CHAPTER-6

6. Two Dimensional Analytical Geometry

6.2 Locus At A Point

Point:

* A Point is an exact position or location on a plane surface.

Locus:

* A Path traced out by a moving point under a certain condition is called the locus of that point.

Example:

* **Circle:**

* A Point moves equal distance from a fixed point 'O'.

* **Angle bisector of the angle $\angle xoy$:**

* A Point P moves such that it is equal distance from two fixed lines ox and oy.

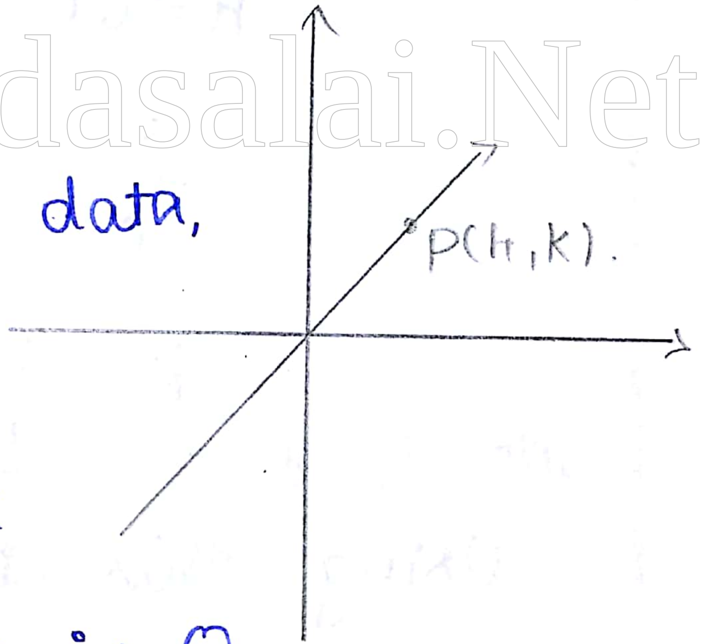
Example: 6.1

Find the locus of a point which moves such that its distance from the x-axis is equal to the distance from the y-axis.

Sol:

Let,

P(h, k) be the point on the locus.



From the given data,

$h = k \rightarrow (1)$

Replace, h by x

k by y in (1)

$x = y$

It is the locus of the point 'p'.

It is a line passing through origin.

Example: 6.2

Find the path traced out by the point $\left(ct, \frac{c}{t}\right)$ here $t \neq 0$ is the parameter and c is a constant.

Sol

Let.

$p(h,k)$ be a point on the locus.

From the given data,

$$h = ct$$

$$\frac{h}{c} = t$$

$$k = \frac{c}{t} \rightarrow (1)$$

Using this in equation 1

$$k = \frac{c}{t}$$

$\left(t = \frac{h}{c}\right)$ substitute the value of t

$$K = \frac{c}{\frac{h}{c}}$$

$$= c \times \frac{c}{h}$$

$$= \frac{c^2}{h}$$

$$k = \frac{c^2}{h}$$

$$hk = c^2$$

Put,

$$h = x, \quad k = y$$

$$xy = c^2$$

$$\therefore xy = c^2$$

$\therefore$ The required locus of the point is $xy = c^2$.

Example: 6.3

Find the locus of a Point P moves such that its distances from two fixed points A(1,0) and B(5,0) are always equal.

Sol:

Let,

P(h,k) be the point on the locus.

From the given data,

$$PA = PB$$

Squaring both sides,

$$(PA)^2 = (PB)^2$$

$$(h-1)^2 + (k-0)^2 = (h-5)^2 + (k-0)^2$$

$$(h-1)^2 + k^2 = (h-5)^2 + k^2$$

$$(h-1)^2 + k^2 - k^2 = (h-5)^2$$

$$(h-1)^2 = (h-5)^2$$

$$h^2 + 1 - 2h = h^2 + 25 - 10h$$

$$\cancel{h^2} - \cancel{h^2} + 1 - 25 - 2h + 10h = 0.$$

$$-24 + 8h = 0.$$

$$-24 + 8h = 0.$$

$-24 = -8h.$

$\frac{\cancel{24}^{3}}{\cancel{8}} = h$

$h = 3.$

Put, $h = 3$ $(h = x).$

$x = 3$

$\therefore$ The Locus of P is $x = 3.$

<u>Example: 6.4</u>

If θ is a parameter, find the equation of the locus of a moving point, whose co-ordinates are $(a\sec\theta, b\tan\theta)$.

