DEPARTMENT OF MATHEMATICS

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UNIT – 1

Matrices and Determinants

1. Matrices and Determinants

Example 1.1 If $A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$ verify that A (adj A) = (adj A) A = | A |. I_3

Solution: $A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$

$$|A| = \begin{vmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{vmatrix}$$

$$= 8\begin{vmatrix} 7 & -4 \\ -4 & 3 \end{vmatrix} +6\begin{vmatrix} -6 & -4 \\ 2 & 3 \end{vmatrix} +2\begin{vmatrix} -6 & 7 \\ 2 & -4 \end{vmatrix}$$

$= 8(21 - 16) +6(- 18 +8) + 2(24 - 14)$

$= 8(5) +6(- 10) + 2(10)$

$= 40 - 60 + 20$

$= 60 - 60$

$|A| = 0$

$$A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix} \begin{array}{cccc} 7 & -4 & -6 & 7 \\ -4 & 3 & 2 & -4 \\ -6 & 2 & 8 & -6 \\ 7 & -4 & -6 & 2 \end{array}$$

Cofactor of A $\quad Aij = \begin{bmatrix} 5 & 10 & 10 \\ 10 & 20 & 20 \\ 10 & 20 & 20 \end{bmatrix}$

$\text{Adj } A = Aij^T$

$$\text{Adj } A = \begin{bmatrix} 5 & 10 & 10 \\ 10 & 20 & 20 \\ 10 & 20 & 20 \end{bmatrix}$$

$$A\,(\text{adj } A) = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix} \begin{bmatrix} 5 & 10 & 10 \\ 10 & 20 & 20 \\ 10 & 20 & 20 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{.........} \quad (1)$$

$$(\text{adj } A)\, A = \begin{bmatrix} 5 & 10 & 10 \\ 10 & 20 & 20 \\ 10 & 20 & 20 \end{bmatrix}\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{ .... } (2)$$

$$|A|.\, I = 0 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{ .... } (3)$$

From (1), (2) and (3)

A (adj A) = (adj A) A = | A | I_3 is verified.

Example 1.2 If A = $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is non singular, find A^{-1}

Solution: A = $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$$|A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

= ad – bc

$$\text{adj } A = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|}\, Adj\ A$$

$$= \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Example 1.3 Find the inverse of the matrix $\begin{bmatrix} 2 & -1 & 3 \\ -5 & 3 & 1 \\ -3 & 2 & 3 \end{bmatrix}$

Solution: A = $\begin{bmatrix} 2 & -1 & 3 \\ -5 & 3 & 1 \\ -3 & 2 & 3 \end{bmatrix}$

$$|A| = \begin{vmatrix} 2 & -1 & 3 \\ -5 & 3 & 1 \\ -3 & 2 & 3 \end{vmatrix}$$

$$= 2\begin{vmatrix} 3 & 1 \\ 2 & 3 \end{vmatrix} + 1\begin{vmatrix} -5 & 1 \\ -3 & 3 \end{vmatrix} + 3\begin{vmatrix} -5 & 3 \\ -3 & 2 \end{vmatrix}$$

= 2(9 – 2) +1(– 15 + 3) + 3(– 10 + 9)

= 2(7) +1(– 12) +3(– 1)

= 14 – 12 – 3

= 14 – 15

= – 1 ≠ 0, hence A^{-1} exists.

$$\begin{matrix} 3 & 1 & -5 & 3 \\ 2 & 3 & -3 & 2 \\ -1 & 3 & 2 & -1 \\ 3 & 1 & -5 & 3 \end{matrix}$$

$$\text{Cofactor of A} \quad A_{ij} = \begin{bmatrix} 7 & 12 & -1 \\ 9 & 15 & -1 \\ -10 & -17 & 1 \end{bmatrix}$$

$$\text{Adj } A = A_{ij}^{T}$$

$$= \begin{bmatrix} 7 & 9 & -10 \\ 12 & 15 & -17 \\ -1 & -1 & 1 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|}\, Adj\ A$$

$$= \frac{1}{-1}\begin{bmatrix} 7 & 9 & -10 \\ 12 & 15 & -17 \\ -1 & -1 & 1 \end{bmatrix}$$

$$= -1\begin{bmatrix} 7 & 9 & -10 \\ 12 & 15 & -17 \\ -1 & -1 & 1 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} -7 & -9 & 10 \\ -12 & -15 & 17 \\ 1 & 1 & -1 \end{bmatrix}$$

Example 1.4 If A is a non-singular matrix of odd order, prove that $|adj\ A|$ is positive.

Solution: Let A be a non-singular matrix of order $2m+1$, where m = 0,1,2,3,. . .

Then, we get $|A| \neq 0$ and, by property (ii), we have $|adj\ A| = |A|^{(2m+1)-1} = |A|^{2m}$.

Since $|A|^{2m}$ is always positive, we get that $|adj\ A|$ is positive.

Example 1.5 Find a matrix A

if $adj\ A = \begin{bmatrix} 7 & 7 & -7 \\ -1 & 11 & 7 \\ 11 & 5 & 7 \end{bmatrix}$

Solution: $adj\ A = \begin{bmatrix} 7 & 7 & -7 \\ -1 & 11 & 7 \\ 11 & 5 & 7 \end{bmatrix}$

$$|adj\ A| = \begin{vmatrix} 7 & 7 & -7 \\ -1 & 11 & 7 \\ 11 & 5 & 7 \end{vmatrix}$$

$$= 7\begin{vmatrix} 11 & 7 \\ 5 & 7 \end{vmatrix} - 7\begin{vmatrix} -1 & 7 \\ 11 & 7 \end{vmatrix} - 7\begin{vmatrix} -1 & 11 \\ 11 & 5 \end{vmatrix}$$

$= 7(77 - 35) - 7\ (-7 - 77) - 7(-5 - 121)$

$= 7(42) - 7\ (-84) - 7(-126)$

$= 7(42 + 84 + 126)$

$= 7(252)$

$= 7(7\times 36)$

$= 7^2 \times 6^2$

$\sqrt{adj\ A} = 7 \times 6 = 42$

We know, $A = \pm\frac{1}{\sqrt{adj\ A}}\ adj(adj\ A)$

Given: $adj\ A = \begin{bmatrix} 7 & 7 & -7 \\ -1 & 11 & 7 \\ 11 & 5 & 7 \end{bmatrix}$

$$\begin{matrix} 11 & 7 & -1 & 11 \\ 5 & 7 & 11 & 5 \\ 7 & -7 & 7 & 7 \\ 11 & 7 & -1 & 11 \end{matrix}$$

Cofactor of adj A $= \begin{bmatrix} 42 & 84 & -126 \\ -84 & 126 & 42 \\ 126 & -42 & 84 \end{bmatrix}$

$$adj(adj\ A) = \begin{bmatrix} 42 & -84 & 126 \\ 84 & 126 & -42 \\ -126 & 42 & 84 \end{bmatrix}$$

$$A = \pm\frac{1}{\sqrt{adj\ A}}\ adj(adj\ A)$$

$$= \pm\frac{1}{42}\begin{bmatrix} 42 & -84 & 126 \\ 84 & 126 & -42 \\ -126 & 42 & 84 \end{bmatrix}$$

$$A = \pm\begin{bmatrix} 1 & -2 & 3 \\ 2 & 3 & -1 \\ -3 & 1 & 2 \end{bmatrix}$$

Example 1.6 If $adj\ A = \begin{bmatrix} -1 & 2 & 2 \\ 1 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$ find A^{-1}

Solution: $adj\ A = \begin{bmatrix} -1 & 2 & 2 \\ 1 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$

$$|adj\ A| = \begin{vmatrix} -1 & 2 & 2 \\ 1 & 1 & 2 \\ 2 & 2 & 1 \end{vmatrix}$$

$$= -1\begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} - 2\begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} + 2\begin{vmatrix} 1 & 1 \\ 2 & 2 \end{vmatrix}$$

$= -1(1 - 4) - 2\ (1 - 4) + 2(2 - 2)$

$= -1(-3) - 2\ (-3) + 2(0)$

$= 3 + 6 + 0$

$= 9 \Rightarrow \sqrt{adj\ A} = 3$

We know, $A^{-1} = \pm\frac{1}{\sqrt{adj\ A}}\ (adj\ A)$

Hence, $A^{-1} = \pm\frac{1}{3}\begin{bmatrix} -1 & 2 & 2 \\ 1 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$

Example1.7 If A is symmetric, prove that then adj A is also symmetric.

Solution: Given A is symmetric. $\therefore A^T = A$

By the property, $adj\ (A^T) = (adj\ A)^T$

So, $adj\ (A) = (adj\ A)^T$ hence adj A is symmetric.

Example 1.8 Verify the property

$(A^T)^{-1} = (A^{-1})^T$ with $A = \begin{bmatrix} 2 & 9 \\ 1 & 7 \end{bmatrix}$

Solution: $A = \begin{bmatrix} 2 & 9 \\ 1 & 7 \end{bmatrix}$

$A^T = \begin{bmatrix} 2 & 1 \\ 9 & 7 \end{bmatrix}$

$|A^T| = \begin{vmatrix} 2 & 1 \\ 9 & 7 \end{vmatrix}$

$= 14 - 9$

$= 5$

$\text{Adj}(A^T) = \begin{bmatrix} 7 & -1 \\ -9 & 2 \end{bmatrix}$

$(A^T)^{-1} = \frac{1}{|A^T|} Adj\ A^T$

$= \frac{1}{5} \begin{bmatrix} 7 & -1 \\ -9 & 2 \end{bmatrix}$ ....(1)

$A = \begin{bmatrix} 2 & 9 \\ 1 & 7 \end{bmatrix}$

$|A| = \begin{vmatrix} 2 & 9 \\ 1 & 7 \end{vmatrix}$

$= 14 - 9$

$= 5$

$\text{adj } A = \begin{bmatrix} 7 & -9 \\ -1 & 2 \end{bmatrix}$

$A^{-1} = \frac{1}{|A|} Adj\ A$

$= \frac{1}{5} \begin{bmatrix} 7 & -9 \\ -1 & 2 \end{bmatrix}$

$(A^{-1})^T = \frac{1}{5} \begin{bmatrix} 7 & -1 \\ -9 & 2 \end{bmatrix}$ ....(1)

From (1) and (2)

$(A^T)^{-1} = (A^{-1})^T$ is verified.

Example1.9 Verify that $(AB)^{-1} = B^{-1} A^{-1}$

with $A = \begin{bmatrix} 0 & -3 \\ 1 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} -2 & -3 \\ 0 & -1 \end{bmatrix}$

$A = \begin{bmatrix} 0 & -3 \\ 1 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} -2 & -3 \\ 0 & -1 \end{bmatrix}$

$AB = \begin{bmatrix} 0 & -3 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} -2 & -3 \\ 0 & -1 \end{bmatrix}$

$AB = \begin{bmatrix} 0+0 & 0+3 \\ -2+0 & -3-4 \end{bmatrix}$

$AB = \begin{bmatrix} 0 & 3 \\ -2 & -7 \end{bmatrix}$

$|AB| = \begin{vmatrix} 0 & 3 \\ -2 & -7 \end{vmatrix}$

$= 0 + 6 = 6$

$\text{adj (AB)} = \begin{bmatrix} -7 & -3 \\ 2 & 0 \end{bmatrix}$

$(AB)^{-1} = \frac{1}{|AB|} Adj\ (AB)$

$= \frac{1}{6} \begin{bmatrix} -7 & -3 \\ 2 & 0 \end{bmatrix}$ .... (1)

$A = \begin{bmatrix} 0 & -3 \\ 1 & 4 \end{bmatrix}$

$|A| = \begin{vmatrix} 0 & -3 \\ 1 & 4 \end{vmatrix}$

$= 0 + 3$

$= 3$

$\text{adj } A = \begin{bmatrix} 4 & 3 \\ -1 & 0 \end{bmatrix}$

$A^{-1} = \frac{1}{|A|} Adj\ A$

$= \frac{1}{3} \begin{bmatrix} 4 & 3 \\ -1 & 0 \end{bmatrix}$

$B = \begin{bmatrix} -2 & -3 \\ 0 & -1 \end{bmatrix}$

$|B| = \begin{vmatrix} -2 & -3 \\ 0 & -1 \end{vmatrix}$

$= 2 + 0$

$= 2$

$\text{adj } B = \begin{bmatrix} -1 & 3 \\ 0 & -2 \end{bmatrix}$

$$B^{-1} = \frac{1}{|B|}\, Adj\ B$$

$$B^{-1} = \frac{1}{2}\begin{bmatrix} -1 & 3 \\ 0 & -2 \end{bmatrix}$$

$$B^{-1}A^{-1} = \frac{1}{2}\begin{bmatrix} -1 & 3 \\ 0 & -2 \end{bmatrix}\frac{1}{2}\begin{bmatrix} 4 & 3 \\ -1 & 0 \end{bmatrix}$$

$$= \frac{1}{6}\begin{bmatrix} -4-3 & -3+0 \\ 0+2 & 0+0 \end{bmatrix}$$

$$= \frac{1}{6}\begin{bmatrix} -7 & -3 \\ 2 & 0 \end{bmatrix} \quad \ldots\ldots (2)$$

From (1) and (2)

$(AB)^{-1} = B^{-1}A^{-1}$ is verified.

Example 1.10 If $A = \begin{bmatrix} 4 & 3 \\ 2 & 5 \end{bmatrix}$, find x and y such that $A^2 + xA + yI_2 = 0_2$.
Hence, find A^{-1}.

Solution: $A = \begin{bmatrix} 4 & 3 \\ 2 & 5 \end{bmatrix}$

$$A^2 = \begin{bmatrix} 4 & 3 \\ 2 & 5 \end{bmatrix}\begin{bmatrix} 4 & 3 \\ 2 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 16+6 & 12+15 \\ 8+10 & 6+25 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 22 & 37 \\ 18 & 31 \end{bmatrix}$$

$$A^2 + xA + yI_2 = 0_2$$

$$\begin{bmatrix} 22 & 37 \\ 18 & 31 \end{bmatrix} + x\begin{bmatrix} 4 & 3 \\ 2 & 5 \end{bmatrix} + y\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 22 & 37 \\ 18 & 31 \end{bmatrix} + \begin{bmatrix} 4x & 3x \\ 2x & 5x \end{bmatrix} + \begin{bmatrix} y & 0 \\ 0 & y \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 22+4x+y & 37+3x+0 \\ 18+2x+0 & 31+5x+y \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$22 + 4x + y = 0,\ \ 37 + 3x = 0,$

$18 + 2x = 0,\ \ 31 + 5x + y = 0$

$18 + 2x = 0$ gives

$2x = -18$

$$x = -\frac{18}{2}$$

$$x = -9$$

Substituting, $x = -9$ in

$22 + 4x + y = 0$

$22 + 4(-9) + y = 0$

$22 - 36 + y = 0$

$-14 + y = 0$

$y = 14$

Hence, Substituting $x = -9$ and $y = 14$ in

$$A^2 + xA + yI_2 = 0_2$$

$$A^2 - 9A + 14I_2 = 0_2$$

Post-multiplying this equation by A^{-1}

We get, $A - 9I + 14A^{-1} = 0_2$

$$14A^{-1} = 9I - A$$

$$A^{-1} = \frac{1}{14}(9I - A)$$

$$= \frac{1}{14}\left[9\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 4 & 3 \\ 2 & 5 \end{pmatrix}\right]$$

$$= \frac{1}{14}\left[\begin{pmatrix} 9 & 0 \\ 0 & 9 \end{pmatrix} - \begin{pmatrix} 4 & 3 \\ 2 & 5 \end{pmatrix}\right]$$

$$A^{-1} = \frac{1}{14}\begin{bmatrix} 5 & -3 \\ -2 & 4 \end{bmatrix}$$

Example 1.11 Prove that $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ is orthogonal.

Solution: Let $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ then,

$$A^T = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

$$AA^T = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} cos^2\theta + sin^2\theta & \cos\theta\sin\theta - \sin\theta\cos\theta \\ \sin\theta\cos\theta - \cos\theta\sin\theta & sin^2\theta + cos^2\theta \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

Similarly we can prove $A^T A = I$.

Hence the given matrix is orthogonal.

Example 1.12

If $A = \frac{1}{7}\begin{bmatrix} 6 & -3 & a \\ b & -2 & 6 \\ 2 & c & 3 \end{bmatrix}$ is orthogonal, find a, b and c and hence A^{-1} .

Solution: Given A is called orthogonal,

hence $AA^T = A^TA = I$

$$AA^T = \frac{1}{7}\begin{bmatrix} 6 & -3 & a \\ b & -2 & 6 \\ 2 & c & 3 \end{bmatrix} \times \frac{1}{7}\begin{bmatrix} 6 & b & 2 \\ -3 & -2 & c \\ a & 6 & 3 \end{bmatrix}$$

$$= \frac{1}{49}\begin{bmatrix} 36+9+a^2 & 6b+6+6a & 12-3c+3a \\ 6b+6+6a & b^2+4+36 & 2b-2c+18 \\ 12-3c+3a & 2b-2c+18 & 4+c^2+9 \end{bmatrix}$$

$$= \frac{1}{49}\begin{bmatrix} 45+a^2 & 6b+6+6a & 12-3c+3a \\ 6b+6+6a & b^2+40 & 2b-2c+18 \\ 12-3c+3a & 2b-2c+18 & c^2+13 \end{bmatrix}$$

$AA^T = I$

$$\frac{1}{49}\begin{bmatrix} 45+a^2 & 6b+6+6a & 12-3c+3a \\ 6b+6+6a & b^2+40 & 2b-2c+18 \\ 12-3c+3a & 2b-2c+18 & c^2+13 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 45+a^2 & 6b+6+6a & 12-3c+3a \\ 6b+6+6a & b^2+40 & 2b-2c+18 \\ 12-3c+3a & 2b-2c+18 & c^2+13 \end{bmatrix} = 49\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 45+a^2 & 6b+6+6a & 12-3c+3a \\ 6b+6+6a & b^2+40 & 2b-2c+18 \\ 12-3c+3a & 2b-2c+18 & c^2+13 \end{bmatrix} = \begin{bmatrix} 49 & 0 & 0 \\ 0 & 49 & 0 \\ 0 & 0 & 49 \end{bmatrix}$$

$45 + a^2 = 49$ gives

$a^2 = 49 - 45 \Longrightarrow a^2 = 4$, and

$b^2 + 40 = 49$ gives

$b^2 = 49 - 40 \Longrightarrow b^2 = 9$ and

$c^2 + 13 = 49$ gives

$c^2 = 49 - 13 \Rightarrow c^2 = 36$

$6b + 6 + 6a = 0$

$6b + 6a = -6$ gives

$b + a = -1$ ....(1)

$12 - 3c + 3a = 0$

$-3c + 3a = -12$ gives

$-c + a = -4$ ....(2)

$2b - 2c + 18 = 0$

$2b - 2c = -18$ gives

$b - c = -9$ ....(3)

From (2) and (3)

$a - c = -4$

$b - c = -9$

$a - b = 5$

$a + b = -1$ ....(1)

2a = 4 which is

a = 2

Substituting a = 2 in $a + b = -1$

$2 + b = -1$

$b = -1 - 2$

$\boldsymbol{b = -3}$

Substituting a = 2 in $a - c = -4$

$2 - c = -4$

$$-c = -4-2$$

$$-c = -6$$

$$\boldsymbol{c = 6}$$

$$\text{So } A = \frac{1}{7}\begin{bmatrix} 6 & -3 & 2 \\ -3 & -2 & 6 \\ 2 & 6 & 3 \end{bmatrix}$$

That is $A^{-1} = A^T$

$$= \frac{1}{7}\begin{bmatrix} 6 & -3 & 2 \\ -3 & -2 & 6 \\ 2 & 6 & 3 \end{bmatrix}$$

EXERCISE 1.1

1. Find the adjoint of the following:

(i) $\begin{bmatrix} -3 & 4 \\ 6 & 2 \end{bmatrix}$

Solution: Let $A = \begin{bmatrix} -3 & 4 \\ 6 & 2 \end{bmatrix}$

$$\text{adj A} = \begin{bmatrix} 2 & -4 \\ -6 & -3 \end{bmatrix}$$

(ii) $\begin{bmatrix} 2 & 3 & 1 \\ 3 & 4 & 1 \\ 3 & 7 & 2 \end{bmatrix}$

Solution:

$$\text{Let } A = \begin{bmatrix} 2 & 3 & 1 \\ 3 & 4 & 1 \\ 3 & 7 & 2 \end{bmatrix} \qquad \begin{matrix} 4 & 1 & 3 & 4 \\ 7 & 2 & 3 & 7 \\ 3 & 1 & 2 & 3 \\ 4 & 1 & 3 & 4 \end{matrix}$$

$$\text{Cofactor of A Aij} = \begin{bmatrix} 1 & -3 & 9 \\ 1 & 1 & -5 \\ -1 & 1 & -1 \end{bmatrix}$$

$$\text{Adj A} = \text{Aij}^T$$

$$\text{Adj A} = \begin{bmatrix} 1 & 1 & -1 \\ -3 & 1 & 1 \\ 9 & -5 & -1 \end{bmatrix}$$

(iii) $\frac{1}{3}\begin{bmatrix} 2 & 2 & 1 \\ -2 & 1 & 2 \\ 1 & -2 & 2 \end{bmatrix}$

Solution: Let $A = \frac{1}{3}\begin{bmatrix} 2 & 2 & 1 \\ -2 & 1 & 2 \\ 1 & -2 & 2 \end{bmatrix}$

$$A = \begin{bmatrix} \frac{2}{3} & \frac{2}{3} & \frac{1}{3} \\ -\frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ \frac{1}{3} & -\frac{2}{3} & \frac{2}{3} \end{bmatrix}$$

$$\begin{matrix} \frac{1}{3} & \frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \\ -\frac{2}{3} & \frac{2}{3} & \frac{1}{3} & -\frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{1}{3} & \frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \end{matrix}$$

$$\text{Cofactor of A Aij} = \begin{bmatrix} \frac{6}{9} & \frac{6}{9} & \frac{3}{9} \\ -\frac{6}{9} & \frac{3}{9} & \frac{6}{9} \\ \frac{3}{9} & -\frac{6}{9} & \frac{6}{9} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{2}{3} & \frac{2}{3} & \frac{1}{3} \\ -\frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ \frac{1}{3} & -\frac{2}{3} & \frac{2}{3} \end{bmatrix}$$

$$= \frac{1}{3}\begin{bmatrix} 2 & 2 & 1 \\ -2 & 1 & 2 \\ 1 & -2 & 2 \end{bmatrix}$$

$$\text{Adj A} = \text{Aij}^T$$

$$= \frac{1}{3}\begin{bmatrix} 2 & -2 & 1 \\ 2 & 1 & -2 \\ 1 & 2 & 2 \end{bmatrix}$$

2. Find the inverse (if it exists) of the following:

(i) $\begin{bmatrix} -2 & 4 \\ 1 & -3 \end{bmatrix}$

$$\text{Let } A = \begin{bmatrix} -2 & 4 \\ 1 & -3 \end{bmatrix}$$

$|A| = \begin{vmatrix} -2 & 4 \\ 1 & -3 \end{vmatrix}$

$= 6 - 4$

$= 2 \neq 0$, hence A^{-1} exists.

adj $A = \begin{bmatrix} -3 & -4 \\ -1 & -2 \end{bmatrix}$

$A^{-1} = \frac{1}{|A|}\, Adj\ A$

$= \frac{1}{2}\begin{bmatrix} -3 & -4 \\ -1 & -2 \end{bmatrix}$

(ii) $\begin{bmatrix} 5 & 1 & 1 \\ 1 & 5 & 1 \\ 1 & 1 & 5 \end{bmatrix}$

Solution: $A = \begin{bmatrix} 5 & 1 & 1 \\ 1 & 5 & 1 \\ 1 & 1 & 5 \end{bmatrix}$

$|A| = \begin{vmatrix} 5 & 1 & 1 \\ 1 & 5 & 1 \\ 1 & 1 & 5 \end{vmatrix}$

$= 5\begin{vmatrix} 5 & 1 \\ 1 & 5 \end{vmatrix} - 1\begin{vmatrix} 1 & 1 \\ 1 & 5 \end{vmatrix} + 1\begin{vmatrix} 1 & 5 \\ 1 & 1 \end{vmatrix}$

$= 5(25 - 1) - 1\,(5 - 1) + 1(1 - 5)$

$= 5(24) - 1(4) + 1(-4)$

$= 120 - 4 - 4$

$= 120 - 8$

$= 112 \neq 0$, hence A^{-1} exists.

$\begin{matrix} 5 & 1 & 1 & 5 \\ 1 & 5 & 1 & 1 \\ 1 & 1 & 5 & 1 \\ 5 & 1 & 1 & 5 \end{matrix}$

Cofactor of A Aij $= \begin{bmatrix} 24 & -4 & -4 \\ -4 & 24 & -4 \\ -4 & -4 & 24 \end{bmatrix}$

Adj A = Aij^T

$= \begin{bmatrix} 24 & -4 & -4 \\ -4 & 24 & -4 \\ -4 & -4 & 24 \end{bmatrix}$

$A^{-1} = \frac{1}{|A|}\, Adj\ A$

$= \frac{1}{112}\begin{bmatrix} 24 & -4 & -4 \\ -4 & 24 & -4 \\ -4 & -4 & 24 \end{bmatrix}$

$= \frac{1(4)}{112}\begin{bmatrix} 6 & -1 & -1 \\ -1 & 6 & -1 \\ -1 & -1 & 6 \end{bmatrix}$

$A^{-1} = \frac{1}{28}\begin{bmatrix} 6 & -1 & -1 \\ -1 & 6 & -1 \\ -1 & -1 & 6 \end{bmatrix}$

(iii) $\begin{bmatrix} 2 & 3 & 1 \\ 3 & 4 & 1 \\ 3 & 7 & 2 \end{bmatrix}$

Solution: $A = \begin{bmatrix} 2 & 3 & 1 \\ 3 & 4 & 1 \\ 3 & 7 & 2 \end{bmatrix}$

$|A| = \begin{vmatrix} 2 & 3 & 1 \\ 3 & 4 & 1 \\ 3 & 7 & 2 \end{vmatrix}$

$= 2\begin{vmatrix} 4 & 1 \\ 7 & 2 \end{vmatrix} - 3\begin{vmatrix} 3 & 1 \\ 3 & 2 \end{vmatrix} + 1\begin{vmatrix} 3 & 4 \\ 3 & 7 \end{vmatrix}$

$= 2(8 - 7) - 3\,(6 - 3) + 1(21 - 12)$

$= 2(1) - 3(3) + 1(9)$

$= 2 - 9 + 9$

$= 2$

$= 2 \neq 0$, hence A^{-1} exists.

$\begin{matrix} 4 & 1 & 3 & 4 \\ 7 & 2 & 3 & 7 \\ 3 & 1 & 2 & 3 \\ 4 & 1 & 3 & 4 \end{matrix}$

Cofactor of A Aij $= \begin{bmatrix} 1 & -3 & 9 \\ 1 & 1 & -5 \\ -1 & 1 & -1 \end{bmatrix}$

Adj A = Aij^T

$= \begin{bmatrix} 1 & 1 & -1 \\ -3 & 1 & 1 \\ 9 & -5 & -1 \end{bmatrix}$

$A^{-1} = \frac{1}{|A|} Adj\ A$

$= \frac{1}{2}\begin{bmatrix} 1 & 1 & -1 \\ -3 & 1 & 1 \\ 9 & -5 & -1 \end{bmatrix}$

$A^{-1} = \frac{1}{2}\begin{bmatrix} 1 & 1 & -1 \\ -3 & 1 & 1 \\ 9 & -5 & -1 \end{bmatrix}$

3. If $F(\alpha) = \begin{bmatrix} \cos\alpha & 0 & \sin\alpha \\ 0 & 1 & 0 \\ -\sin\alpha & 0 & \cos\alpha \end{bmatrix}$,

show that $[F(\alpha)]^{-1} = F(-\alpha)$

Solution: $F(\alpha) = \begin{bmatrix} \cos\alpha & 0 & \sin\alpha \\ 0 & 1 & 0 \\ -\sin\alpha & 0 & \cos\alpha \end{bmatrix}$

$|F(\alpha)| = \begin{vmatrix} \cos\alpha & 0 & \sin\alpha \\ 0 & 1 & 0 \\ -\sin\alpha & 0 & \cos\alpha \end{vmatrix}$

$= \cos\alpha \begin{vmatrix} 1 & 0 \\ 0 & \cos\alpha \end{vmatrix} - 0 \begin{vmatrix} 0 & 0 \\ -\sin\alpha & \cos\alpha \end{vmatrix}$

$+\sin\alpha \begin{vmatrix} 0 & 1 \\ -\sin\alpha & 0 \end{vmatrix}$

$= \cos\alpha\,(\cos\alpha) + \sin\alpha\,(\sin\alpha)$

$= cos^2\alpha + sin^2\alpha$

$= 1 \neq 0$, hence $F(\alpha)^{-1}$ exists.

$$\begin{matrix} 1 & 0 & 0 & 1 \\ 0 & \cos\alpha & -sin\alpha & 0 \\ 0 & \sin\alpha & cos\alpha & 0 \\ 1 & 0 & 0 & 1 \end{matrix}$$

Cofactor of $F(\alpha)$

$= \begin{bmatrix} \cos\alpha & 0 & \sin\alpha \\ 0 & cos^2\alpha + sin^2\alpha & 0 \\ -\sin\alpha & 0 & \cos\alpha \end{bmatrix}$

$= \begin{bmatrix} \cos\alpha & 0 & \sin\alpha \\ 0 & 1 & 0 \\ -\sin\alpha & 0 & \cos\alpha \end{bmatrix}$

Adj $F(\alpha) = F(\alpha)^T$

$= \begin{bmatrix} \cos\alpha & 0 & -\sin\alpha \\ 0 & 1 & 0 \\ \sin\alpha & 0 & \cos\alpha \end{bmatrix}$

$F(\alpha)^{-1} = \frac{1}{|F(\alpha)|} Adj\ F(\alpha)$

$= \frac{1}{1}\begin{bmatrix} \cos\alpha & 0 & -\sin\alpha \\ 0 & 1 & 0 \\ \sin\alpha & 0 & \cos\alpha \end{bmatrix}$

$F(\alpha)^{-1} = \begin{bmatrix} \cos\alpha & 0 & -\sin\alpha \\ 0 & 1 & 0 \\ \sin\alpha & 0 & \cos\alpha \end{bmatrix}$ ......(1)

$F(\alpha) = \begin{bmatrix} \cos\alpha & 0 & \sin\alpha \\ 0 & 1 & 0 \\ -\sin\alpha & 0 & \cos\alpha \end{bmatrix}$

$F(-\alpha) = \begin{bmatrix} \cos(-\alpha) & 0 & \sin(-\alpha) \\ 0 & 1 & 0 \\ -\sin(-\alpha) & 0 & \cos(-\alpha) \end{bmatrix}$

We know $\cos(-\alpha) = \cos\alpha$

and $\sin(-\alpha) = -\sin\alpha$

$F(-\alpha) = \begin{bmatrix} \cos\alpha & 0 & -\sin\alpha \\ 0 & 1 & 0 \\ \sin\alpha & 0 & \cos\alpha \end{bmatrix}$ ......(1)

From (1) *and* (2)

$[F(\alpha)]^{-1} = F(-\alpha)$, hence proved.

4. If $A = \begin{bmatrix} 5 & 3 \\ -1 & -2 \end{bmatrix}$, show that

$A^2 - 3A - 7I_2 = 0_2$. Hence find A^{-1}.

Solution: $A = \begin{bmatrix} 5 & 3 \\ -1 & -2 \end{bmatrix}$

$A^2 = \begin{bmatrix} 5 & 3 \\ -1 & -2 \end{bmatrix}\begin{bmatrix} 5 & 3 \\ -1 & -2 \end{bmatrix}$

$= \begin{bmatrix} 25-3 & 15-6 \\ -5+2 & -3+4 \end{bmatrix}$

$A^2 = \begin{bmatrix} 22 & 9 \\ -3 & 1 \end{bmatrix}$

$A^2 - 3A - 7I_2 =$

$\begin{bmatrix} 22 & 9 \\ -3 & 1 \end{bmatrix} - 3\begin{bmatrix} 5 & 3 \\ -1 & -2 \end{bmatrix} - 7\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$$= \begin{bmatrix} 22 & 9 \\ -3 & 1 \end{bmatrix} - \begin{bmatrix} 15 & 9 \\ -3 & -6 \end{bmatrix} - \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix}$$

$$= \begin{bmatrix} 22 & 9 \\ -3 & 1 \end{bmatrix} - \begin{bmatrix} 22 & 9 \\ -3 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0_2 \text{ Hence Proved.}$$

So, $A^2 - 3A - 7I_2 = 0_2$

Post-multiplying this equation by A^{-1}

We get, $A - 3I - 7A^{-1} = 0_2$

$$A - 3I = 7A^{-1}$$

$$7A^{-1} = A - 3I$$

$$A^{-1} = \frac{1}{7}(A - 3I)$$

$$= \frac{1}{7}\left[\begin{pmatrix} 5 & 3 \\ -1 & -2 \end{pmatrix} - 3\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right]$$

$$= \frac{1}{7}\left[\begin{pmatrix} 5 & 3 \\ -1 & -2 \end{pmatrix} - \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}\right]$$

$$= \frac{1}{7}\left[\begin{pmatrix} 5-3 & 3-0 \\ -1-0 & -2-3 \end{pmatrix}\right]$$

$$A^{-1} = \frac{1}{7}\begin{bmatrix} 2 & 3 \\ -1 & -5 \end{bmatrix}$$

5. If $A = \frac{1}{9}\begin{bmatrix} -8 & 1 & 4 \\ 4 & 4 & 7 \\ 1 & -8 & 4 \end{bmatrix}$ *prove that* $A^{-1} = A^T$

Solution: $A = \frac{1}{9}\begin{bmatrix} -8 & 1 & 4 \\ 4 & 4 & 7 \\ 1 & -8 & 4 \end{bmatrix}$

$$AA^T = \frac{1}{9}\begin{bmatrix} -8 & 1 & 4 \\ 4 & 4 & 7 \\ 1 & -8 & 4 \end{bmatrix}\frac{1}{9}\begin{bmatrix} -8 & 4 & 1 \\ 1 & 4 & -8 \\ 4 & 7 & 4 \end{bmatrix}$$

$$= \frac{1}{81}\begin{bmatrix} 64+1+16 & -32+4+28 & -8-8+16 \\ -32+4+28 & 16+16+49 & 4-32+28 \\ -8-8+16 & 4-32+28 & 1+64+16 \end{bmatrix}$$

$$= \frac{1}{81}\begin{bmatrix} 81 & 0 & 0 \\ 0 & 81 & 0 \\ 0 & 0 & 81 \end{bmatrix}$$

$$= \frac{81}{81}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \text{I.}$$

That is $AA^T = I$ hence A is orthogonal.

