c+a & ca & c^2a^2 \\ a+b & ab & a^2b^2 \end{vmatrix} = 0$

Soln

$$\begin{vmatrix} b+c & bc & b^2c^2 \\ c+a & ca & c^2a^2 \\ a+b & ab & a^2b^2 \end{vmatrix}$$

$$= \frac{1}{abc} \begin{vmatrix} a(b+c) & abc & ab^2c^2 \\ b(c+a) & abc & bc^2a^2 \\ c(a+b) & abc & ca^2b^2 \end{vmatrix}$$ multiply & divided by a, b, c for $R_1 R_2 R_3$

$$= \frac{(abc)(abc)}{abc} \begin{vmatrix} ab+ac & 1 & bc \\ bc+ab & 1 & ca \\ ca+bc & 1 & ab \end{vmatrix}$$ taking abc form c_2 & c_3

$$= abc \begin{vmatrix} ab+bc+ca & 1 & bc \\ ab+bc+ca & 1 & ca \\ ab+bc+ca & 1 & ab \end{vmatrix} \qquad c_1 \to c_1 + c_3$$

$$= abc\,(ab+bc+ca)\begin{vmatrix} 1 & 1 & bc \\ 1 & 1 & ca \\ 1 & 1 & ab \end{vmatrix}$$

$$= abc\,(ab+bc+ca)\,(0) \qquad C_1 \equiv C_2$$

$$= 0.$$

12) Find the value of x if $\begin{vmatrix} x-1 & x & x-2 \\ 0 & x-2 & x-3 \\ 0 & 0 & x-3 \end{vmatrix} = 0$

Soln

$$\begin{vmatrix} x-1 & x & x-2 \\ 0 & x-2 & x-3 \\ 0 & 0 & x-3 \end{vmatrix} = 0$$

$$(x-1)(x-2)(x-3) = 0$$

$$x = 1, 2, 3.$$

13. If a, b, c and x are +ve real no's, then S.T

$$\begin{vmatrix} (a^x+a^{-x})^2 & (a^x-a^{-x})^2 & 1 \\ (b^x+b^{-x})^2 & (b^x-b^{-x})^2 & 1 \\ (c^x+c^{-x})^2 & (c^x-c^{-x})^2 & 1 \end{vmatrix} = 0$$

Soln

$$\begin{vmatrix} (a^x+a^{-x})^2 & (a^x-a^{-x})^2 & 1 \\ (b^x+b^{-x})^2 & (b^x-b^{-x})^2 & 1 \\ (c^x+c^{-x})^2 & (c^x-c^{-x})^2 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} 4 & (a^x-a^{-x})^2 & 1 \\ 4 & (b^x-b^{-x})^2 & 1 \\ 4 & (c^x-c^{-x})^2 & 1 \end{vmatrix} \qquad 4 \to C_1+C_2$$

$$= 0 \qquad (C_1 \equiv 4C_3)$$

14. S.T $|B| = 2|A|$ where $B = \begin{bmatrix} b+c & c+a & a+b \\ c+a & a+b & b+c \\ a+b & b+c & c+a \end{bmatrix}$

$A = \begin{bmatrix} a & b & c \\ b & c & a \\ c & a & b \end{bmatrix}$

Soln

$$|B| = \begin{vmatrix} b+c & c+a & a+b \\ c+a & a+b & b+c \\ a+b & b+c & c+a \end{vmatrix}$$

$$= 2\begin{vmatrix} a+b+c & a+b+c & a+b+c \\ c+a & a+b & b+c \\ a+b & b+c & c+a \end{vmatrix} \quad R_1 \to R_1+R_2+R_3$$

$$= 2\begin{vmatrix} a+b+c & a+b+c & a+b+c \\ -b & -c & -a \\ -c & -a & -b \end{vmatrix} \quad \begin{matrix} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - R_1 \end{matrix}$$

$$= 2\begin{vmatrix} a & b & c \\ -b & -c & -a \\ -c & -a & -b \end{vmatrix} \quad R_1 \to R_1+R_2+R_3$$

$$= 2(-1)(-1)\begin{vmatrix} a & b & c \\ b & c & a \\ a & b & c \end{vmatrix}$$

$$= 2|A|.$$

15) Evaluate $\begin{vmatrix} 2014 & 2017 & 0 \\ 2020 & 2023 & 1 \\ 2023 & 2026 & 0 \end{vmatrix}$

Soln

$$\begin{vmatrix} 2014 & 2017 & 0 \\ 2020 & 2023 & 1 \\ 2023 & 2026 & 0 \end{vmatrix}$$

$$= \begin{vmatrix} 2014 & 3 & 0 \\ 2020 & 3 & 1 \\ 2023 & 3 & 0 \end{vmatrix} \quad C_2 \to C_1 - C_2$$

$$= 3\begin{vmatrix} 2014 & 1 & 0 \\ 2020 & 1 & 1 \\ 2023 & 1 & 0 \end{vmatrix}$$

$$= 3[-2014 + 2023] = 3(9) = 27$$

16) S.T $\begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x^2 & y^2 & z^2 \end{vmatrix} = (x-y)(y-z)(z-x)$

Soln

$$LHS = \begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x^2 & y^2 & z^2 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 0 & 0 \\ x & y-x & z-x \\ x^2 & y^2-x^2 & z^2-x^2 \end{vmatrix} \quad \begin{matrix} c_2 \to c_2 - c_1 \\ c_3 \to c_3 - c_1 \end{matrix}$$

$$= \cancel{(y-x)} (y-x)(z-x) \begin{vmatrix} 1 & 0 & 0 \\ x & 1 & 1 \\ x^2 & y+x & z+x \end{vmatrix}$$

$$= (y-x)(z-x)\left[z+x-(y+x)\right]$$

$$= (y-x)(z-x)(z-y)$$

$$= (x-y)(y-z)(z-x)$$

$= R.H.S.$

17) If $\begin{vmatrix} a & b & a\alpha+b \\ b & c & b\alpha+c \\ a\alpha+b & b\alpha+c & 0 \end{vmatrix} = 0$ P.T a, b, c are in G.P or α is a root of $ax^2+2bx+c=0$.