<u>Sol</u>: Let $P(h,k)$ be a point of the locus.

$h = a\sec\theta$

$\frac{h}{a} = \sec\theta \rightarrow (1)$

$$K = b \tan\theta$$

$$\frac{K}{b} = \tan\theta \rightarrow (2)$$

Squaring both 1 & 2 equations,

$$\frac{H^2}{a^2} = \sec^2\theta \rightarrow (3)$$

$$\frac{K^2}{b^2} = \tan^2\theta \rightarrow (4)$$

Subtracting the equations 3 and 4,

3-4,

$$\frac{H^2}{a^2} - \frac{K^2}{b^2} = \sec^2\theta - \tan^2\theta$$

$$\frac{H^2}{a^2} - \frac{K^2}{b^2} = 1$$

Put,

$$H = x, \quad K = y$$

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$$

$\therefore$ The locus of the point is,

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$$

Example: 6.5

A straight rod of the length 6 units, slides with its ends A and B always on the x and y axes respectively. If O is the origin, then find the locus of the centroid of ΔOAB.

Sol: Let,

P(h,k) be a point on

the locus.

Let the co-ordinates

of O, A and B be, O(0,0)

A(a,0) and B(0,b)

From the data,

P(h,k) is mid point of A and B

$$h = \frac{a+0+0}{3}$$

$$3h = a$$

$$a = 3h$$

$$K = \frac{0+b+0}{3}$$

$$3K = b$$

ΔOAB, $OA^2 + OB^2 = AB^2$

$$OA^2 + OB^2 = AB^2$$

$$a^2 + b^2 = 6^2$$

$$(3h)^2 + (3k)^2 = 6^2$$

$$\cancel{4h^2} + \cancel{4k^2} = \cancel{36}$$

$$9h^2 + 9k^2 = 36$$

$$9(h^2+k^2) = 36$$

$$h^2+k^2 = \frac{36}{9}$$

$$h^2+k^2 = 4$$

Put,

$$h = x, \quad k = y$$

$$x^2+y^2 = 4$$

$\therefore$ Locus of (h,k) is a circle,

$$x^2+y^2 = 4.$$

Example: 6.6

If θ is a parameter, find the equation of the locus of a moving point, whose coordinates are $(a(\theta - \sin\theta), a(1-\cos\theta))$.

Sol:

Let,

$P(h,k)$ be the point on the locus

$$h = a(\theta - \sin\theta) \rightarrow (1)$$

$$\frac{h}{a} = \theta - \sin\theta$$

$$\sin\theta = \theta - \frac{h}{a}$$

$$k = a(1-\cos\theta)$$

$$\frac{k}{a} = 1 - \cos\theta$$

$$\cos\theta = 1 - \frac{k}{a}$$

$$\cos\theta = \frac{a-k}{a}$$

$$\cos\theta = \frac{a-k}{a} \Rightarrow \theta = \cos^{-1}\left(\frac{a-k}{a}\right)$$

$$\sin\theta = \sqrt{1-\cos^2\theta}$$

$$= \sqrt{1-\left(\frac{a-k}{a}\right)^2}$$

$$= \sqrt{1-\frac{(a-k)^2}{a^2}}$$

$$= \sqrt{\frac{a^2-(a^2+k^2-2ak)}{a^2}}$$

$$= \sqrt{\frac{\cancel{a^2}-\cancel{a^2}-k^2+2ak}{a^2}}$$

$$= \sqrt{\frac{2ak-k^2}{a^2}}$$

$$= \frac{\sqrt{2ak-k^2}}{a}$$

From eqn (1)