Therfore $A^{-1} = A^T$

6. If $A = \begin{bmatrix} 8 & -4 \\ -5 & 3 \end{bmatrix}$ verify that

$A(\text{adj } A) = (\text{adj } A)A = |A|.I_2$

Solution: $A = \begin{bmatrix} 8 & -4 \\ -5 & 3 \end{bmatrix}$

$$|A| = \begin{vmatrix} 8 & -4 \\ -5 & 3 \end{vmatrix}$$

$$= 24 - 20$$

$$|A| = 4$$

$$\text{Adj } A = \begin{bmatrix} 3 & 4 \\ 5 & 8 \end{bmatrix}$$

$$A(\text{adj } A) = \begin{bmatrix} 8 & -4 \\ -5 & 3 \end{bmatrix}\begin{bmatrix} 3 & 4 \\ 5 & 8 \end{bmatrix}$$

$$= \begin{bmatrix} 24-20 & 32-32 \\ -15+15 & -20+24 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} \quad \ldots\ldots\ldots \quad (1)$$

$$(\text{adj } A)A = \begin{bmatrix} 3 & 4 \\ 5 & 8 \end{bmatrix}\begin{bmatrix} 8 & -4 \\ -5 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 24-20 & -12+12 \\ 40-40 & -20+24 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} \quad \ldots\ldots\ldots \quad (1)$$

$$|A|.I = 4\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$= \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$ ……… (3)

From (1), (2) and (3)

A (adj A) = (adj A) A = | A | I_2 is verified.

7. If A = $\begin{bmatrix} 3 & 2 \\ 7 & 5 \end{bmatrix}$ and B = $\begin{bmatrix} -1 & -3 \\ 5 & 2 \end{bmatrix}$

Verify that $(AB)^{-1} = B^{-1} A^{-1}$

A = $\begin{bmatrix} 3 & 2 \\ 7 & 5 \end{bmatrix}$ and B = $\begin{bmatrix} -1 & -3 \\ 5 & 2 \end{bmatrix}$

$AB = \begin{bmatrix} 3 & 2 \\ 7 & 5 \end{bmatrix} \begin{bmatrix} -1 & -3 \\ 5 & 2 \end{bmatrix}$

$AB = \begin{bmatrix} -3+10 & -9+4 \\ -7+25 & -21+10 \end{bmatrix}$

$AB = \begin{bmatrix} 7 & -5 \\ 18 & -11 \end{bmatrix}$

$|AB| = \begin{vmatrix} 7 & -5 \\ 18 & -11 \end{vmatrix}$

= – 77 + 90 = 13

$adj\ (AB) = \begin{bmatrix} -11 & 5 \\ -18 & 7 \end{bmatrix}$

$(AB)^{-1} = \frac{1}{|AB|}\ Adj\ (AB)$

$= \frac{1}{13} \begin{bmatrix} -11 & 5 \\ -18 & 7 \end{bmatrix}$ ………. (1)

A = $\begin{bmatrix} 3 & 2 \\ 7 & 5 \end{bmatrix}$

$|A| = \begin{vmatrix} 3 & 2 \\ 7 & 5 \end{vmatrix}$

= 15 – 14

= 1

adj A = $\begin{bmatrix} 5 & -2 \\ -7 & 3 \end{bmatrix}$

$A^{-1} = \frac{1}{|A|}\ Adj\ A$

$= \frac{1}{1} \begin{bmatrix} 5 & -2 \\ -7 & 3 \end{bmatrix}$

$A^{-1} = \begin{bmatrix} 5 & -2 \\ -7 & 3 \end{bmatrix}$

B = $\begin{bmatrix} -1 & -3 \\ 5 & 2 \end{bmatrix}$

$|B| = \begin{vmatrix} -1 & -3 \\ 5 & 2 \end{vmatrix}$

= – 2 +15

= 13

adj B = $\begin{bmatrix} 2 & 3 \\ -5 & -1 \end{bmatrix}$

$B^{-1} = \frac{1}{|B|}\ Adj\ B$

$B^{-1} = \frac{1}{13} \begin{bmatrix} 2 & 3 \\ -5 & -1 \end{bmatrix}$

$B^{-1} A^{-1} = \begin{bmatrix} 2 & 3 \\ -5 & -1 \end{bmatrix} \frac{1}{13} \begin{bmatrix} 5 & -2 \\ -7 & 3 \end{bmatrix}$

$= \frac{1}{13} \begin{bmatrix} 10-21 & -4+9 \\ -25+7 & 10-3 \end{bmatrix}$

$= \frac{1}{13} \begin{bmatrix} -11 & 5 \\ -18 & 7 \end{bmatrix}$ …… (2)

From (1) and (2)

$(AB)^{-1} = B^{-1} A^{-1}$ is verified.

8. If $adj\ A = \begin{bmatrix} 2 & -4 & 2 \\ -3 & 12 & -7 \\ -2 & 0 & 2 \end{bmatrix}$ Find A

Solution: $adj\ A = \begin{bmatrix} 2 & -4 & 2 \\ -3 & 12 & -7 \\ -2 & 0 & 2 \end{bmatrix}$

$$|adj\ A| = \begin{vmatrix} 2 & -4 & 2 \\ -3 & 12 & -7 \\ -2 & 0 & 2 \end{vmatrix}$$

$$= 2\begin{vmatrix} 12 & -7 \\ 0 & 2 \end{vmatrix} + 4\begin{vmatrix} -3 & -7 \\ -2 & 2 \end{vmatrix} + 2\begin{vmatrix} -3 & 12 \\ -2 & 0 \end{vmatrix}$$

$= 2(24 - 0) + 4(-6 - 14) + 2(0 + 24)$

$= 2(24) + 4(-20) + 2(24)$

$= 48 - 80 + 48 = 96 - 80$

$= 16 \Rightarrow \sqrt{adj\ A} = 4$

We know, $A = \pm\frac{1}{\sqrt{adj\ A}}\ adj(adj\ A)$

Given: $adj\ A = \begin{bmatrix} 2 & -4 & 2 \\ -3 & 12 & -7 \\ -2 & 0 & 2 \end{bmatrix}$

$$\begin{matrix} 12 & -7 & -3 & 12 \\ 0 & 2 & -2 & 0 \\ -4 & 2 & 2 & -4 \\ 12 & -7 & -3 & 12 \end{matrix}$$

Cofactor of adj A $= \begin{bmatrix} 24 & 20 & 24 \\ 8 & 8 & 8 \\ 4 & 8 & 12 \end{bmatrix}$

$adj(adj\ A) = \begin{bmatrix} 24 & 8 & 4 \\ 20 & 8 & 8 \\ 24 & 8 & 12 \end{bmatrix}$

$A = \pm\frac{1}{\sqrt{adj\ A}}\ adj(adj\ A)$

$= \pm\frac{1}{4}\begin{bmatrix} 24 & 8 & 4 \\ 20 & 8 & 8 \\ 24 & 8 & 12 \end{bmatrix}$

$A = \pm\frac{1}{4}(4)\begin{bmatrix} 6 & 2 & 1 \\ 5 & 2 & 2 \\ 6 & 2 & 3 \end{bmatrix}$

$A = \pm\begin{bmatrix} 6 & 2 & 1 \\ 5 & 2 & 2 \\ 6 & 2 & 3 \end{bmatrix}$

9. If $adj\ A = \begin{bmatrix} 0 & -2 & 0 \\ 6 & 2 & -6 \\ -3 & 0 & 6 \end{bmatrix}$ find A^{-1}

Solution: $adj\ A = \begin{bmatrix} 0 & -2 & 0 \\ 6 & 2 & -6 \\ -3 & 0 & 6 \end{bmatrix}$

$$|adj\ A| = \begin{vmatrix} 0 & -2 & 0 \\ 6 & 2 & -6 \\ -3 & 0 & 6 \end{vmatrix}$$

$$= 0\begin{vmatrix} 2 & -6 \\ 0 & 6 \end{vmatrix} + 2\begin{vmatrix} 6 & -6 \\ -3 & 6 \end{vmatrix} + 0\begin{vmatrix} 6 & 2 \\ -3 & 0 \end{vmatrix}$$

$= 0 + 2(36 - 18) + 0$

$= 2(36 - 18)$

$= 2(18)$

$= 36$

$\sqrt{adj\ A} = 6$

We know, $A^{-1} = \pm\frac{1}{\sqrt{adj\ A}}(adj\ A)$

Hence, $A^{-1} = \pm\frac{1}{6}\begin{bmatrix} 0 & -2 & 0 \\ 6 & 2 & -6 \\ -3 & 0 & 6 \end{bmatrix}$

10. Find $adj(adj\ A)$ if $adj\ A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ -1 & 0 & 1 \end{bmatrix}$

Given: $adj\ A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ -1 & 0 & 1 \end{bmatrix}$

$$\begin{matrix} 2 & 0 & 0 & 2 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 2 & 0 & 0 & 2 \end{matrix}$$

Cofactor of adj A $= \begin{bmatrix} 2 & 0 & 2 \\ 0 & 2 & 0 \\ 2 & 0 & 2 \end{bmatrix}$

$adj(adj\ A) = (\text{Cofactor of adj } A)^T$

$$adj(adj\ A)=\begin{bmatrix}2 & 0 & 2\\ 0 & 2 & 0\\ 2 & 0 & 2\end{bmatrix}$$

11. If A $=\begin{bmatrix}1 & \tan x\\ -\tan x & 1\end{bmatrix}$ show that

$$A^T A^{-1}=\begin{bmatrix}\cos 2x & -\sin 2x\\ \sin 2x & \cos 2x\end{bmatrix}$$

Solution: A $=\begin{bmatrix}1 & \tan x\\ -\tan x & 1\end{bmatrix}$

$$|A|=\begin{vmatrix}1 & \tan x\\ -\tan x & 1\end{vmatrix}$$

$$=1+tan^2x$$

$$=sec^2x$$

$$\text{adj } A=\begin{bmatrix}1 & -\tan x\\ \tan x & 1\end{bmatrix}$$

We know, $A^{-1}=\frac{1}{|A|}\ Adj\ A$

$$=\frac{1}{sec^2x}\begin{bmatrix}1 & -\tan x\\ \tan x & 1\end{bmatrix}$$

$$=cos^2x\begin{bmatrix}1 & -\tan x\\ \tan x & 1\end{bmatrix}$$

$$=cos^2x\begin{bmatrix}1 & -\frac{\sin x}{\cos x}\\ \frac{\sin x}{\cos x} & 1\end{bmatrix}$$

$$A^{-1}=\begin{bmatrix}cos^2x & -\cos x\sin x\\ \cos x\sin x & cos^2x\end{bmatrix}$$

$$A=\begin{bmatrix}1 & \tan x\\ -\tan x & 1\end{bmatrix}$$

$$A^T=\begin{bmatrix}1 & -\frac{\sin x}{\cos x}\\ \frac{\sin x}{\cos x} & 1\end{bmatrix}$$

A^TA^{-1}

$$=\begin{bmatrix}1 & -\frac{\sin x}{\cos x}\\ \frac{\sin x}{\cos x} & 1\end{bmatrix}\begin{bmatrix}cos^2x & -\cos x\sin x\\ \cos x\sin x & cos^2x\end{bmatrix}$$

$$=\begin{bmatrix}cos^2x-sin^2x & -2\sin x\cos x\\ 2\sin x\cos x & cos^2x-sin^2x\end{bmatrix}$$

$$=\begin{bmatrix}\cos 2x & -\sin 2x\\ \sin 2x & \cos 2x\end{bmatrix}$$

Since $cos^2x-sin^2x=\cos 2x$ and

$2\sin x\cos x=\sin 2x$

12. Find the matrix A for which

$$A\begin{bmatrix}5 & 3\\ -1 & -2\end{bmatrix}=\begin{bmatrix}14 & 7\\ 7 & 7\end{bmatrix}$$

Solution: $A\begin{bmatrix}5 & 3\\ -1 & -2\end{bmatrix}=\begin{bmatrix}14 & 7\\ 7 & 7\end{bmatrix}$

Let A$=\begin{bmatrix}a & b\\ c & d\end{bmatrix}$

$AB=\begin{bmatrix}14 & 7\\ 7 & 7\end{bmatrix}$ gives

$$\begin{bmatrix}a & b\\ c & d\end{bmatrix}\begin{bmatrix}5 & 3\\ -1 & -2\end{bmatrix}=\begin{bmatrix}14 & 7\\ 7 & 7\end{bmatrix}$$

$$\begin{bmatrix}5a-b & 3a-2b\\ 5c-d & 3c-2d\end{bmatrix}=\begin{bmatrix}14 & 7\\ 7 & 7\end{bmatrix}$$

$$5a-b=14\ \ldots\ldots(1)$$

$$3a-2b=7$$

$$5c-d=7$$

$$3c-2d=7$$

Solving $(1)\times 2$

$$10a-2b=28$$

$$3a-2b=7$$

$$7a=21\Longrightarrow a=3$$

Substituting $a=3\ in\ (1)$

$$5a-b=14$$

$$5(3)-b=14$$

$$15-b=14$$

$-b = 14 - 15$

$-b = -1 \Rightarrow b = 1$

Solving (3) × 2

$10c - 2d = 14$

$3c - 2d = 7$

$7c = 7 \Rightarrow c = 1$

Substituting $c = 1$ *in* (3)

$5c - d = 7$

$5(1) - d = 7$

$5 - d = 7$

$-d = 7 - 5$

$-d = 2 \Rightarrow d = -2$

Hence A = $\begin{bmatrix} 3 & 1 \\ 1 & -2 \end{bmatrix}$

13. Given A=$\begin{bmatrix} 1 & -1 \\ 2 & 0 \end{bmatrix}$, B=$\begin{bmatrix} 3 & -2 \\ 1 & 1 \end{bmatrix}$and C=$\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$ find a matrix X such that AXB=C

Solution: AXB=C

Let X=$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then

$$\begin{bmatrix} 1 & -1 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 3 & -2 \\ 1 & 1 \end{bmatrix}=\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$$

$$\begin{bmatrix} a-c & b-d \\ 2a & 2b \end{bmatrix}\begin{bmatrix} 3 & -2 \\ 1 & 1 \end{bmatrix}=\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 3(a-c)+1(b-d) & -2(a-c)+1(b-d) \\ 3(2a)+1(2b) & -2(2a)+1(2b) \end{bmatrix}=\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 3a-3c+b-d & -2a+2c+b-d \\ 6a+2b & -4a+2b \end{bmatrix}=\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$$

$6a + 2b = 2$, and $-4a + 2b = 2$

$3a - 3c + b - d = 1$ and

$-2a + 2c + b - d = 1$

Solving $6a + 2b = 2$

$-4a + 2b = 2$

$10a = 0$ gives $a = 0$

Substituting $a = 0$ in

$-4a + 2b = 2$

$2b = 2$ gives $b = 1$

Substituting $a = 0$ and $b = 1$ in

$3a - 3c + b - d = 1$

$0 - 3c + 1 - d = 1$

$-3c + 1 - d = 1$

$-3c - d = 1 - 1$

$-3c - d = 0$ and

$-2a + 2c + b - d = 1$

$0 + 2c + 1 - d = 1$

$2c + 1 - d = 1$

$2c - d = 1 - 1$

$2c - d = 0$

Solving $-3c - d = 0$ and

$2c - d = 0$ we get

$c = 0$

Substituting $c = 0$ in

$2c - d = 0$

$d = 0$

Substituting $a = 0$, $b = 1$, $c = 0$ and $d = 0$

in X matrix

$$X=\begin{bmatrix} a & b \\ c & d \end{bmatrix} \text{ we get}$$

$$X=\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

14. If $A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$,

show that $A^{-1} = \frac{1}{2}(A^2 - 3I)$

Solution: Given $A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$

$$|A| = \begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix}$$

$$= 0\begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix} - 1\begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} + 1\begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix}$$

$= 0 - 1(0 - 1) + 1(1 - 0)$

$= -1(-1) + 1(1)$

$= 1 + 1$

$|A| = 2 \neq 0$, hence A^{-1} exists.

$$\begin{array}{cccc} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{array}$$

Cofactor of A $\text{Aij} = \begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$

Adj A = Aij^T

$$= \begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|} Adj\ A$$

$$A^{-1} = \frac{1}{2}\begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix} \text{ ...... (1)}$$

$$A^2 = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

$$A^2 - 3I = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} - 3\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} - \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 2-3 & 1-0 & 1-0 \\ 1-0 & 2-3 & 1-0 \\ 1-0 & 1-0 & 2-3 \end{bmatrix}$$

$$\frac{1}{2}(A^2 - 3I) = \frac{1}{2}\begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix} \text{ ...... (2)}$$

From (1)and (2) $A^{-1} = \frac{1}{2}(A^2 - 3I), proved.$

15. Decrypt the received encoded message $[2 \quad -3][20 \quad 4]$with the encryption matrix $\begin{bmatrix} -1 & -1 \\ 2 & 1 \end{bmatrix}$ and the decryption matrix as its inverse, where the system of codes are described by the numbers 1−26 to the letters A−Z respectively, and the number 0 to a blank space.

Solution: Given Encoding matrix

$$A = \begin{bmatrix} -1 & -1 \\ 2 & 1 \end{bmatrix}$$

$$|A| = \begin{vmatrix} -1 & -1 \\ 2 & 1 \end{vmatrix}$$

$= -1 + 2$

$= 1$

$$\text{adj } A = \begin{bmatrix} 1 & 1 \\ -2 & -1 \end{bmatrix}$$

$A^{-1} = \frac{1}{|A|}\, Adj\ A$

$= \frac{1}{1}\begin{bmatrix} 1 & 1 \\ -2 & -1 \end{bmatrix}$

$A^{-1} = \begin{bmatrix} 1 & 1 \\ -2 & -1 \end{bmatrix}$

Decoding matrix

$A^{-1} = \begin{bmatrix} 1 & 1 \\ -2 & -1 \end{bmatrix}$

Given encoded message is

$[2 \quad -3], [20 \quad 4]$

Coded row matrix	Decoding matrix	Decoded row matrix

$[2 \quad -3]\begin{bmatrix} 1 & 1 \\ -2 & -1 \end{bmatrix} = [2+6 \quad 2+3]$

$[20 \quad 4]\begin{bmatrix} 1 & 1 \\ -2 & -1 \end{bmatrix} = [20-8 \quad 20-4]$

So, the sequence of decoded row matrices is

$[8 \quad 5][12 \quad 16]$. Thus, the receiver reads the message as **"HELP"**.

Example 1.13 Reduce the matrix

$\begin{bmatrix} 3 & -1 & 2 \\ -6 & 2 & 4 \\ -3 & 1 & 2 \end{bmatrix}$ to a row – echelon form.

Let $A = \begin{bmatrix} 3 & -1 & 2 \\ -6 & 2 & 4 \\ -3 & 1 & 2 \end{bmatrix}$

$\rightarrow \begin{bmatrix} 3 & -1 & 2 \\ 0 & 0 & 8 \\ 0 & 0 & 4 \end{bmatrix} \begin{matrix} R_1 \\ R_2 = R_2 + 2R_1 \\ R_3 = R_3 + R_1 \end{matrix}$

$\rightarrow \begin{bmatrix} 3 & -1 & 2 \\ 0 & 0 & 8 \\ 0 & 0 & 4 \end{bmatrix} R_3 = 2R_3 - R_2$

$\rightarrow \begin{bmatrix} 3 & -1 & 2 \\ 0 & 0 & 8 \\ 0 & 0 & 0 \end{bmatrix}$

This is also a row-echelon form of the given matrix.

Example 1.14 Reduce the matrix

$\begin{bmatrix} 0 & 3 & 1 & 6 \\ -1 & 0 & 2 & 5 \\ 4 & 2 & 0 & 0 \end{bmatrix}$ to a row – echelon form.

Let $A = \begin{bmatrix} 0 & 3 & 1 & 6 \\ -1 & 0 & 2 & 5 \\ 4 & 2 & 0 & 0 \end{bmatrix}$

$\rightarrow \begin{bmatrix} 1 & 3 & 0 & 6 \\ 2 & 0 & -1 & 5 \\ 0 & 2 & 4 & 0 \end{bmatrix} C_1 \leftrightarrow C_3$

$\rightarrow \begin{bmatrix} 1 & 3 & 0 & 6 \\ 0 & 6 & 1 & 7 \\ 0 & 2 & 4 & 0 \end{bmatrix} R_2 = R_2 - 2R_1$

$\rightarrow \begin{bmatrix} 1 & 3 & 0 & 6 \\ 0 & 6 & 1 & 7 \\ 0 & 0 & 11 & -7 \end{bmatrix} R_3 = 3R_3 - R_2$

This is also a row-echelon form of the given matrix.

Example 1.15

Find the rank of each of the following matrices: (i) $\begin{bmatrix} 3 & 2 & 5 \\ 1 & 1 & 2 \\ 3 & 3 & 6 \end{bmatrix}$

Let $A = \begin{bmatrix} 3 & 2 & 5 \\ 1 & 1 & 2 \\ 3 & 3 & 6 \end{bmatrix}$

$\rightarrow \begin{bmatrix} 1 & 1 & 2 \\ 3 & 2 & 5 \\ 3 & 3 & 6 \end{bmatrix} \begin{matrix} R_1 \leftrightarrow R_2 \\ R_2 \\ R_3 \end{matrix}$

$\rightarrow \begin{bmatrix} 1 & 1 & 2 \\ 0 & -1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{matrix} R_1 \\ R_2 = R_2 - 3R_1 \\ R_3 = R_3 - 3R_1 \end{matrix}$

The Matrix is in Echelon form.

The number of non zero rows are 2.

$\therefore\ \rho(A) = 2$

(ii) $\begin{bmatrix} 4 & 3 & 1 & -2 \\ -3 & -1 & -2 & 4 \\ 6 & 7 & -1 & 2 \end{bmatrix}$

Let $A = \begin{bmatrix} 4 & 3 & 1 & -2 \\ -3 & -1 & -2 & 4 \\ 6 & 7 & -1 & 2 \end{bmatrix}$

$\rightarrow \begin{bmatrix} 1 & 3 & 4 & -2 \\ -2 & -1 & -3 & 4 \\ -1 & 7 & 6 & 2 \end{bmatrix} C_1 \leftrightarrow C_3$

$\rightarrow \begin{bmatrix} 1 & 3 & 4 & -2 \\ 0 & 5 & 5 & 0 \\ -1 & 7 & 6 & 2 \end{bmatrix} R_2 = R_2 + 2R_1$

$\rightarrow \begin{bmatrix} 1 & 3 & 4 & -2 \\ 0 & 5 & 5 & 0 \\ 0 & 10 & 10 & 0 \end{bmatrix} R_3 = R_3 + R_1$

$\rightarrow \begin{bmatrix} 1 & 3 & 4 & -2 \\ 0 & 5 & 5 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} R_3 = R_3 - 2R_2$

The Matrix is in Echelon form.

The number of non zero rows are 2.

$\therefore\ \rho(A) = 2$

Example 1.16 Find the rank of the following matrices which are in row-echelon form:

(i) $\begin{bmatrix} 2 & 0 & -7 \\ 0 & 3 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

The number of non zero rows are 3.

$\therefore\ \rho(A) = 3$

(ii) $\begin{bmatrix} -2 & 2 & -1 \\ 0 & 5 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

The number of non zero rows are 2.

$\therefore\ \rho(A) = 2$

(iii) $\begin{bmatrix} 6 & 0 & -9 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

The number of non zero rows are 2.

$\therefore\ \rho(A) = 2$

Example 1.17

Find the rank of the matrix $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 4 \\ 3 & 0 & 5 \end{bmatrix}$ by reducing it to a row-echelon form.

Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 4 \\ 3 & 0 & 5 \end{bmatrix}$

$\rightarrow \begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -2 \\ 0 & -6 & -4 \end{bmatrix} \begin{matrix} R_1 \\ R_2 = R_2 - 2R_1 \\ R_3 = R_3 - 3R_1 \end{matrix}$

$\rightarrow \begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -2 \\ 0 & 0 & 0 \end{bmatrix} \begin{matrix} R_1 \\ R_2 \\ R_3 = R_3 - 2R_2 \end{matrix}$

The Matrix is in Echelon form.

The number of non zero rows are 2.

$\therefore\ \rho(A) = 2$

Example 1.18 Find the rank of the matrix $\begin{bmatrix} 2 & -2 & 4 & 3 \\ -3 & 4 & -2 & -1 \\ 6 & 2 & -1 & 7 \end{bmatrix}$ by reducing it to a row-echelon form.

Let $A = \begin{bmatrix} 2 & -2 & 4 & 3 \\ -3 & 4 & -2 & -1 \\ 6 & 2 & -1 & 7 \end{bmatrix}$

$\rightarrow \begin{bmatrix} 2 & -2 & 4 & 3 \\ 0 & -2 & 8 & 7 \\ 6 & 2 & -1 & 7 \end{bmatrix} R_2 = 2R_2 + 3R_1$

$\rightarrow \begin{bmatrix} 2 & -2 & 4 & 3 \\ 0 & -2 & 8 & 7 \\ 0 & 8 & -13 & -2 \end{bmatrix} R_3 = R_3 - 3R_1$

$\rightarrow \begin{bmatrix} 2 & -2 & 4 & 3 \\ 0 & -2 & 8 & 7 \\ 0 & 0 & 19 & 26 \end{bmatrix} R_3 = R_3 + 4R_2$

The no. of non zero rows are 3. $\therefore\ \rho(A) = 3$

Example 1.19 Show that the matrix

$\begin{bmatrix}3&1&4\\2&0&-1\\5&2&1\end{bmatrix}$ is non-singular and reduce it to the identity matrix by elementary row transformations.

Let $A=\begin{bmatrix}3&1&4\\2&0&-1\\5&2&1\end{bmatrix}$

$$\rightarrow\begin{bmatrix}1&\frac{1}{3}&\frac{4}{3}\\2&0&-1\\5&2&1\end{bmatrix}\begin{matrix}R_1=R_1\div 3\\R_2\\R_3\end{matrix}$$

$$\rightarrow\begin{bmatrix}1&\frac{1}{3}&\frac{4}{3}\\0&-\frac{2}{3}&-\frac{11}{3}\\0&\frac{1}{3}&-\frac{17}{3}\end{bmatrix}\begin{matrix}R_1\\R_2=R_2-2R_1\\R_3=R_3-5R_1\end{matrix}$$

$$\rightarrow\begin{bmatrix}1&\frac{1}{3}&\frac{4}{3}\\0&1&\frac{11}{2}\\0&\frac{1}{3}&-\frac{17}{3}\end{bmatrix}\begin{matrix}R_1\\R_2=R_2\left(-\frac{3}{2}\right)\\R_3\end{matrix}$$

$$\rightarrow\begin{bmatrix}1&0&-\frac{1}{2}\\0&1&\frac{11}{2}\\0&0&-\frac{15}{2}\end{bmatrix}\begin{matrix}R_1=R_1-\frac{1}{3}R_2\\R_2\\R_3=R_3-\frac{1}{3}R_2\end{matrix}$$

$$\rightarrow\begin{bmatrix}1&0&-\frac{1}{2}\\0&1&\frac{11}{2}\\0&0&-\frac{15}{2}\end{bmatrix}\begin{matrix}R_1\\R_2\\R_3=R_3\left(-\frac{2}{15}\right)\end{matrix}$$

$$\rightarrow\begin{bmatrix}1&0&0\\0&1&0\\0&0&0\end{bmatrix}\begin{matrix}R_1=R_1+\frac{1}{2}R_3\\R_2=R_2-\frac{11}{2}R_3\\R_3\end{matrix}$$

Example 1.20 Find the inverse of the non-singular matrix $A=\begin{bmatrix}0&5\\-1&6\end{bmatrix}$, by Gauss-Jordan method.

Solution: $(A|I_2)=\left(\begin{array}{cc|cc}0&5&1&0\\-1&6&0&1\end{array}\right)$

$$\rightarrow\left(\begin{array}{cc|cc}-1&6&0&1\\0&5&1&0\end{array}\right)R_1\leftrightarrow R_2$$

$$\rightarrow\left(\begin{array}{cc|cc}1&-6&0&-1\\0&5&1&0\end{array}\right)\begin{matrix}R_1=R_1(-1)\\R_2\end{matrix}$$

$$\rightarrow\left(\begin{array}{cc|cc}1&-6&0&-1\\0&1&\frac{1}{5}&0\end{array}\right)\begin{matrix}R_1\\R_2=\frac{1}{5}R_2\end{matrix}$$

$$\rightarrow\left(\begin{array}{cc|cc}1&0&\frac{6}{5}&-1\\0&1&\frac{1}{5}&0\end{array}\right)\begin{matrix}R_1=R_1+6R_2\\R_2\end{matrix}$$

Hence $A^{-1}=\begin{bmatrix}\frac{6}{5}&-1\\\frac{1}{5}&0\end{bmatrix}$

$$A^{-1}=\frac{1}{5}\begin{bmatrix}6&-1\\1&0\end{bmatrix}$$

Example 1.21 Find the inverse of

$A=\begin{bmatrix}2&1&1\\3&2&1\\2&1&2\end{bmatrix}$ by Gauss-Jordan method.

Solution: $(A|I_3)=\left(\begin{array}{ccc|ccc}2&1&1&1&0&0\\3&2&1&0&1&0\\2&1&2&0&0&1\end{array}\right)$

$$\rightarrow\left(\begin{array}{ccc|ccc}1&\frac{1}{2}&\frac{1}{2}&\frac{1}{2}&0&0\\3&2&1&0&1&0\\2&1&2&0&0&1\end{array}\right)R_1=R_1\div 2$$

$$\rightarrow\left(\begin{array}{ccc|ccc}1&\frac{1}{2}&\frac{1}{2}&\frac{1}{2}&0&0\\0&\frac{1}{2}&-\frac{1}{2}&-\frac{3}{2}&1&0\\0&0&1&-1&0&1\end{array}\right)\begin{matrix}R_1\\R_2=R_2-3R_1\\R_3=R_3-2R_1\end{matrix}$$

$$\rightarrow\left(\begin{array}{ccc|ccc}1&\frac{1}{2}&\frac{1}{2}&\frac{1}{2}&0&0\\0&1&-1&-3&2&0\\0&0&1&-1&0&1\end{array}\right)R_2=2R_2$$

$$\rightarrow\left(\begin{array}{ccc|ccc}1&0&1&2&-1&0\\0&1&-1&-3&2&0\\0&0&1&-1&0&1\end{array}\right)R_1=R_1-\frac{1}{2}R_2$$

$$\rightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & -1 & 1 \\ 0 & 1 & 0 & -4 & 2 & 1 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right) \begin{array}{l} R_1 = R_1 - R_3 \\ R_2 = R_2 + R_3 \end{array}$$

Hence $A^{-1} = \begin{bmatrix} 3 & -1 & 1 \\ -4 & 2 & 1 \\ -1 & 0 & 1 \end{bmatrix}$

EXERCISE 1.2

1. Find the rank of the following matrices by minor method:

(i) $\begin{bmatrix} 2 & -4 \\ -1 & 2 \end{bmatrix}$

Let A = $\begin{bmatrix} 2 & -4 \\ -1 & 2 \end{bmatrix}$

$|A| = \begin{vmatrix} 2 & -4 \\ -1 & 2 \end{vmatrix}$

$= 4 - 4$

$= 0$

$\therefore\ \rho(A) = 1$

(ii) $\begin{bmatrix} -1 & 3 \\ 4 & -7 \\ 3 & -4 \end{bmatrix}$

Let A = $\begin{bmatrix} -1 & 3 \\ 4 & -7 \\ 3 & -4 \end{bmatrix}$

A is a matrix order of 3×2,

$\therefore\ \rho(A) \le 2$

We find that there is a second order minor,

$\begin{vmatrix} -1 & 3 \\ 4 & -7 \end{vmatrix}$

$= 7 - 12$

$= -5 \neq 0$

$\therefore\ \rho(A) = 2$

(iii) $\begin{bmatrix} 1 & -2 & -1 & 0 \\ 3 & -6 & -3 & 1 \end{bmatrix}$

Let A = $\begin{bmatrix} 1 & -2 & -1 & 0 \\ 3 & -6 & -3 & 1 \end{bmatrix}$

A is a matrix order of 2×4,

$\therefore\ \rho(A) \le 2$

We find that there is a second order minor,

$\begin{vmatrix} -1 & 0 \\ -3 & 1 \end{vmatrix}$

$= -1 + 0$

$= -1 \neq 0$

$\therefore\ \rho(A) = 2$

(iv) $\begin{bmatrix} 1 & -2 & 3 \\ 2 & 4 & -6 \\ 5 & 1 & -1 \end{bmatrix}$

Let A = $\begin{bmatrix} 1 & -2 & 3 \\ 2 & 4 & -6 \\ 5 & 1 & -1 \end{bmatrix}$

A is a matrix order of 3×3,

$\therefore\ \rho(A) \le 3$

$|A| = \begin{vmatrix} 1 & -2 & 3 \\ 2 & 4 & -6 \\ 5 & 1 & -1 \end{vmatrix}$

$= 1\begin{vmatrix} 4 & -6 \\ 1 & -1 \end{vmatrix} + 2\begin{vmatrix} 2 & -6 \\ 5 & -1 \end{vmatrix} + 3\begin{vmatrix} 2 & 4 \\ 5 & 1 \end{vmatrix}$

$= 1(-4+6) + 2(-2+30) + 3(2-20)$

$= 1(2) + 2(28) + 3(-18)$

$= 2 + 56 - 54$

$= 58 - 54$

$= 4$

$|A| = 4 \neq 0$,

$\therefore\ \rho(A) = 3$

(v) $\begin{bmatrix} 0 & 1 & 2 & 1 \\ 0 & 2 & 4 & 3 \\ 8 & 1 & 0 & 2 \end{bmatrix}$

Let A = $\begin{bmatrix} 0 & 1 & 2 & 1 \\ 0 & 2 & 4 & 3 \\ 8 & 1 & 0 & 2 \end{bmatrix}$

A is a matrix order of 3× 4,

$\therefore\ \rho(A) \leq 3$

We find that there is a third order minor,

$$\begin{vmatrix} 1 & 2 & 1 \\ 2 & 4 & 3 \\ 1 & 0 & 2 \end{vmatrix}$$

$= 1\begin{vmatrix} 4 & 3 \\ 0 & 2 \end{vmatrix} - 2\begin{vmatrix} 2 & 3 \\ 1 & 2 \end{vmatrix} + 1\begin{vmatrix} 2 & 4 \\ 1 & 0 \end{vmatrix}$

$= 1(8 - 0) - 2(4 - 3) + 1(0 - 4)$

$= 1(8) - 2(1) + 1(-4)$

$= 8 - 2 - 4$

$= 8 - 6$

$= 2 \neq 0,$

$\therefore\ \rho(A) = 3$

2. Find the rank of the following matrices by row reduction method:

(i) $\begin{bmatrix} 1 & 1 & 1 & 3 \\ 2 & -1 & 3 & 4 \\ 5 & -1 & 7 & 11 \end{bmatrix}$

Let A = $\begin{bmatrix} 1 & 1 & 1 & 3 \\ 2 & -1 & 3 & 4 \\ 5 & -1 & 7 & 11 \end{bmatrix}$

$\rightarrow \begin{bmatrix} 1 & 1 & 1 & 3 \\ 0 & -3 & 1 & -2 \\ 0 & -6 & 2 & -4 \end{bmatrix} \begin{array}{l} R_1 \\ R_2 = R_2 - 2R_1 \\ R_3 = R_3 - 5R_1 \end{array}$

$\rightarrow \begin{bmatrix} 1 & 1 & 1 & 3 \\ 0 & -3 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{array}{l} R_1 \\ R_2 \\ R_3 = R_3 - 2R_2 \end{array}$

The Matrix is in Echelon form.

