# Application Of Differential Calculus

30.07.2019

## Exercise - 7.1

1. A point moves along a straight line in such a way that after t seconds its distance from the origin is $s = 2t^2 + 3t$ metres.

   1. Find the average velocity of the points between t = 3 and t = 6 seconds.

   II. Find the instantaneous velocity at t = 3 and t = 6 seconds.

$$s(t) = 2t^2 + 3t$$

1. $v = s'(t)$

$= 4t + 3$

$v(3) = 12 + 3 = 15 \text{ m/s}$

$v(6) = 24 + 3 = 27 \text{ m/s}$

$$\text{Average Velocity} = \frac{v(3) + v(6)}{2}$$

$$= \frac{15 + 27}{2} = \frac{42}{2} = 21 \text{ m/s}.$$

II. Instantaneous velocity at t = 3 seconds

$v = 15 \text{ m/s}$.

Instantaneous velocity at t = 6 seconds

$v = 27 \text{ m/s}$

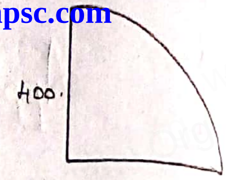
2. A camera is accidently knocked off an edge of a cliff 400 ft high. The camera falls a distance of $s = 16t^2$ in t seconds.

   I. How long does the camera fall before it hits the ground?

   II. What is the average velocity with which the camera falls during the last 2 seconds?

   III. What is the instantaneous velocity ...

$s = 16t^2$.




i. If $s = 400$

$400 = 16t^2$

$t^2 = 25$

$t = 5$ seconds

ii. Average velocity in last 2 seconds

$$= \frac{v(5) + v(3)}{2}$$

$$v = \frac{ds}{dt} = 32t$$

$v(5) = 32(5) = 160$ feet/s.

$v(3) = 32(3) = 96$ feet/s.

Average velocity $= \frac{160 + 96}{2}$

$= \frac{256}{2}$

$= 128$ feet/s.

iii. Instantaneous velocity at the ground $= v(5)$

$= 160$ feet/s.

3. A particle moves along a line according to the law $s(t) = 2t^3 - 9t^2 + 12t - 4$, where $t \geq 0$.

i) At what times the particle changes direction?

ii) Find the total distance travelled by the particle in first 4 seconds.

iii) Find the particle's acceleration each time the velocity is zero.

1. $s(t) = 2t^3 - 9t^2 + 12t - 4$

$v(t) = 6t^2 - 18t + 12$

When change of direction (i.e.,) $v = 0$.

$6t^2 - 18t + 12 = 0$

$t^2 - 3t + 2 = 0$

$(t-1)(t-2) = 0$

$t = 1 \quad t = 2$.

∴ The particle changes the direction at $t = 1$ second and $t = 2$ second.

ii) $s(t) = 2t^3 - 9t^2 + 12t - 4.$

$s(0) = -4$

$s(1) = 2 - 9 + 12 - 4 = 1$

$s(2) = 16 - 36 + 24 - 4 = 40 - 40 = 0$

$s(3) = 54 - 81 + 36 - 4 = 90 - 85 = 5$

$s(4) = 128 - 144 + 48 - 4 = 176 - 148 = 28.$

Total distance travelled by the particle

$= |s(1) - s(0)| + |s(2) - s(1)| + |s(4) - s(2)|$

$= |1 + 4| + |0 - 1| + |28 - 0|$

$= 5 + 1 + 28 = 34$

∴ The total distance travelled by the particle in first 4 second is 34 m.

iii) $s(t) = 2t^3 - 9t^2 + 12t - 4$

$v(t) = 6t^2 - 18t + 12$

$a(t) = 12t - 18$

$a(1) = 12 - 18 = -6\ m/s^2$

$a(2) = 24 - 18 = 6\ m/s^2$

4. If the volume of the cube of side length $x$ is $V = x^3$. Find the rate of change of volume with respect to $x$ when $x = 5$ units.

$V = x^3$, $x = 5$ units.

$\frac{dv}{dx} = 3x^2.$

$= 3(5)^2$

$= 3(25)$

$= 75$ units

5. If the mass $m(x)$ (in kilograms) of a thin rod of length $x$ (in metres) is given by, $m(x) = \sqrt{3x}$ then what is the rate of change of mass with respect to length when it is $x = 3$ and $x = 27$ metres.

$$m(x) = \sqrt{3}\sqrt{x}$$

$$m'(x) = \sqrt{3}\,\frac{1}{2\sqrt{x}}$$

If $x = 3$

$$m'(x) = \sqrt{3}\,\frac{1}{2\sqrt{3}}$$

$$= \frac{1}{2}\ kg/m.$$

If $x = 27$

$$m'(x) = \sqrt{3}\,\frac{1}{2\sqrt{27}}$$

$$= \sqrt{3}\,\frac{1}{2\cdot 3\sqrt{3}}$$

$$= \frac{1}{6}\ kg/m.$$

6. A stone is dropped into a pond causing ripples in the form of concentric circles. The radius $r$ of the outer ripple is increasing at a constant rate at 2 cm per second. When the radius is 5 cm find the rate of changing of the total area of the disturbed water.

$$\frac{dr}{dt} = 2$$

$$r = 5\ cm.$$

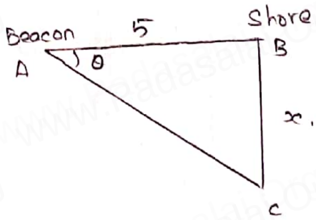
$$A = \pi r^2$$

$$dA = \pi\, 2r\,\frac{dr}{dt}$$

$$dA = \pi \cdot 2 \cdot 5 \cdot 2$$

$$dA = 20\pi\ cm^2/s$$

7. A beacon makes one revolution every 10 seconds. It is located on a ship which is anchored 5 km from the straight shoreline. How fast is the beam moving along the shore line when it makes an angle of 45° with the shore?




$$\frac{d\theta}{dt} = \frac{2\pi}{10}$$

$$= \frac{\pi}{5}$$

$$\theta = 45^\circ.$$

In $\Delta$ ABC,

$$\tan\theta = \frac{x}{5}$$

$$x = 5\tan\theta$$

$$\frac{dx}{dt} = 5\sec^2\theta \frac{d\theta}{dt}$$

$$= 5 \times 2 \times \frac{\pi}{5}$$

$$= 2\pi \text{ km/s}.$$

Example 7.1.

For the function $f(x) = x^2$, $x \in [0, 2]$ compute the average rate of changes in the subintervals $[0, 0.5]$, $[0.5, 1]$, $[1, 1.5]$, $[1.5, 2]$ and the instantaneous rate of changes at the points $x = 0.5, 1, 1.5, 2$

$$f(x) = x^2$$

$$f'(x) = 2x$$

| $a$ | $b$ | $x$ | Average rate is $\frac{f(b)-f(a)}{b-a} = b+a$ | Instantaneous rate is $f'(x) = 2x$ |
|---|---|---|---|---|
| 0 | 0.5 | 0.5 | 0.5 | 1 |
| 0.5 | 1 | 1 | 1.5 | 2 |
| 1 | 1.5 | 1.5 | 2.5 | 3 |
| 1.5 | 2 | 2 | 3.5 | 4 |

Example 7.2

The temperature in celsius in a long rod of length 10m insulted at both ends, is a function of length $x$ given by $T = x(10-x)$. Prove that the rate of change of temperature at the midpoint of the rod is zero.

$$T = 10x - x^2.$$

$$\frac{dT}{dt} = 10 - 2x.$$

length = 10 m.

Midpoint of the rod is at $x = 5$.

Subs $x = 5$, we get

$$\frac{dT}{dt} = 0.$$

Example 7.3

A person learnt 100 words for an English test. The number of words that person remembers in $t$ days after learning is given by $W(t) = 100 \times (1 - 0.1t)^2$, $0 \le t \le 10$. What is the rate at which the person forgets the words 2 days after learning?

$$W(t) = 100 \times (1 - 0.1t)^2$$

$$\frac{d}{dt} W(t) = -20 \times (1 - 0.1t)$$

$\therefore$ at $t = 2$,

$$\frac{d}{dt} W(t) = -20 \times 0.8$$
$$= -16.$$

That is, the person forgets at the rate of 16 words after 2 days of studying.

Example 7.4.

A particle moves so that the distance moved in according to the law $s(t) = \frac{t^3}{3} - t^2 + 3$. At what time the velocity and acceleration are zero respectively?

Distance moved in time 't' is

$$s = \frac{t^3}{3} - t^2 + 3.$$

Velocity at time 't' is

$$v = \frac{ds}{dt} = t^2 - 2t.$$

Acceleration at time 't' is

$$A = \frac{dv}{dt} = 2t - 2.$$

$\therefore$ The velocity is zero when

$$t^2 - 2t = 0$$

i.e $t = 0, 2$.