$$h = a(\theta - \sin\theta)$$

$$h = a\cos^{-1}\left(\frac{a-k}{a}\right) - \sqrt{2ak-k^2}$$

Put, $h=x$, $k=y$

$$x = a\cos^{-1}\left(\frac{a-y}{a}\right) - \sqrt{2ay-y^2}$$

The locus of (h,k) is

$$x = a\cos^{-1}\left(\frac{a-y}{a}\right) - \sqrt{2ay-y^2}.$$

Exercise : 6.1

1) Find the locus of P, if for all values of α, the co-ordinates of a moving point P is

i) $(9\cos\alpha, 9\sin\alpha)$.

Sol Let, $P(h,k)$ be a point on the locus.

From the given data,

$$h = 9\cos\alpha$$

$$h = a\cos\alpha$$

$$\frac{h}{a} = \cos\alpha \rightarrow (1)$$

$$k = a\sin\alpha$$

$$\frac{k}{a} = \sin\alpha \rightarrow (2)$$

Squaring both equations,

$$\frac{h^2}{a^2} = \cos^2\alpha \rightarrow (3)$$

$$\frac{k^2}{a^2} = \sin^2\alpha \rightarrow (4)$$

~~Subtracting~~ Adding the equations 3 and 4,

3-4

$$\frac{h^2}{a^2} + \frac{k^2}{a^2} = \cos^2\alpha + \sin^2\alpha$$

$$\frac{h^2}{a^2} + \frac{k^2}{a^2} = 1.$$

$$h^2 + k^2 = a^2$$

Put, $h = x$, $k = y$

$$x^2 + y^2 = 9^2$$

$$x^2 + y^2 = 81.$$

$\therefore$ The Locus of P is $x^2+y^2=81$.

iii) $(9\cos\alpha, 6\sin\alpha)$.

Sol: Let,

P(h,k) be a point on the Locus.

From the given data,

$$h = 9\cos\alpha$$

$$\frac{h}{9} = \cos\alpha \rightarrow (1)$$

$$k = 6\sin\alpha$$

$$\frac{k}{6} = \sin\alpha \rightarrow (2)$$

Squaring both equations,

$$\frac{h^2}{9^2} = \cos^2\alpha \rightarrow (3)$$

$$\frac{k^2}{6^2} = \sin^2\alpha \rightarrow (4)$$

Adding the equations 3 and 4,

3 + 4,

$$\frac{h^2}{9^2} + \frac{k^2}{6^2} = \cos^2\alpha + \sin^2\alpha$$

$$\frac{h^2}{9^2} + \frac{k^2}{6^2} = 1$$

$$\frac{h^2}{81} + \frac{k^2}{36} = 1.$$

Put, $h = x$ $k = y$

$$\frac{x^2}{81} + \frac{y^2}{36} = 1$$

$\therefore$ The Locus of P is

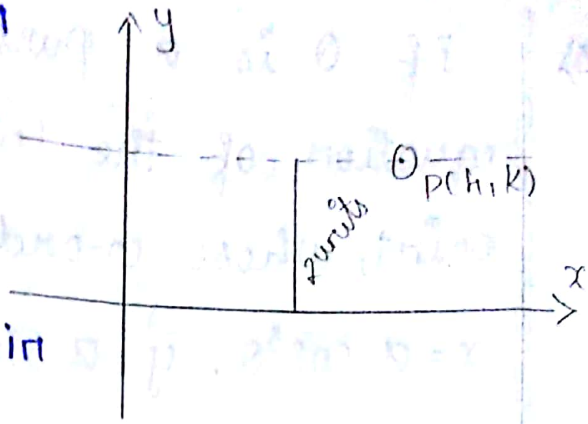
$$\frac{x^2}{81} + \frac{y^2}{36} = 1$$

2) Find the locus of a point P that moves at a constant distant of i) two units from the x-axis.