The number of non zero rows are 2.

$\therefore\ \rho(A) = 2$

(ii) $\begin{bmatrix} 1 & 2 & -1 \\ 3 & -1 & 2 \\ 1 & -2 & 3 \\ 1 & -1 & 1 \end{bmatrix}$

Let A = $\begin{bmatrix} 1 & 2 & -1 \\ 3 & -1 & 2 \\ 1 & -2 & 3 \\ 1 & -1 & 1 \end{bmatrix}$

$\rightarrow \begin{bmatrix} 1 & 2 & -1 \\ 0 & -7 & 5 \\ 0 & 0 & 8 \\ 0 & 0 & 1 \end{bmatrix} \begin{array}{l} R_1 \\ R_2 \\ R_3 = 7R_3 - 4R_2 \\ R_4 = 7R_4 - 3R_2 \end{array}$

$\rightarrow \begin{bmatrix} 1 & 2 & -1 \\ 0 & -7 & 5 \\ 0 & 0 & 8 \\ 0 & 0 & 0 \end{bmatrix} \begin{array}{l} R_1 \\ R_2 \\ R_3 \\ R_4 = 8R_4 - R_3 \end{array}$

The Matrix is in Echelon form.

The number of non zero rows are 3.

$\therefore\ \rho(A) = 3$

(iii) $\begin{bmatrix} 3 & -8 & 5 & 2 \\ 2 & -5 & 1 & 4 \\ -1 & 2 & 3 & -2 \end{bmatrix}$

Let A = $\begin{bmatrix} 3 & -8 & 5 & 2 \\ 2 & -5 & 1 & 4 \\ -1 & 2 & 3 & -2 \end{bmatrix}$

$\rightarrow \begin{bmatrix} 3 & -8 & 5 & 2 \\ 0 & 1 & -7 & 8 \\ -1 & 2 & 3 & -2 \end{bmatrix} R_2 = 3R_2 - 2R_1$

$\rightarrow \begin{bmatrix} 3 & -8 & 5 & 2 \\ 0 & 1 & -7 & 8 \\ 0 & -2 & 14 & -4 \end{bmatrix} R_3 = 3R_3 + R_1$

$\rightarrow \begin{bmatrix} 3 & -8 & 5 & 2 \\ 0 & 1 & -7 & 8 \\ 0 & 0 & 0 & 12 \end{bmatrix} R_3 = R_3 + 2R_2$

The Matrix is in Echelon form, since the number of non zero rows are 3, $\rho(A) = 3$

3. Find the inverse of each of the following by Gauss-Jordan method:

(i) $\begin{bmatrix} 2 & -1 \\ 5 & -2 \end{bmatrix}$

Let A = $\begin{bmatrix} 2 & -1 \\ 5 & -2 \end{bmatrix}$

Solution: $(A|I_2) = \left(\begin{array}{cc|cc} 2 & -1 & 1 & 0 \\ 5 & -2 & 0 & 1 \end{array}\right)$

$$\rightarrow \left(\begin{array}{cc|cc} 1 & -\frac{1}{2} & \frac{1}{2} & 0 \\ 5 & -2 & 0 & 1 \end{array}\right) \begin{array}{l} R_1 = R_1 \div 2 \\ R_2 \end{array}$$

$$\rightarrow \left(\begin{array}{cc|cc} 1 & -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & -\frac{5}{2} & 1 \end{array}\right) \begin{array}{l} R_1 \\ R_2 = R_2 - 5R_1 \end{array}$$

$$\rightarrow \left(\begin{array}{cc|cc} 1 & 0 & -2 & 1 \\ 0 & \frac{1}{2} & -\frac{5}{2} & 1 \end{array}\right) \begin{array}{l} R_1 = R_1 + R_2 \\ R_2 \end{array}$$

$$\rightarrow \left(\begin{array}{cc|cc} 1 & 0 & -2 & 1 \\ 0 & 1 & -5 & 2 \end{array}\right) \begin{array}{l} R_1 \\ R_2 = R_2 \times 2 \end{array}$$

Hence $A^{-1} = \begin{bmatrix} -2 & 1 \\ -5 & 2 \end{bmatrix}$

(ii) $\begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ 6 & -2 & -3 \end{bmatrix}$

Solution: $(A|I_3) = \left(\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 & 1 & 0 \\ 6 & -2 & -3 & 0 & 0 & 1 \end{array}\right)$

$$\rightarrow \left(\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 4 & -3 & -6 & 0 & 1 \end{array}\right) \begin{array}{l} R_2 = R_2 - R_1 \\ R_3 = R_3 - 4R_1 \end{array}$$

$$\rightarrow \left(\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & 1 & -2 & -4 & 1 \end{array}\right) \begin{array}{l} R_2 \\ R_3 = R_3 - 4R_2 \end{array}$$

$$\rightarrow \left(\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -3 & -3 & 1 \\ 0 & 0 & 1 & -2 & -4 & 1 \end{array}\right) \begin{array}{l} R_2 = R_2 + R_1 \\ R_3 \end{array}$$

$$\rightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & -3 & 1 \\ 0 & 1 & 0 & -3 & -3 & 1 \\ 0 & 0 & 1 & -2 & -4 & 1 \end{array}\right) R_1 = R_1 + R_2$$

Hence $A^{-1} = \begin{bmatrix} -2 & -3 & 1 \\ -3 & -3 & 1 \\ -2 & -4 & 1 \end{bmatrix}$

(iii) $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{bmatrix}$

Solution: $(A|I_3) = \left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 5 & 3 & 0 & 1 & 0 \\ 1 & 0 & 8 & 0 & 0 & 1 \end{array}\right)$

$$\rightarrow \left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & -2 & 5 & -1 & 0 & 1 \end{array}\right) \begin{array}{l} R_2 = R_2 - 2R_1 \\ R_3 = R_3 - R_1 \end{array}$$

$$\rightarrow \left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & 0 & -1 & -5 & 2 & 1 \end{array}\right) R_3 = R_3 + 2R_2$$

$$\rightarrow \left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array}\right) R_3 = R_3(-1)$$

$$\rightarrow \left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 0 & 13 & -5 & -3 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array}\right) R_2 = R_2 + 3R_3$$

$$\rightarrow \left(\begin{array}{ccc|ccc} 1 & 2 & 0 & -14 & 6 & 3 \\ 0 & 1 & 0 & 13 & -5 & -3 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array}\right) R_1 = R_1 - 3R_3$$

$$\rightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -40 & 16 & 9 \\ 0 & 1 & 0 & 13 & -5 & -3 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array}\right) R_1 = R_1 - 2R_2$$

Hence $A^{-1} = \begin{bmatrix} -40 & 16 & 9 \\ 13 & -5 & -3 \\ 5 & -2 & -1 \end{bmatrix}$

Example 1.22 Solve the following system of linear equations, using matrix inversion method: $5x + 2y = 3, 3x + 2y = 5$.

Solution: $5x + 2y = 3, 3x + 2y = 5$.

A = $\begin{bmatrix} 5 & 2 \\ 3 & 2 \end{bmatrix}$ X = $\begin{bmatrix} x \\ y \end{bmatrix}$ B = $\begin{bmatrix} 3 \\ 5 \end{bmatrix}$

$|A| = \begin{vmatrix} 5 & 2 \\ 3 & 2 \end{vmatrix}$

= 10 – 6 = 4

$$\text{adj A} = \begin{bmatrix} 2 & -2 \\ -3 & 5 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|} \, Adj\ A$$

$$= \frac{1}{4} \begin{bmatrix} 2 & -2 \\ -3 & 5 \end{bmatrix}$$

$$A^{-1} = \frac{1}{4} \begin{bmatrix} 2 & -2 \\ -3 & 5 \end{bmatrix}$$

$$X = A^{-1} \times B$$

$$= \frac{1}{4} \begin{bmatrix} 2 & -2 \\ -3 & 5 \end{bmatrix} \begin{bmatrix} 3 \\ 5 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 6-10 \\ -9+25 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{4} \begin{bmatrix} -4 \\ 16 \end{bmatrix}$$

$(x, y) = (-1,4)$

Example 1.23 Solve the following system of equations, using matrix inversion method:

$2x_1 + 3x_2 + 3x_3 = 5, x_1 - 2x_2 + x_3 = -4$

and $3x_1 - x_2 - 2x_3 = 3$

Solution: $2x_1 + 3x_2 + 3x_3 = 5$

$$x_1 - 2x_2 + x_3 = -4$$

$$3x_1 - x_2 - 2x_3 = 3$$

$$A = \begin{bmatrix} 2 & 3 & 3 \\ 1 & -2 & 1 \\ 3 & -1 & -2 \end{bmatrix} \quad X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad B = \begin{bmatrix} 5 \\ -4 \\ 3 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 2 & 3 & 3 \\ 1 & -2 & 1 \\ 3 & -1 & -2 \end{vmatrix}$$

$$= 2\begin{vmatrix} -2 & 1 \\ -1 & -2 \end{vmatrix} - 3\begin{vmatrix} 1 & 1 \\ 3 & -2 \end{vmatrix} + 3\begin{vmatrix} 1 & -2 \\ 3 & -1 \end{vmatrix}$$

$$= 2(4+1) - 3(-2-3) + 3(-1+6)$$

$$= 2(5) - 3(-5) + 3(5)$$

$$= 10 + 15 + 15 = 40$$

$$A = \begin{bmatrix} 2 & 3 & 3 \\ 1 & -2 & 1 \\ 3 & -1 & -2 \end{bmatrix} \qquad \begin{array}{cccc} -2 & 1 & 1 & -2 \\ -1 & -2 & 3 & -1 \\ 3 & 3 & 2 & 3 \\ -2 & 1 & 1 & -2 \end{array}$$

$$\text{Cofactor of A} \quad \text{Aij} = \begin{bmatrix} 5 & 5 & 5 \\ 3 & -13 & 11 \\ 9 & 1 & -7 \end{bmatrix}$$

$$\text{Adj A} = \text{Aij}^T$$

$$\text{Adj A} = \begin{bmatrix} 5 & 3 & 9 \\ 5 & -13 & 1 \\ 5 & 11 & -7 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|} \, Adj\ A$$

$$= \frac{1}{40} \begin{bmatrix} 5 & 3 & 9 \\ 5 & -13 & 1 \\ 5 & 11 & -7 \end{bmatrix}$$

$$X = A^{-1} \times B$$

$$= \frac{1}{40} \begin{bmatrix} 5 & 3 & 9 \\ 5 & -13 & 1 \\ 5 & 11 & -7 \end{bmatrix} \begin{bmatrix} 5 \\ -4 \\ 3 \end{bmatrix}$$

$$= \frac{1}{40} \begin{bmatrix} 25-12+27 \\ 25+52+3 \\ 25-44-21 \end{bmatrix}$$

$$= \frac{1}{40} \begin{bmatrix} 52-12 \\ 80 \\ 25-65 \end{bmatrix}$$

$$= \frac{1}{40} \begin{bmatrix} 40 \\ 80 \\ -40 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

$(x_1, x_2, x_3) = (1, 2, -1)$

Example 1.24

If $A = \begin{bmatrix} -4 & 4 & 4 \\ -7 & 1 & 3 \\ 5 & -3 & -1 \end{bmatrix}$ and

$B = \begin{bmatrix} 1 & -1 & 1 \\ 1 & -2 & -2 \\ 2 & 1 & 3 \end{bmatrix}$, find the products

AB and BA and hence solve the system of equations $x - y + z = 4, x - 2y - 2z = 9$ and $2x + y + 3z = 1$.

Solution:

$$A = \begin{bmatrix} -4 & 4 & 4 \\ -7 & 1 & 3 \\ 5 & -3 & -1 \end{bmatrix}, B = \begin{bmatrix} 1 & -1 & 1 \\ 1 & -2 & -2 \\ 2 & 1 & 3 \end{bmatrix}$$

$$AB = \begin{bmatrix} -4 & 4 & 4 \\ -7 & 1 & 3 \\ 5 & -3 & -1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 1 \\ 1 & -2 & -2 \\ 2 & 1 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} -4+4+8 & 4-8+4 & -4-8+12 \\ -7+1+6 & 7-2+3 & -7-2+9 \\ 5-3-2 & -5+6-1 & 5+6-3 \end{bmatrix}$$

$$= \begin{bmatrix} 12-4 & 8-8 & 12-12 \\ -7+7 & 10-2 & -9+9 \\ 5-5 & -6+6 & 11-3 \end{bmatrix}$$

$$AB = \begin{bmatrix} 8 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 8 \end{bmatrix}$$

$$\text{and } BA = \begin{bmatrix} 1 & -1 & 1 \\ 1 & -2 & -2 \\ 2 & 1 & 3 \end{bmatrix} \begin{bmatrix} -4 & 4 & 4 \\ -7 & 1 & 3 \\ 5 & -3 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} -4+7+5 & 4-1-3 & 4-3-1 \\ -4+14-10 & 4-2+6 & 4-6+2 \\ -8-7+15 & 8+1-9 & 8+3-3 \end{bmatrix}$$

$$= \begin{bmatrix} 12-4 & 4-4 & 4-4 \\ -14+14 & 6-6 & 6-6 \\ 15-15 & 9-9 & 11-3 \end{bmatrix}$$

$$BA = \begin{bmatrix} 8 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 8 \end{bmatrix}$$

So, we get $AB = BA = 8I_3$.

That is $\left(\frac{1}{8}A\right)B = B\left(\frac{1}{8}A\right) = I_3$.

$\therefore B^{-1} = \frac{1}{8}(A)$

Writing the given system of equations in matrix form, we get

$$\begin{bmatrix} 1 & -1 & 1 \\ 1 & -2 & -2 \\ 2 & 1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 9 \\ 1 \end{bmatrix}$$

$$\text{So } \begin{bmatrix} x \\ y \\ z \end{bmatrix} = B^{-1} \begin{bmatrix} 4 \\ 9 \\ 1 \end{bmatrix}$$

$$= \frac{1}{8}(A) \begin{bmatrix} 4 \\ 9 \\ 1 \end{bmatrix}$$

$$= \frac{1}{8} \begin{bmatrix} -4 & 4 & 4 \\ -7 & 1 & 3 \\ 5 & -3 & -1 \end{bmatrix} \begin{bmatrix} 4 \\ 9 \\ 1 \end{bmatrix}$$

$$= \frac{1}{8} \begin{bmatrix} -16+36+4 \\ -28+9+3 \\ 20-27-1 \end{bmatrix}$$

$$= \frac{1}{8} \begin{bmatrix} -16+40 \\ -28+12 \\ 20-28 \end{bmatrix}$$

$$= \frac{1}{8} \begin{bmatrix} 24 \\ -16 \\ -8 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \\ -1 \end{bmatrix}$$

EXERCISE 1.3

1. Solve the following system of linear equations by matrix inversion method:

(i) $2x + 5y = -2, \quad x + 2y = -3$.

Solution: $2x + 5y = -2, x + 2y = -3$.

$A = \begin{bmatrix} 2 & 5 \\ 1 & 2 \end{bmatrix}$ $X = \begin{bmatrix} x \\ y \end{bmatrix}$ $B = \begin{bmatrix} -2 \\ -3 \end{bmatrix}$

$$|A| = \begin{vmatrix} 2 & 5 \\ 1 & 2 \end{vmatrix}$$

$= 4 - 5$

$= -1$

adj A $= \begin{bmatrix} 2 & -5 \\ -1 & 2 \end{bmatrix}$

$A^{-1} = \frac{1}{|A|}\, Adj\ A$

$= \frac{1}{-1} \begin{bmatrix} 2 & -5 \\ -1 & 2 \end{bmatrix}$

$A^{-1} = -1 \begin{bmatrix} 2 & -5 \\ -1 & 2 \end{bmatrix}$

$X = A^{-1} \times B$

$= -1 \begin{bmatrix} 2 & -5 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} -2 \\ -3 \end{bmatrix}$

$\begin{bmatrix} x \\ y \end{bmatrix} = -1 \begin{bmatrix} -4+15 \\ 2-6 \end{bmatrix}$

$\begin{bmatrix} x \\ y \end{bmatrix} = -1 \begin{bmatrix} 11 \\ -4 \end{bmatrix}$

(x, y) = $(-11, 4)$

(ii) $2x - y = 8,\ 3x + 2y = -2$.

Solution: $2x - y = 8,\ 3x + 2y = -2$.

A $= \begin{bmatrix} 2 & -1 \\ 3 & 2 \end{bmatrix}$ X $= \begin{bmatrix} x \\ y \end{bmatrix}$ B $= \begin{bmatrix} 8 \\ -2 \end{bmatrix}$

$|A| = \begin{vmatrix} 2 & -1 \\ 3 & 2 \end{vmatrix}$

$= 4 + 3$

$= 7$

adj A $= \begin{bmatrix} 2 & 1 \\ -3 & 2 \end{bmatrix}$

$A^{-1} = \frac{1}{|A|}\, Adj\ A$

$= \frac{1}{7} \begin{bmatrix} 2 & 1 \\ -3 & 2 \end{bmatrix}$

$A^{-1} = \frac{1}{7} \begin{bmatrix} 2 & 1 \\ -3 & 2 \end{bmatrix}$

$X = A^{-1} \times B$

$= \frac{1}{7} \begin{bmatrix} 2 & 1 \\ -3 & 2 \end{bmatrix} \begin{bmatrix} 8 \\ -2 \end{bmatrix}$

$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{7} \begin{bmatrix} 16-2 \\ -24-4 \end{bmatrix}$

$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{7} \begin{bmatrix} 14 \\ -28 \end{bmatrix}$

(x, y) = $(2, -4)$

(iii) $2x + 3y - z = 9, x + y + z = 9$
and $3x - y - z = -1.$
Solution:

A $= \begin{bmatrix} 2 & 3 & -1 \\ 1 & 1 & 1 \\ 3 & -1 & -1 \end{bmatrix}$ X $= \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ B $= \begin{bmatrix} 9 \\ 9 \\ -1 \end{bmatrix}$

$|A| = \begin{vmatrix} 2 & 3 & -1 \\ 1 & 1 & 1 \\ 3 & -1 & -1 \end{vmatrix}$

$= 2 \begin{vmatrix} 1 & 1 \\ -1 & -1 \end{vmatrix} - 3 \begin{vmatrix} 1 & 1 \\ 3 & -1 \end{vmatrix} - 1 \begin{vmatrix} 1 & 1 \\ 3 & -1 \end{vmatrix}$

$= 2(-1+1) - 3(-1-3) - 1(-1-3)$

$= 2(0) - 3(-4) - 1(-4)$

$= 0 + 12 + 4$

$= 16$

A $= \begin{bmatrix} 2 & 3 & -1 \\ 1 & 1 & 1 \\ 3 & -1 & -1 \end{bmatrix}$ $\quad \begin{array}{cccc} 1 & 1 & 1 & 1 \\ -1 & -2 & 3 & -1 \\ 3 & -1 & 2 & 3 \\ 1 & 1 & 1 & 1 \end{array}$

Cofactor of A Aij $= \begin{bmatrix} 0 & 4 & -4 \\ 4 & 1 & 11 \\ 4 & -3 & -1 \end{bmatrix}$

Adj A = Aij^{T}

Adj A $= \begin{bmatrix} 0 & 4 & 4 \\ 4 & 1 & -3 \\ -4 & 11 & -1 \end{bmatrix}$

$A^{-1} = \frac{1}{|A|}\, Adj\ A$

$= \frac{1}{16} \begin{bmatrix} 0 & 4 & 4 \\ 4 & 1 & -3 \\ -4 & 11 & -1 \end{bmatrix}$

$$X = A^{-1} \times B$$

$$= \frac{1}{16}\begin{bmatrix} 0 & 4 & 4 \\ 4 & 1 & -3 \\ -4 & 11 & -1 \end{bmatrix}\begin{bmatrix} 9 \\ 9 \\ -1 \end{bmatrix}$$

$$= \frac{1}{16}\begin{bmatrix} 0+36-4 \\ 36+9+3 \\ -36+99+1 \end{bmatrix}$$

$$= \frac{1}{16}\begin{bmatrix} 36-4 \\ 48 \\ -36+100 \end{bmatrix}$$

$$= \frac{1}{16}\begin{bmatrix} 32 \\ 48 \\ 64 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$$

(x, y, z) = (2, 3, 4)

(iv) $x + y + z - 2 = 0,$
$6x - 4y + 5z - 31 = 0$
$5x + 2y + 2z = 13$

Solution: $x + y + z = 2,$
$6x - 4y + 5z = 31$
$5x + 2y + 2z = 13$

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 6 & -4 & 5 \\ 5 & 2 & 2 \end{bmatrix} \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad B = \begin{bmatrix} 2 \\ 31 \\ 13 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 1 & 1 & 1 \\ 6 & -4 & 5 \\ 5 & 2 & 2 \end{vmatrix}$$

$$= 1\begin{vmatrix} -4 & 5 \\ 2 & 2 \end{vmatrix} - 1\begin{vmatrix} 6 & 5 \\ 5 & 2 \end{vmatrix} + 1\begin{vmatrix} 6 & -4 \\ 5 & 2 \end{vmatrix}$$

$= 1(-8 - 10) - 1(12 - 25) + 1(12+20)$

$= 1(-18) - 1(-13) + 1(32)$

$= -18 + 13 + 32$

$= -18 + 45$

$= 27$

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 6 & -4 & 5 \\ 5 & 2 & 2 \end{bmatrix} \qquad \begin{array}{cccc} -4 & 5 & 6 & -4 \\ 2 & 2 & 5 & 2 \\ 1 & 1 & 1 & 1 \\ -4 & 5 & 6 & -4 \end{array}$$

$$\text{Cofactor of A } A_{ij} = \begin{bmatrix} -18 & 13 & 32 \\ 0 & -3 & 3 \\ 9 & 1 & -10 \end{bmatrix}$$

$\text{Adj } A = A_{ij}^{T}$

$$\text{Adj } A = \begin{bmatrix} -18 & 0 & 9 \\ 13 & -3 & 1 \\ 32 & 3 & -10 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|}\, Adj\ A$$

$$= \frac{1}{27}\begin{bmatrix} -18 & 0 & 9 \\ 13 & -3 & 1 \\ 32 & 3 & -10 \end{bmatrix}$$

$$X = A^{-1} \times B$$

$$= \frac{1}{27}\begin{bmatrix} -18 & 0 & 9 \\ 13 & -3 & 1 \\ 32 & 3 & -10 \end{bmatrix}\begin{bmatrix} 2 \\ 31 \\ 13 \end{bmatrix}$$

$$= \frac{1}{27}\begin{bmatrix} -36+0+117 \\ 26-93+13 \\ 64+93-130 \end{bmatrix}$$

$$= \frac{1}{27}\begin{bmatrix} 117-36 \\ 39-93 \\ 157-130 \end{bmatrix}$$

$$= \frac{1}{27}\begin{bmatrix} 81 \\ -54 \\ 27 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix} \rightarrow (x, y, z) = (3, -2, 1)$$

2.If $A = \begin{bmatrix} -5 & 1 & 3 \\ 7 & 1 & -5 \\ 1 & -1 & 1 \end{bmatrix}$ and

$B = \begin{bmatrix} 1 & 1 & 2 \\ 3 & 2 & 1 \\ 2 & 1 & 3 \end{bmatrix}$, find the products AB and BA and hence solve the system of equations $x + y + 2z = 1, 3x + 2y + z = 7$ and $2x + y + 3z = 2$.

Solution:

$$A = \begin{bmatrix} -5 & 1 & 3 \\ 7 & 1 & -5 \\ 1 & -1 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 & 1 & 2 \\ 3 & 2 & 1 \\ 2 & 1 & 3 \end{bmatrix}$$

$$AB = \begin{bmatrix} -5 & 1 & 3 \\ 7 & 1 & -5 \\ 1 & -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 2 \\ 3 & 2 & 1 \\ 2 & 1 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} -5+3+6 & -5+2+3 & -10+1+9 \\ 7+3-10 & 7+2-5 & 14+1-15 \\ 1-3+2 & 1-2+1 & 2-1+3 \end{bmatrix}$$

$$= \begin{bmatrix} -5+9 & -5+5 & -10+10 \\ 10-10 & 9-5 & 15-15 \\ 3-3 & 2-2 & 3-3 \end{bmatrix}$$

$$AB = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

$$\text{and } BA = \begin{bmatrix} 1 & 1 & 2 \\ 3 & 2 & 1 \\ 2 & 1 & 3 \end{bmatrix}\begin{bmatrix} -5 & 1 & 3 \\ 7 & 1 & -5 \\ 1 & -1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -5+7+2 & 1+1-2 & 3-5+2 \\ -15+14+1 & 3+2-1 & 9-10+1 \\ -10+7+3 & 2+1-3 & 6-5+3 \end{bmatrix}$$

$$= \begin{bmatrix} -5+9 & 2-2 & 5-5 \\ -15+15 & 5-1 & 10-10 \\ -10+10 & 3-3 & 9-5 \end{bmatrix}$$

$$BA = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

So, we get $AB = BA = 4I_3$.

That is $\left(\frac{1}{4}A\right)B = B\left(\frac{1}{4}A\right) = I_3$.

$$\therefore B^{-1} = \frac{1}{4}(A)$$

Writing the given system of equations in matrix form, we get

$$\begin{bmatrix} 1 & 1 & 2 \\ 3 & 2 & 1 \\ 2 & 1 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 7 \\ 2 \end{bmatrix}$$

$$\text{So } \begin{bmatrix} x \\ y \\ z \end{bmatrix} = B^{-1}\begin{bmatrix} 1 \\ 7 \\ 2 \end{bmatrix}$$

$$= \frac{1}{4}(A)\begin{bmatrix} 1 \\ 7 \\ 2 \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} -5 & 1 & 3 \\ 7 & 1 & -5 \\ 1 & -1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 7 \\ 2 \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} -5+7+6 \\ 7+7-10 \\ 1-7+2 \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} -5+13 \\ 14-10 \\ 3-7 \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} 8 \\ 4 \\ -4 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$$

3. A man is appointed in a job with a monthly salary of certain amount and a fixed amount of annual increment. If his salary was Rs. 19,800 per month at the end of the first month after 3 years of service and Rs. 23,400 per month at the end of the first month after 9 years of service, find his starting salary and his annual increment. (Use matrix inversion method to solve the problem.)

Solution:

Let the monthly salary be Rs. x

Let the annual increment be Rs. y

Then from the given data

$$x + 3y = 19{,}800$$

$$x + 9y = 23{,}400$$

Solving by matrix inversion method,

$A = \begin{bmatrix} 1 & 3 \\ 1 & 9 \end{bmatrix}, X = \begin{bmatrix} x \\ y \end{bmatrix}, B = \begin{bmatrix} 19800 \\ 23400 \end{bmatrix}$

$|A| = \begin{vmatrix} 1 & 3 \\ 1 & 9 \end{vmatrix}$

$= 9 - 3$

$= 6$

$\text{adj } A = \begin{bmatrix} 9 & -3 \\ -1 & 1 \end{bmatrix}$

$A^{-1} = \frac{1}{|A|} Adj\ A$

$= \frac{1}{6} \begin{bmatrix} 9 & -3 \\ -1 & 1 \end{bmatrix}$

$A^{-1} = \frac{1}{6} \begin{bmatrix} 9 & -3 \\ -1 & 1 \end{bmatrix}$

$X = A^{-1} \times B$

$= \frac{1}{6} \begin{bmatrix} 9 & -3 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 19{,}800 \\ 23{,}400 \end{bmatrix}$

$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{6} \begin{bmatrix} 1{,}78{,}200 - 70{,}200 \\ -19{,}800 + 23{,}400 \end{bmatrix}$

$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{6} \begin{bmatrix} 1{,}08{,}000 \\ 3{,}600 \end{bmatrix}$

$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 18{,}000 \\ 600 \end{bmatrix}$

Hence,

Monthly salary $x =$ Rs.18,000

Annual increment $y =$ Rs.600

4. 4 men and 4 women can finish a piece of work jointly in 3 days while 2 men and 5 women can finish the same work jointly in 4 days. Find the time taken by one man alone and that of one woman alone to finish the same work by using matrix inversion method.

Solution:

Let 1 man can do the work in $x$days

In 1 day he completes $\frac{1}{x}$ of the work

Let 1 woman can do the work in y days

In 1 day she completes $\frac{1}{y}$ of the work

4 men and 4 women can finish $\frac{1}{3}$ piece of work jointly in 1day is $\frac{4}{x} + \frac{4}{y} = \frac{1}{3}$

2 men and 5 women can finish $\frac{1}{4}$ piece of work jointly in 1day is $\frac{2}{x} + \frac{5}{y} = \frac{1}{4}$

Solving $\frac{4}{x} + \frac{4}{y} = \frac{1}{3}$ and $\frac{2}{x} + \frac{5}{y} = \frac{1}{4}$

Let $\frac{1}{x} = a$ and $\frac{1}{y} = b$ then

$4a + 4b = \frac{1}{3}$

$2a + 5b = \frac{1}{4}$

$A = \begin{bmatrix} 4 & 4 \\ 2 & 5 \end{bmatrix}\ X = \begin{bmatrix} a \\ b \end{bmatrix}\ B = \begin{bmatrix} \frac{1}{3} \\ \frac{1}{4} \end{bmatrix}$

$|A| = \begin{vmatrix} 4 & 4 \\ 2 & 5 \end{vmatrix}$

$= 20 - 8$

$= 12$

$\text{adj } A = \begin{bmatrix} 5 & -4 \\ -2 & 4 \end{bmatrix}$

$A^{-1} = \frac{1}{|A|} Adj\ A$

$= \frac{1}{12} \begin{bmatrix} 5 & -4 \\ -2 & 4 \end{bmatrix}$

$A^{-1} = \frac{1}{12} \begin{bmatrix} 5 & -4 \\ -2 & 4 \end{bmatrix}$

$X = A^{-1} \times B$

$$= \frac{1}{12}\begin{bmatrix} 5 & -4 \\ -2 & 4 \end{bmatrix}\begin{bmatrix} \frac{1}{3} \\ \frac{1}{4} \end{bmatrix}$$

$$\begin{bmatrix} a \\ b \end{bmatrix} = \frac{1}{12}\begin{bmatrix} \frac{5}{3} - \frac{4}{4} \\ -\frac{2}{3} + \frac{4}{4} \end{bmatrix}$$

$$\begin{bmatrix} a \\ b \end{bmatrix} = \frac{1}{12}\begin{bmatrix} \frac{20-12}{12} \\ \frac{-8+12}{12} \end{bmatrix}$$

$$\begin{bmatrix} a \\ b \end{bmatrix} = \frac{1}{12}\begin{bmatrix} \frac{8}{12} \\ \frac{4}{12} \end{bmatrix}$$

$$\mathrm{a} = \frac{1}{12} \times \frac{8}{12} = \frac{1}{12} \times \frac{2}{3} = \frac{1}{6 \times 3} = = \frac{1}{18}$$

$a = \frac{1}{x} = \frac{1}{18}$ gives $x = 18$

$$\mathrm{b} = \frac{1}{12} \times \frac{4}{12} = \frac{1}{12} \times \frac{1}{3} = \frac{1}{12 \times 3} = = \frac{1}{36}$$

$b = \frac{1}{y} = \frac{1}{18}$ gives $y = 36$

1 man can do the work in $x = 18$ days

1 woman can do the work in $y = 36$ days

5. The prices of three commodities A, B and C are Rs x, y, and z per units respectively. A person P purchases 4 units of B and sells two units of A and 5 units of C. Person Q purchases 2 units of C and sells 3 units of A and one unit of B. Person R purchases one unit of A and sells 3 unit of B and one unit of C. In the process, P, Q and R earn Rs 15,000, Rs 1,000 and Rs 4,000 respectively. Find the prices per unit of A, B and C. (Use matrix inversion method to solve the problem.)