The acceleration is zero when

$$2t - 2 = 0$$

ie $t = 1$.

Example 7.5

A particle fired straight up from the ground to reach a height of s feet in t seconds, where

$s(t) = 128t - 16t^2$.

i. Compute the maximum height of the particle reached.

ii. What is the velocity when the particle hits the ground?

1. At maximum height velocity is zero.

$$v(t) = \frac{ds}{dt} = 128 - 32t$$

$$v(t) = 0.$$

$$128 - 32t = 0$$

$$t = 4.$$

After 4 seconds, the particle reaches the maximum height.

The height at $t=4$ is

$$S(4) = 128(4) - 16(4)^2$$
$$= 256 \text{ ft}.$$

ii. When the particle hits the ground then $s=0$.

$$s = 0$$

$$128t - 16t^2 = 0.$$

$$t = 0, 8.$$

The particle hits the ground at $t=8$ seconds.
The velocity when it hits the ground is

$$v(8) = -128 \text{ ft/s}.$$

Example 7.6.

A particle moves along a horizontal line such that its position at any time $t \geq 0$ is given by $s(t) = t^3 - 6t^2 + 9t + 1$, where $s$ is measured in metres and $t$ in seconds.

1) At what time the particle is rest?
2) At what time the particle changes direction?
3) Find the total distance travelled by the particle in the first 2 seconds.

$$s(t) = t^3 - 6t^2 + 9t + 1$$

$$v(t) = 3t^2 - 12t + 9$$

$$a(t) = 6t - 12$$

i) The particle is at rest

$v(t) = 0.$

$3t^2 - 12t + 9 = 0. \Rightarrow t^2 - 4t + 3 = 0.$

$t = 1$ and $t = 3.$

ii) The particle changes direction.

$v(t) = 0$

$3t^2 - 12t + 9 = 0$

$t^2 - 4t + 3 = 0$

$t = 1$ second and $t = 3$ second.

iii) The total distance travelled by first 2 seconds is.

$|s(0) - s(1)| + |s(1) - s(2)|$

$= |1 - 5| + |5 - 3|$

$= 4 + 2 = 6$ metres.

Example 7.7

If we blow air into a balloon of spherical shape ~~of~~ at a rate of 1000 $cm^3$ per second. At what rate the radius of the balloon changes when the radius is 7cm? Also compute the rate at which the surface area changes.

The volume of balloon of radius r is

$$V = \frac{4}{3}\pi r^3.$$

$$\frac{dv}{dt} = 1000\ cm^3/s$$

We find $\frac{dr}{dt}$ when $r = 7$ cm.

$$\frac{dv}{dt} = 3 \times \frac{4}{3}\pi r^2 \frac{dr}{dt}$$

Subs $r = 7$ and $\frac{dv}{dt} = 1000$,

$$\frac{dr}{dt} = \frac{1000}{4 \times 49\pi} = \frac{250}{49\pi}.$$

Surface area of the balloon is

$$S = 4\pi r^2$$

$$\frac{ds}{dt} = 8\pi r \frac{dr}{dt}$$

Subs $\frac{dr}{dt} = \frac{250}{49\pi}$ and $r = 7$

$$\frac{ds}{dt} = 8\pi \times 7 \times \frac{250}{49\pi}$$

$$\frac{ds}{dt} = \frac{2000}{7}$$

Example 7.8

The price of a product is related to the number of units available (supply) by the equation $Px + 3P - 16x = 234$, where $P$ is the price of the product per unit in Rupees and $x$ is the number of units. Find the rate at which the price is changing with respect to time when 90 units are available and the supply is increasing at a rate of 15 units / week.

$$P = \frac{234 + 16x}{x+3}$$

$$\frac{dP}{dt} = \frac{-186}{(x+3)^2} \times \frac{dx}{dt}$$

Sub $x = 90$,

$$\frac{dx}{dt} = 15.$$

we get

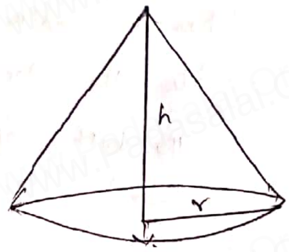
$$\frac{dP}{dt} = \frac{-186}{93^2} \times 15 = \frac{-10}{31} \approx -0.32.$$

$\therefore$ The price is decreasing at a rate of ₹.0.32 per unit

Example 7.9

Salt is poured from a conveyor belt at a rate of 30 cubic meter per minute forming a conical pile with a circular base whose height and diameter of base are always equal. How fast is the height of the pile increasing when the pile is 10 metre high?

 Let h and r be the height and 
the base radius.




$$\therefore h = 2r.$$

$$V = \frac{1}{3}\pi r^3 h. \qquad r = \frac{h}{2}.$$

$$= \frac{1}{12}\pi h^3.$$

$$\frac{dv}{dt} = 30 \text{ metres}^3/\text{min}.$$

$$\frac{dv}{dt} = \frac{1}{4}\pi h^2 \frac{dh}{dt}$$

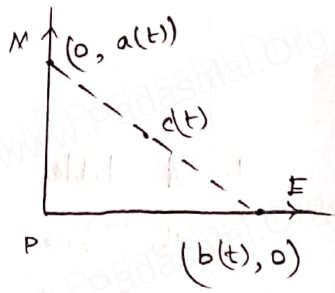
$$\frac{dh}{dt} = 4\frac{dv}{dt} \cdot \frac{1}{\pi h^2}$$

$$= 4 \times 30 \times \frac{1}{100\pi}.$$

$$= \frac{6}{5\pi} \text{ mtr/min}.$$

Example 7.10. (Two variable related rate problem).

A road running north to south crosses a road going east to west at the point P; Car A is driving north along the first road, and car B is driving east along the second road. At a particular time car A is travelling to the north of P and travelling at 80 km/hr, while car B is 15 kms to the east of P and travelling at 100 km/hr. How fast is the distance between the two cars changing?




By Phythagoras theorem,

$$c(t)^2 = a(t)^2 + b(t)^2$$

$$2\,c(t)\,c'(t) = 2\,a(t)\,a'(t) + 2\,b(t)\,b'(t)$$

$$c' = \frac{aa' + bb'}{c}$$

$$= \frac{aa' + bb'}{\sqrt{a^2+b^2}}$$

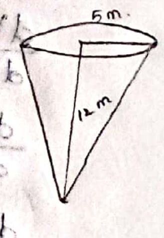
Subs

$$c' = \frac{(10 \times 80) + (15 \times 100)}{\sqrt{100 + 225}}$$

$$= \frac{460}{\sqrt{13}}$$

$$\simeq 127.6 \text{ km/hr}.$$

8. A conical water tank with vertex down of 12 metres height has a radius of 5 metres at the top. If water flows into the tank at the rate of 10 cubic metres per min, how fast is the depth of the water increases when the water is 8 metres deep?




$$\frac{r}{h} = \frac{5}{12}$$

$$r = \frac{5}{12}h$$

$$\frac{dv}{dt} = 10.$$

$h = 8$ metres.

$$v = \frac{1}{3}\pi r^2 h$$

$$= \frac{1}{3}\pi \frac{25}{144}h^2 h$$

$$= \frac{1}{3}\frac{25}{144}\pi h^3.$$

$$\frac{dv}{dt} = \frac{1}{3}\frac{25}{144}\pi 3h^2 \frac{dh}{dt}.$$

$$\frac{dh}{dt} = \frac{144}{25\pi h^2}\frac{dv}{dt}$$

$$= \frac{144}{25\pi \times 64} \times 10$$

$$= \frac{9}{10\pi} \text{ m/min}$$

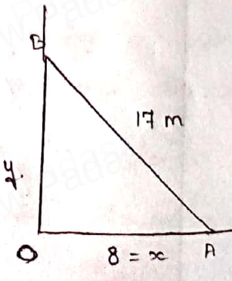
9. A ladder 17 metre long is leaning against the wall. The base of the ladder is pulled away from the wall at a rate of 5 m/s. When the base of the ladder is 8 metres from the wall.

i. How fast is the top of the ladder moving down the wall?

ii. At what rate, the area of the triangle formed by the ladder, wall and the floor is changing?