Sol Let, P(h,k) be a point on the

From the given data,

$k = 2 \rightarrow (1)$



Replace k by y in equation 1,

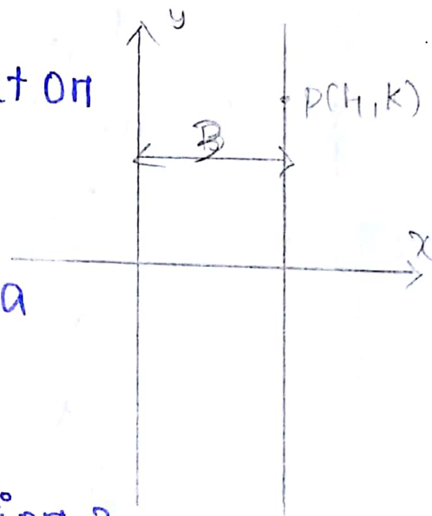
$y = 2$

$\therefore y = 2$ is the required locus of given point 'p' and it is the line parallel to x axis.

ii) Three units from the y-axis.

Sol: Let,

P(h, k) be a point on the locus.



From the given data

$h = 3 \rightarrow (2)$

Put, h = K in equation 2,

$x = 3$

$\therefore x = 3$ is the required locus of given point 'p' and it is the line Parallel to y axis.

31) If θ is a parameter, find the equation of the locus of a moving point, whose co-ordinates are $x = a\cos^3\theta$, $y = a\sin^3\theta$.

Sol: Let,

$P(h,k)$ be a point on the locus

From the given data,

$$h = a\cos^3\theta$$

$$\frac{h}{a} = \cos^3\theta$$

$$\left(\frac{h}{a}\right)^{1/3} = \cos\theta \rightarrow (1)$$

4

$$k = a\sin^3\theta$$

$$\frac{k}{a} = \sin^3\theta$$

$$\left(\frac{k}{a}\right)^{1/3} = \sin\theta \rightarrow (2)$$

Adding and Squaring the both equations,

$$\left(\frac{h}{a}\right)^{2/3} + \left(\frac{k}{a}\right)^{2/3} = \cos^2\theta + \sin^2\theta$$

$$\left(\frac{h}{a}\right)^{2/3} + \left(\frac{k}{a}\right)^{2/3} = 1$$

$$h^{(2/3)} + k^{2/3} = a^{2/3}$$

Put, $h = x, \quad k = y$

$$x^{2/3} + y^{2/3} = a^{2/3}$$

$\therefore$ The required locus is

$$x^{2/3} + y^{2/3} = a^{2/3}.$$

4) Find the value of k and b, if the points $P(-3,1)$ and $Q(2,b)$ lie on the locus of $x^2 - 5x + ky = 0$.

Sol: The Point $P(-3,1)$ lies on the locus of $x^2 - 5x + ky = 0$.

$x = -3$

$y = 1$

$(-3)^2 - 5(-3) + K(1) = 0$

$9 + 15 + K = 0$

$24 + K = 0.$

$K = -24$

$\therefore$ The locus and the Value of $K = -24$.

$Q(2, b)$ is on,

$x^2 - 5x - 24y = 0.$

$(2)^2 - 5(2) - 24(b) = 0.$

$4 - 10 - 24b = 0$

$-6 - 24b = 0$

$-24b = 6$

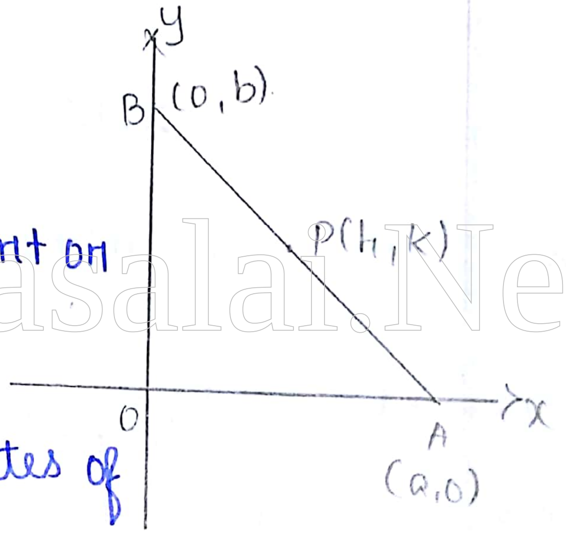
$b = \frac{-6}{24}$

$b = -1/4.$

$\therefore$ The Value of $K = -24$

The Value of $b = \frac{-1}{4}$.