Solution: Given the prices of three commodities A, B and C are Rs x, y, and z per units.

From the data given,

$$-4y + 2x + 5z = 15{,}000$$
$$-2z + 3x + y = 1{,}000$$
$$\text{and } -x + 3y + z = 4{,}000$$

In standard form

$$2x - 4y + 5z = 15{,}000$$
$$3x + y - 2z = 1{,}000$$
$$-x + 3y + z = 4{,}000$$

$$\mathrm{A} = \begin{bmatrix} 2 & -4 & 5 \\ 3 & 1 & -2 \\ -1 & 3 & 1 \end{bmatrix} \quad \mathrm{X} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad \mathrm{B} = \begin{bmatrix} 15000 \\ 1000 \\ 4000 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 2 & -4 & 5 \\ 3 & 1 & -2 \\ -1 & 3 & 1 \end{vmatrix}$$

$$= 2\begin{vmatrix} 1 & -2 \\ 3 & 1 \end{vmatrix} + 4\begin{vmatrix} 3 & -2 \\ -1 & 1 \end{vmatrix} + 5\begin{vmatrix} 3 & 1 \\ -1 & 3 \end{vmatrix}$$

$$= 2(1+6) + 4\,(3-2) + 5\,(9+1)$$

$$= 2(7) + 4(1) + 5(10)$$

$$= 14 + 4 + 50$$

$$= 68$$

$$\mathrm{A} = \begin{bmatrix} 2 & -4 & 5 \\ 3 & 1 & -2 \\ -1 & 3 & 1 \end{bmatrix} \qquad \begin{array}{cccc} 1 & -2 & 3 & 1 \\ 3 & 1 & -1 & 3 \\ -4 & 5 & 2 & -4 \\ 1 & -2 & 3 & 1 \end{array}$$

$$\text{Cofactor of A} \quad \mathrm{Aij} = \begin{bmatrix} 7 & -1 & 10 \\ 19 & 7 & -2 \\ 3 & 19 & 14 \end{bmatrix}$$

$$\mathrm{Adj\ A} = \mathrm{Aij}^{\mathrm{T}}$$

$$\text{Adj } A = \begin{bmatrix} 7 & 19 & 3 \\ -1 & 7 & 19 \\ 10 & -2 & 14 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|} Adj\ A$$

$$= \frac{1}{68}\begin{bmatrix} 7 & 19 & 3 \\ -1 & 7 & 19 \\ 10 & -2 & 14 \end{bmatrix}$$

$$X = A^{-1} \times B$$

$$= \frac{1}{68}\begin{bmatrix} 7 & 19 & 3 \\ -1 & 7 & 19 \\ 10 & -2 & 14 \end{bmatrix}\begin{bmatrix} 15000 \\ 1000 \\ 4000 \end{bmatrix}$$

$$= \frac{1}{68}\begin{bmatrix} 1{,}05{,}000 + 19{,}000 + 12{,}000 \\ -15{,}000 + 7{,}000 + 76{,}000 \\ 15{,}000 - 2{,}000 + 56{,}000 \end{bmatrix}$$

$$= \frac{1}{68}\begin{bmatrix} 1{,}36{,}000 \\ 68{,}000 \\ 2{,}04{,}000 \end{bmatrix}$$

$$= \begin{bmatrix} 2{,}000 \\ 1{,}000 \\ 3{,}000 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2{,}000 \\ 1{,}000 \\ 3{,}000 \end{bmatrix}$$

The price of the commodity A = Rs. 2,000

The price of the commodity B = Rs. 1,000

The price of the commodity C = Rs. 3,000

Example 1.25 Solve, by Cramer's rule, the system of equations $x_1 - x_2 = 3$, $2x_1 + 3x_2 + 4x_3 = 17$ and $x_2 + 2x_3 = 7$.

Solution: Given $x_1 - x_2 + 0x_3 = 3$,

$$2x_1 + 3x_2 + 4x_3 = 17$$

$$0x_1 + x_2 + 2x_3 = 7$$

$$\Delta = \begin{vmatrix} 1 & -1 & 0 \\ 2 & 3 & 4 \\ 0 & 1 & 2 \end{vmatrix}$$

$$= 1\begin{vmatrix} 3 & 4 \\ 1 & 2 \end{vmatrix} + 1\begin{vmatrix} 2 & 4 \\ 0 & 2 \end{vmatrix} + 0\begin{vmatrix} 2 & 3 \\ 0 & 1 \end{vmatrix}$$

$$= 1(6 - 4) + 1(4 - 0) + 0$$

$$= 1(2) + 1(4) + 0$$

$$= 2 + 4 + 0$$

$$= 6 \neq 0$$

$$\Delta_{x_1} = \begin{vmatrix} 3 & -1 & 0 \\ 17 & 3 & 4 \\ 7 & 1 & 2 \end{vmatrix}$$

$$= 3\begin{vmatrix} 3 & 4 \\ 1 & 2 \end{vmatrix} + 1\begin{vmatrix} 17 & 4 \\ 7 & 2 \end{vmatrix} + 0\begin{vmatrix} 17 & 3 \\ 7 & 1 \end{vmatrix}$$

$$= 3(6 - 4) + 1(34 - 28) + 0$$

$$= 3(2) + 1(6) + 0$$

$$= 6 + 6 + 0$$

$$= 12$$

$$\Delta_{x_2} = \begin{vmatrix} 1 & 3 & 0 \\ 2 & 17 & 4 \\ 0 & 7 & 2 \end{vmatrix}$$

$$= 1\begin{vmatrix} 17 & 4 \\ 7 & 2 \end{vmatrix} - 3\begin{vmatrix} 2 & 4 \\ 0 & 2 \end{vmatrix} + 0\begin{vmatrix} 2 & 17 \\ 0 & 7 \end{vmatrix}$$

$$= 1(34 - 28) - 3(4 - 0) + 0$$

$$= 1(6) - 3(4) + 0$$

$$= 6 - 12 + 0$$

$$= -6$$

$$\Delta_{x_3} = \begin{vmatrix} 1 & -1 & 3 \\ 2 & 3 & 17 \\ 0 & 1 & 7 \end{vmatrix}$$

$$= 1\begin{vmatrix} 3 & 17 \\ 1 & 7 \end{vmatrix} + 1\begin{vmatrix} 2 & 17 \\ 0 & 7 \end{vmatrix} + 3\begin{vmatrix} 2 & 3 \\ 0 & 1 \end{vmatrix}$$

$$= 1(21 - 17) + 1(14 - 0) + 3(2 - 0)$$

$= 1(4) + 1(14) + 3(2)$

$= 4 + 14 + 6$

$= 24$

By Cramer's rule, we get

$$x_1 = \frac{\Delta_{x_1}}{\Delta} = \frac{12}{6} = 2$$

$$x_2 = \frac{\Delta_{x_2}}{\Delta} = \frac{-6}{6} = -1$$

$$x_3 = \frac{\Delta_{x_3}}{\Delta} = \frac{24}{6} = 4$$

Example 1.26 In a T20 match, Chennai Super Kings needed just 6 runs to win with 1 ball left to go in the last over. The last ball was bowled and the batsman at the crease hit it high up. The ball traversed along a path in a vertical plane and the equation of the path is $y = ax^2 + bx + c$ with respect to a xy-coordinate system in the vertical plane and the ball traversed through the points (10,8), (20,16), (40,22), can you conclude that Chennai Super Kings won the match? Justify your answer. (All distances are measured in metres and the meeting point of the plane of the path with the farthest boundary line is (70,0).)

Solution: The path $y = ax^2 + bx + c$ passes through the points (10,8), (20,16), (40,22). So, we get the system of equations

$$100a + 10b + c = 8$$

$$400a + 20b + c = 16$$

$$1600a + 40b + c = 22$$

To apply Cramer's rule, we find

$$\Delta = \begin{vmatrix} 100 & 10 & 1 \\ 400 & 20 & 1 \\ 1600 & 40 & 1 \end{vmatrix}$$

$$= 100\begin{vmatrix} 20 & 1 \\ 40 & 1 \end{vmatrix} - 10\begin{vmatrix} 400 & 1 \\ 1600 & 1 \end{vmatrix} + 1\begin{vmatrix} 400 & 20 \\ 1600 & 40 \end{vmatrix}$$

$= 100(20 - 40) - 10(400 - 1600)$
$+ 1(16000 - 32000)$

$= 100(-20) - 10(-1200) + 1(-16000)$

$= -2000 + 12000 - 16000$

$= -18000 + 12000$

$= -6000$

$$\Delta_a = \begin{vmatrix} 8 & 10 & 1 \\ 16 & 20 & 1 \\ 22 & 40 & 1 \end{vmatrix}$$

$$= 8\begin{vmatrix} 20 & 1 \\ 40 & 1 \end{vmatrix} - 10\begin{vmatrix} 16 & 1 \\ 22 & 1 \end{vmatrix} + 1\begin{vmatrix} 16 & 20 \\ 22 & 40 \end{vmatrix}$$

$= 8(20 - 40) - 10(16 - 22) + 1(640 - 440)$

$= 8(-20) - 10(-6) + 1(200)$

$= -160 + 60 + 200$

$= -160 + 260$

$= 100$

$$\Delta_b = \begin{vmatrix} 100 & 8 & 1 \\ 400 & 16 & 1 \\ 1600 & 22 & 1 \end{vmatrix}$$

$$= 100\begin{vmatrix} 16 & 1 \\ 22 & 1 \end{vmatrix} - 8\begin{vmatrix} 400 & 1 \\ 1600 & 1 \end{vmatrix} + 1\begin{vmatrix} 400 & 16 \\ 1600 & 22 \end{vmatrix}$$

$= 100(16 - 22) - 8(400 - 1600) + 1(8800 - 25600)$

$= 100(-6) - 8(-1200) + 1(-16800)$

$= -600 + 9600 - 16800$

$= -17400 + 9600$

$= -7800$

$$\Delta_c = \begin{vmatrix} 100 & 10 & 8 \\ 400 & 20 & 16 \\ 1600 & 40 & 22 \end{vmatrix}$$

$$= 100\begin{vmatrix} 20 & 16 \\ 40 & 22 \end{vmatrix} - 10\begin{vmatrix} 400 & 16 \\ 1600 & 22 \end{vmatrix}$$

$$+ 8\begin{vmatrix} 400 & 20 \\ 1600 & 40 \end{vmatrix}$$

$=100(440-640)-10(8800-25600)$

$+8(16000-32000)$

$=100(-200)-10(-16800)+8(-16000)$

$=-20000+168000-128000$

$=168000-148000$

$=20000$

By Cramer's rule, we get

$$a=\frac{\Delta_a}{\Delta}=\frac{100}{-6000}=-\frac{1}{60}$$

$$b=\frac{\Delta_b}{\Delta}=\frac{-7800}{-6000}=\frac{78}{60}=\frac{13}{10}$$

$$c=\frac{\Delta_c}{\Delta}=\frac{20000}{-6000}=-\frac{20}{6}=-\frac{10}{3}$$

So, the equation of the path is $y=ax^2+bx+c$ becomes

$$y=-\frac{1}{60}x^2+\frac{13}{10}x-\frac{10}{3}$$

substituting the point (70,0)

When $x=70$, in

$$y=-\frac{1}{60}x^2+\frac{13}{10}x-\frac{10}{3}$$

$$=-\frac{1}{60}\times 70\times 70+\frac{13}{10}\times 70-\frac{10}{3}$$

$$=-\frac{490}{6}+91-\frac{10}{3}$$

$$=-\frac{245}{3}+91-\frac{10}{3}$$

$$=-\frac{255}{3}+91$$

$$=-85+91$$

$$=6$$

We get $y=6$. So, the ball went by 6 metres high over the boundary line and it is impossible for a fielder standing even just before the boundary line to jump and catch the ball. Hence the ball went for a super six and the Chennai Super Kings won the match.

EXERCISE 1.4

1. Solve the following systems of linear equations by Cramer's rule:

(i) $5x-2y+16=0, x+3y-7=0$

Solution: $5x-2y+16=0$

$x+3y-7=0$ gives

$$5x-2y=-16$$

$$x+3y=7$$

$$\Delta=\begin{vmatrix}5 & -2\\ 1 & 3\end{vmatrix}$$

$=15+2$

$=17\neq 0$

$$\Delta_x=\begin{vmatrix}-16 & -2\\ 7 & 3\end{vmatrix}$$

$=-48+14$

$=-34$

$$\Delta_y=\begin{vmatrix}5 & -16\\ 1 & 7\end{vmatrix}$$

$=35+16$

$=51$

By Cramer's rule, we get

$$x=\frac{\Delta_x}{\Delta}=\frac{-34}{17}=-2$$

$$y=\frac{\Delta_y}{\Delta}=\frac{51}{17}=3$$

ii) $\frac{3}{x}+2y=12, \frac{2}{x}+3y=13$

Solution: $\frac{3}{x}+2y=12$

$\frac{2}{x}+3y=13$

Let $\frac{1}{x} = a$, then equation is

$$3a + 2y = 12$$

$$2a + 3y = 13$$

$$\Delta = \begin{vmatrix} 3 & 2 \\ 2 & 3 \end{vmatrix}$$

$$= 9 - 4$$

$$= 5 \neq 0$$

$$\Delta_a = \begin{vmatrix} 12 & 2 \\ 13 & 3 \end{vmatrix}$$

$$= 36 - 26$$

$$= 10$$

$$\Delta_y = \begin{vmatrix} 3 & 12 \\ 2 & 13 \end{vmatrix}$$

$$= 39 - 24$$

$$= 15$$

By Cramer's rule, we get

$$a = \frac{\Delta_a}{\Delta} = \frac{10}{5} = 2$$

$$y = \frac{\Delta_y}{\Delta} = \frac{15}{5} = 3$$

$$a = 2 = \frac{1}{x}\ gives\ x = \frac{1}{2}$$

$$y = 3$$

(iii) $3x + 3y - z = 11, 2x - y + 2z = 9$

and $4x + 3y + 2z = 25$

Solution : $3x + 3y - z = 11$

$$2x - y + 2z = 9$$

$$4x + 3y + 2z = 25$$

To apply Cramer's rule, we find

$$\Delta = \begin{vmatrix} 3 & 3 & -1 \\ 2 & -1 & 2 \\ 4 & 3 & 2 \end{vmatrix}$$

$$= 3\begin{vmatrix} -1 & 2 \\ 3 & 2 \end{vmatrix} - 3\begin{vmatrix} 2 & 2 \\ 4 & 2 \end{vmatrix} - 1\begin{vmatrix} 2 & -1 \\ 4 & 3 \end{vmatrix}$$

$$= 3(-2 - 6) - 3(4 - 8) - 1(6 + 4)$$

$$= 3(-8) - 3(-4) - 1(10)$$

$$= -24 + 12 - 10$$

$$= -34 + 12$$

$$= -22 \neq 0$$

$$\Delta_x = \begin{vmatrix} 11 & 3 & -1 \\ 9 & -1 & 2 \\ 25 & 3 & 2 \end{vmatrix}$$

$$= 11\begin{vmatrix} -1 & 2 \\ 3 & 2 \end{vmatrix} - 3\begin{vmatrix} 9 & 2 \\ 25 & 2 \end{vmatrix} - 1\begin{vmatrix} 9 & -1 \\ 25 & 3 \end{vmatrix}$$

$$= 11(-2 - 6) - 3(18 - 50) - 1(27 + 25)$$

$$= 11(-8) - 3(-32) - 1(52)$$

$$= -88 + 96 - 52$$

$$= -140 + 96$$

$$= -44$$

$$\Delta_y = \begin{vmatrix} 3 & 11 & -1 \\ 2 & 9 & 2 \\ 4 & 25 & 2 \end{vmatrix}$$

$$= 3\begin{vmatrix} 9 & 2 \\ 25 & 2 \end{vmatrix} - 11\begin{vmatrix} 2 & 2 \\ 4 & 2 \end{vmatrix} - 1\begin{vmatrix} 2 & 9 \\ 4 & 25 \end{vmatrix}$$

$$= 3(18 - 50) - 11(4 - 8) - 1(50 - 36)$$

$$= 3(-32) - 11(-4) - 1(14)$$

$$= -96 + 44 - 14$$

$$= -110 + 44$$

$$= -66$$

$$\Delta_z = \begin{vmatrix} 3 & 3 & 11 \\ 2 & -1 & 9 \\ 4 & 3 & 25 \end{vmatrix}$$

$$= 3\begin{vmatrix} -1 & 9 \\ 3 & 25 \end{vmatrix} - 3\begin{vmatrix} 2 & 9 \\ 4 & 25 \end{vmatrix} + 11\begin{vmatrix} 2 & -1 \\ 4 & 3 \end{vmatrix}$$

$= 3(-25 - 27) -3(50 - 36) +11(6 + 4)$

$= 3(-52) -3(14) +11(10)$

$= -156 - 42 + 110$

$= -198 + 110$

$= -88$

By Cramer's rule, we get

$x = \frac{\Delta_x}{\Delta} = \frac{-44}{-22} = 2$

$y = \frac{\Delta_y}{\Delta} = \frac{-66}{-22} = 3$

$z = \frac{\Delta_z}{\Delta} = \frac{-88}{-22} = 4$

(iv) $\frac{3}{x} - \frac{4}{y} - \frac{2}{z} - 1 = 0, \frac{1}{x} + \frac{2}{y} + \frac{1}{z} - 2 = 0$

$\frac{2}{x} - \frac{5}{y} - \frac{4}{z} + 1 = 0$

Solution: $\frac{3}{x} - \frac{4}{y} - \frac{2}{z} - 1 = 0$

$\frac{1}{x} + \frac{2}{y} + \frac{1}{z} - 2 = 0$

$\frac{2}{x} - \frac{5}{y} - \frac{4}{z} + 1 = 0$

Let $\frac{1}{x} = a, \frac{1}{y} = b$ and $\frac{1}{z} = c$

$3a - 4b - 2c = 1$

$a + 2b + c = 2$

$2a - 5b - 4c = -1$

To apply Cramer's rule, we find

$\Delta = \begin{vmatrix} 3 & -4 & -2 \\ 1 & 2 & 1 \\ 2 & -5 & -4 \end{vmatrix}$

$= 3\begin{vmatrix} 2 & 1 \\ -5 & -4 \end{vmatrix} +4\begin{vmatrix} 1 & 1 \\ 2 & -4 \end{vmatrix} -2\begin{vmatrix} 1 & 2 \\ 2 & -5 \end{vmatrix}$

$= 3(-8 +5) +4(-4 - 2) -2(-5 - 4)$

$= 3(-3) + 4(-6) -2(-9)$

$= -9 - 24 +18$

$= -33 +18$

$= -15 \neq 0$

$\Delta_a = \begin{vmatrix} 1 & -4 & -2 \\ 2 & 2 & 1 \\ -1 & -5 & -4 \end{vmatrix}$

$= 1\begin{vmatrix} 2 & 1 \\ -5 & -4 \end{vmatrix} +4\begin{vmatrix} 2 & 1 \\ -1 & -4 \end{vmatrix} -2\begin{vmatrix} 2 & 2 \\ -1 & -5 \end{vmatrix}$

$= 1(-8 +5) +4(-8+1) -2(-10 + 2)$

$= 1(-3) + 4(-7) -2(-8)$

$= -3 - 28 +16$

$= -31 +16$

$= -15$

$\Delta_b = \begin{vmatrix} 3 & 1 & -2 \\ 1 & 2 & 1 \\ 2 & -1 & -4 \end{vmatrix}$

$= 3\begin{vmatrix} 2 & 1 \\ -1 & -4 \end{vmatrix} -1\begin{vmatrix} 1 & 1 \\ 2 & -4 \end{vmatrix} -2\begin{vmatrix} 1 & 2 \\ 2 & -1 \end{vmatrix}$

$= 3(-8 +1) -1(-4 - 2) -2(-1 - 4)$

$= 3(-7) -1(-6) -2(-5)$

$= -21+6 +10$

$= -21 +16$

$= -5$

$\Delta_c = \begin{vmatrix} 3 & -4 & 1 \\ 1 & 2 & 2 \\ 2 & -5 & -1 \end{vmatrix}$

$= 3\begin{vmatrix} 2 & 2 \\ -5 & -1 \end{vmatrix} +4\begin{vmatrix} 1 & 2 \\ 2 & -1 \end{vmatrix} +1\begin{vmatrix} 1 & 2 \\ 2 & -5 \end{vmatrix}$

$= 3(-2 +10) +4(-1 - 4) +1(-5 - 4)$

$= 3(8) +4(-5) +1(-9)$

$= 24 - 20 -9$

$= 24 - 29$

$= -5$

By Cramer's rule, we get

$$a = \frac{\Delta_a}{\Delta} = \frac{-15}{-15} = 1 = \frac{1}{x} \; gives \; x = 1$$

$$b = \frac{\Delta_b}{\Delta} = \frac{-5}{-15} = \frac{1}{3} = \frac{1}{y} \; gives \; y = 3$$

$$c = \frac{\Delta_c}{\Delta} = \frac{-5}{-15} = \frac{1}{3} = \frac{1}{z} \; gives \; z = 3$$

2. In a competitive examination, one mark is awarded for every correct answer while $\frac{1}{4}$ mark is deducted for every wrong answer. A student answered 100 questions and got 80 marks. How many questions did he answer correctly? (Use Cramer's rule to solve the problem).

Solution: Let the number of correctly answered questions $= x$

Let the number of not correctly answered questions $= y$

Given mark for each correct answers =1

Mark for each wrong answers $= -\frac{1}{4}$

$\therefore x + y = 100$ and

$$x - \frac{y}{4} = 80 \Rightarrow \frac{4x - y}{4} = 80$$

$$4x - y = 320$$

Solving $x + y = 100$

$$4x - y = 320$$

$$\Delta = \begin{vmatrix} 1 & 1 \\ 4 & -1 \end{vmatrix}$$

$= -1 - 4$

$= -5 \neq 0$

$$\Delta_x = \begin{vmatrix} 100 & 1 \\ 320 & -1 \end{vmatrix}$$

$= -100 - 320$

$= -420$

$$\Delta_y = \begin{vmatrix} 1 & 100 \\ 4 & 320 \end{vmatrix}$$

$= 320 - 400$

$= -80$

By Cramer's rule, we get

$$x = \frac{\Delta_x}{\Delta} = \frac{-420}{-5} = 84$$

$$y = \frac{\Delta_y}{\Delta} = \frac{-80}{-5} = 16$$

The student answered 84 questions correctly.

3. A chemist has one solution which is 50% acid and another solution which is 25% acid. How much each should be mixed to make 10 litres of a 40% acid solution ? (Use Cramer's rule to solve the problem).

Solution:

Let A be the solutions which is 50% acid has x litres and B be the solutions which is 25% acid has y litres.

$$\left[x \times 50\% = x \times \frac{50}{100} = \frac{x}{2}\right]$$

$$\left[y \times 25\% = y \times \frac{25}{100} = \frac{y}{4}\right]$$

$$\left[10 \times 40\% = 10 \times \frac{40}{100} = 4\right]$$

$\therefore x + y = 10$ and

$$\frac{x}{2} + \frac{y}{4} = 4$$

$$\Delta = \begin{vmatrix} 1 & 1 \\ \frac{1}{2} & \frac{1}{4} \end{vmatrix}$$

$$=\frac{1}{4}-\frac{1}{2}$$

$$=\frac{2-4}{8}$$

$$=\frac{-2}{8}$$

$$=-\frac{1}{4}\neq 0$$

$$\Delta_x=\begin{vmatrix}10 & 1\\ 4 & \frac{1}{4}\end{vmatrix}$$

$$=\frac{10}{4}-4$$

$$=\frac{10-16}{4}$$

$$=-\frac{6}{4}$$

$$\Delta_y=\begin{vmatrix}1 & 10\\ \frac{1}{2} & 4\end{vmatrix}$$

$$=4-\frac{10}{2}$$

$$=4-5$$

$$=-1$$

By Cramer's rule, we get

$$x=\frac{\Delta_x}{\Delta}=\frac{-\frac{6}{4}}{-\frac{1}{4}}=\frac{6}{4}\times\frac{4}{1}=6$$

$$y=\frac{\Delta_y}{\Delta}=\frac{-1}{-\frac{1}{4}}=1\times\frac{4}{1}=4$$

4. A fish tank can be filled in 10 minutes using both pumps A and B simultaneously. However, pump B can pump water in or out at the same rate. If pump B is inadvertently run in reverse, then the tank will be filled in 30 minutes. How long would it take each pump to fill the tank by itself ? (Use Cramer's rule to solve the problem).

Solution:

Let the pump A can fill the tank in x minutes and the pump B can fill the tank in y minutes.

In 1 minute pump A can fill $\frac{1}{x}$ part

In 1 minute pump B can fill $\frac{1}{y}$ part

By the data $\frac{1}{x}+\frac{1}{y}=\frac{1}{10}$

$$\frac{1}{x}-\frac{1}{y}=\frac{1}{30}$$

To solve let $\frac{1}{x}=a,\ \frac{1}{y}=b$

$$a+b=\frac{1}{10}$$

$$a-b=\frac{1}{30}$$

$$\Delta=\begin{vmatrix}1 & 1\\ 1 & -1\end{vmatrix}$$

$$=-1-1$$

$$=-2\neq 0$$

$$\Delta_a=\begin{vmatrix}\frac{1}{10} & 1\\ \frac{1}{30} & -1\end{vmatrix}$$

$$=-\frac{1}{10}-\frac{1}{30}$$

$$=\frac{-3-1}{30}$$

$$=-\frac{4}{30}$$

$$=-\frac{2}{15}$$

$$\Delta_b=\begin{vmatrix}1 & \frac{1}{10}\\ 1 & \frac{1}{30}\end{vmatrix}$$

$$=\frac{1}{30}-\frac{1}{10}$$

$$= \frac{1-3}{30}$$

$$= -\frac{2}{30}$$

$$= -\frac{1}{15}$$

By Cramer's rule, we get

$$a = \frac{\Delta_a}{\Delta} = \frac{-\frac{2}{15}}{-2} = \frac{2}{15} \times \frac{1}{2} = \frac{1}{15}$$

$$b = \frac{\Delta_b}{\Delta} = \frac{-\frac{1}{15}}{-2} = \frac{1}{15} \times \frac{1}{2} = \frac{1}{30}$$

$$a = \frac{1}{15} = \frac{1}{x} \text{ gives } x = 15$$

$$b = \frac{1}{30} = \frac{1}{x} \text{ gives } y = 30$$

Hence the pump A can fill the tank in 15 minutes and the pump B can fill the tank in 30 minutes.

5. A family of 3 people went out for dinner in a restaurant. The cost of two dosai, three idlies and two vadais is Rs.150. The cost of the two dosai, two idlies and four vadais is Rs.200. The cost of five dosai, four idlies and two vadais is Rs. 250. The family has Rs. 350 in hand and they ate 3 dosai and six idlies and six vadais. Will they be able to manage to pay the bill within the amount they had?

Solution: Let the cost of a dosai = Rs. x

The cost of a idly = Rs. y

The cost of a vadai = Rs. z

Given $2x + 3y + 2z = 150$

$2x + 2y + 4z = 200$

$5x + 4y + 2z = 250$

To apply Cramer's rule, we find

$$\Delta = \begin{vmatrix} 2 & 3 & 2 \\ 2 & 2 & 4 \\ 5 & 4 & 2 \end{vmatrix}$$

$$= 2\begin{vmatrix} 2 & 4 \\ 4 & 2 \end{vmatrix} - 3\begin{vmatrix} 2 & 4 \\ 5 & 2 \end{vmatrix} + 2\begin{vmatrix} 2 & 2 \\ 5 & 4 \end{vmatrix}$$

$$= 2(4-16) - 3(4-20) + 2(8-10)$$

$$= 2(-12) - 3(-16) + 2(-2)$$

$$= -24 + 48 - 4$$

$$= -28 + 48$$

$$= 20 \neq 0$$

$$\Delta_x = \begin{vmatrix} 150 & 3 & 2 \\ 200 & 2 & 4 \\ 250 & 4 & 2 \end{vmatrix}$$

$$= 150\begin{vmatrix} 2 & 4 \\ 4 & 2 \end{vmatrix} - 3\begin{vmatrix} 200 & 4 \\ 250 & 2 \end{vmatrix} + 2\begin{vmatrix} 200 & 2 \\ 250 & 4 \end{vmatrix}$$

$$= 150(4-16) - 3(400-1000)$$

$$+2(800-500)$$

$$= 150(-12) - 3(-600) + 2(300)$$

$$= -1800 + 1800 + 600$$

$$= -1800 + 2400$$

$$= 600$$

$$\Delta_y = \begin{vmatrix} 2 & 150 & 2 \\ 2 & 200 & 4 \\ 5 & 250 & 2 \end{vmatrix}$$

$$= 2\begin{vmatrix} 200 & 4 \\ 250 & 2 \end{vmatrix} - 150\begin{vmatrix} 2 & 4 \\ 5 & 2 \end{vmatrix} + 2\begin{vmatrix} 2 & 200 \\ 5 & 250 \end{vmatrix}$$

$$= 2(400-1000) - 150(4-20) + 2(500-1000)$$

$$= 2(-600) - 150(-16) + 2(-500)$$

$$= -1200 + 2400 - 1000$$

$$= -2200 + 2400$$

$$= 200$$

$$\Delta_z = \begin{vmatrix} 2 & 3 & 150 \\ 2 & 2 & 200 \\ 5 & 4 & 250 \end{vmatrix}$$

$$= 2\begin{vmatrix} 2 & 200 \\ 4 & 250 \end{vmatrix} - 3\begin{vmatrix} 2 & 200 \\ 5 & 250 \end{vmatrix} + 150\begin{vmatrix} 2 & 2 \\ 5 & 4 \end{vmatrix}$$

$= 2(500 - 800) - 3(500 - 1000) + 150(8 - 10)$

$= 2(-300) - 3(-500) + 150(-2)$

$= -600 + 1500 - 300$

$= -900 + 1500$

$= 600$

By Cramer's rule, we get

$$x = \frac{\Delta_x}{\Delta} = \frac{600}{20} = 30$$

$$y = \frac{\Delta_y}{\Delta} = \frac{200}{20} = 10$$

$$z = \frac{\Delta_z}{\Delta} = \frac{600}{20} = 30 \text{ that is}$$

The cost of a dosai = Rs. 30

The cost of a idly = Rs. 10

The cost of a vadai = Rs. 30

The family ate 3 dosai and six idlies and six vadais.

$3x + 6y + 6z = 3(30) + 6(10) + 6(30)$

$= 90 + 60 + 180$

$= 330$ Rs.

The family has Rs. 350 so they are able to manage to pay the bill.

Example 1.27 Solve the following system of linear equations, by Gaussian elimination method: $4x + 3y + 6z = 25$, $x + 5y + 7z = 13$, $2x + 9y + z = 1$

Solution: Given

$$4x + 3y + 6z = 25$$

$$x + 5y + 7z = 13$$

$$2x + 9y + z = 1$$

Transforming the augmented matrix to echelon form, we get

$$[AB] = \begin{bmatrix} 4 & 3 & 6 & 25 \\ 1 & 5 & 7 & 13 \\ 2 & 9 & 1 & 1 \end{bmatrix} R_1 \leftrightarrow R_2$$

$$\rightarrow \begin{bmatrix} 1 & 5 & 7 & 13 \\ 4 & 3 & 6 & 25 \\ 2 & 9 & 1 & 1 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & 5 & 7 & 13 \\ 0 & -17 & -22 & -27 \\ 0 & -1 & -13 & -25 \end{bmatrix} \begin{matrix} R_2 = R_2 - 4R_1 \\ R_3 = R_3 - 2R_1 \end{matrix}$$

$$\rightarrow \begin{bmatrix} 1 & 5 & 7 & 13 \\ 0 & -17 & -22 & -27 \\ 0 & 0 & -199 & -398 \end{bmatrix} R_3 = 17R_3 - R_2$$

The equivalent system is written by using the echelon form: $x + 5y + 7z = 13$

$$-17y - 22z = -27$$

$$-199z = -398$$

We get, $199z = 398$

$$z = \frac{398}{199}$$

$$= 2$$

Substituting z = 2 in $-17y - 22z = -27$

$$-17y - 22(2) = -27$$

$$-17y - 44 = -27$$

$$-17y = -27 + 44$$

$$-17y = 17$$

$$y = \frac{17}{-17} = -1$$

Substituting z = 2 and y = −1 in

$$x + 5y + 7z = 13$$

$$x + 5(-1) + 7(2) = 13$$

$$x - 5 + 14 = 13$$

$$x + 9 = 13$$

$$x = 13 - 9$$

$$x = 4$$

So the solution is $x = 4, y = -1$ *and* $z = 2$

Example 1.28 The upward speed v (t) of a rocket at time t is approximated by $v(t) = at^2 + bt + c, 0 \le t \le 100$ where a, b and c are constants. It has been found that the speed at times t = 3. t = 6 and t = 9 seconds are respectively, 64, 133, and 208 miles per second respectively. Find the speed at time t =15 seconds. (Use Gaussian elimination method.)