18. S.T $\begin{vmatrix} a^2+x^2 & ab & ac \\ ab & b^2+x^2 & bc \\ ac & bc & c^2+x^2 \end{vmatrix}$ is divisible by x^4

Soln

$$\begin{vmatrix} a^2+x^2 & ab & ac \\ ab & b^2+x^2 & bc \\ ac & bc & c^2+x^2 \end{vmatrix}$$

$$= \frac{1}{abc}\begin{vmatrix} a(a^2+x^2) & a^2b & a^2c \\ ab^2 & b(b^2+x^2) & b^2c \\ ac^2 & bc^2 & c(c^2+x^2) \end{vmatrix}$$ multiply & divide a, b, c for R_1, R_2, R_3

$$= \frac{abc}{abc}\begin{vmatrix} a^2+x^2 & a^2 & a^2 \\ b^2 & b^2+x^2 & b^2 \\ c^2 & c^2 & c^2+x^2 \end{vmatrix}$$ Taking a, b, c from C_1, C_2, C_3

$$= \begin{vmatrix} a^2+b^2+c^2+x^2 & a^2+b^2+c^2+x^2 & a^2+b^2+c^2+x^2 \\ b^2 & b^2+x^2 & b^2 \\ c^2 & c^2 & c^2+x^2 \end{vmatrix}$$ $R_1 \Rightarrow R_1+R_2+R_3$

$$= (a^2+b^2+c^2+x^2)\begin{vmatrix} 1 & 1 & 1 \\ b^2 & b^2+x^2 & b^2 \\ c^2 & c^2 & c^2+x^2 \end{vmatrix}$$

$$= (a^2+b^2+c^2+x^2)\begin{vmatrix} 1 & 0 & 0 \\ b^2 & x^2 & 0 \\ c^2 & 0 & x^2 \end{vmatrix}$$ $C_2 \rightarrow C_2 - C_1$, $C_3 \rightarrow C_3 - C_1$

$$= (a^2+b^2+c^2+x^2)\,(1)\,(x^2)\,(x^2)$$

$$= x^4\,(a^2+b^2+c^2+x^2)$$

$\therefore \begin{vmatrix} a^2+x^2 & ab & ac \\ ab & b^2+x^2 & bc \\ ac & bc & c^2+x^2 \end{vmatrix}$ is divisible by x^4.

7.3 (3)

19. S.T $\begin{vmatrix} 1+a & 1 & 1 \\ 1 & 1+b & 1 \\ 1 & 1 & 1+c \end{vmatrix} = abc\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)$

Soln

$$\begin{vmatrix} 1+a & 1 & 1 \\ 1 & 1+b & 1 \\ 1 & 1 & 1+c \end{vmatrix}$$

$$= \begin{vmatrix} 1+a & 0 & -a \\ 1 & b & 0 \\ 1 & -c & c \end{vmatrix} \qquad \begin{matrix} C_2 \to C_2 - C_3 \\ C_3 \to C_3 - C_1 \end{matrix}$$

$$= (1+a)(bc+0) - 0 - a(-c-b)$$

$$= bc + abc + ac + abc$$

$$= abc\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)$$

20) S.T $\begin{vmatrix} a^2 & abc & ac+c^2 \\ a^2+ab & b^2 & ac \\ ab & b^2+bc & c^2 \end{vmatrix} = 4a^2b^2c^2$

Soln

Theorem: Factor Theorem:

If each element of matrix A is a polynomial in x and if $|A|$ vanishes for $x=a$ then $(x-a)$ is a factor of $|A|$

Note:

1. When $x=a$, if two rows (columns) become an identical then $\Delta=0$ & $(x-a)$ is a factor of Δ
2. When $x=a$, if three rows (columns) become an identical then $\Delta=0$ & $(x-a)^2$ is a factor of Δ.
3. In general if r rows becomes identical then $\Delta=0$ & $(x-a)^{r-1}$ is a factor of Δ.

Cyclic Symmetric form:

A square matrix (or its determinant) is said to be symmetric form if each row is obtained from the first row by changing the variables cyclically.