$$\frac{dx}{dt} = 5$$

$$x = 8$$



1. From Phythagoras theorem,

$$x^2 + y^2 = 17^2 \quad (1)$$

$$8^2 + y^2 = 17^2$$

$$y^2 = 289 - 64$$

$$y^2 = 225$$

$$y = 15.$$

(1) ⇒

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0.$$

$$y\frac{dy}{dt} = -x\frac{dx}{dt}$$

$$\frac{dy}{dt} = -\frac{x}{y}\frac{dx}{dt}$$

$$= -\frac{8}{15}(5)$$

$$= -\frac{8}{3} \text{ m/s}.$$

11. $A = \frac{1}{2}xy$

$$\frac{dA}{dt} = \frac{1}{2}\left(x\frac{dy}{dt} + y\frac{dx}{dt}\right)$$

$$= \frac{1}{2}\left(8\left(\frac{-8}{3}\right) + 15(5)\right)$$

$$= \frac{1}{2}\left(\frac{-64}{3} + 75\right)$$

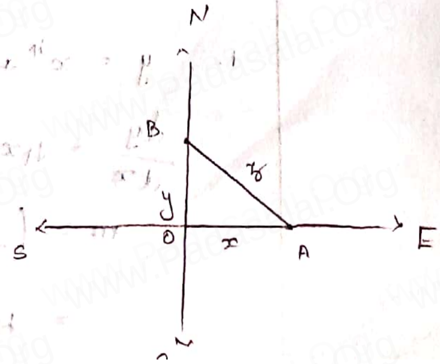
$$= \frac{1}{2}\left(\frac{-64+225}{3}\right)$$

$$= \frac{1}{2}\left(\frac{161}{3}\right)$$

$$= \frac{161}{6}$$

$$= 26.83 \text{ m}^2/\text{s}.$$

10. A police jeep, approaching an orthogonal intersection from the northern direction is chasing a speeding car that has turned and moving straight east. When the jeep is 0.6 km north of the intersection and the car is 0.8 km to the east. The police determine with a radar that the distance between them and the car is increasing at 20 km/hr. If the jeep is moving at 60 km/hr at the instant of measurement, what is the speed of the car?




$y = 0.6$

$x = 0.8$

$$\frac{dy}{dt} = -60$$

$$\frac{dz}{dt} = 20$$

$x^2 + y^2 = z^2$ (By Phythagoras Theorem)

$(0.8)^2 + (0.6)^2 = z^2$

$0.64 + 0.36 = z^2$

$z^2 = 1$

$z = 1.$

$1 \Rightarrow \quad 2x \frac{dx}{dt} + 2y \frac{dy}{dt} = 2z \frac{dz}{dt}$

$x \frac{dx}{dt} = z \frac{dz}{dt} - y \frac{dy}{dt}$

$\frac{dx}{dt} = \frac{1}{x} \left( z \frac{dz}{dt} - y \frac{dy}{dt} \right)$

$= \frac{1}{0.8} \left( 1(20) - 0.6(-60) \right)$

$= \frac{1}{0.8} (20 + 36)$

$= \frac{56}{0.8}$

$= 70$ km/hr.

Equations of Tangent and Normal :

The tangent to a plane curve at a point $(x_1, y_1)$ is $y - y_1 = m(x - x_1)$, where $m = \frac{dy}{dx}$ at $(x_1, y_1)$

A normal at a point on the curve $(x_1, y_1)$ is $y - y_1 = -\frac{1}{m}(x - x_1)$.

Exercise - 7.2.

1. Find the slope of the tangent to the curves at the respective given points.

i. $y = x^4 + 2x^2 - x$ at $x = 1$.

ii. $x = a\cos^3 t$, $y = b\sin^3 t$ at $t = \pi/2$.

i. $y = x^4 + 2x^2 - x$

$\frac{dy}{dx} = 4x^3 + 4x - 1$

$m = \left(\frac{dy}{dx}\right)$ at $x = 1$.

$= 4(1)^3 + 4(1) - 1$

$= 7.$

11. $x = a\cos^3 t$ $\qquad$ $y = b\sin^3 t$

$$\frac{dx}{dt} = a\, 3\cos^2 t(-\sin t) \qquad \frac{dy}{dt} = b\, 3\sin^2 t\,(\cos t)$$

$$\frac{dx}{dt} = -3a\cos^2 t\,\sin t \qquad \frac{dy}{dt} = 3b\sin^2 t\,(\cos t).$$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

$$= \frac{3b\sin^{\cancel{2}} t\,\cancel{\cos t}}{-\cancel{3}a\cos^{\cancel{2}} t\,\cancel{\sin t}}$$

$$= -\frac{b}{a}\tan t$$

$$m = \left(\frac{dy}{dx}\right) \text{ at } t = \frac{\pi}{2}.$$

$$= \frac{-b}{a}\tan\frac{\pi}{2} \qquad = -\frac{b}{a}\infty$$

$$= -\infty = \infty.$$

2. Find the point on the curve $y = x^2 - 5x + 4$ at which the tangent is parallel to the line $3x + y = 7$.

$y = x^2 - 5x + 4.$ $\qquad$ (1)

$\parallel^{rl}$ line is $3x + y = 7$ $\qquad$ (2)

Slope of tangent,

$m = -3.$ $\qquad \left(-\frac{\text{coefficient of } x}{\text{coefficient of } y}\right)$

But $\frac{dy}{dx} = 2x - 5$

$$m = \left(\frac{dy}{dx}\right)_{(x_1, y_1)} = 2x_1 - 5.$$

$$2x_1 - 5 = -3$$

$$2x_1 = 2$$

$$x_1 = 1$$

$$① \Rightarrow y_1 = 1^2 - 5(1) + 4$$

$$= 0.$$

$\therefore$ The points $(x_1, y_1) = (1, 0)$.

3. Find the points on the curve $y = x^3 - 6x^2 + x + 3$, where the normal is parallel to the line $x + y = 1729$.

$$y = x^3 - 6x^2 + x + 3$$



$\parallel^{el}$ to the normal is $x+y = 1729$.

$$-\frac{1}{m} = -1,$$

$$m = 1.$$

But

$$\frac{dy}{dx} = 3x^2 - 12x + 1.$$

$$m = \left(\frac{dy}{dx}\right)_{(x_1, y_1)}$$

$$= 3x_1^2 - 12x_1 + 1.$$

i.e.,

$$3x_1^2 - 12x_1 + 1 = 1$$

$$3x_1^2 - 12x_1 = 0.$$

$$3x_1(x_1 - 4) = 0$$

$$x_1 = 0 \qquad x_1 = 4.$$

If $x_1 = 0$

$$y_1 = 0 - 0 + 0 + 3 = 3.$$

If $x_1 = 4$

$$y_1 = 64 - 96 + 4 + 3$$

$$= -25.$$

$\therefore$ The points are $(0,3)$ and $(4,-25)$.

4. Find the points on the curve $y^2 - 4xy = x^2 + 5$ for which the tangent is horizontal.

$$y^2 - 4xy = x^2 + 5$$

$$y^2 = x^2 + 5 + 4xy. \quad (1)$$

Tangent is horizontal.

$$m = 0.$$

But

$$2y\frac{dy}{dx} = 2x + 4\left(x\frac{dy}{dx} + y\right)$$

$$y\frac{dy}{dx} = x + 2\left(x\frac{dy}{dx} + y\right)$$

$$y\frac{dy}{dx} - 2x\frac{dy}{dx} - 2y = x.$$

$$\frac{dy}{dx}(y-2x) = x + 2y.$$

$$\frac{dy}{dx} = \frac{x+2y}{y-2x}.$$

$$m = \left(\frac{dy}{dx}\right) (x_1, y_1).$$

$$= \frac{x_1 + 2y_1}{y_1 - 2x_1}.$$

Equating the m.

$$\frac{x_1 + 2y_1}{y_1 - 2x_1} = 0.$$

$$x_1 + 2y_1 = 0.$$

$$x_1 = -2y_1. \quad (2)$$

From (1).

$$y_1^2 - 4(-2y_1)y_1 = (-2y_1)^2 + 5$$

$$y_1^2 + 8y_1^2 = 4y_1^2 + 5$$

$$5y_1^2 = 5$$

$$y_1^2 = 1$$

$$y_1 = \pm 1$$

If $y_1 = 1$,

$$x_1 = -2.$$

If $y_1 = -1$

$$x_1 = 2.$$

$\therefore$ The points are $(-2, 1)$ and $(2, -1)$.