Solution: $v(t) = at^2 + bt + c$

At t = 3, $v(3) = a(3)^2 + b(3) + c = 64$

$$a(9) + b(3) + c = 64$$

$$9a + 3b + c = 64 \quad \ldots\ldots(1)$$

At t = 6, $v(6) = a(6)^2 + b(6) + c = 133$

$$a(36) + b(6) + c = 133$$

$$36a + 6b + c = 133 \quad \ldots\ldots(2)$$

At t = 9, $v(9) = a(9)^2 + b(9) + c = 208$

$$a(81) + b(9) + c = 208$$

$$81a + 9b + c = 208 \quad \ldots\ldots(3)$$

Transforming the augmented matrix to echelon form, we get

$$[AB] = \begin{bmatrix} 9 & 3 & 1 & 64 \\ 36 & 6 & 1 & 133 \\ 81 & 9 & 1 & 208 \end{bmatrix}$$

$$= \begin{bmatrix} 9 & 3 & 1 & 64 \\ 0 & -6 & -3 & -123 \\ 0 & -18 & -8 & -368 \end{bmatrix} \begin{matrix} R_2 = R_2 - 4R_1 \\ R_3 = R_3 - 9R_1 \end{matrix}$$

$$= \begin{bmatrix} 9 & 3 & 1 & 64 \\ 0 & -6 & -3 & -123 \\ 0 & 0 & 1 & 1 \end{bmatrix} R_3 = R_3 - 3R_2$$

The equivalent system is written by using the echelon form: $9a + 3b + c = 64$

$$-6b - 3c = -123$$

$$c = 1$$

Substituting c = 1 in $-6b - 3c = -123$

$$-6b - 3(1) = -123$$

$$-6b - 3 = -123$$

$$-6b = -123 + 3$$

$$-6b = -120$$

$$6b = 120$$

$$b = \frac{120}{6}$$

$$= 20$$

Substituting c = 1 and b = 20 in

$$9a + 3y + z = 64$$

$$9a + 3(20) + 1 = 64$$

$$9a + 60 + 1 = 64$$

$$9a + 61 = 64$$

$$9a = 64 - 61$$

$$9a = 3$$

$$a = \frac{3}{9}$$

$$= \frac{1}{3}$$

So substituting $a = \frac{1}{3}$, $b = 20$ and c =1 in

$v(t) = at^2 + bt + c$ we get

$$v(t) = \frac{1}{3}t^2 + 20t + 1$$

The speed at t = 15 minutes

$$v(15) = \frac{1}{3}(15)^2 + 20(15) + 1$$

$$= \frac{1}{3}(225) + 300 + 1$$

$= 75 + 301$

$= 376.$

EXERCISE 1.5

1. Solve the following systems of linear equations by Gaussian elimination method:

(i) $2x - 2y + 3z = 2,\ x + 2y - z = 3,$
$3x - y + 2z = 1$

Solution: $2x - 2y + 3z = 2,$

$x + 2y - z = 3,$

$3x - y + 2z = 1$

Transforming the augmented matrix to echelon form, we get

$$[AB] = \begin{bmatrix} 2 & -2 & 3 & 2 \\ 1 & 2 & -1 & 3 \\ 3 & -1 & 2 & 1 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 2 & -1 & 3 \\ 2 & -2 & 3 & 2 \\ 3 & -1 & 2 & 1 \end{bmatrix} R_{1\leftrightarrow}R_2$$

$$\to \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & -6 & 5 & -4 \\ 0 & -7 & 5 & -8 \end{bmatrix} \begin{matrix} R_2 = R_2 - 2R_1 \\ R_3 = R_3 - 3R_1 \end{matrix}$$

$$\to \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & -6 & 5 & -4 \\ 0 & 0 & -5 & -20 \end{bmatrix} R_3 = 6R_3 - 7R_2$$

The equivalent system is written by using the echelon form: $x + 2y - z = 3$

$-6y + 5z = -4$

$-5z = -20$

We get, $5z = 20$

$$z = \frac{20}{5}$$

$= 4$

Substituting z = 4 in $-6y + 5z = -4$

$-6y + 5(4) = -4$

$-6y + 20 = -4$

$-6y = -4 - 20$

$-6y = -24$

$6y = 24$

$$y = \frac{24}{6}$$

$= 4$

Substituting

z = 4 and y = 4 in $x + 2y - z = 3$

$x + 2(4) - 4 = 3$

$x + 8 - 4 = 3$

$x + 4 = 3$

$x = 3 - 4$

$x = -1$

So the solution is $x = -1, y = 4\ and\ z = 4$

(ii) $2x + 4y + 6z = 22, 3x + 8y + 5z = 27,$
$-x + y + 2z = 2$

Solution: $2x + 4y + 6z = 22$

$3x + 8y + 5z = 27$

$-x + y + 2z = 2$

(1) $is \div 2$, then

$$x + 2y + 3z = 11$$
$$3x + 8y + 5z = 27$$
$$-x + y + 2z = 2$$

Transforming the augmented matrix to echelon form, we get

$$[AB] = \begin{bmatrix} 1 & 2 & 3 & 11 \\ 3 & 8 & 5 & 27 \\ -1 & 1 & 2 & 2 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & 2 & 3 & 11 \\ 0 & 2 & -4 & -6 \\ 0 & 3 & 5 & 13 \end{bmatrix} \begin{matrix} R_2 = R_2 - 3R_1 \\ R_3 = R_3 + R_1 \end{matrix}$$

$$\rightarrow \begin{bmatrix} 1 & 2 & 3 & 11 \\ 0 & 2 & -4 & -6 \\ 0 & 0 & 22 & 44 \end{bmatrix} R_3 = 2R_3 - 3R_2$$

The equivalent system is written by using the echelon form: $x + 2y + 3z = 11$

$$2y - 4z = -6$$
$$22z = 44$$

We get, $z = \frac{44}{22}$

$=2$

Substituting z = 2 in $2y - 4z = -6$

$$2y - 4(2) = -6$$
$$2y - 8 = -6$$
$$2y = -6 + 8$$
$$2y = 2$$

$y = \frac{2}{2}$

$=1$

Substituting

z = 2 and y = 1 in $x + 2y + 3z = 11$

$$x + 2(1) + 3(2) = 11$$
$$x + 2 + 6 = 11$$
$$x + 8 = 11$$
$$x = 11 - 8$$
$$x = 3$$

So the solution is $x = 3, y = 1$ *and* $z = 2$

2. If $ax^2 + bx + c$ is divided by x+3, x – 5 and x −1, the remainders are 21,61 and 9 ,and respectively. Find a, b and c. (Use Gaussian elimination method.)

Solution: Let $P(x) = ax^2 + bx + c$ is divided by x+3, x – 5 and x −1 then the remainder is $P(-3)$, $P(5)$ and $P(1)$.

From the data given,

$$P(-3) = a(-3)^2 + b(-3) + c = 21$$
$$9a - 3b + c = 21 \quad \ldots..(1)$$
$$P(5) = a(5)^2 + b(5) + c = 61$$
$$25a + 5b + c = 61 \quad \ldots..(2)$$
$$P(1) = a(1)^2 + b(1) + c = 9$$
$$a + b + c = 9 \quad \ldots..(3)$$

Transforming the augmented matrix to echelon form, we get

$$[AB] = \begin{bmatrix} 9 & -3 & 1 & 21 \\ 25 & 5 & 1 & 61 \\ 1 & 1 & 1 & 9 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & 1 & 1 & 9 \\ 25 & 5 & 1 & 61 \\ 9 & -3 & 1 & 21 \end{bmatrix} R_1 \leftrightarrow R_3$$

$$\rightarrow \begin{bmatrix} 1 & 1 & 1 & 9 \\ 0 & -20 & -24 & -164 \\ 0 & -12 & -8 & -60 \end{bmatrix} \begin{matrix} R_2 = R_2 - 25R_1 \\ R_3 = R_3 - 9R_1 \end{matrix}$$

$$\rightarrow \begin{bmatrix} 1 & 1 & 1 & 9 \\ 0 & 5 & 6 & 41 \\ 0 & 0 & -8 & -48 \end{bmatrix} \begin{matrix} R_2 = R_2 \div (-4) \\ R_3 = R_3 \div (-4) \end{matrix}$$

The equivalent system is written by using the echelon form: $a + b + c = 9$

$$5b + 6c = 41$$

$$-8c = -48$$

We get, $8c = 48$

$$c = \frac{48}{8}$$

$$= 6$$

Substituting c = 6in $5b + 6c = 41$

$$5b + 6(6) = 41$$

$$5b + 36 = 41$$

$$5b = 41 - 36$$

$$5b = 5$$

$$b = \frac{5}{5} = 1$$

Substituting

c = 6 and b = 1 in $a + b + c = 9$

$$a + (1) + (6) = 9$$

$$a + 1 + 6 = 9$$

$$a + 7 = 9$$

$$a = 9 - 7$$

$$a = 2$$

So the solution is $a = 2, b = 1$ *and* $c = 6$

3. An amount of Rs.65,000 is invested in three bonds at the rates of 6%, 8%, and 9 % per annum respectively. The total annual income is Rs. 4,800. The income from the third bond is Rs. 600 more than that from the second bond. Determine the price of each bond. (Use Gaussian elimination method.)

Solution:

Let the prices of each bond is Rs. x, y, z.

By the data, $x + y + z = 65,000$ ....(1)

$$\frac{6x}{100} + \frac{8y}{100} + \frac{9z}{100} = 4800 \textit{ gives}$$

$$6x + 8y + 9z = 4,80,000 \quad .....(2)$$

$$-\frac{8y}{100} + \frac{9z}{100} = 600 \textit{ gives}$$

$$-8y + 9z = 60,000 \quad .....(3)$$

Transforming the augmented matrix to echelon form, we get

$$[AB] = \begin{bmatrix} 1 & 1 & 1 & 65,000 \\ 6 & 8 & 9 & 4,80,000 \\ 0 & -8 & 9 & 60,000 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & 1 & 1 & 65,000 \\ 0 & 2 & 3 & 90.000 \\ 0 & -8 & 9 & 60,000 \end{bmatrix} R_2 = R_2 - 6R_1$$

$$\rightarrow \begin{bmatrix} 1 & 1 & 1 & 65,000 \\ 0 & 2 & 3 & 90.000 \\ 0 & 0 & 21 & 4,20,000 \end{bmatrix} R_3 = R_3 + 4R_2$$

The equivalent system is written by using the echelon form: $x + y + z = 65,000$

$$2y + 3z = 90,000$$

$$21z = 4,20,000$$

We get, $z = \frac{4,20,000}{21}$

$$= 20,000$$

Substituting z = 20,000 in $2y + 3z = 90,000$

$$2y + 3(20,000) = 90,000$$

$$2y + 60,000 = 90,000$$

$2y = 90{,}000 - 60{,}000$

$2y = 30{,}000$

$y = \frac{30{,}000}{2}$

$=15{,}000$

Substituting

$z = 20{,}000$ and $y = 15{,}000$ in

$x + y + z = 65{,}000$

$x + 15{,}000 + 20{,}000 = 65{,}000$

$x + 35{,}000 = 65{,}000$

$x = 65{,}000 - 35{,}000$

$x = 30{,}000$

The prices of each bond is Rs.30,000

Rs.15,000and Rs.20,000 respectively.

4. A boy is walking along the path

$y = ax^2 + bx + c$ through the points

$(-6, 8), (-2, -12)$ and $(3, 8)$. He wants to meet his friend at $P(7, 60)$. Will he meet his friend? (Use Gaussian elimination method.)

Solution: $y = ax^2 + bx + c$

Substituting the given points $(-6, 8)$,

$(-2, -12)$ and $(3, 8)$ in the above equation

we get $8 = a(-6)^2 + b(-6) + c$

$8 = 36a - 6b + c$

$36a - 6b + c = 8 \quad .....(1)$

Similarly

$4a - 2b + c = -12 \quad .....(2)$

$9a + 3b + c = 8 \quad .....(3)$

Transforming the augmented matrix to echelon form, we get

$$[AB] = \begin{bmatrix} 36 & -6 & 1 & 8 \\ 4 & -2 & 1 & -12 \\ 9 & 3 & 1 & 8 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 36 & -6 & 1 & 8 \\ 0 & -12 & 8 & -116 \\ 0 & 18 & 3 & 24 \end{bmatrix} \begin{matrix} R_2 = 9R_2 - R_1 \\ R_3 = 4R_3 - R_1 \end{matrix}$$

$$\rightarrow \begin{bmatrix} 36 & -6 & 1 & 8 \\ 0 & 6 & -4 & 58 \\ 0 & 6 & 1 & 8 \end{bmatrix} \begin{matrix} R_2 \div (-2) \\ R_3 \div (3) \end{matrix}$$

$$\rightarrow \begin{bmatrix} 36 & -6 & 1 & 8 \\ 0 & 6 & -4 & 58 \\ 0 & 0 & -5 & 50 \end{bmatrix} R_3 = R_3 - R_2$$

The equivalent system is written by using the echelon form: $36\,a - 6b + c = 8$

$6b - 4c = 58$

$-5c = 50$

We get, $c = \frac{50}{-5}$

$= -10$

Substituting $c = -10$ in $6b - 4c = 58$

$6b - 4(-10) = 58$

$6b + 40 = 58$

$6b = 58 - 40$

$6b = 18$

$b = \frac{18}{6}$

$b = 3$

Substituting $c = -10$ and $b = 3$ in

$36\,a - 6b + c = 8$

$36\,a - 6(3) + (-10) = 8$

$36\,a - 18 - 10 = 8$

$$36\,a - 28 = 8$$

$$36\,a = 8 + 28$$

$$36\,a = 36$$

$$a = \frac{36}{36}$$

$$a = 1$$

Substituting $a = 1, b = 3$ and $c = -10$ in

$y = ax^2 + bx + c$ it becomes

$$y = x^2 + 3x - 10$$

substituting the point $P(7, 60)$

$$60 = (7)^2 + 3(7) - 10$$

$$60 = 49 + 21 - 10$$

$$60 = 70 - 10$$

$$60 = 60$$

So the boy meets his friend.

Example 1.29 Test for consistency of the following system of linear equations and if possible solve:

$x + 2y - z = 3,\ \ 3x - y + 2z = 1$

$x - 2y + 3z = 3$ and $x - y + z + 1 = 0$

Solution

Here the number of unknowns is 3.

The matrix form of the system is $AX = B$, where

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 3 & -1 & 2 \\ 1 & -2 & 3 \\ 1 & -1 & 1 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, B = \begin{bmatrix} 3 \\ 1 \\ 3 \\ -1 \end{bmatrix}$$

The augmented matrix to echelon form,

$$\text{we get } [AB] = \begin{bmatrix} 1 & 2 & -1 & 3 \\ 3 & -1 & 2 & 1 \\ 1 & -2 & 3 & 3 \\ 1 & -1 & 1 & -1 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & -7 & 5 & -8 \\ 0 & -4 & 4 & 0 \\ 0 & -3 & 2 & -4 \end{bmatrix} \begin{matrix} R_2 = R_2 - 3R_1 \\ R_3 = R_3 - R_1 \\ R_4 = R_4 - R_1 \end{matrix}$$

$$\to \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & -7 & 5 & -8 \\ 0 & 4 & -4 & 0 \\ 0 & 3 & -2 & 4 \end{bmatrix} \begin{matrix} R_3 \times (-1) \\ R_4 \times (-1) \end{matrix}$$

$$\to \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & -7 & 5 & -8 \\ 0 & 0 & -8 & -32 \\ 0 & 0 & 4 & 16 \end{bmatrix} \begin{matrix} R_3 = 7R_3 + 4R_1 \\ R_4 = 4R_3 - 3R_1 \end{matrix}$$

$$\to \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & -7 & 5 & -8 \\ 0 & 0 & -8 & -32 \\ 0 & 0 & 0 & 0 \end{bmatrix} R_4 = 2R_4 + R_3$$

There are three non-zero rows in the row-echelon form of $[AB]$. So, $\rho[AB] = \rho[A] = 3$

$\rho(AB) = \rho(A) = 3, n = 3$ The given equation is consistent, has unique solution.

$$-8z = -32$$

$$z = \frac{-32}{-8} = 4$$

sub z = 4 in

$$-7y + 5z = -8$$

$$-7y + 5(4) = -8$$

$$-7y + 20 = -8$$

$$-7y = -8 - 20$$

$$-7y = -28$$

$$7y = 28$$

$$y = \frac{28}{7} = 4$$

sub z = 4 and y = 4 in $x + 2y - z = 3$

$$x + 2(4) - 4 = 3$$

$$x + 8 - 4 = 3$$

$$x + 4 = 3$$

$$x = 3 - 4$$

$$x = -1$$

So the solution is $x = -1, y = 4$ *and* $z = 4$

Example 1.30

Test for consistency of the following system of linear equations and if possible solve:

$4x - 2y + 6z = 8,\ x + y - 3z = -1$

$15x - 3y + 9z = 21.$

Solution

Here the number of unknowns is 3.

The matrix form of the system is $AX = B$, where

$$A = \begin{bmatrix} 4 & -2 & 6 \\ 1 & 1 & -3 \\ 15 & -3 & 9 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, B = \begin{bmatrix} 8 \\ -1 \\ 21 \end{bmatrix}$$

The augmented matrix to echelon form,

$$\text{we get } [AB] = \begin{bmatrix} 4 & -2 & 6 & 8 \\ 1 & 1 & -3 & -1 \\ 15 & -3 & 9 & 21 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & 1 & -3 & -1 \\ 4 & -2 & 6 & 8 \\ 15 & -3 & 9 & 21 \end{bmatrix} R_1 \leftrightarrow R_2$$

$$\rightarrow \begin{bmatrix} 1 & 1 & -3 & -1 \\ 0 & -6 & 18 & 12 \\ 0 & -18 & 54 & 36 \end{bmatrix} \begin{matrix} R_2 = R_2 - 4R_1 \\ R_3 = R_3 - 15R_1 \end{matrix}$$

$$\rightarrow \begin{bmatrix} 1 & 1 & -3 & -1 \\ 0 & -6 & 18 & 12 \\ 0 & 0 & 0 & 0 \end{bmatrix} R_3 = R_3 - 3R_2$$

There are two non-zero rows in the row-echelon form of $[AB]$. So, $\rho[AB] = \rho[A] = 2$

$\rho(AB) = \rho(A) = 2, n = 3$ The given equation is consistent, has infinitely many solutions.

$$x + y - 3z = -1 \quad ....(1)$$

$$-6y + 18z = 12 \quad ....(2)$$

$$(2) \div -6$$

$$y - 3z = -2 \quad ....(2)$$

To solve the equations let z = t,

then $\quad y - 3t = -2$

$$y = -2 + 3t \Rightarrow 3t - 2$$

substituting

$$z = t, y = 3t - 2 \text{ in } x + y - 3z = -1$$

$$x + 3t - 2 - 3t = -1$$

$$x + 6t - 2 = -1$$

$$x = -1 - 6t + 2$$

$$x = 1 - 6t$$

So the solution

$$x = 1 - 6t, y = 3t - 1, z = t \text{ where } t \in R$$

Example 1.31 Test for consistency of the following system of linear equations and if possible solve:

$x - y + z = -9,\ 2x - 2y + 2z = -18$

$3x - 3y + 3z + 27 = 0.$

The augmented matrix to echelon form,

$$\text{we get } [AB] = \begin{bmatrix} 1 & -1 & 1 & -9 \\ 2 & -2 & 2 & -18 \\ 3 & -3 & 3 & -27 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & -1 & 1 & -9 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} R_2 = R_2 - 2R_1 \\ R_3 = R_3 - 3R_1 \end{matrix}$$

There is one non-zero row in the row-echelon form of $[AB]$. So, $\rho[AB] = \rho[A] = 1$

$\rho(AB) = \rho(A) = 1, n = 3$ The given equation is consistent, has infinitely many solutions.

$$x - y + z = -9$$

To solve the equations let $y = s, z = t,$

then $x - s + t = -9$

$$x = -9 + s - t$$

So the solution

$x = -9 + s - t, y = s, z = t$ $where$ $s, t \in R$

Example 1.32
Test for consistency of the following system of linear equations and if possible solve:

$x - y + z = -9,\ 2x - y + z = 4$

$3x - y + z = 6$ and $4x - y + 2z = 7$

Solution
Here the number of unknowns is 3.
The matrix form of the system is $AX = B$, where

$$A = \begin{bmatrix} 1 & -1 & 1 \\ 2 & -1 & 1 \\ 3 & -1 & 1 \\ 4 & -1 & 2 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, B = \begin{bmatrix} -9 \\ 4 \\ 6 \\ 7 \end{bmatrix}$$

The augmented matrix to echelon form,

$$\text{we get } [AB] = \begin{bmatrix} 1 & -1 & 1 & -9 \\ 2 & -1 & 1 & 4 \\ 3 & -1 & 1 & 6 \\ 4 & -1 & 2 & 7 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & -1 & 1 & -9 \\ 0 & 1 & -1 & 22 \\ 0 & 2 & -2 & 33 \\ 0 & 3 & -2 & 43 \end{bmatrix} \begin{matrix} R_2 = R_2 - 2R_1 \\ R_3 = R_3 - 3R_1 \\ R_4 = R_4 - 4R_1 \end{matrix}$$

$$\rightarrow \begin{bmatrix} 1 & -1 & 1 & -9 \\ 0 & 1 & -1 & 22 \\ 0 & 0 & 0 & -11 \\ 0 & 0 & 1 & -23 \end{bmatrix} \begin{matrix} R_3 = R_3 - 2R_2 \\ R_4 = R_4 - 3R_2 \end{matrix}$$

$$\rightarrow \begin{bmatrix} 1 & -1 & 1 & -9 \\ 0 & 1 & -1 & 22 \\ 0 & 0 & 1 & -23 \\ 0 & 0 & 0 & -11 \end{bmatrix} R_{3 \leftrightarrow} R_4$$

$\rho(AB) = 4$ and $\rho(A) = 3$

$\rho(AB) \neq \rho(A)$ The given equation is

inconsistent, has **no solutions**.

Example 1.33
Find the condition on a,b and c so that the following system of linear equations has one parameter family of solutions:

$x + y + z = a,\ x + 2y + 3z = b$

$3x + 5y + 7z = c$

Solution:
Here the number of unknowns is 3.

The matrix form of the system is $AX = B$, where

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 3 & 5 & 7 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, B = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

The augmented matrix to echelon form,

$$\text{we get } [AB] = \begin{bmatrix} 1 & 1 & 1 & a \\ 1 & 2 & 3 & b \\ 3 & 5 & 7 & c \end{bmatrix}$$

$$\rightarrow$$

$$\begin{bmatrix} 1 & 1 & 1 & a \\ 0 & 1 & 2 & b - a \\ 0 & 2 & 4 & c - 3a \end{bmatrix} \begin{matrix} R_2 = R_2 - R_1 \\ R_3 = R_3 - 3R_1 \end{matrix}$$

$$\rightarrow \begin{bmatrix} 1 & 1 & 1 & a \\ 0 & 1 & 2 & b - a \\ 0 & 0 & 0 & c - 2b - a \end{bmatrix} R_3 = R_3 - 2R_2$$

In order that the system should have one parameter family of solutions, we must have $\rho(AB) = \rho(A) = 2$. So, the third row in the echelon form should be a zero row. So, $c - 2b - a = 0$ hence $c = 2b + a = 0$.

Example 1.34

Investigate for what values of λ and μ the system of linear equations $x + 2y + z = 7$, $x + y + \lambda z = \mu, x + 3y - 5z = 5$ has

(i) no solution (ii) a unique solution

(iii) an infinite number of solutions.

Solution:

Here the number of unknowns is 3.

The matrix form of the system is $AX = B$, where

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & \lambda \\ 1 & 3 & -5 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, B = \begin{bmatrix} 7 \\ \mu \\ 5 \end{bmatrix}$$

The augmented matrix to echelon form,

$$\text{we get } [AB] = \begin{bmatrix} 1 & 2 & 1 & 7 \\ 1 & 1 & \lambda & \mu \\ 1 & 3 & -5 & 5 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 1 & 1 & a \\ 0 & -1 & \lambda - 1 & \mu - 7 \\ 0 & 1 & -6 & -2 \end{bmatrix} \begin{matrix} R_2 = R_2 - R_1 \\ R_3 = R_3 - R_1 \end{matrix}$$

$$\to \begin{bmatrix} 1 & 1 & 1 & a \\ 0 & 2 & -6 & -2 \\ 0 & 0 & \lambda - 7 & \mu - 9 \end{bmatrix} R_3 = R_3 + R_2$$

(i) If $\lambda = 7$ and $\mu \neq 9$, then $\rho(AB) = 3$ *and*

$\rho(A) = 2. \rho(AB) \neq \rho(A)$

The given equation is **inconsistent**, has

nosolution.

(ii) If $\lambda \neq 7$ and μ has any value, then

$\rho(AB) = \rho(A) = 3, n = 3.$

The given equation is **consistent**,

has **unique solution.**

(iii) If $\lambda = 7$ and $\mu = 9$, then

$\rho(AB) = \rho(A) = 2, n < 3.$

The given equation is **consistent**,

has **infinite solutions.**

EXERCISE 1.6

1. Test for consistency and if possible, solve the following systems of equations by rank method.

(i) $x - y + 2z = 2,\ 2x + y + 4z = 7$

$4x - y + z = 4$

Solution:

Here the number of unknowns is 3.

The matrix form of the system is $AX = B$,

where

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 2 & 1 & 4 \\ 4 & -1 & 1 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, B = \begin{bmatrix} 2 \\ 7 \\ 4 \end{bmatrix}$$

The augmented matrix to echelon form,

$$\text{we get } [AB] = \begin{bmatrix} 1 & -1 & 2 & 2 \\ 2 & 1 & 4 & 7 \\ 4 & -1 & 1 & 4 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & -1 & 2 & 2 \\ 0 & 3 & 0 & 3 \\ 0 & 3 & -7 & -4 \end{bmatrix} \begin{matrix} R_2 = R_2 - 2R_1 \\ R_3 = R_3 - 4R_1 \end{matrix}$$

$$\to \begin{bmatrix} 1 & -1 & 2 & 2 \\ 0 & 3 & 0 & 3 \\ 0 & 0 & -7 & -7 \end{bmatrix} R_3 = R_3 - R_2$$

There are three non-zero rows in the row-echelon form of $[AB]$. So, $\rho[AB] = \rho[A] = 3$

$\rho(AB) = \rho(A) = 3, n = 3$ The given equation is consistent, has unique solution.

$-7z = -7 \Rightarrow 7z = 7$

$z = \frac{7}{7} = 1$

sub z = 1 in

3y + 0z = 3

3y = 3

$y = \frac{3}{3} = 1$

sub z = 1and y = 1 in $x - y + 2z = 2$

$x - 1 + 2(1) = 2$

$x - 1 + 2 = 2$

$x + 1 = 2$

$x = 2 - 1$

$x = 1$

So the solution is $x = 1.\ y = 1$, and $z = 1$.

(ii) $3x + y + z = 2,\ x - 3y + 2z = 1$

$7x - y + 4z = 5$

Solution:
Here the number of unknowns is 3.

The matrix form of the system is $AX = B$,

where

$$A = \begin{bmatrix} 3 & 1 & 1 \\ 1 & -3 & 2 \\ 7 & -1 & 4 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, B = \begin{bmatrix} 2 \\ 1 \\ 5 \end{bmatrix}$$

The augmented matrix to echelon form,

we get $[AB] = \begin{bmatrix} 3 & 1 & 1 & 2 \\ 1 & -3 & 2 & 1 \\ 7 & -1 & 4 & 5 \end{bmatrix}$

$$\rightarrow \begin{bmatrix} 1 & -3 & 2 & 1 \\ 3 & 1 & 1 & 2 \\ 7 & -1 & 4 & 5 \end{bmatrix} R_1 \leftrightarrow R_2$$

$$\rightarrow \begin{bmatrix} 1 & -3 & 2 & 1 \\ 0 & 10 & -5 & -1 \\ 0 & 20 & -10 & -2 \end{bmatrix} \begin{matrix} R_2 = R_2 - 3R_1 \\ R_3 = R_3 - 7R_1 \end{matrix}$$

$$\rightarrow \begin{bmatrix} 1 & -3 & 2 & 1 \\ 0 & 10 & -5 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} R_3 = R_3 - 2R_2$$

There are two non-zero rows in the row-echelon form of $[AB]$. So, $\rho[AB] = \rho[A] = 2$

$\rho(AB) = \rho(A) = 2, n = 3$ The given equation is consistent, has infinitely many solutions.

$x - 3y + 2z = 1 \quad ....(1)$

$10y - 5z = -1 \quad ....(2)$

To solve the equations let z = t,

then $10y - 5t = -1$

$10\ y = -1 + 5t \Rightarrow 5t - 1$

$y = \frac{5t-1}{10}$

substituting

$z = t, y = \frac{5t-1}{10}\ in\ x - 3y + 2z = 1$

$x - 3\left(\frac{5t-1}{10}\right) + 2t = 1$

$x = 1 - 2t + 3\left(\frac{5t-1}{10}\right)$

$= 1 - 2t + \frac{15t-3}{10}$

$= \frac{10-20t+15t-3}{10}$

$x = \frac{7-5t}{10}$

Solution:

$x = \frac{7-5t}{10}, y = \frac{5t-1}{10}$ and z = t where $t \in R$

(iii) $2x + 2y + z = 5,\ x - y + z = 1$

$3x + y + 2z = 4$

Solution:
Here the number of unknowns is 3.

The matrix form of the system is $AX = B$,

where

$$A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & -1 & 1 \\ 3 & 1 & 2 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, B = \begin{bmatrix} 5 \\ 1 \\ 4 \end{bmatrix}$$

The augmented matrix to echelon form,

$$\text{we get } [AB] = \begin{bmatrix} 2 & 2 & 1 & 5 \\ 1 & -1 & 1 & 1 \\ 3 & 1 & 2 & 4 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & -1 & 1 & 1 \\ 2 & 2 & 1 & 5 \\ 3 & 1 & 2 & 4 \end{bmatrix} R_1 \leftrightarrow R_2$$

$$\rightarrow \begin{bmatrix} 1 & -1 & 1 & 1 \\ 0 & 4 & -1 & 3 \\ 0 & 4 & -1 & 1 \end{bmatrix} \begin{matrix} R_2 = R_2 - 2R_1 \\ R_3 = R_3 - 3R_1 \end{matrix}$$

$$\rightarrow \begin{bmatrix} 1 & -1 & 1 & 1 \\ 0 & 4 & -1 & 3 \\ 0 & 0 & 0 & -2 \end{bmatrix} R_3 = R_3 - R_2$$

$$\rho(AB) = 3 \; and \; \rho(A) = 2$$

$\rho(AB) \neq \rho(A)$ The given equation is **inconsistent**, has **no solutions.**

(iv) $2x - y + z = 2,\; 6x - 3y + 3z = 6$

$4x - 2y + 2z = 4$

Solution:

Here the number of unknowns is 3.

The matrix form of the system is $AX = B$,

where

$$A = \begin{bmatrix} 2 & -1 & 1 \\ 6 & -3 & 3 \\ 4 & -2 & 2 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, B = \begin{bmatrix} 2 \\ 6 \\ 4 \end{bmatrix}$$

The augmented matrix to echelon form,

$$\text{we get } [AB] = \begin{bmatrix} 2 & -1 & 1 & 2 \\ 6 & -3 & 3 & 6 \\ 4 & -2 & 2 & 4 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 2 & -1 & 1 & 2 \\ 2 & -1 & 1 & 2 \\ 2 & -1 & 1 & 2 \end{bmatrix} \begin{matrix} R_2 = R_2 \div 3 \\ R_3 = R_3 \div 2 \end{matrix}$$

$$\rightarrow \begin{bmatrix} 2 & -1 & 1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} R_2 = R_2 - R_1 \\ R_3 = R_3 - R_1 \end{matrix}$$

There is one non-zero row in the row-echelon form of $[AB]$. So, $\rho[AB] = \rho[A] = 1$

$\rho(AB) = \rho(A) = 1, n = 3$ The given equation is consistent, has infinitely many solutions.

$$2x - y + z = 2$$

To solve the equations let $y = s, z = t$,

$$2x - s + t = 2$$

$$2x = 2 + s - t$$

$$x = \frac{2+s-t}{2}$$

So the solution

$x = \frac{2+s-t}{2}, y = s, z = t \; where \; s, t \in R$

2. Find the value of k for which the equations $kx - 2y + z = 1$, $x - 2ky + z = -2, x - 2y + kz = 1$, have (i) no solution (ii) unique solution

(iii) infinitely many solution.

Solution: Here the number of unknowns is 3.The matrix form of the system is $AX = B$, where

$$A = \begin{bmatrix} k & -2 & 1 \\ 1 & -2k & 1 \\ 1 & -2 & k \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, B = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

The augmented matrix to echelon form,

$$\text{we get } [AB] = \begin{bmatrix} k & -2 & 1 & 1 \\ 1 & -2k & 1 & -2 \\ 1 & -2 & k & 1 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & -2 & k & 1 \\ 1 & -2k & 1 & -2 \\ k & -2 & 1 & 1 \end{bmatrix} R_1 \leftrightarrow R_3$$

$$\rightarrow \begin{bmatrix} 1 & -2 & k & 1 \\ 0 & 2-2k & 1-k & -3 \\ 0 & -2+2k & 1-k^2 & 1-k \end{bmatrix} \begin{matrix} R_2 = R_2 - R_1 \\ R_3 = R_3 - kR_1 \end{matrix}$$

$$\rightarrow \begin{bmatrix} 1 & -2 & k & 1 \\ 0 & 2-2k & 1-k & -3 \\ 0 & 0 & 2-k-k^2 & -2-k \end{bmatrix} R_3 = R_3 + R_2$$

$$\rightarrow \begin{bmatrix} 1 & -2 & k & 1 \\ 0 & 2(1-k) & (1-k) & -3 \\ 0 & 0 & (2+k)(1-k) & -(2+k) \end{bmatrix}$$

(i)When k =1, $\rho(AB) = 3$ *and* $\rho(A) = 2$

$\rho(AB) \neq \rho(A)$ The given equation is

inconsistent, has **no solutions.**

(ii) $k \neq 1$ *and* $k \neq -2$ $\rho(AB) = \rho(A) = 3$

and $n = 3$ The given equation is

consistent, has unique solution.