Result:

If m is the difference between the product of factors and the degree of the product of the leading diagonals

1. $m=0 \Rightarrow$ The required factor is a constant k
2. $m=1 \Rightarrow$ The required factor is $k(a+b+c)$
3. $m=2 \Rightarrow$ The required factor is $k(a^2+b^2+c^2)+l(ab+bc+ca)$

Problems

1. Using factor theorem prove that

$$\begin{vmatrix} x+1 & 3 & 5 \\ 2 & x+2 & 5 \\ 2 & 3 & x+4 \end{vmatrix} = (x-1)^2(x+9)$$

Soln Let $|A| = \begin{vmatrix} x+1 & 3 & 5 \\ 2 & x+2 & 5 \\ 2 & 3 & x+4 \end{vmatrix} \longrightarrow (1)$

put $x=1$ in $|A|$

$$|A| = \begin{vmatrix} 2 & 3 & 5 \\ 2 & 3 & 5 \\ 2 & 3 & 5 \end{vmatrix}$$

$$= 0 \qquad R_1 = R_2 = R_3$$

$\therefore$ Three rows are identical

$\Rightarrow (x-1)^2$ is a factor of $|A|$

put $x=-9$ in (1)

$$|A| = \begin{vmatrix} -8 & 3 & 5 \\ 2 & -7 & 5 \\ 2 & 3 & -5 \end{vmatrix}$$

$$= \begin{vmatrix} 0 & 3 & 5 \\ 0 & -7 & 5 \\ 0 & 3 & -5 \end{vmatrix} \qquad C_1 \Rightarrow C_1 + C_2 + C_3$$

$$= 0 \qquad C_1 \equiv 0$$

$\therefore (x+9)$ is a factor of $|A|$.

$\therefore$ product of factors is $(x-1)^2 (x+9)$

$|A|$ is a cubic polynomial in x

$\therefore$ The remaining factor must be a constant k

$$\therefore \begin{vmatrix} x+1 & 3 & 5 \\ 2 & x+2 & 5 \\ 2 & 3 & x+4 \end{vmatrix} = k\,(x-1)^2 (x+9)$$

equ. co. eff of x^3

$$k = 1$$

$$\therefore \begin{vmatrix} x+1 & 3 & 5 \\ 2 & x+2 & 5 \\ 2 & 3 & x+4 \end{vmatrix} = (x-1)^2 + (x+9)$$

7.3.3 -02

2 using factor theorem S.T $\begin{vmatrix} x & a & a \\ a & x & a \\ a & a & x \end{vmatrix} = (x-a)^2(x+2a)$

Soln

$$|A| = \begin{vmatrix} x & a & a \\ a & x & a \\ a & a & x \end{vmatrix} \rightarrow (1)$$

Put $x = a$ in $|A|$

$$|A| = \begin{vmatrix} a & a & a \\ a & a & a \\ a & a & a \end{vmatrix}$$

$$= 0 \qquad R_1 = R_2 = R_3$$

$\therefore$ Three rows are identical

$\Rightarrow (x-a)^2$ is a factor of $|A|$

Put $x = -2a$

$$|A| = \begin{vmatrix} -2a & a & a \\ a & -2a & a \\ a & a & -2a \end{vmatrix}$$

$$= \begin{vmatrix} 0 & a & a \\ 0 & -2a & a \\ 0 & a & -2a \end{vmatrix} \qquad C_1 \rightarrow C_1 + C_2 + C_3$$

$$= 0.$$

$\therefore (x+2a)$ is a factor of $|A|$

$\therefore$ product of factors is $(x-a)^2(x+2a)$

$|A|$ is a cubic polynomial of x

$\therefore$ The reaming factor must be k.

$$\therefore \begin{vmatrix} x & a & a \\ a & x & a \\ a & a & x \end{vmatrix} = k(x-a)^2(x+2a)$$

equ. the co.off of x^3

$$1 = k$$

$$\therefore \begin{vmatrix} x & a & a \\ a & x & a \\ a & a & x \end{vmatrix} = (x-a)^2(x+2a)$$

9. Solve $\begin{vmatrix} x+a & b & c \\ a & x+b & c \\ a & b & x+c \end{vmatrix} = 0$

Soln

$|A| = \begin{vmatrix} x+a & b & c \\ a & x+b & c \\ a & b & x+c \end{vmatrix}$

put $x=0$

$|A| = \begin{vmatrix} a & b & c \\ a & b & c \\ a & b & c \end{vmatrix}$

$= 0$ $R_1=R_2=R_3$

∴ Three rows are identical

$\Rightarrow (x-0)^2$ is a factor of $|A|$

$\Rightarrow x^2$ is a factor of $|A|$

Put $x = -(a+b+c)$

$|A| = 0$

∴ $(x-(a+b+c))$ is a factor of $|A|$

product of factor is $x^2(x-(a+b+c))$

$|A|$ is a cubic polynomial in x.

$|A|$ The remaining factor must be a constant k

$\begin{vmatrix} x+a & b & c \\ a & x+b & c \\ a & b & x+c \end{vmatrix} = k\,x^2(x-(a+b+c)) \rightarrow (1)$

eq. the co. eff of x^3

$1 = k$

$\begin{vmatrix} x^2+a^2 & b & c \\ a & x+b & c \\ a & b & x+c \end{vmatrix} = 0$