5. Find the tangent and normal to the following curves at the given points on the curve.

i. $y = x^2 - x^4$ at $(1, 0)$

ii. $y = x^4 + 2e^x$ at $(0, 2)$

iii. $y = x\sin x$ at $\left(\frac{\pi}{2}, \frac{\pi}{2}\right)$

iv. $x = \cos t$, $y = 2\sin^2 t$ at $t = \frac{\pi}{3}$

i. $y = x^2 - x^4$ at $(1,0)$.

$$\frac{dy}{dx} = 2x - 4x^3$$

$$m = \left(\frac{dy}{dx}\right)_{(1,0)}$$

$$= 2 - 4 = -2.$$

Tangent is

$$y - y_1 = m(x - x_1)$$

$$y - 0 = -2(x - 1)$$

$$2x + y - 2 = 0.$$

Normal is

$$y - y_1 = \frac{-1}{m}(x - x_1)$$

$$y - 0 = \frac{1}{2}(x - 1)$$

$$x - 2y - 1 = 0.$$

ii. $y = x^4 + 2e^x$ at $(0,2)$

$$\frac{dy}{dx} = 4x^3 + 2e^x.$$

$$m = \left(\frac{dy}{dx}\right)_{(0,2)}$$

$$= 4(0) + 2e^0$$

$$= 2.$$

Tangent is

$$y - y_1 = m(x - x_1)$$

$$y - 2 = 2(x - 0)$$

$$2x - y + 2 = 0.$$

Normal is

$$y - y_1 = \frac{-1}{m}(x - x_1)$$

$$y - 2 = \frac{-1}{2}(x - 0)$$

$$2y - 4 = -x$$

$$x + 2y - 4 = 0.$$

iii. $y = x\sin x$ at $\left(\frac{\pi}{2}, \frac{\pi}{2}\right)$

$$\frac{dy}{dx} = x(\cos x) + \sin x.$$

$$m = \left(\frac{dy}{dx}\right)_{\left(\frac{\pi}{2}, \frac{\pi}{2}\right)}$$

$$= \frac{\pi}{2}\left(\cos\frac{\pi}{2}\right) + \sin\frac{\pi}{2}$$

$$= 1.$$

Tangent is

$$y - y_1 = m(x - x_1)$$

$$y - \frac{\pi}{2} = 1\left(x - \frac{\pi}{2}\right)$$

$$y - \frac{\pi}{2} = x - \frac{\pi}{2}$$

$$x - y = 0.$$

Normal is

$$y - y_1 = -\frac{1}{m}(x - x_1)$$

$$y - \frac{\pi}{2} = -\frac{1}{1}\left(x - \frac{\pi}{2}\right)$$

$$y - \frac{\pi}{2} = -x + \frac{\pi}{2}$$

$$x + y = \pi.$$

iv. $x = \cos t$ $\qquad$ $y = 2\sin^2 t$

$$\frac{dx}{dt} = -\sin t \qquad \frac{dy}{dt} = 2 \cdot 2\sin t(\cos t)$$

$$= 4\sin t\cos t.$$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{4\sin t\cos t}{-\sin t}$$

$$= -4\cos t.$$

$$m = \left(\frac{dy}{dx}\right)_{\left(\frac{\pi}{3}\right)}$$

$$= -4\cos\frac{\pi}{3}$$

$$= -4\cdot\frac{1}{2} = -2.$$

$$y_1 = 2\sin^2\left(\frac{\pi}{3}\right) = 2\cdot\frac{3}{4} = \frac{3}{2}. \qquad x_1 = \cos\frac{\pi}{3} = \frac{1}{2}$$

Tangent is

$$y - y_1 = m(x - x_1)$$

$$y - \frac{3}{2} = -2\left(x - \frac{1}{2}\right)$$

$$-\frac{3}{2} + y = -2x + 2/2$$

$$-\frac{3}{2} + y = -2x + 1.$$

$$2y - 3 = -4x + 2.$$

$$4x + 2y - 5 = 0.$$

Normal is

$$y - y_1 = -\frac{1}{m}(x - x_1)$$

$$y - \frac{3}{2} = \frac{1}{2}\left(x - \frac{1}{2}\right)$$

$$y - \frac{3}{2} = \frac{x}{2} - \frac{1}{4}$$

$$4y - 6 = 2x - 1$$

$$2x - 4y + 5 = 0.$$

6. Find the equation of the tangent to the curve $y = 1 + x^3$ for which the tangent is orthogonal with the line $x + 12y = 12$.

$$y = 1 + x^3.$$

Line is $x + 12y = 12$.

$$-\frac{1}{m} = -\frac{1}{12}$$

$$\frac{-1}{m} = \frac{-1}{12}$$

$$-\frac{1}{m} = -1/12. \quad m = 12.$$

But

$$\frac{dy}{dx} = 3x^2$$

$$m = \left(\frac{dy}{dx}\right)_{(x_1, y_1)}$$

$$= 3x_1^2.$$

i.e.,

$$3x_1^2 = 12$$

$$x_1^2 = 4$$

$$x_1 = \pm 2.$$

If $x_1 = 2$

$$y_1 = 1 + 8$$

$$= 9$$

If $x_1 = -2$

$$y_1 = 1 - 8$$

$$= -7$$

Eqn of tangent at $(2,9)$.

$$y - y_1 = m(x - x_1)$$
$$y - 9 = 12(x - 2)$$
$$y - 9 = 12x - 24$$
$$12x - y - 15 = 0.$$

Eqn of tangent at $(-2,-7)$

$$y - y_1 = m(x - x_1)$$
$$y + 7 = 12(x + 2)$$
$$y + 7 = 12x + 24$$
$$12x - y + 17 = 0.$$

7. Find the equations of the tangents to the curve $y = \frac{x+1}{x-1}$ which are parallel to the line $x + 2y = 6$.

$$y = \frac{x+1}{x-1}$$

parallel to the line $x + 2y = 6$.

$$m = \frac{-1}{2}.$$

But
$$\frac{dy}{dx} = \frac{1(x-1) - 1(x+1)}{(x-1)^2}$$
$$= \frac{x-1-x-1}{x^2+1-2x}$$
$$= \frac{-2}{(x-1)^2}.$$

$$m = \left(\frac{dy}{dx}\right)_{(x_1, y_1)}$$
$$= \frac{-2}{(x_1-1)^2}$$

$$\frac{-2}{x_1^2+1-2x_1} = -\frac{1}{2}$$
$$4 = x_1^2 + 1 - 2x_1;$$

$$x_1^2 - 2x_1 - 3 = 0.$$
$$(x_1 - 3)(x_1 + 1) = 0.$$
$$x_1 = 3 \qquad x_1 = -1.$$

If $x_1 = 3$

$$y_1 = \frac{3+1}{3-1} = \frac{4}{2} = 2$$

If $x_1 = -1$

$$y_1 = \frac{-1+1}{-1-1} = 0.$$

Eqn of tangent at $(3,2)$

$$y - y_1 = m(x - x_1)$$

$$y - 2 = -\frac{1}{2}(x-3)$$

$$y - 2 = \frac{-x}{2} + \frac{3}{2}$$

$$2y - 4 = -x + 3$$

$$x + 2y - 7 = 0.$$

Eqn of tangent at $(-1,0)$.

$$y - y_1 = m(x - x_1)$$

$$y - 0 = \frac{-1}{2}(x+1)$$

$$y = \frac{-x}{2} - \frac{1}{2}$$

$$2y = -x - 1$$

$$x + 2y + 1 = 0.$$

8. Find the equation of the tangent and normal to the curve given by $x = 7\cos t$ and $y = 2\sin t$, $t \in R$ at any point on the curve.

$x = 7\cos t$

$$\frac{dx}{dt} = 7(-\sin t)$$

$y = 2\sin t$

$$\frac{dy}{dt} = 2\cos t.$$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2\cos t}{-7\sin t} = \frac{-2}{7}\cot t.$$

$$\left(\frac{dy}{dx}\right)_{(x_1, y_1)} = \frac{-2}{7}\frac{\cos t}{\sin t}$$

Eqn. of tangent :

$$y - y_1 = m(x - x_1)$$

$$y - 2\sin t = -\frac{2\cos t}{7\sin t}(x - 7\cos t)$$

$$7\sin t\, y - 14\sin^2 t = -2\cos t\, x + 14\cos^2 t$$

$$7\sin t\, y + 2\cos t\, x = 14.$$

Eqn. of normal is

$$y - y_1 = \frac{-1}{m}(x - x_1)$$

$$y - 2\sin t = \frac{7\sin t}{2\cos t}(x - 7\cos t)$$

$$2\cos t\, y - 4\sin t\cos t = 7\sin t\, x - 49\sin t\cos t$$

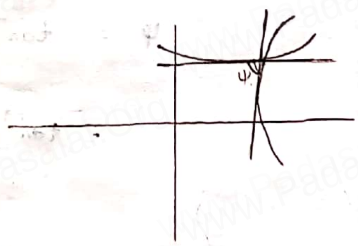
$$2\cot y - 7\sin t\, x = -45\sin t\cos t$$

$$7\sin t\, x - 2\cot y - 45\sin t\cos t = 0.$$

8. Angle between the curves :

Let $m_1, m_2$ be the slopes of the curves,

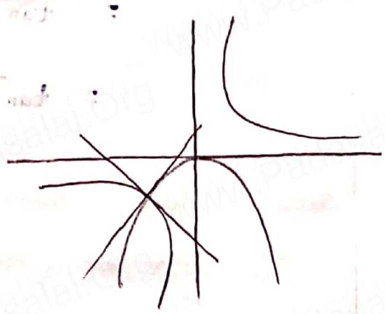
then $\psi = \tan^{-1}\left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$




Note :

If two curves cut orthogonally $m_1 m_2 = -1$.