(iii) $k = -2, \rho(AB) = \rho(A) = 2$

and $n = 3$ The given equation is

consistent, has infinitely many solutions.

3. Investigate the values of λ and μ the system of linear equations

$2x + 3y + 5z = 9, 7x + 3y - 5z = 8,$

$2x + 3y + \lambda z = \mu$, have (i) no solution

(ii) a unique solution (iii) an infinite number of solutions.

Solution: Here the number of unknowns is 3. The matrix form of the system is $AX = B$ where

$$A = \begin{bmatrix} 2 & 3 & 5 \\ 7 & 3 & -5 \\ 2 & 3 & \lambda \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, B = \begin{bmatrix} 9 \\ 8 \\ \mu \end{bmatrix}$$

The augmented matrix to echelon form,

we get $[AB] = \begin{bmatrix} 2 & 3 & 5 & 9 \\ 7 & 3 & -5 & 8 \\ 2 & 3 & \lambda & \mu \end{bmatrix}$

$$\rightarrow \begin{bmatrix} 2 & 3 & 5 & 9 \\ 0 & -15 & -45 & -47 \\ 0 & 0 & \lambda-5 & \mu-9 \end{bmatrix} \begin{matrix} R_2 = 2R_2 - 7R_1 \\ R_3 = R_3 - R_1 \end{matrix}$$

(i)When λ =5, $\rho(AB) = 3$ *and* $\rho(A) = 2$

$\rho(AB) \neq \rho(A)$ The given equation is

inconsistent, has **no solutions.**

(ii) $\lambda \neq 5$ *and* $\mu \neq 9$ $\rho(AB) = \rho(A) = 3$

and $n = 3$ The given equation is

consistent, has **unique solution.**

(iii) $\lambda = 5$ *and* $\mu = 9$, $\rho(AB) = \rho(A) = 2$

and $n = 3$ The given equation is

consistent, has **infinitely many solutions.**

Example 1.35 Solve the following system:

$x + 2y + 3z = 0, 3x + 4y + 4z = 0,$

$7x + 10y + 12z = 0.$

Solution

Here the number of equations is equal to the number of unknowns.

Transforming into echelon form, the augmented matrix becomes

$$\begin{bmatrix} 1 & 2 & 3 & 0 \\ 3 & 4 & 4 & 0 \\ 7 & 10 & 12 & 0 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & -2 & -5 & 0 \\ 0 & -4 & -9 & 0 \end{bmatrix} \begin{matrix} R_2 = R_2 - 3R_1 \\ R_3 = R_3 - 7R_1 \end{matrix}$$

$$\rightarrow \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & -2 & -5 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} R_3 = R_3 - 2R_2$$

So, $\rho(AB) = \rho(A) = 3$ and $n = 3$.

Hence, the system has a unique solution.

Since $x = 0, y = 0, z = 0$, is always a solution of the homogeneous system, the only solution is the trivial solution $x = 0, y = 0, z = 0$.

Example 1.36 Solve the following system:

$x + 3y - 2z = 0, 2x - y + 4z = 0,$

$x - 11y + 14z = 0.$

Solution

Here the number of equations is equal to the number of unknowns.

Transforming into echelon form, the augmented matrix becomes

$$\begin{bmatrix} 1 & 3 & -2 & 0 \\ 2 & -1 & 4 & 0 \\ 1 & -11 & 14 & 0 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & 3 & -2 & 0 \\ 0 & -7 & 8 & 0 \\ 0 & -14 & 16 & 0 \end{bmatrix} \begin{matrix} R_2 = R_2 - 2R_1 \\ R_3 = R_3 - R_1 \end{matrix}$$

$$\rightarrow \begin{bmatrix} 1 & 3 & -2 & 0 \\ 0 & -7 & 8 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} R_3 = R_3 - 2R_2$$

So, $\rho(AB) = \rho(A) = 2$ and $n = 3$.

Hence, the system has a one parameter family of solutions. Writing the equations using the echelon form, we get

$x + 3y - 2z = 0$ and $-7y + 8z = 0$

To solve the equations let z = t,

then $-7y + 8t = 0$

$$-7y = -8t$$

$$7y = 8t$$

$$y = \frac{8t}{7}$$

substituting

$$z = t, y = \frac{8t}{7} \; in \; x + 3y - 2z = 0$$

$$x + 3\left(\frac{8t}{7}\right) - 2t = 0$$

$$x + \frac{24t}{7} - 2t = 0$$

$$x = 2t - \frac{24t}{7}$$

$$x = \frac{14t - 24t}{7}$$

$$x = -\frac{10t}{7}$$

So the solution is

$x = -\frac{10t}{7}, y = \frac{8t}{7}$ and z = t where $t \in R$

Example 1.37 Solve the following system:

$x + y - 2z = 0, 2x - 3y + z = 0,$

$3x - 7y + 10z = 0, 6x - 9y + 10z = 0.$

Solution

Here the number of equations is equal to the number of unknowns.

Transforming into echelon form, the augmented matrix becomes

$$\begin{bmatrix} 1 & 1 & -2 & 0 \\ 2 & -3 & 1 & 0 \\ 3 & -7 & 10 & 0 \\ 6 & -9 & 10 & 0 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & 1 & -2 & 0 \\ 0 & -5 & 5 & 0 \\ 0 & -10 & 16 & 0 \\ 0 & -15 & 22 & 0 \end{bmatrix} \begin{matrix} R_2 = R_2 - 2R_1 \\ R_3 = R_3 - 3R_1 \\ R_4 = R_4 - 6R_1 \end{matrix}$$

$$\rightarrow \begin{bmatrix} 1 & 1 & -2 & 0 \\ 0 & -5 & 5 & 0 \\ 0 & 0 & 6 & 0 \\ 0 & 0 & 7 & 0 \end{bmatrix} \begin{matrix} R_3 = R_3 - 2R_2 \\ R_4 = R_4 - 3R_2 \end{matrix}$$

$$\rightarrow \begin{bmatrix} 1 & 1 & -2 & 0 \\ 0 & -5 & 5 & 0 \\ 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} R_4 = 6R_4 - 7R_3$$

So, $\rho(AB) = \rho(A) = 3$ and $n = 3$.

Hence, the system has a unique solution.

Since $x = 0, y = 0, z = 0$, is always a solution of the homogeneous system, the only solution is the trivial solution $x = 0, y = 0, z = 0$.

Example 1.38 Determine the values of λ for which the following system of equations

$(3\lambda - 8)x + 3y + 3z = 0,$

$3x + (3\lambda - 8)y + 3z = 0,$

$3x + 3y + (3\lambda - 8)z = 0.$

Solution

Here the number of unknowns is 3. So, if the system is consistent and has a non-trivial solution, then the rank of the coefficient matrix is equal to the rank of the augmented matrix and is less than 3. So the determinant of the coefficient matrix should be 0.

Hence we get

$$\begin{vmatrix} (3\lambda-8) & 3 & 3 \\ 3 & (3\lambda-8) & 3 \\ 3 & 3 & (3\lambda-8) \end{vmatrix} = 0$$

$R_1 = R_1 + R_2 + R_3$

$$\rightarrow \begin{vmatrix} 3\lambda-2 & 3\lambda-2 & 3\lambda-2 \\ 3 & (3\lambda-8) & 3 \\ 3 & 3 & (3\lambda-8) \end{vmatrix} = 0$$

Taking $(3\lambda - 2)$from R_1

$$\rightarrow (3\lambda-2)\begin{vmatrix} 1 & 1 & 1 \\ 3 & (3\lambda-8) & 3 \\ 3 & 3 & (3\lambda-8) \end{vmatrix} = 0$$

$C_2 = C_1$ and $C_3 = C_1$

$$\rightarrow (3\lambda-2)\begin{vmatrix} 1 & 0 & 1 \\ 3 & 3\lambda-11 & 0 \\ 3 & 0 & 3\lambda-11 \end{vmatrix} = 0$$

$$(3\lambda - 2)(3\lambda - 11)^2 = 0$$

$$3\lambda - 2 = 0 \text{ and } 3\lambda - 11 = 0$$

Gives

$$3\lambda = 2 \Longrightarrow \lambda = \tfrac{2}{3} \text{ and } 3\lambda = 11 \Longrightarrow \lambda = \tfrac{11}{3}$$

Example 1.39 By using Gaussian elimination method, balance the chemical reaction equation: $C_5 + H_8 \rightarrow CO_2 + H_2O$

Solution

We are searching for positive integers x_1, x_2, x_3 and x_4 such that

$x_1\, C_5 + x_2 H_8 \rightarrow x_3 CO_2 + x_4 H_2 O$ ....(1)

The number of carbon atoms on the left-hand side of (1) should be equal to the number of carbon atoms on the right-hand side of (1). So we get a linear homogenous equation

$5x_1 = x_3$ gives $5x_1 - x_3 = 0$ ....(2)

Similarly, considering hydrogen and oxygen atoms, we get respectively,

$8x_2 = 2x_4$ gives $4x_2 - x_4 = 0$ ....(3)

$2x_2 = 2x_3 + x_4$ gives

$2x_2 - 2x_3 - x_4 = 0$ ....(4)

Equations (2), (3), and (4) constitute a homogeneous system of linear equations in four unknowns. The augmented matrix is

$$[AB] = \begin{bmatrix} 5 & 0 & -1 & 0 & 0 \\ 4 & 0 & 0 & -1 & 0 \\ 0 & 2 & -2 & -1 & 0 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 5 & 0 & -1 & 0 & 0 \\ 0 & 0 & 4 & -5 & 0 \\ 0 & 2 & -2 & -1 & 0 \end{bmatrix} R_2 = 5R_2 - 4R_1$$

$$\rightarrow \begin{bmatrix} 5 & 0 & -1 & 0 & 0 \\ 0 & 2 & -2 & -1 & 0 \\ 0 & 0 & 4 & -5 & 0 \end{bmatrix} R_2 \leftrightarrow R_3$$

$\rho(AB) = \rho(A) = 3, n = 4$ The given equation is consistent, has infinitely many solutions. Writing the equations using the echelon form, we get

$5x_1 - x_3 = 0, 2x_2 - 2x_3 - x_4 = 0$ and

$4x_3 - 5x_4 = 0.$

Substituting $x_4 = t$ in $4x_3 - 5x_4 = 0$

$$4x_3 - 5t = 0$$

$$4x_3 = 5t$$

$$x_3 = \frac{5t}{4}$$

Substituting $x_4 = t$ and $x_3 = \frac{5t}{4}$ in

$$2x_2 - 2x_3 - x_4 = 0$$

$$2x_2 - 2\left(\frac{5t}{4}\right) - t = 0$$

$$2x_2 - \frac{5t}{2} - t = 0$$

$$2x_2 = \frac{5t}{2} + t$$

$$= \frac{5t+2t}{2}$$

$$= \frac{7t}{2}$$

$$x_2 = \frac{7t}{4}$$

Substituting $x_3 = \frac{5t}{4}$ in

$$5x_1 - x_3 = 0$$

$$5x_1 - \frac{5t}{4} = 0$$

$$5x_1 = \frac{5t}{4} \text{ gives}$$

$$x_1 = \frac{t}{4}$$

So, $x_1 = \frac{t}{4}, x_2 = \frac{7t}{4}, x_3 = \frac{5t}{4}$ and $x_4 = t$

Let us choose $t = 4$. Then

$x_1 = 1, x_2 = 7, x_3 = 5$ and $x_4 = 4$

So the balanced equation is

$$C_5 + 7H_8 \rightarrow 5CO_2 + 4H_2O.$$

Example 1.40 If the system of equations

$px + by + cz = 0, ax + qy + cz = 0$ and

$ax + by + rz = 0$, has a non-trivial solution and $p \neq a, q \neq b$ and $r \neq c$, prove that

$$\frac{p}{p-a} + \frac{q}{q-b} + \frac{r}{r-c} = 2$$

Solution

The system $px + by + cz = 0$, $ax + qy + cz = 0$ and $ax + by + rz = 0$ has a non-trivial solution. So we have

$$\begin{vmatrix} p & b & c \\ a & q & c \\ a & b & r \end{vmatrix} = 0$$

$R_2 = R_2 - R_1, R_3 = R_3 - R_1$

$$\begin{vmatrix} p & b & c \\ a-p & q-b & 0 \\ a-p & 0 & r-c \end{vmatrix} = 0$$

$$\begin{vmatrix} p & b & c \\ -(p-a) & q-b & 0 \\ -(p-a) & 0 & r-c \end{vmatrix} = 0$$

Dividing C_1 by $(p-a)$, C_2 by $(q-b)$ and C_3 by $(r-c)$

$$\begin{vmatrix} \frac{p}{p-a} & \frac{b}{q-b} & \frac{c}{r-c} \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{vmatrix} = 0$$

$$\frac{p}{p-a}\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} - \frac{b}{q-b}\begin{vmatrix} -1 & 0 \\ -1 & 1 \end{vmatrix} + \frac{c}{r-c}\begin{vmatrix} -1 & 1 \\ -1 & 0 \end{vmatrix} = 0$$

$$\frac{p}{p-a}(1-0) - \frac{b}{q-b}(-1-0) + \frac{c}{r-c}(0+1) = 0$$

$$\frac{p}{p-a}(1) - \frac{b}{q-b}(-1) + \frac{c}{r-c}(1) = 0$$

$$\frac{p}{p-a}+\frac{b}{q-b}+\frac{c}{r-c}=0$$

$$\frac{p}{p-a}+\frac{q-(q-b)}{q-b}+\frac{r-(r-c)}{r-c}=0$$

$$\frac{p}{p-a}+\frac{q}{q-b}-\frac{q-b}{q-b}+\frac{r}{r-c}-\frac{r-c}{r-c}=0$$

$$\frac{p}{p-a}+\frac{q}{q-b}-1+\frac{r}{r-c}-1=0$$

$$\frac{p}{p-a}+\frac{q}{q-b}+\frac{r}{r-c}-2=0$$

$\frac{p}{p-a}+\frac{q}{q-b}+\frac{r}{r-c}=2$. Hence proved.

EXERCISE 1.7

1. Solve the following system of homogenous equations.

(i) $3x+2y+7z=0,\ 4x-3y-2z=0$

$5x+9y+23z=0,$

Solution

Here the number of equations is equal to the number of unknowns.

Transforming into echelon form, the augmented matrix becomes

$$\begin{bmatrix}3 & 2 & 7 & 0\\ 4 & -3 & -2 & 0\\ 5 & 9 & 23 & 0\end{bmatrix}$$

$$\rightarrow\begin{bmatrix}3 & 2 & 7 & 0\\ 0 & -17 & -34 & 0\\ 0 & 17 & 34 & 0\end{bmatrix}\begin{matrix}R_2=3R_2-4R_1\\ R_3=3R_3-5R_1\end{matrix}$$

$$\rightarrow\begin{bmatrix}3 & 2 & 7 & 0\\ 0 & -17 & -34 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}R_3=R_3+R_2$$

So, $\rho(AB)=\rho(A)=2$ and $n=3$.

Hence, the system has a one parameter family of solutions. Writing the equations using the echelon form, we get

$3x+2y+7z=0$ and $-17y-34z=0$

To solve the equations let z = t,

then $-17y-34t=0$

$$-17y=34t$$

$$y=-\frac{34t}{17}$$

$$y=-2t$$

Substituting $z=t$ and $y=-2t$ in

$$3x+2y+7z=0$$

We get $3x+2(-2t)+7t=0$

$$3x-4t+7t=0$$

$$3x+3t=0$$

$$3x=-3t$$

$$x=-\frac{3t}{3}$$

$$x=-t$$

So the solution is

$x=-t,\ y=-2t$ and z = t where $t\in R$

(ii) $2x+3y-z=0,\ x-y-2z=0$

$3x+y+3z=0,$

Solution

Here the number of equations is equal to the number of unknowns. Transforming into echelon form, the augmented matrix becomes

$$\begin{bmatrix}2 & 3 & -1 & 0\\ 1 & -1 & -2 & 0\\ 3 & 1 & 3 & 0\end{bmatrix}$$

$$\rightarrow\begin{bmatrix}1 & -1 & -2 & 0\\ 2 & 3 & -1 & 0\\ 3 & 1 & 3 & 0\end{bmatrix}R_1\leftrightarrow R_2$$

$$\rightarrow\begin{bmatrix}1 & -1 & -2 & 0\\ 0 & 5 & 3 & 0\\ 0 & 4 & 9 & 0\end{bmatrix}\begin{matrix}R_2=R_2-2R_1\\ R_3=R_3-3R_1\end{matrix}$$

$$\rightarrow \begin{bmatrix} 1 & -1 & -2 & 0 \\ 0 & 5 & 3 & 0 \\ 0 & 0 & 33 & 0 \end{bmatrix} R_3 = 5R_3 - 4R_2$$

So, $\rho(AB) = \rho(A) = 3$ and $n = 3$.

Hence, the system has a unique solution.

Since $x = 0, y = 0, z = 0$, is always a solution of the homogeneous system, the only solution is the trivial solution $x = 0, y = 0, z = 0$.

2. Determine the values of λ for which the following system of equations $x + y + 3z = 0, \quad 4x + 3y + \lambda z = 0,$ $2x + y + 2z = 0$,has (i) a unique solution (ii) a non-trivial solution.

Solution :Here the number of equations is equal to the number of unknowns.

Transforming into echelon form, the augmented matrix becomes

$$\begin{bmatrix} 1 & 1 & 3 & 0 \\ 4 & 3 & \lambda & 0 \\ 2 & 1 & 2 & 0 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & 1 & 3 & 0 \\ 0 & -1 & \lambda - 12 & 0 \\ 0 & -1 & -4 & 0 \end{bmatrix} \begin{matrix} R_2 = R_2 - 4R_1 \\ R_3 = R_3 - 2R_1 \end{matrix}$$

$$\rightarrow \begin{bmatrix} 1 & 1 & 3 & 0 \\ 0 & -1 & \lambda - 12 & 0 \\ 0 & 0 & \lambda - 8 & 0 \end{bmatrix} R_3 = R_3 - R_2$$

(i) $\lambda \neq 8, \ \rho(AB) = \rho(A) = 3$ and $n = 3$

The given equation is **consistent**, has **unique solution.**

(ii) $\lambda = 8, \ \rho(AB) = \rho(A) = 2$ and $n = 3$

The given equation is **consistent**, has a non-trivial solution.

3. By using Gaussian elimination method, balance the chemical reaction equation:

$C_2H_6 + O_2 \rightarrow H_2O + CO_2$.

Solution

We are searching for positive integers x_1, x_2, x_3 and x_4 such that

$x_1 \, C_2H_6 + x_2O_2 \rightarrow x_3H_2O + x_4CO_2$. ....(1)

The number of carbon atoms on the left-hand side of (1) should be equal to the number of carbon atoms on the right-hand side of (1). So we get a linear homogenous equation

$2x_1 = x_4$ gives $2x_1 - x_4 = 0$ ....(2)

Similarly, considering hydrogen and oxygen atoms, we get respectively,

$6x_1 = 2x_3$ gives $6x_1 - 2x_3 = 0$ ....(3)

$2x_2 = x_3 + 2x_4$ gives

$$2x_2 - x_3 - 2x_4 = 0 \quad ....(4)$$

Equations (2), (3), and (4) constitute a homogeneous system of linear equations in four unknowns. The augmented matrix is

$$[AB] = \begin{bmatrix} 2 & 0 & 0 & -1 & 0 \\ 3 & 0 & -1 & 0 & 0 \\ 0 & 2 & -1 & -2 & 0 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 2 & 0 & 0 & -1 & 0 \\ 0 & 0 & -2 & 3 & 0 \\ 0 & 2 & -1 & -2 & 0 \end{bmatrix} \mathrm{R}_2 = 2\mathrm{R}_2 - 3\mathrm{R}_1$$

$$\rightarrow \begin{bmatrix} 2 & 0 & 0 & -1 & 0 \\ 0 & 2 & -1 & -2 & 0 \\ 0 & 0 & -2 & 3 & 0 \end{bmatrix} R_2 \leftrightarrow R_3$$

$\rho(AB) = \rho(A) = 3, n = 4$ The given equation is consistent, has infinitely many solutions. Writing the equations using the echelon form, we get

$-2x_3 + 3x_4 = 0$, $2x_2 - x_3 - 2x_4 = 0$ and $2x_1 - x_4 = 0$.

Substituting $x_4 = t$ in $2x_1 - x_4 = 0$

$$2x_1 - t = 0$$
$$2x_1 = t$$
$$x_1 = \frac{t}{2}$$

Substituting $x_4 = t$ in $-2x_3 + 3x_4 = 0$

$$-2x_3 + 3t = 0$$
$$-2x_3 = -3t$$
$$2x_3 = 3t$$
$$x_3 = \frac{3t}{2}$$

Substituting $x_1 = \frac{t}{2}, x_3 = \frac{3t}{2}$ and $x_4 = t$ in

$$2x_2 - x_3 - 2x_4 = 0$$
$$2x_2 - \frac{3t}{2} - 2t = 0$$
$$2x_2 = \frac{3t}{2} + 2t$$
$$2x_2 = \frac{3t+4t}{2}$$
$$2x_2 = \frac{7t}{2}$$
$$x_2 = \frac{7t}{4}$$

So, $x_1 = \frac{t}{2}, x_2 = \frac{7t}{4}, x_3 = \frac{3t}{2}$ and $x_4 = t$

Let us choose $t = 4$. Then

$x_1 = 2, x_2 = 7, x_3 = 6$ and $x_4 = 4$

So the balanced equation is

$2\,C_2H_6 + 7O_2 \rightarrow 6H_2O + 4CO_2$.

EXERCISE 1.8

Choose the Correct answer:

1. If $|adj(adjA)| = |A|^9$ then the order of the square matrix A is .....

(1) 3 **(2) 4** (3) 2 (4) 5

2. If A is a 3×3 non-singular matrix such that A $A^T = A^T A$ and B $= A^{-1}A^T$, then $BB^T = \ldots\ldots$

(1) A (2) B **(3) I** (4) B^T

3. If A $= \begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix}$, B = adj A and C = 3A, the $\frac{|adjB|}{|C|} = \ldots\ldots$

(1) $\frac{1}{3}$ **(2) $\frac{1}{9}$** (3) $\frac{1}{4}$ (4) 1

4. If A $\begin{bmatrix} 1 & -2 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} 6 & 0 \\ 0 & 6 \end{bmatrix}$, then $A = \ldots\ldots$

(1) $\begin{bmatrix} 1 & -2 \\ 1 & 4 \end{bmatrix}$ (2) $\begin{bmatrix} 1 & 2 \\ -1 & 4 \end{bmatrix}$

(3) $\begin{bmatrix} 4 & 2 \\ -1 & 1 \end{bmatrix}$ (4) $\begin{bmatrix} 4 & -1 \\ 2 & 1 \end{bmatrix}$

5. If A $= \begin{bmatrix} 7 & 3 \\ 4 & 2 \end{bmatrix}$, then 9I − A =

(1) A^{-1} (2) $\frac{A^{-1}}{2}$ (3) $3A^{-1}$ **(4) $2\,A^{-1}$**

6. If $A = \begin{bmatrix} 2 & 0 \\ 1 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 4 \\ 2 & 0 \end{bmatrix}$ then

| adj (AB) | =

(1) −40 **(2) −80** (3) −60 (4) −20

7. If P $= \begin{bmatrix} 1 & x & 0 \\ 1 & 3 & 0 \\ 2 & 4 & -2 \end{bmatrix}$ is the adjoint of 3×3 matrix A and | A | = 4, then x is

(1) 15 (2) 12 (3) 14 **(4) 11**

8. If A $= \begin{bmatrix} 3 & 1 & -1 \\ 2 & -2 & 0 \\ 1 & 2 & -1 \end{bmatrix}$ and

$A^{-1} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ then the value of a_{23} is .....

(1) 0 (2) -2 (3) -3 **(4) -1**

9. If A, B and C are invertible matrices of some order, then which one of the following is not true?

(1) adj A = | A| $A^{-1}1$

(2) adj (AB) = adj (A)adj (B)

(3) det $A^{-1} = (detA)^{-1}$

(4) $(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$

10. If $(AB)^{-1} = \begin{bmatrix} 12 & -17 \\ -19 & 27 \end{bmatrix}$ and $A^{-1} = \begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix}$, then B^{-1} = ...

(1) $\begin{bmatrix} 2 & -5 \\ -3 & 8 \end{bmatrix}$ (2) $\begin{bmatrix} 8 & 5 \\ 3 & 2 \end{bmatrix}$

(3) $\begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}$ (4) $\begin{bmatrix} 8 & -5 \\ -3 & 2 \end{bmatrix}$

11. If A^TA^{-1} is symmetric, then A^2 =......

(1) A^{-1} **(2) $(A^T)^2$**

(3) A^T (4) $(A^{-1})^2$

12. If A is a non-singular matrix such that $A^{-1} = \begin{bmatrix} 5 & 3 \\ -2 & -1 \end{bmatrix}$, then $(A^T)^{-1}$ = ...

(1) $\begin{bmatrix} -5 & 3 \\ 2 & 1 \end{bmatrix}$ (2) $\begin{bmatrix} 5 & 3 \\ -2 & -1 \end{bmatrix}$

(3) $\begin{bmatrix} -1 & -3 \\ 2 & 5 \end{bmatrix}$ **(4) $\begin{bmatrix} 5 & -2 \\ 3 & -1 \end{bmatrix}$**

13. If A = $\begin{bmatrix} \frac{3}{5} & \frac{4}{5} \\ x & \frac{3}{5} \end{bmatrix}$ and $A^T = A^{-1}$, then the value of x is .....

(1) $-\frac{4}{5}$ (2) $-\frac{3}{5}$ (3) $\frac{3}{5}$ (4) $\frac{4}{5}$

14. If A = $\begin{bmatrix} 1 & \tan\frac{\theta}{2} \\ -\tan\frac{\theta}{2} & 1 \end{bmatrix}$ and AB = I, then B= ....

(1) $\left(cos^2\frac{\theta}{2}\right)A$ **(2) $\left(cos^2\frac{\theta}{2}\right)A^T$**

(3) $(cos^2\theta)I$ (4) $\left(sin^2\frac{\theta}{2}\right)A$

15. If A = $\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$ and A$(adjA) = \begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}$, then k is ....

(1) 0 (2) sinθ (3) cosθ **(4) 1**

16. If A = $\begin{bmatrix} 2 & 3 \\ 5 & -2 \end{bmatrix}$ be such that A^{-1}, then λ is .....

(1) 17 (2) 14 **(3) 19** (4) 21

17. If adj A = $\begin{bmatrix} 2 & 3 \\ 4 & -1 \end{bmatrix}$ and adjB = $\begin{bmatrix} 1 & -2 \\ -3 & 1 \end{bmatrix}$ then adj(AB) is .....

(1) $\begin{bmatrix} -7 & -1 \\ 7 & -9 \end{bmatrix}$ **(2) $\begin{bmatrix} -6 & 5 \\ -2 & -10 \end{bmatrix}$**

(3) $\begin{bmatrix} -7 & 7 \\ -1 & -9 \end{bmatrix}$ (4) $\begin{bmatrix} -6 & -2 \\ 5 & -10 \end{bmatrix}$

18. The rank of the matrix $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 6 & 8 \\ -1 & -2 & -3 & -4 \end{bmatrix}$ is ......

(1) 1 (2) 2 (3) 4 (4) 3

19. If $x^ay^b = e^m, x^cy^d = e^n$, $\Delta_1 = \begin{vmatrix} m & b \\ n & d \end{vmatrix}$, $\Delta_2 = \begin{vmatrix} a & m \\ c & n \end{vmatrix}$, $\Delta_3 = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$ then

the values of x and y are respectively

(1) $e^{\Delta_2/\Delta_1}, e^{\Delta_3/\Delta_1}$

(2) $\log(\Delta_1/\Delta_3), \log(\Delta_2/\Delta_3)$

(3) $\log(\Delta_2/\Delta_1), \log(\Delta_3/\Delta_1)$

(4) $e^{\Delta_1/\Delta_3}, e^{\Delta_2/\Delta_3}$

20. Which of the following is/are correct?

(i) Adjoint of a symmetric matrix is also a symmetric matrix.

(ii) Adjoint of a diagonal matrix is also a diagonal matrix.

(iii) If A is a square matrix of order n and λ is a scalar, then $\operatorname{adj}(\lambda A) = \lambda^n adj(A)$.

(iv) $A(\text{adj}A) = (\text{adj}A) = |A|\,I$

(1) Only (i) (2) (ii) and (iii)

(3) (iii) and (iv) **(4) (i), (ii) and (iv)**

21. If $\rho(A) = (AB)$, then the system $AX=B$ of linear equations is

(1) consistent and has a unique solution

(2) consistent

(3) consistent and has infinitely many solution

(4) inconsistent

22. If $0 \le \theta \le \pi$ and the system of equations $x + (\sin\theta)y - (\cos\theta)z = 0$, $(\cos\theta)x - y + z = 0$,$(\sin\theta)x + y - z = 0$ has a non-trivial solution then θ is . . . .

(1) $\frac{2\pi}{3}$ (2) $\frac{3\pi}{4}$ (3) $\frac{5\pi}{6}$ **(4) $\frac{\pi}{4}$**

23. The augmented matrix of a system of linear equations is $\begin{bmatrix} 1 & 2 & 7 & 3 \\ 0 & 1 & 4 & 6 \\ 0 & 0 & \lambda-7 & \mu+5 \end{bmatrix}$

The system has infinitely many solutions if . . . . . .

(1) $= 7, \mu \ne -5$, (2) $\lambda = -7, \mu = 5$,

(3) $\lambda \ne 7, \mu \ne -5$, **(4) $\lambda = 7, \mu = -5$**

24. Let $A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$ and $4B = \begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & x \\ -1 & 1 & 3 \end{bmatrix}$. If B is the inverse of A, then the value of x is . . . . . .

(1) 2 (2) 4 (3) 3 **(4) 1**

25. If $A = \begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}$ then $\operatorname{adj}(adjA)$ is . . . .

(1) $\begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}$ (2) $\begin{bmatrix} 6 & -6 & 8 \\ 4 & -6 & 8 \\ 0 & -2 & 2 \end{bmatrix}$

(3) $\begin{bmatrix} -3 & 3 & -4 \\ -2 & 3 & -4 \\ 0 & 1 & -1 \end{bmatrix}$ (4) $\begin{bmatrix} 3 & -3 & 4 \\ 0 & -1 & 1 \\ 2 & -3 & 4 \end{bmatrix}$

Formulae:

1. For 2x 2 matrix, adj A is obtained by

i) Interchange of main diagonal elements

ii) Change the sign of other elements.

2. Cofactor Matrix of A = A_{ij}

3. Adjoint Matrix of A = Aij^T

4. $A^{-1} = \frac{1}{|A|}\, Adj\ A$

5. A (adj A) = (adj A) A = | A | I_n

6. $A = \pm\frac{1}{\sqrt{adj\ A}}\, adj(adj\ A)$ and

$A^{-1} = \pm\frac{1}{\sqrt{adj\ A}}\,(adj\ A)$

7. $(A^T)^{-1} = (A^{-1})^T$

8. $(AB)^{-1} = B^{-1}A^{-1}$ and $(A^{-1})^{-1} = A$

9. If the Matrix is in Echelon form, then the number of non zero rows is the rank of the matrix and it is denoted by $\rho(A)$.

10. A square matrix A is called orthogonal

if $AA^T = A^TA = I$

11. A is called orthogonal if and only if

A is non singular and $A^{-1} = A^T$

12. $\cos 2x = cos^2x - sin^2x$ and

$\sin 2x = 2\sin x \cos x$

14. (i) $|A^{-1}| = \frac{1}{|A|}$ (ii) $(A^T)^{-1} = (A^{-1})^T$

(iii) $(\lambda A)^{-1} = \frac{1}{\lambda} A^{-1}$, λ is a scalar.

15. Methods to solve the system of linear

equations A X B =

(i) By matrix inversion $X = A^{-1}B$

(ii) By Cramer's rule if $\Delta \neq 0$

$x = \frac{\Delta_x}{\Delta}, y = \frac{\Delta_y}{\Delta}$ and $z = \frac{\Delta_z}{\Delta}$

(iii) By Gaussian elimination method

16. If $\rho(AB) = \rho(A)$ then the given equation

is consistent.

17. If $\rho(AB) \neq \rho(A)$ then the given equation

is inconsistent.

18. If $\rho(AB) = \rho(A) = n$, the number of

unknowns then the given equation is

consistent and has unique solution.

19. If $\rho(AB) = \rho(A) \neq n$, then the given

equation is consistent and has

infinitely many solutions.

20. The homogenous system of linear

equations AX = 0

(i) has the trivial solution, if | A | $\neq$ 0 .

(ii) has a non trivial solution, if | A | =0 .