$k\,x^2(x-(a+b+c)) = 0$

$x^2 = 0$ | $x-(a+b+c) = 0$

$x = 0, 0$ | $x = a+b+c$

4. Solve $\begin{vmatrix} 4-x & 4+x & 4+x \\ 4+x & 4-x & 4+x \\ 4+x & 4+x & 4-x \end{vmatrix} = 0$

Soln

$$|A| = \begin{vmatrix} 4-x & 4+x & 4+x \\ 4+x & 4-x & 4+x \\ 4+x & 4+x & 4-x \end{vmatrix}$$

put $x = 0$

$$|A| = \begin{vmatrix} 4 & 4 & 4 \\ 4 & 4 & 4 \\ 4 & 4 & 4 \end{vmatrix}$$

$= 0$ $\quad R_1 = R_2 = R_3$

Three rows are identical

$\Rightarrow (x-0)^2$ is a factor of $|A|$

$\Rightarrow x^2$ is a factor of $|A|$

put $x = -12$

$|A| = 0.$

$\therefore (x+12)$ is a factor of $|A|$

$\therefore$ product is $x^2(x+12)$

$|A|$ is a cubic polynomial of x

the remaining factor must be a constant k

$$\begin{vmatrix} 4-x & 4+x & 4+x \\ 4+x & 4-x & 4+x \\ 4+x & 4+x & 4-x \end{vmatrix} = k\, x^2(x+12)$$

$= -x^2(x+12)$ $\quad$ Equ. coff of x^3

$k = -1$

now

$$\begin{vmatrix} 4-x & 4+x & 4+x \\ 4+x & 4-x & 4+x \\ 4+x & 4+x & 4-x \end{vmatrix} = 0$$

$-x^2(x+12) = 0$

$x^2 = 0$ $\quad\Big|\quad$ $x+12 = 0$

$x = 0, 0$ $\quad$ $x = -12$

5. Show that $\begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x^2 & y^2 & z^2 \end{vmatrix} = (x-y)(y-z)(z-x)$

Soln

$$|A| = \begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x^2 & y^2 & z^2 \end{vmatrix}$$

put $x=y$

$$|A| = \begin{vmatrix} 1 & 1 & 1 \\ x & x & z \\ x^2 & x^2 & z \end{vmatrix}$$

$= 0$ $\quad C_1 = C_2$

$\therefore (x-y)$ is a factor of $|A|$

The ~~given~~ $|A|$ is cyclic $\therefore (y-z)(z-x)$ also factor of $|A|$

$\therefore$ product of factor is $(x-y)(y-z)(z-x)$

The degree of product of factor is 3

The degree of product of leading diagonal element is 3

$\therefore$ other factor is K.

$$\begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x^2 & y^2 & z^2 \end{vmatrix} = K(x-y)(y-z)(z-x) \longrightarrow (1)$$

Let $x=0$, $y=1$, $c=2$

$$\begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 1 & 4 \end{vmatrix} = K(-1)(-1)(2)$$

$1(2) - 1(0) + 1(0) = 2K$

$2 = 2K$

$K = 1$

(1) becomes

$$\begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x^2 & y^2 & z^2 \end{vmatrix} = (x-y)(y-z)(z-x)$$

6) Prove by factor method $\begin{vmatrix} b+c & a-c & a-b \\ b-c & c+a & b-a \\ c-b & c-a & a+b \end{vmatrix} = 8abc$

Soln

Let $|A| = \begin{vmatrix} b+c & a-c & a-b \\ b-c & c+a & b-a \\ c-b & c-a & a+b \end{vmatrix}$

put $a=0$ $|A| = \begin{vmatrix} b+c & -c & -b \\ b-c & c & b \\ c-b & c & b \end{vmatrix}$

$= cb \begin{vmatrix} b+c & -1 & -1 \\ b-c & 1 & 1 \\ c-b & 1 & 1 \end{vmatrix}$

$= cb(0)$ $\quad C_2 = C_3$

$= 0$

$\therefore (a-0)$ is a factor of $|A|$

a is a factor of $|A|$

III^{ly} b, c are factor of $|A|$

$\therefore$ product of factor is abc

The degree of product of factor is 3

The degree of product of leading Co.eff is 3

$\therefore$ The remaing factor is k.

$\therefore \begin{vmatrix} b+c & a-c & a-b \\ b-c & c+a & b-a \\ c-b & c-a & a+b \end{vmatrix} = k\,abc$

$a=1, b=1, c=1$

$\begin{vmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{vmatrix} = kabc$

$8 = k$ $\quad (abc = 1)$

$\begin{vmatrix} b+c & a-c & a-b \\ b-c & c+a & b-a \\ c-b & c-a & a+b \end{vmatrix} = 8abc.$

7. Show that $\begin{vmatrix} b+c & a & a^2 \\ c+a & b & b^2 \\ a+b & c & c^2 \end{vmatrix} = (a+b+c)(a-b)(b-c)(c-a)$

Soln

$$|A| = \begin{vmatrix} b+c & a & a^2 \\ c+a & b & b^2 \\ a+b & c & c^2 \end{vmatrix}$$

put $a=b$ $\quad |A| = \begin{vmatrix} b+c & b & b^2 \\ c+b & b & b^2 \\ 2b & c & c^2 \end{vmatrix}$

$= 0 \quad (R_1 \equiv R_2$

$\therefore (a-b)$ is a factor of $|A|$

$|A|$ is cyclic $\Rightarrow (b-c), (c-a)$ are also a factor

$\therefore$ degree of factors $(a-b)(b-c)(c-a)$ is 3

degree of $|A|$ is 4

$\therefore$ other requ. factor is $k(a+b+c)$

$$\begin{vmatrix} b+c & a & a^2 \\ c+a & b & b^2 \\ a+b & c & c^2 \end{vmatrix} = k(a+b+c)(a-b)(b-c)(c-a)$$