9. Find the angle b/w the rectangular hyperbola $xy = 2$ and the parabola $x^2 + 4y = 0$.



$$xy = 2$$

$$x^2 + 4y = 0$$

Use (1) in (2).

$$x^2 + 4\left(\frac{2}{x}\right) = 0$$

$$x^3 + 8 = 0$$

$$x = -2.$$

If $x = -2$.

$$-2y = 2$$

$$y = -1.$$

$\therefore$ The point of intersection is $(-2, -1)$

① $\Rightarrow$ $xy = 2$

$$x\frac{dy}{dx} + y = 0.$$

$$x\frac{dy}{dx} = -y$$

$$\frac{dy}{dx} = \frac{-y}{x}.$$

$$m_1 = \left(\frac{dy}{dx}\right)_{(-2,-1)}$$

$$= \frac{+1}{-2} = \frac{-1}{2}.$$

② $\Rightarrow$ $x^2 + 4y = 0.$

$$2x + 4\frac{dy}{dx} = 0.$$

$$4\frac{dy}{dx} = -2x.$$

$$\frac{dy}{dx} = -\frac{1}{2}x.$$

$$m_2 = \left(\frac{dy}{dx}\right)_{(-2,-1)}$$

$$= \frac{+2}{2} = 1.$$

$$\varphi = \tan^{-1}\left|\frac{m_1 - m_2}{1+m_1 m_2}\right|$$

$$= \tan^{-1}\left|\frac{\frac{-1}{2} - 1}{1 + \frac{-1}{2}\cdot +1}\right|$$

$$= \tan^{-1}\left|\frac{\frac{-3}{2}}{1 - \frac{1}{2}}\right|$$

$$= \tan^{-1}\left|\frac{\frac{-3}{2}}{1/2}\right|$$

$$= \tan^{-1}\left|\frac{3}{2} \times \frac{2}{1}\right|$$

$$= \tan^{-1}(3).$$

10. Show that the two curves $x^2 - y^2 = r^2$ and $xy = c^2$ where $c, r$ are constants cut orthogonally.

$x^2 - y^2 = r^2$

$xy = c^2$

(1) $\Rightarrow$ $2x - 2y\dfrac{dy}{dx} = 0.$

$y\dfrac{dy}{dx} = x$

$\dfrac{dy}{dx} = \dfrac{x}{y}$

$m_1 = \left(\dfrac{dy}{dx}\right)_{(x_1, y_1)}$

$= \dfrac{x_1}{y_1}.$

(2) $\Rightarrow$ $1y + x\dfrac{dy}{dx} = 0$

$x\dfrac{dy}{dx} = -y$

$\dfrac{dy}{dx} = \dfrac{-y}{x}.$

$m_2 = \left(\dfrac{dy}{dx}\right)(x_1, y_1)$

$= \dfrac{-y_1}{x_1}$

$m_1 m_2 = \dfrac{x_1}{y_1}\left(\dfrac{-y_1}{x_1}\right)$

$= -1.$

$\therefore$ The curves cut orthogonally.

Example 7.11.

Find the equation of tangent and normal to the curve $y = x^2 + 3x - 2$ at the point $(1,2)$

$\dfrac{dy}{dx} = 2x + 3.$

$m = \left(\dfrac{dy}{dx}\right)_{(1,2)}$

$= 5.$

Eqn. of tangent

$y - y_1 = m(x - x_1)$

$y - 2 = 5(x - 1)$

$5x - y - 3 = 0.$

Eqn. of normal is.

$$y - y_1 = \frac{-1}{m}(x - x_1)$$

$$y - 2 = -\frac{1}{5}(x - 1)$$

$$y - 2 = \frac{-x}{5} + \frac{1}{5}$$

$$x + 5y - 11 = 0.$$

Example 7.12.

For what value of $x$ the tangent of the curve $y = x^3 - 3x^2 + x - 2$ is parallel to the line $y = x$.

The line $y = x$

Slope $m = 1$.

$y = x^3 - 3x^2 + x - 2$

$$\frac{dy}{dx} = 3x^2 - 6x + 1.$$

$$m = \left(\frac{dy}{dx}\right)_{(x_1, y_1)}$$

$$= 3x^2 - 6x + 1.$$

i.e.,

$$3x^2 - 6x + 1 = 1.$$

$$3x^2 - 6x = 0.$$

$$3x(x - 2) = 0.$$

$$x = 0 \qquad x = 2.$$

If $x = 0$

$y = -2$.

If $x = 2$

$y = -4$.

$\therefore$ at $(0, -2)$ and $(2, -4)$ the tangent is parallel to the line $y = x$.

Example 7.13.

Find the equation of the tangent and normal to the Lissajous curve given by $x = 2\cos 3t$ and $y = 3\sin 2t$, $t \in R$.

$x = 2\cos 3t$

$$\frac{dx}{dt} = 2 \cdot -3\sin 3t$$

$$= -6\sin 3t$$

$y = 3\sin 2t$

$$\frac{dy}{dt} = 3 \cdot 2\cos 2t$$

$$= 6\cos 2t$$

$$\frac{dy}{dx} = \frac{6\cos 2t}{-6\sin 3t}$$

$$= -\frac{\cos 2t}{\sin 3t}$$

Eqn. of tangent.

$$y - y_1 = m(x - x_1).$$

$$y - 3\sin 2t = \frac{-\cos 2t}{\sin 3t}(x - 2\cos 3t)$$

$$x\cos 2t + y\sin 3t = 3\sin 2t\sin 3t + 2\cos 2t\cos 3t.$$

Eqn. of normal.

$$y - y_1 = \frac{-1}{m}(x - x_1)$$

$$y - 3\sin 2t = \frac{\sin 3t}{\cos 2t}(x - 2\cos 3t)$$

$$x\sin 3t - y\cos 2t = 2\sin 3t\cos 3t - 3\sin 2t\cos 2t.$$

Example 7.14.

Find the acute angle between $y = x^2$ and $y = (x-3)^2$.

$$y = x^2.$$

$$\frac{dy}{dx} = 2x.$$

Point of intersection.

$$x^2 = (x-3)^2.$$

$$x^2 = x^2 - 6x + 9$$

$$x = 3/2.$$

$\therefore$ The point of intersection is $\left(\frac{3}{2}, \frac{9}{4}\right)$

$$m_1 = \left(\frac{dy}{dx}\right)_{\left(\frac{3}{2}, \frac{9}{4}\right)}$$

$$= 3.$$

$$y = (x-3)^2$$

$$\frac{dy}{dx} = 2(x-3)$$

$$m_2 = \left(\frac{dy}{dx}\right)_{\left(\frac{3}{2}, \frac{9}{4}\right)}$$

$$= -3.$$

$$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$$

$$= \left|\frac{3-(-3)}{1-9}\right|$$

$$= \frac{3}{4}$$

$$\theta = \tan^{-1}\left(\frac{3}{4}\right)$$

Example 7.15.

Find the acute angle b/w the curves $y = x^2$ and $x = y^2$ at their points of intersections $(0,0)$, $(1,1)$.

$$y = x^2.$$

$$\frac{dy}{dx} = 2x.$$

$$m_1 = 2x.$$

$$x = y^2$$

$$\frac{dy}{dx} = \frac{1}{2y}.$$

$$m_2 = \frac{1}{2y}.$$

At $(0,0)$

$$\tan\theta_1 = \left|\frac{2x - \frac{1}{2y}}{1 + (2x)\left(\frac{1}{2y}\right)}\right|$$

$$\tan\theta_1 = \lim_{(x,y)\to(0,0)} \left|\frac{2x - \frac{1}{2y}}{1 + 2x\left(\frac{1}{2y}\right)}\right|.$$

$$= \lim_{(x,y)\to(0,0)} \left|\frac{4xy - 1}{2(y+x)}\right|$$

$$= \infty.$$

which gives $\theta_1 = \tan^{-1}\infty$

$$= \frac{\pi}{2}.$$

A (1,1)

$m_1 = 2.$ $m_2 = \frac{1}{2}.$

$$\tan \theta_2 = \left| \frac{2 - \frac{1}{2}}{1 + \frac{1}{2}(2)} \right|$$

$$= \frac{3}{4}.$$

$$\theta_2 = \tan^{-1}\left(\frac{3}{4}\right)$$

Example 7.16.