Chapter2- Complex Numbers

Example 2.1 Simplify the following

(i)$i^7 = i^4 i^3 = i^3 = i^2 i = -i$

(ii)$i^{1729} = i^{1728} i = i^{4(432)} i = i$

(iii)$i^{-1924} + i^{2018}$

$= i^{-4(482)+0} + i^{4(504)+2}$

$= i^0 + i^2$

$= 1 - 1$

$= 0$

(iv) $\sum_{n=1}^{102} i^n$

$= (i + i^2 + i^3 + i^4) + (i^5 + i^6 + i^7 + i^8)$

$+ \ldots + (i^{97} + i^{98} + i^{99} + i^{100})$

$+ i^{101} + i^{102}$

$= i^{100+1} + i^{100+2}$

$= i^1 + i^2$

$= i - 1$

(v)$i\, i^2 i^3 i^4 \ldots i^{40}$

$= i^{1+2+3+\cdots+40}$

$= i^{\frac{40\times41}{2}}$

$= i^{820}$

$= i^{4(205)+0}$

$= i^0$

$= 1$

EXERCISE 2.1

Simplify the following:

1. $i^{1947} + i^{1650} = i^{1944+3} + i^{1948+2}$

$= i^3 + i^2$

$= i^2 i + i^2$

$= -i - 1$

2. $i^{1948} - i^{-1869}$

$= i^{1948+0} - i^{-1868-1}$

$= i^0 - i^{-1}$

$= 1 + i \quad (i^{-1} = -i)$

3. $\sum_{n=1}^{12} i^n$

$= (i + i^2 + i^3 + i^4) + (i^5 + i^6 + i^7 + i^8)$

$+ (i^9 + i^{10} + i^{11} + i^{12})$

$= 0$

4. $i^{59} + \frac{1}{i^{59}} = i^{56+3} + \frac{1}{i^{56+3}}$

$= i^3 + \frac{1}{i^3}$

$= i^2 i + \frac{1}{i^3 i}$

$= -i - \frac{1}{i}$

$= -i - (i^{-1})$

$= -i - (-i)$

$= -i + i$

$= 0$

5. $i\, i^2 i^3 i^4 \ldots i^{2000}$

$= i^{1+2+3+\cdots+2000}$

$= i^{\frac{2000\times2001}{2}}$

$= i^{1000\times2001}$

$= i^{4(250)(2001)}$

$= 1$

6. $\sum_{n=1}^{10} i^{n+50}$

$= (i^{51} + i^{52} + i^{53} + i^{54})$

$+ (i^{55} + i^{56} + i^{57} + i^{58}) + (i^{59} + i^{60})$

$= i^{59} + i^{60}$

$= i^{56+3} + i^{60+0}$

$= i^3 + i^0$

$= i^2 i + 1$

$= -i + 1$

Example 2.2 Find the value of the real numbers x and y, if the complex number $(2+i)x + (1-i)y + 2i - 3$ and $x + (-1+2i)y + 1 + i$ are equal

Solution:

$(2+i)x + (1-i)y + 2i - 3$

$= 2x + xi + y - yi + 2i - 3$

$= 2x + y - 3 + xi - yi + 2i$

$= (2x + y - 3) + (x - y + 2)i$ ..(1)

$x + (-1+2i)y + 1 + i$

$= x - y + 2yi + 1 + i$

$= x - y + 1 + 2yi + i$

$= (x - y + 1) + (2y + 1)i$ ..(2)

Given (1) = (2)

Equating the real part

$2x + y - 3 = x - y + 1$

$2x - x + y + y = 1 + 3$

$x + 2y = 4$ ...(3)

Equating the imaginary part

$x - y + 2 = 2y + 1$

$x - y - 2y = 1 - 2$

$x - 3y = -1$ ....(4)

Solving (3) and (4)

$(3) \times 3 \Rightarrow 3x + 6y = 12$

$(4) \times 2 \Rightarrow 2x - 6y = -2$

$5x = 10 \Rightarrow x = 2$

Substituting $x = 2$ in $x + 2y = 4$

$2 + 2y = 4$

$2y = 4 - 2$

$2y = 2 \Rightarrow y = 1$

So the solution is $x = 2$ and $y = 1$

EXERCISE 2.2

1. Evaluate the following if $Z = 5 - 2i$ and $W = -1 + 3i$

(i) $z + w = 5 - 2i - 1 + 3i$

$= 4 + i$

(ii) $z - i\,w = 5 - 2i - i(-1 + 3i)$

$= 5 - 2i + i - 3i^2$

$= 5 - i - 3(-1)$

$= 5 - i + 3$

$= 8 - i$

(iii) $2z + 3\,w$

$= 2(5 - 2i) + 3(-1 + 3i)$

$= 10 - 4i - 3 + 9i$

$= 7 + 5i$

(iv) $zw = (5 - 2i)(-1 + 3i)$

$= -5 + 15i + 2i - 6i^2$

$= -5 + 17i - 6(-1)$

$= -5 + 17i + 6$

$= 1 + 17i$

(v) $Z^2 + 2ZW + W^2$

$Z^2 = (5 - 2i)(5 - 2i)$

$= 25 - 10i - 10i + 4(i^2)$

$= 25 - 20i + 4(-1)$

$= 25 - 20i - 4$

$= 21 - 20i$

$2ZW = 2(5 - 2i)(-1 + 3i)$

$= 2(-5 + 15i + 2i - 6i^2)$

$= 2[-5 + 17i - 6(-1)]$

$= 2(-5 + 17i + 6)$

$= 2(1 + 17i)$

$= 2 + 34i$

$W^2 = (-1 + 3i)(-1 + 3i)$

$= 1 - 3i - 3i + 9i^2$

$= 1 - 6i + 9(-1)$

$= 1 - 6i - 9$

$= -8 - 6i$

$Z^2 + 2ZW + W^2$

$= 21 - 20i + 2 + 34i - 8 - 6i$

$= 23 - 8 + 34i - 26i$

$= 15 + 8i$

(vi) $(Z+W)^2$.

$z+w = 5-2i-1+3i$

$= 4+i$

$(Z+W)^2 = (4+i)^2$

$= 16+8i+i^2$

$= 16+8i+(-1)$

$= 16+8i-1$

$= 15+8i$

2. Given the complex number $Z = 2+3i$, represent the complex numbers in Argand diagram.

(i) z, iz, and $z+iz$

(ii) z, $-iz$, and $z-iz$.

3. Find the values of the real numbers x and y, if the complex numbers

$(3-i)x-(2-i)y+2i+5$ and

$2x+(-1+2i)y+3+2i$ are equal

Solution:

$(3-i)x-(2-i)y+2i+5$

$= 3x-xi-2y+yi+2i+5$

$= 3x-2y+5-xi+yi+2i$

$= (3x-2y+5)+i(-x+y+2)$ ..(1)

$2x+(-1+2i)y+3+2i$

$= 2x-y+2yi+3+2i$

$= 2x-y+3+2yi+2i$

$= (2x-y+3)+(2y+2)i$ ..(2)

Given (1) = (2)

Equating the real part

$3x-2y+5 = 2x-y+3$

$3x-2x-2y+y = 3-5$

$x-y = -2$ ....(3)

Equating the imaginary part

$-x+y+2 = 2y+2$

$-x+y-2y = 2-2$

$-x-y = 0$

$x+y = 0$ ....(4)

Solving (3) and (4)

$x-y = -2$

$x+y = 0$

$2x = -2 \Rightarrow x = -1$

Substituting $x=-1$ in $x+y=0$

$-1+y = 0 \Rightarrow y = 1$

So the solution is $x=-1$ and $y=1$

EXERCISE 2.3

1. If $Z_1 = 1-3i, Z_2 = -4i$ and $Z_3 = 5$, show that

(i) $(Z_1+Z_2)+Z_3 = Z_1+(Z_2+Z_3)$

$(Z_1+Z_2)+Z_3 = (1-3i-4i)+5$

$= (1-7i)+5$

$= 6-7i$ ....(LHS)

$Z_1+(Z_2+Z_3)$

$= 1-3i+(-4i+5)$

$= 1-3i-4i+5$

$= 6-7i$ ....(RHS)

Hence $(LHS) = (RHS)$

(ii) $(Z_1Z_2)Z_3 = Z_1(Z_2Z_3)$

$(Z_1Z_2)Z_3 = [(1-3i)(-4i)]5$

$= (-4i+12i^2)5$

$= (-4i+12(-1))5$

$= (-4i-12)5$

$= -20i-60$ ....(LHS)

$Z_1(Z_2Z_3) = (1-3i)(-4i\times 5)$

$= (1-3i)(-20i)$

$= -20i+60(i^2)$

$= -20i+60(-1)$

$= -20i-60$ ....(RHS)

Hence $(LHS) = (RHS)$

2. If $Z_1 = 3, Z_2 = -7i$ and $Z_3 = 5+4i$, show that

(i) $Z_1(Z_2+Z_3) = Z_1Z_2+Z_1Z_3$

$Z_1(Z_2+Z_3) = 3(-7i+5+4i)$

$= 3(-3i+5)$

$= 3(5-3i)$

$= 15-9i$....(LHS)

$Z_1Z_2+Z_1Z_3$

$= (3)(-7i)+(3)(5+4i)$

$= -21i+15+12i$

$= 15-9i$....(RHS)

Hence $(LHS) = (RHS)$

(ii) $(Z_1+Z_2)Z_3 = Z_1Z_3 + Z_2Z_3$.

$$(Z_1+Z_2)Z_3 = (3-7i)(5+4i)$$
$$= 15+12i-35i-28i^2$$
$$= 15-23i-28i^2$$
$$= 15-23i-28(-1)$$
$$= 15-23i+28$$
$$= 43-23i \; ....(LHS)$$

$Z_1Z_3 + Z_2Z_3$

$$= 3(5+4i)+(-7i)(5+4i)$$
$$= 15+12i-35i-28i^2$$
$$= 15-23i-28i^2$$
$$= 15-23i-28(-1)$$
$$= 15-23i+28$$
$$= 43-23i \; ....(RHS)$$

Hence $(LHS) = (RHS)$

3. If $Z_1 = 2+5i, Z_2 = -3-4i$ and $Z_3 = 1+i$, find the additive and multiplicative inverse of $Z_1,\ Z_2\ and\ Z_3$.

(i) Additive inverse of $Z_1 = -Z_1$

$$= -(2+5i)$$
$$= -2-5i$$

(ii) Additive inverse of $Z_2 = -Z_2$

$$= -(-3-4i)$$
$$= 3+4i$$

(iii) Additive inverse of $Z_3 = -Z_3$

$$= -(1+i)$$
$$= -1-i$$

(i) Multiplicative inverse of $Z_1 = \frac{1}{Z_1}$

$$= \frac{1}{2+5i}$$
$$= \frac{1}{2+5i}\left(\frac{2-5i}{2-5i}\right)$$
$$= \frac{2-5i}{4+25}$$
$$= \frac{1}{29}(2-5i)$$

(ii) Multiplicative inverse of $Z_2 = \frac{1}{Z_2}$

$$= \frac{1}{-3-4i}$$
$$= \frac{1}{-3-4i}\left(\frac{-3+4i}{-3+4i}\right)$$
$$= \frac{-3+4i}{9+16}$$
$$= \frac{1}{25}(-3+4i)$$

(iii) Multiplicative inverse of $Z_3 = \frac{1}{Z_3}$

$$= \frac{1}{1+i}$$
$$= \frac{1}{1+i}\left(\frac{1-i}{1-i}\right)$$
$$= \frac{1-i}{1+1}$$
$$= \frac{1}{2}(1-i)$$

Example 2.3 Write $\frac{3+4i}{5-12i}$ in the $x+iy$ form, hence find its real and imaginary parts.

Solution:

$$\text{Let } Z = \frac{3+4i}{5-12i}$$
$$= \frac{3+4i}{5-12i} \times \frac{5+12i}{5+12i}$$
$$= \frac{15+36i+20i+48i^2}{25+144}$$
$$= \frac{15+56i\ +48(-1)}{169}$$
$$= \frac{15+56i-48}{169}$$
$$= \frac{-33+56i}{169}$$
$$\frac{3+4i}{5-12i} = \frac{-33}{169} + \frac{56i}{169}$$

$Re(Z) = -\frac{33}{169}$ and $Img(Z) = \frac{56}{169}$

Example 2.4 Simplify $\left(\frac{1+i}{1-i}\right)^3 - \left(\frac{1-i}{1+i}\right)^3$

Solution:

$$\frac{1+i}{1-i} = \frac{1+i}{1-i} \times \frac{1+i}{1+i}$$
$$= \frac{1+i+i+i^2}{1+1}$$
$$= \frac{1+2i+(-1)}{1+1}$$
$$= \frac{1+2i-1}{2}$$

$$= \frac{2i}{2}$$
$$= i$$
$$\left(\frac{1+i}{1-i}\right)^3 = i^3$$
$$= i^2 i$$
$$= -i$$
$$\left(\frac{1-i}{1+i}\right)^3 = \left(\frac{1+i}{1-i}\right)^{-3}$$
$$= i^{-3}$$
$$= \frac{1}{i^3} \times \frac{i}{i}$$
$$= \frac{i}{i^4}$$
$$= i$$
$$\left(\frac{1+i}{1-i}\right)^3 - \left(\frac{1-i}{1+i}\right)^3 = -i - i = -2i$$

Example 2.5 If $\frac{Z+3}{Z-5i} = \frac{1+4i}{2}$, find the complex number.

Solution:

$$\frac{Z+3}{Z-5i} = \frac{1+4i}{2}$$
$$2(z+3) = (z-5i)(1+4i)$$
$$2z+6 = z+4zi-5i-20i^2$$
$$2z+6 = z+4zi-5i-20(-1)$$
$$2z+6 = z+4zi-5i+20$$
$$2z-z-4zi = -5i+20-6$$
$$z-4zi = -5i+14$$
$$z(1-4i) = 14-5i$$
$$z = \frac{14-5i}{1-4i} \times \frac{1+4i}{1+4i}$$
$$= \frac{14+56i-5i-20i^2}{1+16}$$
$$= \frac{14+51i-20(-1)}{17}$$
$$= \frac{14+51i+20}{17}$$
$$= \frac{34+51i}{17}$$
$$= \frac{17(2+3i)}{17}$$
$$= 2+3i$$

Example 2.6 If $Z_1 = 3-2i$ *and* $Z_2 = 6+4i$, find $\frac{Z_1}{Z_2}$

Solution: $\frac{Z_1}{Z_2} = \frac{3-2i}{6+4i}$

$$= \frac{3-2i}{6+4i} \times \frac{6-4i}{6-4i}$$
$$= \frac{18-12i-12i+8i^2}{36+16}$$
$$= \frac{18-24i+8(-1)}{36+16}$$
$$= \frac{18-24i-8}{52}$$
$$= \frac{10-24i}{52}$$
$$= \frac{2(5-12i)}{52}$$
$$= \frac{(5-12i)}{26}$$
$$= \frac{5}{26} - \frac{12i}{26}$$
$$= \frac{5}{26} - \frac{6i}{13}$$

Example 2.7 Find Z^{-1} if $Z = (2+3i)(1-i)$.

Solution:

$$Z^{-1} = \frac{1}{Z}$$
$$= \frac{1}{(2+3i)(1-i)}$$
$$= \frac{1}{2-2i+3i-3i^2}$$
$$= \frac{1}{2+i-3(-1)}$$
$$= \frac{1}{2+i+3}$$
$$= \frac{1}{5+i} \times \frac{5-i}{5-i}$$
$$= \frac{5-i}{25+1}$$
$$= \frac{5-i}{26}$$
$$= \frac{5}{26} - \frac{i}{26}$$

Example 2.8 (i) Show that

$\left(2+i\sqrt{3}\right)^{10}+\left(2-i\sqrt{3}\right)^{10}$ is real and (ii)

$\left(\frac{19+9i}{5-3i}\right)^{15}-\left(\frac{8+i}{1+2i}\right)^{15}$ is purely imaginary.

Solution:

$$\text{Let } Z=\left(2+i\sqrt{3}\right)^{10}+\left(2-i\sqrt{3}\right)^{10}$$

$$\text{Then } \bar{Z}=\overline{\left(2+i\sqrt{3}\right)^{10}+\left(2-i\sqrt{3}\right)^{10}}$$

$$=\overline{\left(2+i\sqrt{3}\right)^{10}}+\overline{\left(2-i\sqrt{3}\right)^{10}}$$

$$=\left(\overline{2+i\sqrt{3}}\right)^{10}+\left(\overline{2-i\sqrt{3}}\right)^{10}$$

$$=\left(2-i\sqrt{3}\right)^{10}+\left(2+i\sqrt{3}\right)^{10}$$

$= Z$, so

$\left(2+i\sqrt{3}\right)^{10}+\left(2-i\sqrt{3}\right)^{10}$ is real.

(ii) Let $Z=\left(\frac{19+9i}{5-3i}\right)^{15}-\left(\frac{8+i}{1+2i}\right)^{15}$

$$\frac{19+9i}{5-3i}=\frac{19+9i}{5-3i}\times\frac{5+3i}{5+3i}$$

$$=\frac{95+57i+45i+27i^2}{25+9}$$

$$=\frac{95+102i+27(-1)}{34}$$

$$=\frac{95+102i-27}{34}$$

$$=\frac{68+102i}{34}$$

$$=\frac{34(2+3i)}{34}$$

$$=2+3i$$

$$\frac{8+i}{1+2i}=\frac{8+i}{1+2i}\times\frac{1-2i}{1-2i}$$

$$=\frac{8-16i+i-2i^2}{1+4}$$

$$=\frac{8-15i\ -2(-1)}{5}$$

$$=\frac{8-15i+2}{5}$$

$$=\frac{10-15i}{5}=\frac{5(2-3i)}{5}$$

$$=2-3i$$

So $Z=\left(\frac{19+9i}{5-3i}\right)^{15}-\left(\frac{8+i}{1+2i}\right)^{15}$ gives

$$Z=(2+3i)^{15}-(2-3i)^{15}$$

$$\text{So, } \bar{Z}=\overline{(2+3i)^{15}-(2-3i)^{15}}$$

$$=\overline{(2+3i)^{15}}-\overline{(2-3i)^{15}}$$

$$=(\overline{2+3i})^{15}-(\overline{2-3i})^{15}$$

$$=(2-3i)^{15}-(2+3i)^{15}$$

$$=-\left[(2+3i)^{15}-(2-3i)^{15}\right]$$

$$\bar{Z}=-Z$$

$$Z=\left(\frac{19+9i}{5-3i}\right)^{15}-\left(\frac{8+i}{1+2i}\right)^{15}$$

is purely imaginary.

EXERCISE 2.4

1. Write the following in the rectangular form:

(i) $\overline{(5+9i)+(2-4i)}$

Solution:

$$\text{Let } Z=\overline{(5+9i)+(2-4i)}$$

$$=\overline{(5+9i+2-4i)}$$

$$=\overline{(7+5i)}$$

$$=7-5i$$

(ii) $\frac{10-5i}{6+2i}$

$$\text{Let } Z=\frac{10-5i}{6+2i}$$

$$=\frac{10-5i}{6+2i}\times\frac{6-2i}{6-2i}$$

$$=\frac{60-20i-30i+10i^2}{36+4}$$

$$=\frac{60-50i\ +10(-1)}{40}$$

$$=\frac{60-50i-10}{40}$$

$$=\frac{50-50i}{40}$$

$$=\frac{50(1-i)}{40}$$

$$=\frac{5}{4}(1-i)$$

(iii) $\overline{3i} + \frac{1}{2-i}$

Let $Z = \overline{3i} + \frac{1}{2-i}$

$= -3i + \frac{1}{2-i} \times \frac{2+i}{2+i}$

$= -3i + \frac{2+i}{4+1}$

$= -3i + \frac{2+i}{5}$

$= \frac{5(-3i)+2+i}{5}$

$= \frac{-15i+2+i}{5}$

$= \frac{-14i+2}{5}$

$= \frac{2-14i}{5}$

$= \frac{2}{5} - \frac{14i}{5}$

2. If $Z = x + iy$, find the following in rectangular form.

(i) Re $\left(\frac{1}{Z}\right)$

$Z = x + iy$

$\frac{1}{Z} = \frac{1}{x+iy}$

$= \frac{1}{x+iy} \times \frac{x-iy}{x-iy}$

$= \frac{x-iy}{x^2+y^2}$

$= \frac{x}{x^2+y^2} - \frac{iy}{x^2+y^2}$

Re $\left(\frac{1}{Z}\right) = \frac{x}{x^2+y^2}$

(ii) Re $(i\bar{Z})$

$Z = x + iy$

$\bar{Z} = x - iy$

$(i\bar{Z}) = i(x - iy)$

$= xi - i^2y$

$= xi - (-1)y \quad = xi + y$

Re $(i\bar{Z}) = y$

(iii) Im $(3Z + 4\bar{Z} - 4i)$

$Z = x + iy$

$3Z + 4\bar{Z} - 4i$

$= 3(x+iy) + 4(x-iy) - 4i$

$= 3x + 3yi + 4x - 4yi - 4i$

$= 7x - yi - 4i$

$= 7x + (-y-4)i$

Im $(3Z + 4\bar{Z} - 4i) = -y - 4$

3. If $Z_1 = 2 - i$ and $Z_2 = -4 + 3i$,

find the inverse of Z_1Z_2 and $\frac{Z_1}{Z_2}$.

Solution:

$Z_1Z_2 = (2-i)(-4+3i)$

$= -8 + 6i + 4i - 3i^2$

$= -8 + 10i - 3i^2$

$= -8 + 10i - 3(-1)$

$= -8 + 10i + 3$

$= -5 + 10i$

(i) inverse of $Z_1Z_2 = \frac{1}{Z_1Z_2}$

$= \frac{1}{(-5+10i)} \times \frac{-5-10i}{(-5-10i)}$

$= \frac{-5-10i}{25+100}$

$= \frac{-5-10i}{125}$

$= \frac{-5\,(1+2i)}{125}$

$= -\frac{(1+2i)}{25}$

$= -\frac{1}{25}(1+2i)$

(ii)Inverse of $\frac{Z_1}{Z_2} = \frac{Z_2}{Z_1}$

$\frac{Z_2}{Z_1} = \frac{-4+3i}{2-i} \times \frac{2+i}{2+i}$

$= \frac{-8-4i+6i+3i^2}{4+1}$

$= \frac{-8+2i+3(-1)}{5}$

$$= \frac{-8+2i-3}{5}$$

$$= \frac{-11+2i}{5}$$

$$= \frac{1}{5}(-11+2i)$$

4. The complex numbers u, v, and w are related by $\frac{1}{u} = \frac{1}{v} + \frac{1}{w}$.
If v = 3 − $4i$ and w = 4 + $3i$, find u in rectangular form.

Solution: $\frac{1}{u} = \frac{1}{v} + \frac{1}{w}$

$$= \frac{1}{3-4i} + \frac{1}{4+3i}$$

$$= \frac{4+3i+3-4i}{(3-4i)(4+3i)}$$

$$= \frac{7-i}{12+9i-16i-12i^2}$$

$$= \frac{7-i}{12-7i-12(-1)}$$

$$= \frac{7-i}{12-7i+12}$$

$$= \frac{7-i}{24-7i}$$

$$u = \frac{24-7i}{7-i} \times \frac{7+i}{7+i}$$

$$= \frac{168+24i-49i-7i^2}{49+1}$$

$$= \frac{168-25i-7(-1)}{50}$$

$$= \frac{168-25i+7}{50}$$

$$= \frac{175-25i}{50}$$

$$= \frac{25(7-i)}{50}$$

$$= \frac{(7-i)}{2}$$

$$u = \frac{1}{2}(7-i)$$

5. Prove the following properties:

(i) z is real if and only if $z = \bar{z}$

Solution: Let $z = x + iy$

then $\bar{z} = x - iy$

if $z = \bar{z}$

$$x + iy = x - iy$$

It is true if and only if $y = 0$

that is z is real

(ii) Re(Z) $= \frac{z+\bar{z}}{2}$ and Im(Z) $= \frac{z-\bar{z}}{2i}$

$$z + \bar{z} = x + iy + x - iy$$

$$= 2x$$

$$= 2Re(Z)$$

$$\frac{z+\bar{z}}{2} = Re(Z)$$

$$z - \bar{z} = x + iy - (x - iy)$$

$$= x + iy - x + iy$$

$$= 2iy$$

$$= 2Im(Z)$$

$$\frac{z-\bar{z}}{2i} = Im(Z)$$

6. Find the least value of the positive integer n for which $\left(\sqrt{3}+i\right)^n$
(i) real (ii) purely imaginary.

Solution:

Let $\sqrt{3} + i = r(\cos\theta + i\sin\theta)$

$$\text{r} = \sqrt{a^2+b^2}$$

$$= \sqrt{\sqrt{3}^2 + 1^2}$$

$$= \sqrt{3+1}$$

$$= \sqrt{4}$$

$$= 2$$

$$\tan\theta = \frac{b}{a}$$

$$= \frac{1}{\sqrt{3}}$$

$$\theta = \frac{\pi}{6}$$

$$\sqrt{3}+i = 2\left(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right)$$

$$\left(\sqrt{3}+i\right)^n = 2^n\left(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right)^n$$

$$= 2^n\left(\cos\frac{n\pi}{6}+i\sin\frac{n\pi}{6}\right)$$

(i) When n = 3

$$\left(\sqrt{3}+i\right)^n = 2^n\left(\cos\frac{n\pi}{6}+i\sin\frac{n\pi}{6}\right) \Rightarrow$$

$$\left(\sqrt{3}+i\right)^3 = 2^3\left(\cos\frac{3\pi}{6}+i\sin\frac{3\pi}{6}\right)$$

$$= 2^3\left(\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}\right)$$

$$= 2^3(0+i\,(1))$$

$$= 2^3(0+i)$$

$= 2^3 i$, purely imaginary.

(ii) When n = 6

$$\left(\sqrt{3}+i\right)^n = 2^n\left(\cos\frac{n\pi}{6}+i\sin\frac{n\pi}{6}\right) \Rightarrow$$

$$\left(\sqrt{3}+i\right)^6 = 2^6\left(\cos\frac{6\pi}{6}+i\sin\frac{6\pi}{6}\right)$$

$$= 2^3(\cos\pi+i\sin\pi)$$

$$= 2^3((-1)+i\,(0))$$

$$= 2^3(-1+0)$$

$= -2^3$, real.

7. Show that

(i) Show that $\left(2+i\sqrt{3}\right)^{10}-\left(2-i\sqrt{3}\right)^{10}$

is purely imaginary and

(ii) $\left(\frac{19-7i}{9+i}\right)^{15}+\left(\frac{20-5i}{7-6i}\right)^{15}$ is real.

Solution:

Let $Z = \left(2+i\sqrt{3}\right)^{10}-\left(2-i\sqrt{3}\right)^{10}$

Then $\bar{Z} = \overline{\left(2+i\sqrt{3}\right)^{10}-\left(2-i\sqrt{3}\right)^{10}}$

$$= \overline{\left(2+i\sqrt{3}\right)^{10}}-\overline{\left(2-i\sqrt{3}\right)^{10}}$$

$$= \left(\overline{2+i\sqrt{3}}\right)^{10}-\left(\overline{2-i\sqrt{3}}\right)^{10}$$

$$= \left(2-i\sqrt{3}\right)^{10}-\left(2+i\sqrt{3}\right)^{10}$$

$$= -\left[\left(2+i\sqrt{3}\right)^{10}-\left(2-i\sqrt{3}\right)^{10}\right]$$

$= -Z$, so

$\left(2+i\sqrt{3}\right)^{10}-\left(2-i\sqrt{3}\right)^{10}$ is imaginary.

(ii) Let $Z = \left(\frac{19-7i}{9+i}\right)^{15}+\left(\frac{20-5i}{7-6i}\right)^{15}$

$$\frac{19-7i}{9+i} = \frac{19-7i}{9+i}\times\frac{9-i}{9-i}$$

$$= \frac{171-19i-63i+7i^2}{81+1}$$

$$= \frac{171-82i+7(-1)}{81+1}$$

$$= \frac{171-82i-7}{82}$$

$$= \frac{164-82i}{82}$$

$$= \frac{82(2-i)}{82}$$

$$= 2-i$$

$$\frac{20-5i}{7-6i} = \frac{20-5i}{7-6i}\times\frac{7+6i}{7+6i}$$

$$= \frac{140+120i-35i-30i^2}{49+36}$$

$$= \frac{140+85i-30(-1)}{85}$$

$$= \frac{140+85i+30}{85}$$

$$= \frac{170+85i}{85}$$

$$= \frac{85(2+i)}{85}$$

$$= 2+i$$

$$Z = \left(\frac{19-7i}{9+i}\right)^{15}+\left(\frac{20-5i}{7-6i}\right)^{15}$$

$$= (2-i)^{15}+(2+i)^{15}$$

So, $\bar{Z} = \overline{(2-i)^{15}+(2+i)^{15}}$

$$= \overline{(2-i)^{15}}+\overline{(2+i)^{15}}$$

$$= (\overline{2-i})^{15}+(\overline{2+i})^{15}$$

$$= (2+i)^{15}+(2-i)^{15}$$

$$= [(2+i)^{15}+(2-i)^{15}]$$

$\bar{Z} = Z$

$\left(\frac{19-7i}{9+i}\right)^{15}+\left(\frac{20-5i}{7-6i}\right)^{15}$ is real.

Example 2.9 If $Z_1 = 3+4i, Z_2 = 5-12i$ and $Z_3 = 6+8i$ find $|Z_1|, |Z_2|, |Z_3|, |Z_1+Z_2|, |Z_2-Z_3|$ and $|Z_1+Z_3|$.

Solution: $(i)\ Z_1 = 3 + 4i$

$|Z_1| = |3 + 4i|$

$= \sqrt{3^2 + 4^2}$

$= \sqrt{9 + 16}$

$= \sqrt{25}$

$= 5$

$(ii)\ Z_2 = 5 - 12i$

$|Z_2| = |5 - 12i|$

$= \sqrt{5^2 + (-12)^2}$

$= \sqrt{25 + 144}$

$= \sqrt{169}$

$= 13$

$(iii)\ Z_3 = 6 + 8i$

$|Z_3| = |6 + 8i|$

$= \sqrt{6^2 + 8^2}$

$= \sqrt{36 + 64}$

$= \sqrt{100}$

$= 10$

$(iv)\ Z_1 + Z_2 = 3 + 4i + 5 - 12i$

$= 8 - 8i$

$|Z_1 + Z_2| = |8 - 8i|$

$= \sqrt{8^2 + (-8)^2}$

$= \sqrt{64 + 64}$

$= \sqrt{2(64)}$

$= 8\sqrt{2}$

$(v)\ Z_2 - Z_3 = 5 - 12i - (6 + 8i)$

$= 5 - 12i - 6 - 8i$

$= -1 - 20i$

$|Z_2 - Z_3| = |-1 - 20i|$

$= \sqrt{(-1)^2 + (-20)^2}$

$= \sqrt{1 + 400}$

$= \sqrt{401}$

$(vi)\ Z_1 + Z_3 = 3 + 4i + 6 + 8i$

$= 9 + 12i$

$|Z_1 + Z_2| = |9 + 12i|$

$= \sqrt{9^2 + 12^2}$

$= \sqrt{81 + 144}$

$= \sqrt{225}$

$= 15$

Example 2.10 Find the following

$(i)\ \left|\frac{2+i}{-1+2i}\right|$

$(ii)\ |(\overline{1+i})(2+3i)(4i-3)|$

$(iii)\ \left|\frac{i(2+i)^3}{(1+i)^2}\right|$

Solution: $(i)\ \left|\frac{2+i}{-1+2i}\right| = \frac{|2+i|}{|-1+2i|}$

$= \frac{\sqrt{2^2+1^2}}{\sqrt{(-1)^2+2^2}}$

$= \frac{\sqrt{4+1}}{\sqrt{1+4}}$

$=1$

$(ii)\ |(\overline{1+i})(2+3i)(4i-3)|$

$= |\overline{1+i}||2+3i||4i-3|$

$= |1+i||2+3i||4i-3|$

$= \left(\sqrt{1^2+1^2}\right)\left(\sqrt{2^2+3^2}\right)\left(\sqrt{4^2+(-3)^2}\right)$

$= \left(\sqrt{1+1}\right)\left(\sqrt{4+9}\right)\left(\sqrt{16+9}\right)$

$= \left(\sqrt{2}\right)\left(\sqrt{13}\right)\left(\sqrt{25}\right)$

$= 5\left(\sqrt{2}\right)\left(\sqrt{13}\right)$

$= 5\left(\sqrt{26}\right)$

$(iii)\ \left|\frac{i(2+i)^3}{(1+i)^2}\right| = \frac{|i||2+i|^3}{|1+i|^2}$

$= \frac{\sqrt{1^2}\left(\sqrt{2^2+1^2}\right)^3}{\left(\sqrt{1^2+1^2}\right)^2}$

$= \frac{\sqrt{1}\left(\sqrt{4+1}\right)^3}{\left(\sqrt{1+1}\right)^2}$

$= \frac{1\left(\sqrt{5}\right)^3}{\left(\sqrt{2}\right)^2}$

$= \frac{5\sqrt{5}}{2}$

Example 2.11 Which one of the points $i, -2+i$, and 3 is farthest from the origin?

Solution:

The distance from the origin to

$Z = i, -2+i$, and 3are

$|Z| = |i| = \sqrt{1^2} = \sqrt{1} = 1$

$|Z| = |-2+i| = \sqrt{(-2)^2 + 1^2}$

$= \sqrt{4+1} = \sqrt{5}$

$|Z| = |3| = 3$

The farthest point from the origin is 3

Example 2.12 If Z_1, Z_2 and Z_3 are complex numbers such that

$|Z_1| = |Z_2| = |Z_3| = |Z_1 + Z_2 + Z_3| = 1$,

find the value of $\left|\frac{1}{Z_1} + \frac{1}{Z_2} + \frac{1}{Z_3}\right|$

Solution: $|Z_1| = |Z_2| = |Z_3| = 1$

Hence $|Z_1|^2 = 1 \Rightarrow Z_1\overline{Z_1}$

$\overline{Z_1} = \frac{1}{Z_1}$

Similarly $\overline{Z_2} = \frac{1}{Z_2}$ and $\overline{Z_3} = \frac{1}{Z_3}$

So, $\left|\frac{1}{Z_1} + \frac{1}{Z_2} + \frac{1}{Z_3}\right| = |\overline{Z_1} + \overline{Z_2} + \overline{Z_3}|$

$= |\overline{Z_1 + Z_2 + Z_3}|$

$= |Z_1 + Z_2 + Z_3|$

$= 1$

Example 2.13 If $|Z| = 2$ show that $3 \le |z + 3 + 4i| \le 7$

Solution:

We know that $|Z_1 + Z_2| \le |Z_1| + |Z_2|$

Hence, $|Z + (3+4i)| \le |Z| + |3+4i|$

$\le 2 + |3+4i|$

$\le 2 + \sqrt{3^2 + 4^2}$

$\le 2 + \sqrt{9+16}$

$\le 2 + \sqrt{25}$

$\le 2 + 5$

$\le 7 \quad \ldots.(1)$

We know that $|Z_1 + Z_2| \ge ||Z_1| - |Z_2||$

Hence, $|Z + (3+4i)| \ge ||Z| - |3+4i||$

$\ge |2 - |3+4i||$

$\ge |2 - \sqrt{3^2+4^2}|$

$\ge |2 - \sqrt{9+16}|$

$\ge |2 - \sqrt{25}|$

$\ge |2 - 5|$

$\ge |-5|$

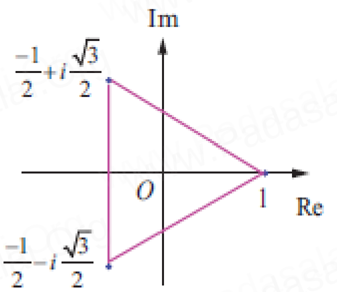
$\ge 3 \quad \ldots.(2)$

From (1) *and* (2)

$3 \le |z + 3 + 4i| \le 7$ is proved.