5) A straight rod of length 8 units slides with its ends A and B always on the x and y axes respectively. Find the locus of the midpoint of the line segment AB.



Sol: Let,

P(h, k) be a point on the locus.

Let the co-ordinates of A and B be,

A(a, 0) and B(0, b)

From the data,

P(h, k) is mid point of A and B

$$h = \frac{a+0}{2}$$

$$2h = a$$

$$K = \frac{0+b}{2}$$

$$2K = b$$

ΔOAB,

$$OA^2 + OB^2 = AB^2$$

$$a^2 + b^2 = 8^2$$

$$4h^2 + 4K^2 = 64$$

$$4(h^2 + K^2) = 64$$

$$h^2 + K^2 = \frac{64}{4}$$

$$h^2 + K^2 = 16$$

Put,

$h = x$, $K = y$

$$x^2 + y^2 = 16.$$

$\therefore$ The locus of the midpoint is $x^2 + y^2 = 16$.

6) Find the equation of the locus of a point such that the sum of the squares of the distance from the points (3,5), (1,-1) is equal to 20.

Sol: Let,

P(H,K) be any point on the locus.

From the given data,

$$(PA)^2 + (PB)^2 = 20.$$

$$[(H-3)^2+(K-5)^2] + [(H-1)^2+(K+1)^2] = 20$$

$$(H^2+9+K-25)+(H^2+1-2H+K^2+1+2K)=20$$

$$H^2+9-6H+K^2+25-10K+H^2+1-2H+K^2+1+2K=20$$

$$2H^2+2K^2-8H-8K+16=0.$$

Divided by 2,

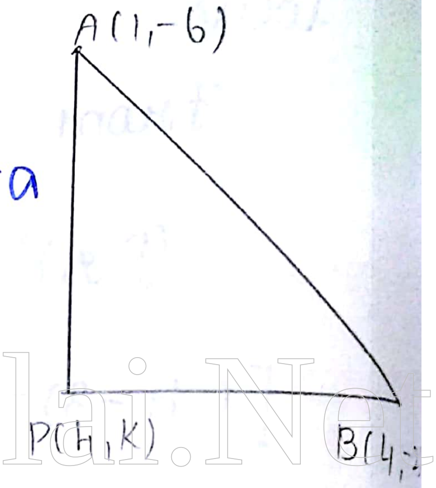
$$H^2+K^2-4H-4K+8=0.$$

Put, $H=x$, $K=y$

$$x^2+y^2-4x-4y+8=0.$$

7) Find the equation of the locus of the point P such that the line segment AB, joining the points A(1,-6) and B(4,-2), subtends a right angle at P.

Sol:
Let, $P(h,k)$ be a point on the locus.