$a=0, b=1, c=2$

$$\begin{vmatrix} 3 & 0 & 0 \\ 2 & 1 & 1 \\ 1 & 2 & 4 \end{vmatrix} = k(0+1+2)((-1)(-1)(2))$$

$3(4-2) = 3k(2)$

$3k = 3$

$k = 1$

$\therefore$

$$\begin{vmatrix} b+c & a & a^2 \\ c+a & b & b^2 \\ a+b & c & c^2 \end{vmatrix} = (a+b+c)(a-b)(b-c)(c-a)$$

7.3.9.5

8. Prove that $A = \begin{vmatrix} (q+r)^2 & p^2 & q^2 \\ q^2 & (r+p)^2 & q^2 \\ r^2 & r^2 & (p+q)^2 \end{vmatrix} = 2pr(p+q+r)^3$

soln

$$|A| = \begin{vmatrix} (q+r)^2 & p^2 & q^2 \\ q^2 & (r+p)^2 & q^2 \\ r^2 & r^2 & (p+q)^2 \end{vmatrix}$$

put $p=0$

$$|A| = \begin{vmatrix} (q+r)^2 & p^2 & q^2 \\ q^2 & r^2 & q^2 \\ r^2 & r^2 & q^2 \end{vmatrix}$$

$= 0$ $\quad R_2 = R_3$

$\therefore (p-0)$ is a factor of $|A|$

$\Rightarrow p$ is a factor of $|A|$

$|A|$ is symmetric form in p, q, r

$\therefore q, r$ are also a factor

puting $p+q+r=0 \Rightarrow q+r=-p$ $r+p=-q$, $p+r=-r$

$$|A| = \begin{vmatrix} p^2 & p^2 & q^2 \\ q^2 & q^2 & q^2 \\ r^2 & r^2 & r^2 \end{vmatrix}$$

$= 0$

Since 3 colums are identical.

$\therefore (p+q+r)^2$ is a factor of $|A|$

$\therefore$ degree of factors $pqr(p+q+r)^2$ is 5

degree of $|A|$ is 6.

$\therefore$ other required factor is $k(p+q+r)$

$$\begin{vmatrix} (q+r)^2 & p^2 & q^2 \\ q^2 & (r+p)^2 & q^2 \\ r^2 & r^2 & (p+q)^2 \end{vmatrix} = k(p+q+r)(p+q+r)^2 \times pqr$$

puting $p=q=r=1$

$$\begin{vmatrix} 4 & 1 & 1 \\ 1 & 4 & 1 \\ 1 & 1 & 4 \end{vmatrix} = k(3)^3(1) \Rightarrow k=2$$

$$\therefore \begin{vmatrix} (q+r)^2 & p^2 & q^2 \\ q^2 & (r+p)^2 & q^2 \\ r^2 & r^2 & (p+q)^2 \end{vmatrix} = 2(pqr)(p+q+r)^3$$

7.33. (5)

4. P.T $\begin{vmatrix} 1 & x^2 & x^3 \\ 1 & y^2 & y^3 \\ 1 & z^2 & z^3 \end{vmatrix} = (x-y)(y-z)(z-x)(xy+yz+zx)$

Soln

$|A| = \begin{vmatrix} 1 & x^2 & x^3 \\ 1 & y^2 & y^3 \\ 1 & z^2 & z^3 \end{vmatrix}$

Put $x=y$ $|A| = \begin{vmatrix} 1 & y^2 & y^3 \\ 1 & y^2 & y^3 \\ 1 & z^2 & z^3 \end{vmatrix}$

$= 0 \quad (R_1 = R_2)$

$\therefore (x-y)$ is a factor of $|A|$

$|A|$ is cyclic $\therefore (y-z)(z-x)$ also a factor of $|A|$.

$\therefore$ product of factor is $(x-y)(y-z)(z-x)$

The degree of the factors $(x-y)(z-x)(z-x)$ is 3

The degree of leading ~~coeff~~ diagonal elem elements $|xy^2x z^3|$ is 5

$\therefore$ other factor is $k(x^2+y^2+z^2)+l(xy+yz+zx)$

$\therefore \begin{vmatrix} 1 & x^2 & x^3 \\ 1 & y^2 & y^3 \\ 1 & z^2 & z^3 \end{vmatrix} = k(x^2+y^2+z^2)+l(xy+yz+zx)$
$(x-y)(y-z)(z-x)$

put $x=0, y=1, z=2$,

$\begin{vmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 4 & 8 \end{vmatrix} = (k(0+1+4)+l(0+2+0))(-1(1-2)(2-0))$

$8-4 = (5k+2l)(2)$

$4 = (5k+2l)2$

$5k+2l = 2 \longrightarrow (1)$

put $x=0, y=-1, z=1$

$\begin{vmatrix} 1 & 0 & 0 \\ 1 & 1 & -1 \\ 1 & 1 & 1 \end{vmatrix} = [k(2)+l(-1)]((1)(-2)(1))$

$2 = (2k-l)(-2)$

$2k-l = -1 \longrightarrow (2)$

(1) & (2) $\Rightarrow k=0, l=1$

$\therefore \begin{vmatrix} 1 & x^2 & x^3 \\ 1 & y^2 & y^3 \\ 1 & z^2 & z^3 \end{vmatrix} = (x-y)(y-z)(z-x)(xy+yz+zx)$

10) In a triangle ΔABC, if $\begin{vmatrix} 1 & 1 & 1 \\ 1+\sin A & 1+\sin B & 1+\sin C \\ \sin A(1+\sin A) & \sin B(1+\sin B) & \sin C(1+\sin C) \end{vmatrix} = 0$.

prove that ΔABC is isosceles triangle.