Find the angle of intersection of the curve $y = \sin x$ with the positive $x$ axis.

$$\frac{dy}{dx} = \cos x$$

The slope at $x$ is $= n\pi$.

$$\cos(n\pi) = (-1)^n.$$

$$\tan^{-1}(-1)^n = \begin{cases} \frac{\pi}{4}, & \text{when } n \text{ is even} \\ \frac{3\pi}{4}, & \text{when } n \text{ is odd.} \end{cases}$$

Example 7.17.

If the curves $ax^2 + by^2 = 1$ and $cx^2 + dy^2 = 1$ intersect each other orthogonally then $\frac{1}{a} - \frac{1}{b} = \frac{1}{c} - \frac{1}{d}$.

The two intersect at point $(x_0, y_0)$ if $(a-c)x_0^2 + (b-d)y_0^2 = 0$.

$ax^2 + by^2 = 1.$

$$\frac{dy}{dx} = \frac{-ax}{by}$$

$cx^2 + dy^2 = 1$

$$\frac{dy}{dx} = \frac{-cx}{dy}.$$

Two curves are orthogonally,

$$\left(-\frac{ax_0}{by_0}\right) \times \left(\frac{-cx_0}{dy_0}\right) = -1.$$

i.e $acx_0^2 + bdy_0^2 = 0.$

$(a-c)x_0^2 + (b-d)y_0^2 = 0.$

give $\frac{a-c}{ac} = \frac{b-d}{bd}$

That is

$$\frac{1}{c} - \frac{1}{a} = \frac{1}{d} - \frac{1}{b}.$$

$$\frac{1}{a} - \frac{1}{b} = \frac{1}{c} - \frac{1}{d}.$$

Example 7.18

Prove that the elipse $x^2+4y^2=8$ and the hyperbola $x^2-2y^2=4$ intersect orthogonally.

Point of intersection $(a,b)$.

$a^2+4b^2=8$ $\qquad$ $a^2-2b^2=4$

$\Rightarrow$ $x^2+4y^2=8$

$$2x+8y\frac{dy}{dx} = 8.$$

$$\frac{dy}{dx} = -\frac{x}{4y}.$$

$$m_1 = \left(\frac{dy}{dx}\right)_{(a,b)} = \frac{-a}{4b}.$$

$\Rightarrow$ $x^2-2y^2=4$

$$2x-4y\frac{dy}{dx} = 0.$$

$$\frac{dy}{dx} = \frac{x}{2y}.$$

$$m_2 = \left(\frac{dy}{dx}\right)_{(a,b)} = \frac{a}{2b}.$$

$$m_1 \times m_2 = \frac{-a}{4b} \times \frac{a}{2b} = \frac{-a^2}{8b^2}.$$

$$\frac{a^2}{-16-16} = \frac{b^2}{-8+4} = \frac{1}{-2-4}.$$

$$\therefore \frac{a^2}{-32} = \frac{\cancel{32}b^2}{-4} = \frac{1}{-6}$$

$$\frac{a^2}{b^2} = \frac{32}{4} = 8.$$

Subs in $\frac{-a^2}{8b^2}$, we get $m_1 \times m_2 = -1$. Hence the curves cut orthogonally

Hence Proved.

Example

Compute the values of 'c' satisfied by the Rolle's theorem for the function.

$$f(x) = x^2(1-x)^2,\ x \in [0,1]$$

$f(0) = 0 = f(1)$.

$f(x)$ is continuous in the interval $[0,1]$

$f'(x) = 2x(1-x)^2 + 2(1-x)x^2$

$= 2x(1-x)(1-2x)$

$f'(c) = 0$ gives $c = 0, 1$ and $\frac{1}{2}$.

which $c \Rightarrow \frac{1}{2} \in (0,1)$.

Example 7.20

Find the values in the interval $\left(\frac{1}{2}, 2\right)$ satisfied by the Rolle's theorem for the function

$f(x) = x + \frac{1}{x},\ x \in \left[\frac{1}{2}, 2\right]$

$f(x)$ is continuous in $\left[\frac{1}{2}, 2\right]$

$f\left(\frac{1}{2}\right) = \frac{5}{2} = f(2)$

By rolle's theorem;

$c \in \left(\frac{1}{2}, 2\right)$

$f'(c) = 1 - \frac{1}{c^2} = 0$

$c^2 = 1$

$c = \pm 1$.

As $1 \in \left(\frac{1}{2}, 2\right)$

we choose $c = 1$.

Example 7.21.

Compute the value of 'c' satisfied by Rolle's theorem for the function $f(x) = \log\left(\frac{x^2+6}{5x}\right)$ in the interval $[2,3]$.

 $f(2) = 0 = f(3)$ 

$f(x)$ is continuous in the interval $[2,3]$

$$f'(x) = \frac{x^2-6}{x(x^2+6)}$$

$$f'(c) = 0.$$

$$\frac{c^2-6}{c(c^2+6)} = 0.$$

$$c = \pm\sqrt{6}$$

$$c = +\sqrt{6} \in (2,3)$$

Example 7.22.

Without actually solving, show that the equation $x^4 + 2x^3 - 2 = 0$ has only one real root in the interval $(0,1)$.

$$f(x) = x^4 + 2x^3 - 2$$

$f(x)$ is continuous in $[0,1]$

$$f'(x) = 4x^3 + 6x^2$$

$$f'(x) = 0.$$

$$2x^2(2x+3) = 0.$$

$$x = 0 \qquad x = -3/2 \notin (0,1)$$

$$f'(x) > 0, \forall x \in (0,1)$$

$\therefore$ The equation ~~of~~ has only one real root in the interval $(0,1)$.

Hence Proved.

Example 7.23.

Prove using the Rolle's theorem that b/w any two distinct real roots of the polynomial $a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \ldots\ldots + a_1 x + a_0$ there is zero of a polynomial.

$$na_n x^{n-1} + (n-1)a_{n-1}x^{n-2} + \ldots\ldots a_1.$$

Let $P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1$.

Let $\alpha$ & $\beta$ be two real roots of $P(x)$

$\therefore P(\alpha) = P(\beta) = 0$.

$P(x)$ is continuous in $[\alpha, \beta]$.

$$P'(x) = n a_n x^{n-1} + (n-1) a_{n-1} x^{n-2} + \cdots + a_1$$

which completes the proof.

Example 7.24.

Prove that there is a zero of a polynomial, $2x^3 - 9x^2 - 11x + 12$ in the interval $(2, 7)$ given that 2 and 7 are the zeros of the polynomial $x^4 - 6x^3 - 11x^2 + 24x + 28$.

$$P(x) = x^4 - 6x^3 - 11x^2 + 24x + 28.$$

$$\alpha = 2, \quad \beta = 7.$$

$$P'(x) = 4x^3 - 18x^2 - 22x + 24.$$

$$\frac{P'(x)}{2} = 2x^3 - 9x^2 - 11x + 12 = Q(x) \text{ (say)}.$$

$\Rightarrow$ there is a zero of a polynomial $Q(x)$ in the interval $(2, 7)$

Example 7.25.

Find the values in the interval $(1, 2)$ of the mean value theorem satisfied by the function

$f(x) = x - x^2$ for $1 \le x \le 2$.

$$f(1) = 0.$$
$$f(2) = -2.$$

$f(x)$ is defined in the interval $(1, 2)$.

$$f'(c) = \frac{f(2) - f(1)}{2 - 1}$$

$$= 1 - 2c.$$

$$1 - 2c = -2.$$

$$c = 3/2.$$

Example 7.26.

A truck travels on a toll road with a speed limit 80 km/hr. The truck completes a 164 km journey in 2 hrs. At the end of the toll road the trucker is issued with a speed violation ticket. Justify this using the Mean Value theorem.

$f(t)$ is the distance travelled by the truck in 't' hrs. This is continuous function in $[0,2]$

$f(0) = 0 \qquad f(2) = 164.$

$$f'(c) = \frac{164-0}{2-0} = 82 > 80.$$

$\therefore$ At some point of time, during the travel in 2 hrs the trucker must have travelled with a speed more than 80 km which justifies the issuance of a speed violation ticket.

Example 7.27.

Suppose $f(x)$ is a differentiable function for all $x$ with $f'(x) \leq 29$ and $f(2) = 17$. What is the maximum value of $f(7)$.