From the given data and the picture,

$$AB^2 = AP^2 + BP^2$$

$$(1-4)^2 + (-6+2)^2 = (h-1)^2 + (k+6)^2 + (h-4)^2 + (k+2)^2$$

$$(3)^2 + (4)^2 = h^2 + 1 + 2h + k^2 + 36 + 12k + h^2 + 16 - 8h + k^2 + 4 + 4k$$

$$9 + 16 = h^2 + 1 + 2h + k^2 + 36 + 12k + h^2 + 16 - 8h + k^2 + 4 + 4k$$

$$2h^2 + 2k^2 - 10h + 16k + 32 = 0.$$

Divided by 2,

$$h^2 + k^2 - 5h + 8k + 16 = 0$$

Put,

$$h = x$$

$$k = y$$

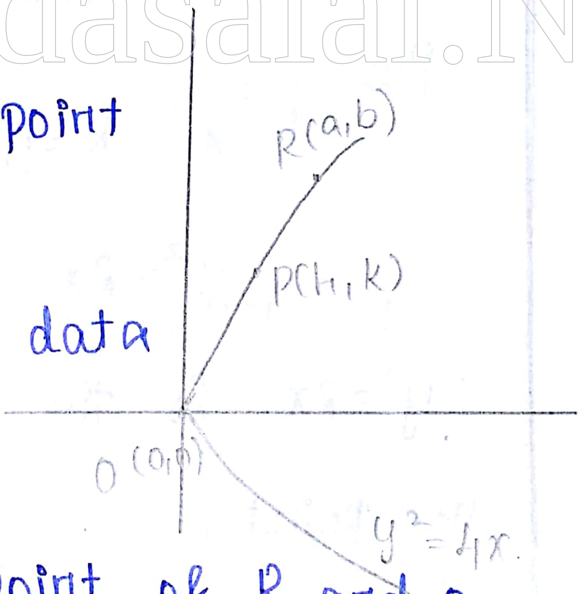
$$x^2 + y^2 - 5\overset{x}{y} + 8k + 16 = 0.$$

$$\therefore x^2 + y^2 - 5x + 8y + 16 = 0.$$

If O is origin and R is a variable point on $y^2 = 4x$, then find the equation of the locus of the mid-point of the line segment OR.

Sol:

Let, P be a point on the locus.



From the given data and Picture,

P(h,k) is midpoint of R and O.

$$h = \frac{a+0}{2}$$

$$2h = a.$$

$$K = \frac{0+b}{2}$$

$$2K = b$$

R(a, b) is the point on,

$$y^2 = 4x$$

$$b^2 = 4a$$

$$4K^2 = 8h$$

$$K^2 = \frac{8h}{4}$$

$$K^2 = 2h$$

Put, $h = x \quad K = y$

$$y^2 = 2x.$$

$\therefore y^2 = 2x$ is the required locus of the point P.

9) The coordinates of a moving point P are $\left(\frac{a}{2}(\operatorname{cosec}\theta + \sin\theta), \frac{b}{2}(\operatorname{cosec}\theta - \sin\theta)\right)$ where θ is a variable parameter. Show that the equation the locus P is $b^2x^2 - a^2y^2 = a^2b^2$.

Let, $P(h,k)$ be the point on the locus.

$$h = \frac{a}{2}(\operatorname{cosec}\theta + \sin\theta)$$

$$\frac{2h}{a} = \operatorname{cosec}\theta + \sin\theta \rightarrow (1)$$

$$k = \frac{b}{2}(\operatorname{cosec}\theta - \sin\theta)$$

$$\frac{2k}{b} = (\operatorname{cosec}\theta - \sin\theta) \rightarrow (2)$$

Squaring the both equations,

$$\left(\frac{2h}{a}\right)^2 = \operatorname{cosec}^2\theta + \sin^2\theta$$

$$\frac{4h^2}{a^2} = \operatorname{cosec}^2\theta + \sin^2\theta \rightarrow (3)$$

$$\left(\frac{2k}{b}\right)^2 = \operatorname{cosec}^2\theta - \sin^2\theta$$

$$\frac{4k^2}{b^2} = \operatorname{cosec}^2\theta - \sin^2\theta \rightarrow (4)$$

Subtracting the equations 3 and 4,

3-4,

$$\frac{4h^2}{a^2} - \frac{4k^2}{b^2} = (\operatorname{cosec}^2\theta + \sin^2\theta) - (\operatorname{cosec}^2\theta - \sin^2\theta)$$

$$\frac{4h^2}{a^2} - \frac{4k^2}{b^2} = \operatorname{cosec}^4\theta + \sin^4\theta + 2\operatorname{cosec}^2\theta \sin^2\theta + (-\operatorname{cosec}^4\theta - \sin^4\theta + 2\operatorname{cosec}^2\theta\sin^2\theta)$$

$$\frac{4h^2}{a^2} - \frac{4k^2}{b^2} = 4 \operatorname{cosec}^2\theta \sin\theta$$