Soln

$$|A| = \begin{vmatrix} 1 & 1 & 1 \\ 1+\sin A & 1+\sin B & 1+\sin C \\ \sin A(1+\sin A) & \sin B(1+\sin B) & \sin C(1+\sin C) \end{vmatrix}$$

Put $\sin A = \sin B$

$|A| = 0 \quad \because C_1 = C_2$

$\therefore (\sin A - \sin B)$ is a factor of $|A|$

$|A|$ is cyclic $\therefore (\sin B - \sin C), (\sin C - \sin A)$ are factor of $|A|$

$\therefore$ degree of product of ~~all~~ factors is 3

degree of product of leading diagonal elements of $|A|$ is 4.

$\therefore$ other factor is $k(A+B+C)$

$\therefore |A| = k(A+B+C)(\sin A - \sin B)(\sin B - \sin C)(\sin C - \sin A)$

$\therefore$ Since $|A| = 0$.

$\therefore \sin A = \sin B$	$\sin B = \sin C$	$\sin C = \sin A$
$A = B$	$B = C$	$C = A$.

$\therefore A = B = C$

$\therefore \Delta ABC$ is isceles triangle.

7.3.4 Product of determinants

1. Row by column multiplication Rule
2. Row by Row multiplication Rule
3. column by column " "
4. column by row " "

Result:

1. $|AB| = |A||B|$
2. $|AB| = |BA|$

Problems:

1. verify that $|AB| = |A||B|$ if $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ $B = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$

Soln

$$AB = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos^2\theta + \sin\theta & \cos\theta\sin\theta - \sin\theta\cos\theta \\ \sin\theta\cos\theta - \cos\theta\sin\theta & \cos^2\theta + \sin^2\theta \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$|AB| = 1$

$|A| = \cos^2\theta + \sin^2\theta = 1$

$|B| = \cos^2\theta + \sin^2\theta = 1$

$|A||B| = 1 \times 1 = 1$

$= |AB|$

2. Show that $\begin{vmatrix} 0 & c & b \\ c & 0 & a \\ b & a & 0 \end{vmatrix}^2 = \begin{vmatrix} b^2+c^2 & ab & ac \\ ab & c^2+a^2 & bc \\ ac & bc & a^2+b^2 \end{vmatrix}$

Soln LHS $= \begin{vmatrix} 0 & c & b \\ c & 0 & a \\ b & a & 0 \end{vmatrix}^2$

$= \begin{vmatrix} 0 & c & b \\ c & 0 & a \\ b & a & 0 \end{vmatrix} \times \begin{vmatrix} 0 & c & b \\ c & 0 & a \\ b & a & 0 \end{vmatrix}$

$$= \begin{vmatrix} 0+c^2+b^2 & 0+0+ab & 0+ac+0 \\ 0+0+ab & c^2+0+a^2 & bc+0+0 \\ 0+ac+0 & bc+0+0 & b^2+a^2+0 \end{vmatrix}$$

$$= \begin{vmatrix} b^2+c^2 & ab & ac \\ ab & a^2+c^2 & bc \\ ac & bc & a^2+b^2 \end{vmatrix}$$

$= RHS.$

3. Show that $\begin{vmatrix} 2bc-a^2 & c^2 & b^2 \\ c^2 & 2ca-b^2 & a^2 \\ b^2 & a^2 & 2ab-c^2 \end{vmatrix} = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}^2$

Soln

$$RHS = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} \times \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$$

$$= \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} \times (-1) \begin{vmatrix} a & b & c \\ c & a & b \\ b & c & a \end{vmatrix} \quad R_2 \leftrightarrow R_3$$

$$= \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} \times \begin{vmatrix} -a & -b & -c \\ c & a & b \\ b & c & a \end{vmatrix}$$

$$= \begin{vmatrix} -a^2+bc+ca & -ab+ab+c^2 & -ac+b^2+ac \\ -ab+c^2+ab & -b^2+ac+ac & -bc+bc+a^2 \\ -ac+ac+b^2 & -bc+a^2+bc & -c^2+ab+ab \end{vmatrix}$$

$$= \begin{vmatrix} 2bc-a^2 & c^2 & b^2 \\ c^2 & 2ca & a^2 \\ a^2 & a^2 & 2ab-c^2 \end{vmatrix}$$

$= RHS$

4. S.T $\begin{vmatrix} 1 & x & x \\ x & 1 & x \\ x & x & 1 \end{vmatrix}^2 = \begin{vmatrix} 1-2x^2 & -x^2 & -x^2 \\ -x^2 & -1 & x^2-2x \\ -x^2 & x^2-2x & -1 \end{vmatrix}$

Soln

$$LHS = \begin{vmatrix} 1 & x & x \\ x & 1 & x \\ x & x & 1 \end{vmatrix} \times \begin{vmatrix} 1 & x & x \\ x & 1 & x \\ x & x & 1 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & x & x \\ x & 1 & x \\ x & x & 1 \end{vmatrix} \times (-1)(-1) \begin{vmatrix} 1 & x & x \\ -x & -1 & -x \\ -x & -x & -1 \end{vmatrix}$$