$'c' \in (2,7)$.

$$\frac{f(7) - f(2)}{7-2} = f'(c) \leq 29.$$

Hence

$$f(7) \leq 5 \times 29 + 17 = 162.$$

$\therefore$ The max. value of $f(7)$ is 162.

Example 7.28

Prove, using mean value theorem, that

$$|\sin\alpha - \sin\beta| \leq |\alpha-\beta|, \quad \alpha,\beta \in R.$$

Let $f(x) = \sin x$ which is differentiable function in any open interval.

$c \in (\alpha,\beta)$

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$$\frac{\sin\beta - \sin\alpha}{\beta - \alpha} = f'(c) = \cos(c)$$

$$\therefore \left|\frac{\sin\alpha - \sin\beta}{\alpha - \beta}\right| = |\cos(c)| \leq 1.$$

Hence $|\sin\alpha - \sin\beta| \leq |\alpha - \beta|$.

Hence Proved.

Example 7.29

A thermometer was taken from a freezer and placed in a boiling water. It tooks 22 seconds for the thermometer to raise from $-10^\circ c$ to $100^\circ c$. show that the rate of change of temperature at some time $t$ is $5^\circ c$ per second.

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

$$= \frac{100 - (10)}{22}$$

$$= \frac{110}{22}$$

$$= 5^\circ c \text{ per second.}$$

Hence proved.

Exercise - 7.3.

1. Explain why Rolle's theorem is not applicable to the following functions in the respective intervals.

i) $f(x) = \left|\frac{1}{x}\right|$, $x \in [-1, 1]$

ii) $f(x) = \tan x$, $x \in [0, \pi]$

iii) $f(x) = x - 2\log x$, $x \in [2, 7]$.

i. $f(x)$ is not continuous at ... so that Rolle's theorem is not applicable.

ii. $f(x)$ is not continuous at $x = \frac{\pi}{2}$. So that Rolle's theorem is not applicable.

iii. a. $f(x)$ is continuous in $[2,7]$

b. $f'(x)$ exist on $[2,7]$

c. If $f(2) = 2 - 2\log 2$

$f(7) = 7 - 2\log 7$

$f(2) \neq f(7)$

$f(a) \neq f(b)$.

Hence the Rolle's theorem is not applicable.

2. Using the Rolle's theorem, determine the values of $x$ at which tangent is parallel to the $x$ axis for the following functions

i. $f(x) = x^2 - x$, $x \in [0,1]$.

ii. $f(x) = \dfrac{x^2 - 2x}{x+2}$, $x \in [-1,6]$.

iii. $f(x) = \sqrt{x} - \dfrac{x}{3}$, $x \in [0,9]$.

i. a. $f(x)$ continuous exists in $[0,1]$

b. $f'(x)$ exists in $(0,1)$

c. $f(0) = 0$

$f(1) = 0$.

$f(0) = f(1)$.

$c \in (0,1)$

$f'(c) = 0$

$f'(x) = 2x - 1$.

$2x - 1 = 0$.

$f(c) = 0$.

$2c - 1 = 0$.

$c = 1/2 \in (0,1)$

ii. a. f(x) is continuous at $[-1, 6]$

b. $f'(x)$ exists in $(-1, 6)$.

c. $f(-1) = \frac{+1+2}{-1+2} = +3.$

$f(6) = \frac{36-12}{6+2} = +3.$

$f'(x) = \frac{(2x-2)(x+2) - (x^2-2x)(1)}{(x+2)^2}$

$= \frac{2x^2 + 4x - 2x - 4 - x^2 + 2x}{(x+2)^2}$

$= \frac{x^2 + 4x - 4}{(x+2)^2}$

$f'(c) = 0$

$\frac{c^2 + 4c - 4}{(c+2)^2} = 0.$

$c^2 + 4c - 4 = 0.$

$c = \frac{-4 \pm \sqrt{16+16}}{2}$

$= \frac{-4 \pm 4\sqrt{2}}{2}$

$= -2 \pm 2\sqrt{2}.$

The suitable c is $-2 + 2\sqrt{2}. \in (-1, 6)$

iii. a. f(x) continuous in $[0, 9]$

b. $f'(x)$ exists in $(0, 9)$

c. $f(0) = 0$

$f(9) = 0.$

$f(0) = f(9)$

$f'(x) = \frac{1}{2\sqrt{x}} - \frac{1}{3}$

$f'(c) = 0.$

$\Rightarrow \frac{1}{2\sqrt{c}} - \frac{1}{3} = 0$

$\frac{1}{2\sqrt{c}} = \frac{1}{3}.$

$\sqrt{c} = \frac{3}{2}$

$c = \frac{9}{4}. \in (0, 9).$

3. Explain why Lagrange Mean Value theorem is not applicable for the following conditions in the respective intervals.

i. $f(x) = \frac{x+1}{x}$, $x \in [-1,2]$

ii. $f(x) = |3x+1|$, $x \in [-1,3]$

i. $f(x)$ is not continuous at $x = 0$.

$\therefore$ Lagrange Mean Value theorem fails or not applicable.

ii.

a. $f(x)$ is continuous in $[-1,3]$.

b. $f'(x)$ not exists in $(-1,3)$ at $x = -1/3$

$\therefore$ Lagrange Mean Value theorem is not applicable.

4. Using the Lagrange's Mean Value theorem, determine the values of $x$ at which the tangent is parallel to the secant line at the end points of the given inte

i. $f(x) = x^3 - 3x + 2$, $x \in [-2,2]$

ii. $f(x) = (x-2)(x-7)$, $x \in [3,11]$

i. a. $f(x)$ is continuous in $[-2,2]$

b. $f'(x)$ is exists in $(-2,2)$

c. $f(-2) = -8+6+2$

$= 0.$

$f(2) = 8 - 6 + 2$

$= 4.$

W.K.T. $f'(c) = \frac{f(b)-f(a)}{b-a}$

$= \frac{4-0}{2+2}$

$= \frac{4}{4}$

$= 1.$

$f'(x) = 3x^2 - 3.$

$f'(c) = 3c^2 - 3.$

$3c^2 - 3 = 1$

$3c^2 = 4$

$c^2 = \frac{4}{3}.$

$c = \pm \frac{2}{\sqrt{3}} \in (-2, 2)$

ii. a. $f(x)$ is continuous in $[3, 11]$

b. $f'(x)$ exists in $(3, 11).$

c. $f(3) = (3-2)(3-7)$

$= (1)(-4) = -4.$

$f(11) = (11-2)(11-7)$

$= (9)(4) = 36.$

W.K.T $f'(c) = \frac{f(b) - f(a)}{b-a}$

$= \frac{36+4}{11-3}$

$= \frac{40}{8} = 5.$

$f'(x) = (x-2)(1) + (x-7)(1)$

$= (x-2) + (x-7). = 2x - 9$

$f(x) = x^2 - 7x - 2x + 14)$

$= x^2 - 9x + 14$

$f'(x) = 2x - 9.$

$f'(c) = 2c - 9.$

$2c - 9 = 5$

$2c = 14$

$c = 7 \in (3, 11).$

5. Show that the value of mean value theorem for positive numbers $[a,b]$ is roots of $ab$

i. $f(x) = \frac{1}{x}$ on a

ii. $f(x) = Ax^2 + Bx + C$ on any interval $[a,b]$ is $\frac{a+b}{2}$.

i. a. $f(x)$ is continuous on $[a,b]$

b. $f'(x)$ exists on $(a,b).$

$f(a) = \frac{1}{a}$

$f(b) = \frac{1}{b}.$

It belongs to Lagrange's Mean Value Theorem.

$f'(c) = \frac{f(b) - f(a)}{b-a}$

$= \frac{\frac{1}{b} - \frac{1}{a}}{b-a} = \frac{a-b}{ab(b-a)}$

$= \frac{-1}{ab}.$

$f'(x) = \frac{-1}{x^2}$

$f'(c) = \frac{-1}{c^2}.$

$\frac{-1}{c^2} = \frac{-1}{ab}.$

$c^2 = ab.$

$c = \sqrt{ab}$

Hence the proof.

11. $f(x) = Ax^2 + Bx + c \quad , \quad x \in [a,b]$

a. $f(x)$ continuous on $[a,b]$

b. $f(x)$ exists on $(a,b)$

c. $f(a) = Aa^2 + Ba + c$

$f(b) = Ab^2 + Bb + c.$

It belongs to Lagrange's Mean Value theorem.