$$\frac{4(h^2-k^2)}{a^2-b^2} = 4$$

$$\frac{h^2}{a^2} - \frac{k^2}{b^2} = \frac{4}{4}$$

$$\frac{h^2}{a^2} - \frac{k^2}{b^2} = 1.$$

$$h^2b^2 - a^2k^2 = a^2b^2$$

Put, $h = x$, $k = y$

$$bx^2x^2 - a^2y^2 = a^2b^2$$

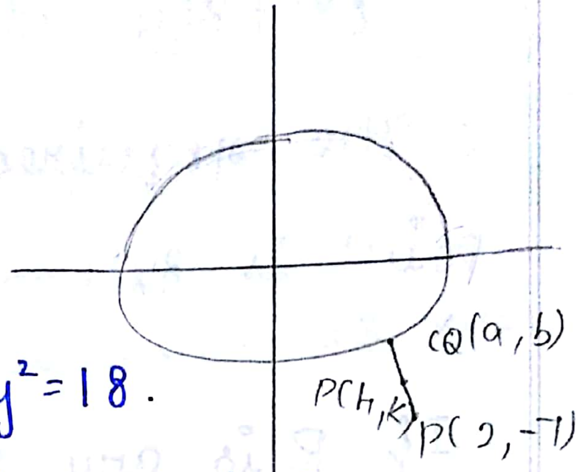
$\therefore$ It is the required locus of the Point.

If $P(2,-7)$ is a given point and Q is a point on $2x^2+9y^2=18$, then find the equations of the locus of the mid-point of PQ.

Sol

Let,

$Q(a,b)$ a point on $2x^2+9y^2=18$.



$M(h,k)$ be the mid point of PQ.

$$h = \frac{a+2}{2}$$

$$2h = a+2$$

$$2h-2 = a$$

$$k = \frac{b-7}{2}$$

$$2k = b-7$$

$$2k+7 = b.$$

$Q(a,b)$ lies on,

$$2x^2+9y^2=18.$$

$$2(2h-2)^2+9(2k+7)^2=18$$

$8h^2+36k^2-16h+252k+431=0.$

Put, $h=x$ $k=y$

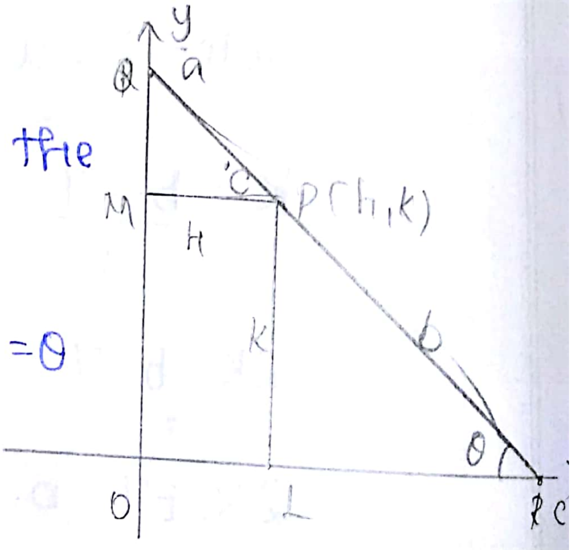
$8x^2+36y^2-16x+252y+431=0.$

$\therefore$ The required locus of the point is $8x^2+36y^2-16x+252y+431=0.$

If R is any point on the x-axis and Q is any point on the y-axis and P is a variable point on RQ with RP=b, PQ=a, then find the equation of locus of P.

Sol: Let P(h,k) be the point on PQ.

Such that $\angle ORQ=\theta$



ΔPLR $\sin\theta=\frac{k}{b}$

$PR=b$, $PQ=a$

ΔPMR $\cos\theta=\frac{h}{a}$

$\cos^2\theta+\sin^2\theta=1.$

$$\frac{h^2}{a^2} + \frac{k^2}{b^2} = 1$$

Put, $h = x$ $k = y$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

$\therefore$ The required equation of the locus of P is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

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12) If the point $P(6,2)$ and $Q(-2,1)$ and R are the vertices of a ΔPQR and R is the point on the locus $y = x^2 - 3x + 4$, then find the equation of the locus of centroid of ΔPQR.