7.3.4 (2)

$$= \begin{vmatrix} 1 & x & x \\ x & 1 & x \\ x & x & 1 \end{vmatrix} \times \begin{vmatrix} 1 & x & x \\ -x & -1 & -x \\ -x & -x & -1 \end{vmatrix}$$

$$= \begin{vmatrix} 1-x^2-x^2 & x-x-x^2 & x-x^2-x \\ x-x-x^2 & x^2-1-x^2 & x^2-x-x \\ x-x^2-x & x^2-x-x & x^2-x^2-1 \end{vmatrix}$$

$$= \begin{vmatrix} 1-2x^2 & -x^2 & -x^2 \\ -x^2 & -1 & x^2-2x \\ -x^2 & x^2-2x & -1 \end{vmatrix}$$

$= RHS.$

5 Find the product $\begin{vmatrix} \log_3 64 & \log_4 3 \\ \log_3 8 & \log_4 9 \end{vmatrix} \times \begin{vmatrix} \log_2 3 & \log_8 3 \\ \log_3 4 & \log_3 4 \end{vmatrix}$

Soln

$$\begin{vmatrix} \log_3 64 & \log_4 3 \\ \log_3 8 & \log_4 9 \end{vmatrix} \times \begin{vmatrix} \log_2 3 & \log_8 3 \\ \log_3 4 & \log_3 4 \end{vmatrix}$$

$$= \begin{vmatrix} \log_3 64 \log_2 3 + \log_4 3 \log_3 4 & \log_3 64 \log_8 3 + \log_4 3 \log_3 4 \\ \log_3 8 \log_2 3 + \log_4 9 \log_3 4 & \log_3 8 \log_8 3 + \log_4 9 \log_3 4 \end{vmatrix}$$

$$= \begin{vmatrix} \log_2 64 + 1 & \log_8 64 + 1 \\ \log_2 8 + \log_3 9 & 1 + \log_3 9 \end{vmatrix}$$

$$= \begin{vmatrix} 6+1 & 2+1 \\ 3+2 & 1+2 \end{vmatrix} \qquad \because \log_x y = \frac{\log_b y}{\log_b x}$$

$$= \begin{vmatrix} 7 & 3 \\ 5 & 3 \end{vmatrix}$$

$= 21 - 15$

$= 6.$

6. If $\cos 2\theta = 0$ determine $\begin{vmatrix} 0 & \cos\theta & \sin\theta \\ \cos\theta & \sin\theta & 0 \\ \sin\theta & 0 & \cos\theta \end{vmatrix}^2$

Soln

$$\begin{vmatrix} 0 & \cos\theta & \sin\theta \\ \cos\theta & \sin\theta & 0 \\ \sin\theta & 0 & \cos\theta \end{vmatrix} = 0 - \cos^3\theta + \sin\theta(-\sin^2\theta)$$

$= -(\cos^3\theta + \sin^3\theta)$

$=$

$= \frac{-1}{2}(\cos\theta + \sin\theta)$

$\cos^2\theta = \frac{1+\cos 2\theta}{2}$

$\cos\theta = \frac{1}{2}$

$\sin^2\theta = \sqrt{1-\cos^2\theta}$

$= 1 - \frac{1}{2}$

$= \frac{1}{2}$

$$\begin{vmatrix} 0 & \cos\theta & \sin\theta \\ \cos\theta & \sin\theta & 0 \\ \sin\theta & 0 & \cos\theta \end{vmatrix}^2 = \frac{1}{4}(\cos\theta + \sin\theta)^2$$

$$= \frac{1}{4}(\cos^2\theta + \sin^2\theta + 2\sin\theta\cos\theta)$$

$$= \frac{1}{4}(1 + \sin 2\theta)$$

$$= \frac{1}{4}\left(1 + \sqrt{1-\cos^2 2\theta}\right)$$

$$= \frac{1}{4}(1+1) \qquad \because \cos 2\theta = 0.$$

$$= \frac{1}{2}$$

7.3.5 Relation between a determinant and its cofactor Determinant:

Let $|A| = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

Let $A_1, B_1, C_1 \ldots$ be cofactors of $a_1, b_1, c_1, \ldots$ in $|A|$

Then $|A| = a_1A_1 + b_1B_1 + c_1C_1$

$|A| = a_2A_2 + b_2B_2 + c_2C_2$

$|A| = a_3A_3 + b_3B_3 + c_3C_3$

$a_1A_3 + b_1B_3 + c_1C_3 = 0 = a_2A_1 + b_2B_1 + c_2C_1$

$a_3A_2 + b_2B_2 + c_3C_2 = 0 = a_3A_1 + b_3B_1 + c_3C_1$

$a_1A_2 + b_1B_2 + c_1C_2 = 0.$

problem:

1. A_i, B_i, C_i are cofactors of $a_i, b_i, c_i \ldots$

$|A| = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ show that $\begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix} = |A|^2$

7.3.4 (3)