$f'(c). = \frac{f(b) - f(a)}{b-a}$

$= \frac{Ab^2 + Bb + c - Aa^2 - Ba - c}{b-a}$

$= \frac{A(b^2-a^2) + B(b-a)}{b-a}$

$= \frac{A(b+a)(b-a) + B(b-a)}{b-a}.$

$= A(b+a) + B.$

$f'(x) = 2Ax + B.$

$f'(c) = 2Ac + B$

$$= A(b+a) + B.$$

$$2Ac = A(b+a)$$

$$2c = a+b$$

$$c = \frac{a+b}{2}.$$

Hence the proof.

6. A race car driver racing at $20^{th}$ km. if his speed never exceeds 150 km/hr, what is the maximum distance he can cover in the next 2 hours.

Let $f(x)$ be the function of displacement. $f(x)$ is continuous $[0,2]$ and differentiable on $(0,2)$.

$f(a) = 20$

$f(b) = ?$

$a = 0$, $b = 2$.

$f'(c) = 150$.

By L.M.V,

$$\frac{f(b) - f(a)}{b-a} = f'(c)$$

$$\frac{f(b) - 20}{2-0} \leq 150$$

$$f(b) - 20 \leq 300$$

$$f(b) \leq 320 \text{ km}$$

Then the car cover maximum 320 km in next 2 hours.

7. Suppose that for a function $f(x)$, $f'(x) \leq 1$ for all $1 \leq x \leq 4$. Show that $f(4) - f(1) \leq 3$

$f(x)$ is continuous on $[1,4]$

$f'(x)$ exists on $(1,4)$.

$f'(x) \leq 1$.

By LMV theorem,

$$\frac{f(b) - f(a)}{b-a} = f'(c)$$

$$\frac{f(4) - f(1)}{4-1} \leq 1$$

$$f(4) - f(1) \leq 3.$$

Hence the Proof.

 

8. Does there exist a differentiable function $f(x)$ such that $f(0) = -1$, $f(2) = 4$ and $f'(x) \leq 2$ for all $x$. justify your answer.

Let $a = 0$

$b = 0$

$f'(c) \leq 0.$

By, L.M.V.

$$f'(c) = \frac{f(b) - f(a)}{b-a}$$

$$= \frac{4-(-1)}{2-0}$$

$$= \frac{5}{2}$$

$$= 2.5$$

here $f'(c) \not\leq 2$

$\therefore$ There is no a function satisfying the given condition.

10.

9. Show that there lies a point on the curve $f(x) = x(x+3)e^{-\pi/2}$, $-3 \leq x \leq 0$ where tangent drawn is parallel to the $x$-axis.

$$f(x) = x(x+3)e^{-\pi/2}.$$

a. $f(x)$ is continuous on $[-3, 0]$

b. $f'(x)$ exists on $(-3, 0)$.

c. $f(-3) = 0.$

$f(0) = 0.$

$f(-3) = f(0)$

Hence Hypothesis of Rolle's Theorem satisfied.

Then,

$f'(c) = 0.$

Hence there is a point on the curve.

$f(x) = (x^2 + 3x)e^{-\pi/2}$

$f'(x) = (2x+3)e^{-\pi/2}$

$f'(c) = (2c+3)e^{-\pi/2}.$

$(2c+3)e^{-\pi/2} = 0$

$2c+3 = 0$

$2c = -3$

$c = -3/2 \in (-3,0)$.

Hence there exist a point $c = -3/2 \in (-3,0)$ such that tangent at the point is parallel to $x$-axis.

10. Using mean value theorem, prove that for $a>0,\ b>0$, $|e^{-a}-e^{-b}| \le |a-b|$.

Let $f(x) = e^{-x}$, $x \in [a,b]$

a. $f(x)$ is continuous on $[a,b]$

b. $f'(x)$ exists on $(a,b)$.

c. $f(a) = e^{-a}$

$f(b) = e^{-b}$.

$$\frac{f(b)-f(a)}{b-a} = f'(c).$$

since $x \in [a,b]$

and $a>0,\ b>0$.

$|f'(x)| \le 1$

$$\left|\frac{f(b)-f(a)}{b-a}\right| = |f'(c)|$$

$$\frac{|f(b)-f(a)|}{|b-a|} \le |f'(c)|$$

$|f(b)-f(a)| \le 1\,|b-a|$

$|e^{-b}-e^{-a}| \le |b-a|$

$|e^{-a}-e^{-b}| \le |a-b|$

Series Expansions.

1. Taylor's Series.

Let $f(x)$ be a function infinitely differentiable at $x = a$. Then $f(x)$ can be expanded as a series, in an interval $(x-a, x+a)$

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

$$= f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots$$

II. Maclaurin's Series.

Put $a = 0$ in the Taylor's Series.

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

$$= f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots$$

Example 7.30

Expand $\log(1+x)$ as a Maclaurin's series upto 4 non-zero terms for $-1 < x \le 1$

$$f(x) = \log(1+x)$$

Maclaurin's Series of $f(x) = \sum_{n=0}^{\infty} a_n x^n$ where

$a_n = \dfrac{f^{(n)}(0)}{n!}$  $x = 0$.

$f(x) = \log(1+x)$  $f(0) = 0$.

$f'(x) = \dfrac{1}{1+x}$  $f'(0) = 1$

$f''(x) = -\dfrac{1}{(1+x)^2}$  $f''(0) = -1$

$f'''(x) = \dfrac{2}{(1+x)^3}$  $f'''(0) = 2$

$f^{(iv)}(x) = \dfrac{-6}{(1+x)^4}$  $f^{(iv)}(0) = -6$.

Then the Maclaurin's series,

$$f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{(iv)}(0)}{4!}x^4 + \ldots\ldots$$

$$\log(1+x) = 0 + \frac{1}{1}x + \frac{-1}{2!}x^2 + \frac{2}{3!}x^3 + \frac{-6}{4!}x^4 + \ldots$$

$$= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots\ldots ; -1 < x \leq 1$$

Example 7.31.

Expand $\tan x$ in ascending powers of $x$ upto $5^{th}$ powers for $-\frac{\pi}{2} < x < \frac{\pi}{2}$.

Let $f(x) = \tan x$.

$f(x) = \tan x \qquad f(0) = 0$

$f'(x) = \sec^2 x \qquad f'(0) = 1$

$f''(x) = 2\sec^2 x \tan x \qquad f''(0) = 0.$

$f'''(x) = 2\sec^4 x + 4\sec^2 x \tan^2 x \qquad f'''(0) = 2.$

$f^{(iv)}(x) = 16\sec^4 x \cdot \tan x + 8\sec^2 x \cdot \tan^2 x \qquad f^{(iv)}(0) = 0.$

$f^{(v)}(x) = 16\sec^6 x + 88\sec^4 x \cdot \tan^2 x + 16\sec^2 x \cdot \tan^4 x \qquad f^{(v)}(0) = 16.$

Then the Maclaurin's series.

$$\tan x = 0 + x + 0 + \frac{2}{3!}x^3 + 0 + \frac{16}{5!}x^5 + \ldots\ldots$$

$$\tan x = x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \ldots ; \quad -\frac{\pi}{2} < x < \frac{\pi}{2}.$$

Example 7.32.

Write the Taylor series expansion of $\frac{1}{x}$ about $x=2$ by finding the first three non-zero terms.

Let $f(x) = \frac{1}{x}$.

$f(x) = \frac{1}{x}$ $\quad f(2) = \frac{1}{2}$.

$f'(x) = -\frac{1}{x^2}$ $\quad f'(2) = \frac{-1}{4}$

$f''(x) = \frac{2}{x^3}$ $\quad f''(2) = \frac{1}{4}$

$f'''(x) = \frac{-6}{x^4}$ $\quad f'''(2) = \frac{-3}{8}$

Then the Taylor's series,

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \dots$$

$$\frac{1}{x} = \frac{1}{2} - \frac{1}{4}\frac{x-2}{1!} + \frac{1}{4}\frac{(x-2)^2}{2!} + \frac{-3}{8}\frac{(x-2)^3}{3!} + \dots$$

$$= \frac{1}{2} - \frac{x-2}{4} + \frac{(x-2)^2}{8} - \frac{(x-2)^3}{16} + \dots$$

Exercise - 7.4.

1. Write the Maclaurin series expansion of the following functions.

i. $e^x$ $\quad$ ii. $\sin x$ $\quad$ iii. $\cos x$.

iv. $\log(1-x)$ ; $-1 \le x < 1$ $\quad$ v. $\tan^{-1}(x)$ ; $-1 \le x \le 1$

vi. $\cos^2 x$.

1. Let $f(x) = e^x$.

$f(x) = e^x$ $\quad f(0) = 1$

$f'(x) = e^x$ $\quad f(0) = 1$

$f''(x) = e^x$ $\quad f(0) = 1$.