Sol. Let,
$P(h,k)$ be the centroid of Δ PQ.

Let, $P(6,2)$, $Q(-2,1)$

$$h = \frac{6-2+a}{3}$$

$$h = \frac{6-2+a}{3}$$

$$h = \frac{4+a}{3}$$

$$3h = 4+a$$

$$\boxed{3h-4 = a}$$

$$k = \frac{2+1+b}{3}$$

$$k = \frac{3+b}{3}.$$

$$3k = 3+b$$

$$\boxed{3k-3 = b}$$

$R(a,b)$ is the point on the locus,

$$y = x^2 - 3x + 4$$

$$3k-3 = (3h+4)^2 - 3(3h-4) + 4$$

$$3k-3 = 9h^2 - 24h + 16 - 9h - 12 + 4$$

$$9h^2 - 33h - 3k + 35 = 0.$$

∴ The Required locus of the point.

If Q is a point on the locus of $x^2+y^2+4x-3y+7=0$, then find the equation of locus of P which divides segment OQ externally in the ratio 3:4, where O is origin.

Sol. Let, Q(a,b) be the point on

$$x^2+y^2+4x-3y+7=0.$$

Let, P(h,k) be a point on locus.

Given,

P divides OQ externally in the ratio 3:4

$$(h,k) = \frac{3a-4(0)}{-1}, \frac{3b-4(0)}{-1}$$

$$h = \frac{3a-0}{-1}$$

$$h = -3a$$

$$a = \frac{-h}{3}$$

$k = -3b$

$b = \frac{-k}{3}$

$Q(a,b)$ lies on $x^2+y^2+4x-3y+7=0$.

$$\left(\frac{h}{3}\right)^2 + \left(\frac{k}{3}\right)^2 - \frac{4h}{3} + \frac{\cancel{3}k}{\cancel{3}} + 7 = 0.$$

$$\frac{h^2}{9} + \frac{k^2}{9} - \frac{4h}{3} + k + 7 = 0.$$

Multiplied 9,

$$h^2 + k^2 - 12h + 9k + 63 = 0.$$

put $h = x$, $k = y$

$$x^2 + y^2 - 12\overset{x}{\cancel{y}} + 9y + 63 = 0.$$

$\therefore$ The Required Locus of the point is $x^2+y^2-12x+9y+63=0$.

The Sum of the distance of a moving point from the points (4,0) and (-4,0) is always 10 units, Find the equation of the locus of the moving point.

Sol:

Let,

P(h,k) be a point on the locus.

$$PA + PB = 10$$

$$\sqrt{(h-4)^2+(k-0)^2} + \sqrt{(h+4)^2+(k-0)^2} = 10$$

$$\sqrt{(h-4)^2+k^2} = 10 - \sqrt{(h+4)^2+k^2}$$

Squaring both sides,

$$(h-4)^2+k^2 = 100+(h+4)^2+k^2$$

$$\cancel{h^2}+16-8h+\cancel{k^2} = 100+\cancel{h^2}+\cancel{16}+8h+\cancel{k^2}$$

$$-20\sqrt{(h+4)^2+k^2}$$

$$16h^2+625+200h = 25(h^2+k^2+8h+16)$$

$$16h^2+625+200h = 25h^2+25k^2+200h+400$$

$$16h^2+625-25h^2-25k^2 = 400.$$

$$9h^2-25k^2+625-400 = 0.$$

$$9h^2-25k^2-225 = 0.$$

$$9h^2+25k^2 = 225.$$

Divided by 225,

$$\frac{h^2}{25}+\frac{k^2}{9} = 1.$$

Put, $h = x$, $k = y$

$$\frac{x^2}{25}+\frac{y^2}{9} = 1.$$

$\therefore$ The Locus of Point is $\frac{x^2}{25}+\frac{y^2}{9} = 1.$

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