Soln

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \times \begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix}$$

$$= \begin{vmatrix} \cancel{a_1A_1+b_1A_2+c_1A_3} & \cancel{a_1B_1+b_1B_2+c_1B_3} & \cancel{a_1c_1+b_1c_2+c_1c_3} \\ \cancel{a_2A_1+b_2A_2+c_2A_3} & \cancel{a_2B_1} & \end{vmatrix}$$

$$= \begin{vmatrix} a_1A_1+b_1B_1+c_1C_1 & a_1A_2+b_1B_2+c_1C_2 & a_1A_3+b_1B_3+c_1C_3 \\ a_2A_1+b_2B_1+c_2C_1 & a_2A_2+B_2b_2+c_2C_2 & a_2A_3+b_2B_3+c_2C_3 \\ a_3A_1+b_3B_1+c_3C_1 & a_3A_2+b_3B_3+c_3C_3 & a_3A_3+b_3B_3+c_3C_3 \end{vmatrix}$$

$$= \begin{vmatrix} |A| & 0 & 0 \\ 0 & |A| & 0 \\ 0 & 0 & |A| \end{vmatrix}$$

$$|A| \times \begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix} = |A|^3$$

$$\begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix} = |A|^2$$

-H

7.3.6 Area of triangle :

The area of triangle whose vertices are (x_1, y_1) (x_2, y_2) (x_3, y_3) is

$$\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

Note:

Area is zero then the points (x_1, y_1) (x_2, y_2) (x_3, y_3) are collinear.

problem

1. Find the area of triangle whose vertices are $(-2,-3)$ $(3,2)$ & $(-1,-8)$

$$\text{Area of triangle} = \frac{1}{2}\begin{vmatrix} -2 & -3 & 1 \\ 3 & 2 & 1 \\ -1 & -8 & 1 \end{vmatrix}$$

$$= \frac{1}{2}(-20+12-22)$$

$$= -15$$

$$= 15 \text{ sq. units}$$

2. Find the area of triangle whose vertices are (0,0) (1,2) and (4,3)

Soln

$$\text{Area of triangle} = \frac{1}{2}\begin{vmatrix} 0 & 0 & 1 \\ 1 & 2 & 1 \\ 4 & 3 & 1 \end{vmatrix}$$

$$= \frac{1}{2}(1(3-8))$$

$$= -5/2$$

$$= 5/2 \text{ unit (sq)}$$

3. If (k,2) (2,4) (3,2) are vertices of the triangle of area 4 sq. units then find k.

Soln

$$\text{Area of triangle} = 4$$

$$\frac{1}{2}\begin{vmatrix} k & 2 & 1 \\ 2 & 4 & 1 \\ 3 & 2 & 1 \end{vmatrix} = 4$$

$$k(2) - 2(-1) + 1(4-12) = 8$$

$$2k + 2 - 8 = 8$$

$$2k = 8+6$$

$$2k = 14$$

$$k = 7$$

$$|k| = 7$$

$$k = \pm 7$$

4. If the area of triangle with vertices (-3,0) (3,0) (0,k) is sq. units, find the value of k.

Soln

7.3.4 (4)

Area of triangle = absolute value of $\frac{1}{2}\begin{vmatrix} -3 & 0 & 1 \\ 3 & 0 & 1 \\ 0 & k & 1 \end{vmatrix}$

$$9 = \left|\frac{1}{2}(-k)(-3-3)\right|$$

$$9 = |3k|$$

$$|k| = 3$$

$$k = \pm 3.$$

5. Show that the points $(a, b+c)$, $(b, c+a)$ & $(c, a+b)$ are collinear.

Soln

$$\begin{vmatrix} a & b+c & 1 \\ b & c+a & 1 \\ c & a+b & 1 \end{vmatrix}$$

$$= \begin{vmatrix} a+b+c & b+c & 1 \\ a+b+c & a+c & 1 \\ a+b+c & a+b & 1 \end{vmatrix} \quad C_1 \to C_1 + C_2$$

$$= (a+b+c)\begin{vmatrix} 1 & b+c & 1 \\ 1 & a+c & 1 \\ 1 & a+b & 1 \end{vmatrix}$$

$$= (a+b+c)(0)$$

$$= 0.$$

∴ given points are collinear.

7.2.11 Singular and nonsingular matrix

A square matrix 'A' is called

(i) singular if $|A| = 0$

(ii) non singular if $|A| \neq 0$.

Results:

A & B are non singular

⇒ AB & BA are non singular

Problems

1. Identity the singular & nonsingular matrix.

(i) $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$

$$|A| = \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 2 & 3 \\ 3 & 3 & 3 \\ 3 & 3 & 3 \end{vmatrix} \qquad \begin{array}{l} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - R_1 \end{array}$$

$= 0$

$\therefore$ A is singular

(ii) $|A| = \begin{vmatrix} 2 & -3 & 5 \\ 6 & 0 & 4 \\ 1 & 5 & -7 \end{vmatrix}$

$= 2(-20) + 3(-42-4) + 5(35)$ 150

$= -40 - 132 + 175$

$\neq 0$

A is non-singular

(iii) $|A| = \begin{vmatrix} 0 & a-b & k \\ b-a & 0 & 5 \\ -k & -5 & 0 \end{vmatrix}$

$= 0$ (A is skew).

A is singular.

2. Determine the value of a & b so that the following matrices are singular.

Soln

(i) $A = \begin{bmatrix} 7 & 3 \\ -2 & a \end{bmatrix}$

$|A| = 0$

$7a - 6 = 0$

$7a = 6$

$a = 6/7$

(ii) $|B| = 0$

$(b-1)8 - 2(10) + 3(-7) = 0$

$8b - 8 - 20 - 21 = 0$

$8b = 49$

$b = \frac{49}{8}$.