Example 2.14 Show that the points $1, \frac{-1}{2} + i\frac{\sqrt{3}}{2}$ and $\frac{-1}{2} - i\frac{\sqrt{3}}{2}$ are the vertices of an equilateral triangle.

Solution:



Let $Z_1 = 1$

$Z_2 = \frac{-1}{2} + i\frac{\sqrt{3}}{2}$

$Z_3 = \frac{-1}{2} - i\frac{\sqrt{3}}{2}$

The lengths of the sides of the triangles are

$|Z_1 - Z_2| = \left|1 - \left(\frac{-1}{2} + i\frac{\sqrt{3}}{2}\right)\right|$

$= \left|1 + \frac{1}{2} - i\frac{\sqrt{3}}{2}\right|$

$= \left|\frac{2+1}{2} - i\frac{\sqrt{3}}{2}\right|$

$= \left|\frac{3}{2} - i\frac{\sqrt{3}}{2}\right|$

$= \sqrt{\left(\frac{3}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}$

$$=\sqrt{\frac{9}{4}+\frac{3}{4}}$$

$$=\sqrt{\frac{9+3}{4}}$$

$$=\sqrt{\frac{12}{4}}$$

$$=\sqrt{3}$$

$$|Z_2-Z_3|=\left|\left(\frac{-1}{2}+i\frac{\sqrt{3}}{2}\right)-\left(\frac{-1}{2}-i\frac{\sqrt{3}}{2}\right)\right|$$

$$=\left|\frac{-1}{2}+i\frac{\sqrt{3}}{2}+\frac{1}{2}+i\frac{\sqrt{3}}{2}\right|$$

$$=\left|2i\frac{\sqrt{3}}{2}\right|$$

$$=|\sqrt{3}i|$$

$$=\sqrt{\left(\sqrt{3}\right)^2}$$

$$=\sqrt{3}$$

$$|Z_3-Z_1|=\left|\left(\frac{-1}{2}-i\frac{\sqrt{3}}{2}\right)-(1)\right|$$

$$=\left|\frac{-1}{2}-i\frac{\sqrt{3}}{2}-1\right|$$

$$=\left|\frac{-1}{2}-1-i\frac{\sqrt{3}}{2}\right|$$

$$=\left|\frac{-1-2}{2}-i\frac{\sqrt{3}}{2}\right|$$

$$=\left|\frac{-3}{2}-i\frac{\sqrt{3}}{2}\right|$$

$$=\sqrt{\left(-\frac{3}{2}\right)^2+\left(-\frac{\sqrt{3}}{2}\right)^2}$$

$$=\sqrt{\frac{9}{4}+\frac{3}{4}}$$

$$=\sqrt{\frac{9+3}{4}}$$

$$=\sqrt{\frac{12}{4}}$$

$$=\sqrt{3}$$

Since the sides are equal, the given points form an equilateral triangle.

Example 2.15 If Z_1, Z_2 and Z_3 are complex numbers such that $|Z_1|=|Z_2|=|Z_3|=r>0$ and $Z_1+Z_2+Z_3\neq 0$, prove that $\left|\frac{Z_1Z_2+Z_2Z_3+Z_3Z_1}{Z_1+Z_2+Z_3}\right|$=r

Solution: $|Z_1|=|Z_2|=|Z_3|=r$

$$|Z_1|^2=r^2$$

$$Z_1\overline{Z_1}=r^2$$

$$Z_1=\frac{r^2}{\overline{Z_1}}$$

Similarly $Z_2=\frac{r^2}{\overline{Z_2}}$ and $Z_3=\frac{r^2}{\overline{Z_3}}$

So, $Z_1+Z_2+Z_3=\frac{r^2}{\overline{Z_1}}+\frac{r^2}{\overline{Z_2}}+\frac{r^2}{\overline{Z_3}}$

$$=r^2\left(\frac{1}{\overline{Z_1}}+\frac{1}{\overline{Z_2}}+\frac{1}{\overline{Z_3}}\right)$$

$$=r^2\left(\frac{\overline{Z_2}\,\overline{Z_3}+\overline{Z_1}\,\overline{Z_3}+\overline{Z_1}\,\overline{Z_2}}{\overline{Z_1}\,\overline{Z_2}\,\overline{Z_3}}\right)$$

$$|Z_1+Z_2+Z_3|=|r^2|\left|\frac{\overline{Z_2Z_3+Z_1Z_3+Z_1Z_2}}{\overline{Z_1Z_2Z_3}}\right|$$

$$=|r^2|\frac{|Z_2Z_3+Z_1Z_3+Z_1Z_2|}{|Z_1||Z_2||Z_3|}$$

$$=r^2\frac{|Z_2Z_3+Z_1Z_3+Z_1Z_2|}{(r)(r)(r)}$$

$$=r^2\frac{|Z_2Z_3+Z_1Z_3+Z_1Z_2|}{r^3}$$

$$=\frac{|Z_2Z_3+Z_1Z_3+Z_1Z_2|}{r}$$

$$\mathrm{r}=\frac{|Z_2Z_3+Z_1Z_3+Z_1Z_2|}{|Z_1+Z_2+Z_3|}$$

Hence $\left|\frac{Z_1Z_2+Z_2Z_3+Z_3Z_1}{Z_1+Z_2+Z_3}\right|$=r

Example 2.16 Show that the equation $Z^2=\bar{Z}$ has four solutions.

Solution: $Z^2=\bar{Z}$

$$|Z|^2=|\bar{Z}|$$

$$|Z|^2=|Z|$$

$$|Z|^2-|Z|=0$$

$$|Z|(|Z|-1)=0$$

So, $|Z|=0$ and $|Z|-1=0$

$|Z|=0$ ⇒gives $Z=0$

$|Z|-1=0$ gives $|Z|=1$

$$Z\,\bar{Z} = 1$$

$$\bar{Z} = \frac{1}{Z}$$

$$\text{Given } Z^2 = \bar{Z}$$

$$= \frac{1}{Z}$$

$$Z^3 = 1$$

It has 3 non-zero solutions. Hence including zero solution, there are four solutions.

Example 2.17

Find the square root of $6 - 8i$

Solution: $\sqrt{a+ib} = \pm\sqrt{\frac{|Z|+a}{2}} + i\frac{b}{|b|}\sqrt{\frac{|Z|-a}{2}}$

Let $Z = 6 - 8i$

$$|Z| = |6 - 8i|$$

$$= \sqrt{6^2 + (-8)^2}$$

$$= \sqrt{36 + 64}$$

$$= \sqrt{100}$$

$$= 10$$

Here a = 6 and b = – 8

$$\sqrt{6 - i8} = \pm\left(\sqrt{\frac{10+6}{2}} + \frac{(-8)}{|-8|}i\sqrt{\frac{10-6}{2}}\right)$$

$$= \pm\left(\sqrt{\frac{16}{2}} - i\sqrt{\frac{4}{2}}\right)$$

$$= \pm(\sqrt{8} - i\sqrt{2})$$

$$= \pm(\sqrt{4\times 2} - i\sqrt{2})$$

$$= \pm(2\sqrt{2} - i\sqrt{2})$$

EXERCISE 2.5

1. Find the modulus of the following complex numbers

(i) $\frac{2i}{3+4i}$

$$\left|\frac{2i}{3+4i}\right| = \frac{|2i|}{|3+4i|}$$

$$= \frac{\sqrt{2^2}}{\sqrt{3^2+4^2}}$$

$$= \frac{\sqrt{4}}{\sqrt{9+16}}$$

$$= \frac{\sqrt{4}}{\sqrt{25}}$$

$$= \frac{2}{5}$$

(ii) $\frac{2-i}{1+i} + \frac{1-2i}{1-i}$

$$\frac{2-i}{1+i} + \frac{1-2i}{1-i}$$

$$= \frac{(2-i)(1-i)+(1-2i)(1+i)}{(1+i)(1-i)}$$

$$= \frac{(2-2i-i+i^2)+(1+i-2i-2i^2)}{(1-i+i-i^2)}$$

$$= \frac{(2-3i+(-1))+(1-i-2(-1))}{(1-(-1))}$$

$$= \frac{(2-3i-1)+(1-i+2)}{(1+1)}$$

$$= \frac{(1-3i)+(3-i)}{2}$$

$$= \frac{1-3i+3-i}{2}$$

$$= \frac{4-4i}{2}$$

$$= \frac{2(2-2i)}{2}$$

$$= 2 - 2i$$

$$\left|\frac{2-i}{1+i} + \frac{1-2i}{1-i}\right| = |2-2i|$$

$$= \sqrt{2^2 + (-2)^2}$$

$$= \sqrt{4+4}$$

$$= \sqrt{2(4)}$$

$$= 2\sqrt{2}$$

(iii) $|(1-i)^{10}| = |(1-i)|^{10}$

$$= \sqrt{1^2 + (-1)^2}^{\,10}$$

$$= \sqrt{1+1}^{\,10}$$

$$= \sqrt{2}^{\,10}$$

$$= \left(2^{\frac{1}{2}}\right)^{10}$$

$$= 2^5 = 32$$

(iv) $|2i(3-4i)(4-3i)|$

$= |2i||3-4i||4-3i|$

$= \sqrt{2^2}\sqrt{3^2+(-4)^2}\ \sqrt{4^2+(-3)^2}$

$= \sqrt{4}\sqrt{9+16}\ \sqrt{16+9}$

$= \sqrt{4}\ \sqrt{25}\ \sqrt{25}$

$= (2)(5)(5)$

$=50$

2. For any two complex numbers Z_1 and Z_2 such that $|Z_1| = |Z_2| = 1$ and $Z_1Z_2 \neq -1$, then show that $\frac{Z_1+Z_2}{1+Z_1Z_2}$ is a real number.

Solution: Given: $|Z_1| = 1$

$\Rightarrow Z_1\overline{Z_1} = 1$

$Z_1 = \frac{1}{\overline{Z_1}}$

Similarly $Z_2 = \frac{1}{\overline{Z_2}}$

$$\frac{Z_1+Z_2}{1+Z_1Z_2} = \frac{\frac{1}{\overline{Z_1}}+\frac{1}{\overline{Z_2}}}{1+\frac{1}{\overline{Z_1}}\frac{1}{\overline{Z_2}}}$$

$$= \frac{\frac{\overline{Z_2}+\overline{Z_1}}{\overline{Z_1}\overline{Z_2}}}{\frac{\overline{Z_1}\overline{Z_2}+1}{\overline{Z_1}\overline{Z_2}}}$$

$$= \frac{\overline{Z_1}+\overline{Z_2}}{1+\overline{Z_1}\overline{Z_2}}$$

$$= \frac{\overline{Z_1Z_2}}{1+\overline{Z_1Z_2}}$$

$$= \overline{\left(\frac{Z_1Z_2}{1+Z_1Z_2}\right)}$$

Since, $Z = \bar{Z},\ \ Z\ is\ real$

$\therefore \frac{Z_1+Z_2}{1+Z_1Z_2}\ is\ real$

3. Which one of the points $10-8i$, $11+6i$ is closest to $1+i$.

Solution:

Let A, B and C be the points.

Then the distance

$|AC| = |(10-8i)-(1+i)|$

$= |10-8i-1-i|$

$= |9-9i|$

$= \sqrt{9^2+(-9)^2}$

$= \sqrt{81+81} \quad = \sqrt{162}$

$|BC| = |(11+6i)-(1+i)|$

$= |11+6i-1-i|$

$= |10+5i|$

$= \sqrt{10^2+(5)^2}$

$= \sqrt{100+25}$

$= \sqrt{125}$

The point $10-8i$ is closest to $1+i$.

4. If $|Z| = 3$ show that $7 \leq |z+6-8i| \leq 13$

Solution:

We know that $|Z_1+Z_2| \leq |Z_1|+|Z_2|$

Hence,

$|Z+(6-8i)| \leq |Z|+|6-8i|$

$\leq 3+|6-8i|$

$\leq 3+\sqrt{6^2+(-8)^2}$

$\leq 3+\sqrt{36+64}$

$\leq 3+\sqrt{100}$

$\leq 3+10$

$\leq 13 \quad \ldots.(1)$

We know that $|Z_1+Z_2| \geq \big||Z_1|-|Z_2|\big|$

Hence,

$|Z+(6-8i)| \geq \big||Z|-|6-8i|\big|$

$\geq \big|3-|6-8i|\big|$

$\geq \left|3-\sqrt{6^2+(-8)^2}\right|$

$\geq |3-\sqrt{36+64}|$

$\geq |3-\sqrt{100}|$

$\geq |3-10|$

$\geq |-7|$

$\geq 7 \quad \ldots.(2)$

From $(1)\ and\ (2)$

$7 \leq |z+6-8i| \leq 13$ is proved.

5. If $|Z| = 1$, show that $2 \leq |z^2-3| \leq 4$

Solution:

We know that $|Z_1 + Z_2| \le |Z_1| + |Z_2|$

Hence,

$$|Z^2 + (-3)| \le |Z|^2 + |-3|$$
$$\le 1 + 3$$
$$\le 4 \quad \ldots . (1)$$

We know that $|Z_1 + Z_2| \ge \big||Z_1| - |Z_2|\big|$

Hence,

$$|Z^2 - 3| \ge \big||Z|^2 - |3|\big|$$
$$\ge |1 - 3|$$
$$\ge |-2|$$
$$\ge 2 \quad \ldots . (2)$$

From (1) *and* (2)

$2 \le |z^2 - 3| \le 4$ is proved.

6. If $\left|z - \frac{2}{z}\right| = 2$, show that the greatest and least value of $|z|$ are $\sqrt{3} + 1$ and $\sqrt{3} - 1$ respectively.

Solution: Given $\left|z - \frac{2}{z}\right| = 2$

$$|Z| = \left|z - \frac{2}{z} + \frac{2}{z}\right|$$

By triangle law of inequality,

$$\left|z - \frac{2}{z} + \frac{2}{z}\right| \le \left|z - \frac{2}{z}\right| + \left|\frac{2}{z}\right|$$
$$\le 2 + \left|\frac{2}{z}\right|$$
$$\le 2 + \frac{2}{|Z|}$$
$$|Z| \le \frac{2|Z| + 2}{|Z|}$$
$$|Z|^2 \le 2|Z| + 2$$
$$|Z|^2 - 2|Z| \le 2$$

Adding 1 on either side

$$|Z|^2 - 2|Z| + 1 \le 2 + 1$$
$$[|Z| - 1]^2 \le 3$$
$$(|Z| - 1) \le \pm\sqrt{3}$$
$$|Z| \le \pm\sqrt{3} + 1$$

The greatest and least value of $|z|$ are $\sqrt{3} + 1$ and $\sqrt{3} - 1$ respectively.

7. If Z_1, Z_2 and Z_3 are three complex numbers such that $|Z_1| = 1, |Z_2| = 2$, $|Z_3| = 3$ *and* $|Z_1 + Z_2 + Z_3| = 1$, show that $|9z_1z_2 + 4z_1z_3 + z_2z_3| = 6$

Solution: Given $|Z_1| = 1$

$$\therefore {Z_1}^2 = 1$$
$$Z_1\overline{Z_1} = 1$$
$$Z_1 = \frac{1}{\overline{Z_1}}$$
$$|Z_2| = 2$$
$$\therefore {Z_2}^2 = 4$$
$$Z_2\overline{Z_2} = 4$$
$$Z_2 = \frac{4}{\overline{Z_2}} \text{ and}$$
$$|Z_3| = 3$$
$$\therefore {Z_3}^2 = 9$$
$$Z_3\overline{Z_3} = 9$$
$$Z_3 = \frac{9}{\overline{Z_3}}$$
$$Z_1 + Z_2 + Z_3 = \frac{1}{\overline{Z_1}} + \frac{4}{\overline{Z_2}} + \frac{9}{\overline{Z_3}}$$
$$= \frac{\overline{Z_2}\,\overline{Z_3} + 4\overline{Z_1}\,\overline{Z_3} + 9\,\overline{Z_1}\,\overline{Z_2}}{\overline{Z_1}\,\overline{Z_2}\,\overline{Z_3}}$$
$$|Z_1 + Z_2 + Z_3| = \left|\frac{\overline{Z_2}\,\overline{Z_3} + 4\overline{Z_1}\,\overline{Z_3} + 9\,\overline{Z_1}\,\overline{Z_2}}{\overline{Z_1}\,\overline{Z_2}\,\overline{Z_3}}\right|$$
$$= \left|\frac{\overline{Z_2Z_3 + 4Z_1Z_3 + 9\,Z_1Z_2}}{\overline{Z_1Z_2Z_3}}\right|$$
$$= \frac{\left|\overline{Z_2Z_3 + 4Z_1Z_3 + 9\,Z_1Z_2}\right|}{\left|\overline{Z_1Z_2Z_3}\right|}$$
$$= \frac{|Z_2Z_3 + 4\,Z_1Z_3 + 9\,Z_1Z_2|}{|Z_1||Z_2||Z_3|}$$
$$1 = \frac{|Z_2Z_3 + 4\,Z_1Z_3 + 9\,Z_1Z_2|}{(1)(2)(3)}$$
$$1 = \frac{|Z_2Z_3 + 4\,Z_1Z_3 + 9\,Z_1Z_2|}{6}$$

$|9z_1z_2 + 4z_1z_3 + z_2z_3| = 6$

8. If the area of the triangle formed by the vertices $z, iz, and\ z+iz$ is 50 square units, find the value if $|Z|$.

Solution:

Let A $= Z = x+iy$

B $= iZ = i(x+iy)$

$= xi + yi^2$

$= xi + y(-1)$

$= xi - y$

$= -y + xi$

C $= Z + iZ = (x+iy) + i(x+iy)$

$= x + iy + xi + yi^2$

$= x + iy + xi + y(-1)$

$= x + iy + xi - y$

$= (x-y) + i(x+y)$

are the vertices of the triangle.

Length of the sides

$|AB| = |(x+iy) - (-y+xi)|$

$= |x + iy + y - xi|$

$= |(x+y) + i(y-x)|$

$= \sqrt{(x+y)^2 + (y-x)^2}$

$(AB)^2 = (x+y)^2 + (y-x)^2$

$= x^2 + y^2 + 2xy + x^2 + y^2 - 2xy$

$= x^2 + y^2 + x^2 + y^2$

$= 2x^2 + 2y^2$

$= 2(x^2 + y^2)$

$|BC| = |(-y+xi) - [(x-y) + i(x+y)]|$

$= |(-y+xi) - (x-y) - i(x+y)|$

$= |-y + xi - x + y - xi - yi|$

$= |-x - yi|$

$= |-(x+yi)|$

$= |x+yi|$

$= \sqrt{x^2+y^2}$

$(BC)^2 = x^2 + y^2$

$|AC| = |(x+yi) - [(x-y) + i(x+y)]|$

$= |(x+yi) - (x-y) - i(x+y)|$

$= |x + yi - x + y - xi - yi|$

$= |y - xi|$

$= \sqrt{y^2 + x^2}$

$(AC)^2 = x^2 + y^2$

Since $(AB)^2 = (BC)^2 + (AC)^2$, the given vertices form a right angled triangle with AB as hypotenuse, BC and AC as the base and height.

Area of the triangle $= \frac{1}{2} \times b \times h$

$50 = \frac{1}{2} \times b \times h$

$100 = \sqrt{(x^2+y^2)} \times \sqrt{(x^2+y^2)}$

$100 = (x^2 + y^2)$

$\therefore \sqrt{(x^2+y^2)} = 10$

Since $Z = x + iy$

$|Z| = \sqrt{(x^2+y^2)}$

$\therefore |Z| = 10$

9. Show that the equation $Z^3 + 2\bar{Z} = 0$ has five solutions.

Solution: Given $Z^3 + 2\bar{Z} = 0$

$Z^3 = -2\bar{Z}$

Taking modulus $|Z^3| = |-2\bar{Z}|$

$|Z|^3 = 2|\bar{Z}|$

$|Z|^3 = 2|Z|$

$|Z|^3 - 2|Z| = 0$

$|Z|(|Z|^2 - 2) = 0$

Gives $|Z| = 0\ and\ (|Z|^2 - 2) = 0$

$(|Z|^2 - 2) = 0$

$|Z|^2 = 2$

$(Z\bar{Z})^2 = 2$

$Z\bar{Z} = \sqrt{2}$

$\bar{Z} = \frac{\sqrt{2}}{Z}$

Given $Z^3 + 2\bar{Z} = 0$

$Z^3 + 2\frac{\sqrt{2}}{Z} = 0$

$Z^4 + 2\sqrt{2} = 0$

It has 4 non-zero solutions. Hence including zero solution, there are five solutions.

10. Find the square root of

(i) $4+3i$

Solution: $\sqrt{a+ib} = \pm\sqrt{\frac{|Z|+a}{2}} + i\frac{b}{|b|}\sqrt{\frac{|Z|-a}{2}}$

Let $Z = 4+3i$

$|Z| = |4+3i|$

$= \sqrt{4^2+3^2}$

$= \sqrt{16+9}$

$= \sqrt{25}$

$= 5$

Here a = 4 and b = 3

$\sqrt{4+3i} = \pm\left(\sqrt{\frac{5+4}{2}} + \frac{(3)}{|3|}i\sqrt{\frac{5-4}{2}}\right)$

$= \pm\left(\sqrt{\frac{9}{2}} + i\sqrt{\frac{1}{2}}\right)$

$= \pm\left(\frac{3}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right)$

(ii) $-6+8i$

Solution: $\sqrt{a+ib} = \pm\sqrt{\frac{|Z|+a}{2}} + i\frac{b}{|b|}\sqrt{\frac{|Z|-a}{2}}$

Let $Z = -6+8i$

$|Z| = |-6+8i|$

$= \sqrt{(-6)^2+8^2}$

$= \sqrt{36+64}$

$= \sqrt{100}$

$= 10$

Here a = - 6 and b = 8

$\sqrt{-8+6i} = \pm\left(\sqrt{\frac{10-6}{2}} + \frac{(8)}{|8|}i\sqrt{\frac{10+6}{2}}\right)$

$= \pm\left(\sqrt{\frac{4}{2}} + i\sqrt{\frac{16}{2}}\right)$

$= \pm\left(\sqrt{2} + i\sqrt{8}\right)$

$= \pm\left(\sqrt{2} + i\sqrt{4\times 2}\right)$

$= \pm\left(\sqrt{2} + 2\sqrt{2}i\right)$

(iii) $-5-12i$.

Solution: $\sqrt{a+ib} = \pm\sqrt{\frac{|Z|+a}{2}} + i\frac{b}{|b|}\sqrt{\frac{|Z|-a}{2}}$

Let $Z = -5-12i$

$|Z| = |-5-12i\,|$

$= \sqrt{(-5)^2+(-12)^2}$

$= \sqrt{25+144}$

$= \sqrt{169}$

$= 13$

Here a = - 5 and b = - 12

$\sqrt{-5-12i} = \pm\left(\sqrt{\frac{13-5}{2}} + \frac{(-12)}{|-12|}i\sqrt{\frac{13+5}{2}}\right)$

$= \pm\left(\sqrt{\frac{13-5}{2}} + \frac{(-12)}{12}i\sqrt{\frac{13+5}{2}}\right)$

$= \pm\left(\sqrt{\frac{8}{2}} - i\sqrt{\frac{18}{2}}\right)$

$= \pm\left(\sqrt{4} - i\sqrt{9}\right)$

$= \pm(2-3i)$

Example 2.18 Given the complex number $Z = 3+2i$, represent the complex numbers $z, iz, and\ z+iz$ in one Argand diagram. Show that these complex numbers form the vertices of an isosceles right triangle.

Let $A = Z = 3+2i$

$B = iZ = i(3+2i\,)$

$= 3i+2i^2$

$= 3i+2(-1)$

$= 3i-2$

$= -2+3i$

$C = Z + iZ = (3+2i\,)+i(3+2i\,)$

$= 3+2i+3i+2i^2$

$= 3+2i+3i+2(-1)$

$= 3+2i+3i-2$

$= 1+5i$

are the vertices of the triangle.

Length of the sides

$$|AB| = |(3+2i)-(-2+3i)|$$
$$= |3+2i+2-3i|$$
$$= |5-i|$$
$$= \sqrt{(5)^2+(-1)^2}$$
$$= \sqrt{25+1}$$
$$= \sqrt{26}$$
$$(AB)^2 = 26$$
$$|BC| = |(-2+3i)-(1+5i)|$$
$$= |-2+3i-1-5i|$$
$$= |-3-2i|$$
$$= \sqrt{(-3)^2+(-2)^2}$$
$$= \sqrt{9+4}$$
$$= \sqrt{13}$$
$$(BC)^2 = 13$$
$$|AC| = |(3+2i)-(1+5i)|$$
$$= |3+2i-1-5i|$$
$$= |2-3i|$$
$$= \sqrt{2^2+(-3)^2}$$
$$= \sqrt{4+9}$$
$$= \sqrt{13}$$
$$(AC)^2 = 13$$

Since $(AB)^2 = (BC)^2 + (AC)^2$, the given vertices form a right angled triangle with AB as hypotenuse, and BC = AC, the vertices form an isosceles right triangle.

Example 2.19 Show that $|3z-5+i| = 4$ represents a circle, and, find its centre and radius.

Solution: Given $|3z-5+i| = 4$

$$3\left|z-\frac{(5-i)}{3}\right| = 4$$
$$\left|z-\frac{(5-i)}{3}\right| = \frac{4}{3}$$
$$\left|z-\left(\frac{5}{3}-\frac{i}{3}\right)\right| = \frac{4}{3}$$

Comparing with $|z-z_0| = \mathrm{r}$

represents a circle, whose centre and radius are $\left(\frac{5}{3},-\frac{1}{3}\right)$, and $\frac{4}{3}$ respectively

Example 2.20 Show that $|z+2-i| < 2$ represents interior points of a circle. Find its centre and radius.

Solution: Given $|z+2-i| = 2$

$$|z-(-2+i)| = 2$$

Comparing with $|z-z_0| = \mathrm{r}$

The above equation represents the circle with centre $Z_0 = -2+i$ and radius r = 2. Therefore $|z+2-i| < 2$ represents all points inside the circle with centre at $-2+i$ and radius 2

Example 2.21 Obtain the Cartesian form of the locus of z in each of the following cases.

(i) Let $Z = x+yi$

Then $|Z| = |Z-i|$ gives

$$|x+yi| = |(x+yi)-i|$$
$$|x+yi| = |x+yi-i|$$
$$|x+yi| = |x+(y-1)i|$$
$$\sqrt{x^2+y^2} = \sqrt{x^2+(y-1)^2}$$

Squaring,

$$x^2+y^2 = x^2+(y-1)^2$$
$$y^2 = (y-1)^2$$
$$y^2 = y^2+1-2y$$
$$0 = 1-2y$$

$2y-1 = 0$, is the locus of Z.

(ii) $|2z-3-i| = 3$ gives

$$|2(x+yi)-3-i| = 3$$
$$|2x+2yi-3-i| = 3$$
$$|(2x-3)+(2y-1)i| = 3$$
$$\sqrt{(2x-3)^2+(2y-1)^2} = 3$$

Squaring,

$$(2x-3)^2+(2y-1)^2 = 9$$
$$4x^2+9-12x+4y^2+1-4y = 9$$

$$4x^2+4y^2-12x-4y+10=9$$
$$4x^2+4y^2-12x-4y+10-9=0$$
$$4x^2+4y^2-12x-4y+1=0,$$
is the locus of Z.

EXERCISE 2.6

1. If $z=x+iy$ is a complex such that $\left|\frac{Z-4i}{Z+4i}\right|=1$ show that the locus of Z is real axis.

Solution: $z=x+iy$

$$\left|\frac{Z-4i}{Z+4i}\right|=1$$
$$\frac{|Z-4i|}{|Z+4i|}=1$$
$$|z-4i|=|z+4i|$$
$$|(x+yi)-4i|=|(x+yi)+4i|$$
$$|x+yi-4i|=|x+yi+4i|$$
$$|x+(y-4)i|=|x+(y+4)i|$$
$$\sqrt{x^2+(y-4)^2}=\sqrt{x^2+(y+4)^2}$$

Squaring

$$x^2+(y-4)^2=x^2+(y+4)^2$$
$$x^2+y^2+16-8y=x^2+y^2+16+8y$$
$$x^2+y^2+16-8y-x^2-y^2-16-8y=0$$
$$-8y-8y=0$$
$$-16y=0 \quad \div(-16)$$
$$y=0$$
$$\therefore z=x+iy, gives$$

$z=x$, the locus of Z is real axis.

2. If $z=x+iy$ is a complex such that $Img\left(\frac{2Z+1}{iZ+1}\right)=0$, show that the locus of Z is $2x^2+2y^2+x-2y=0$.

Solution: Given $z=x+iy$

$$2z+1=2(x+iy)+1$$
$$=2x+2yi+1$$
$$=(2x+1)+2yi$$
$$iz+1=i(x+iy)+1$$
$$=xi+yi^2+1$$
$$=xi+y(-1)+1$$
$$=xi-y+1$$
$$=(1-y)+xi$$
$$\frac{2Z+1}{iZ+1}=\frac{(2x+1)+2yi}{(1-y)+xi}$$
$$=\frac{(2x+1)+2yi}{(1-y)+xi}\times\frac{(1-y)-xi}{(1-y)-xi}$$
$$Img\left(\frac{2Z+1}{iZ+1}\right)=\frac{(2x+1)(-xi)+2yi(1-y)}{(1-y)^2+x^2}$$
$$Img\left(\frac{2Z+1}{iZ+1}\right)=0, \text{ gives}$$
$$\frac{(2x+1)(-x)+2y(1-y)}{(1-y)^2+x^2}=0$$
$$(2x+1)(-x)+2y(1-y)=0$$
$$-2x^2-x+2y-2y^2=0 \quad \times(-1)$$
$$2x^2+2y^2+x-2y=0.$$

3. Obtain the Cartesian form of the locus of $z=x+iy$ in each of the following cases:

(i) $[Re(iz)]^2=3$

Solution: $z=x+iy$

$$iz=i(x+iy)$$
$$=xi+yi^2$$
$$=xi+y(-1)$$
$$=xi-y$$
$$=-y+xi$$
$$Re(iz)=-y$$
$$[Re(iz)]^2=(-y)^2=y^2$$
$$[Re(iz)]^2=3, gives$$
$$y^2=3$$

The Cartesian form of the locus of Z is $y^2=3$.

(ii) $Im[(1-i)z+1]=0$

Solution: $z=x+iy$

$$(1-i)z+1=(1-i)(x+iy)+1$$
$$=x+yi-xi-yi^2+1$$

$= x + yi - xi - y(-1) + 1$

$= x + yi - xi + y + 1$

$= (x + y + 1) + (y - x)i$

$Im[(1 - i)z + 1] = 0, gives$

$y - x = 0 \quad \times (-1)$

$x - y = 0$

The Cartesian form of the locus of

Z is $x - y = 0$

(iii) $|z + i| = |z - 1|$

Solution: $z = x + iy$

$|z + i| = |z - 1|$

$|(x + yi) + i| = |(x + yi) - 1|$

$|x + yi + i| = |x + yi - 1|$

$|x + (y + 1)i| = |(x - 1) + yi|$

$\sqrt{x^2 + (y + 1)^2} = \sqrt{(x - 1)^2 + y^2}$

Squaring

$x^2 + (y + 1)^2 = (x - 1)^2 + y^2$

$x^2 + y^2 + 1 + 2y = x^2 + 1 - 2x + y^2$

$x^2 + y^2 + 1 + 2y - x^2 - 1 + 2x - y^2 = 0$

$2x + 2y = 0 \quad \div 2$

$x + y = 0$

The Cartesian form of the locus of

Z is $x + y = 0$

(iv) $\bar{z} = z^{-1}$

Solution: $\bar{z} = z^{-1}$

$\bar{z} = \frac{1}{z}$

$z\bar{z} = 1$

$|Z|^2 = 1$

$x^2 + y^2 = 1$

The Cartesian form of the locus of

Z is $x^2 + y^2 = 1$

4. Show that the following equations represent a circle, and, find its centre and radius.

(i) $|z + 2 - i| = 3$

Solution: $|z + 2 - i| = 3$

$|z - (2 + i)| = 3$

It is of the form $|z - z_0| = \mathrm{r}$, so it represents a circle.

Comparing with $|z - z_0| = \mathrm{r}$

centre is $(2 + i)$ and radius is 3

(ii) $|2z + 2 - 4i| = 2$

Solution: $|2z + 2 - 4i| = 2 \quad \div 2$

$|z + 1 - 2i| = 1$

$|z - (-1 + 2i)| = 1$

It is of the form $|z - z_0| = \mathrm{r}$, so it represents a circle.

Comparing with $|z - z_0| = \mathrm{r}$

centre is $(-1 + 2i)$ and radius is 1

(iii) $|3z - 6 + 12i| = 8$

Solution: $|3z - 6 + 12i| = 8 \quad \div 3$

$|z - (2 - 4i)| = \frac{8}{3}$

It is of the form $|z - z_0| = \mathrm{r}$, so it represents a circle.

Comparing with $|z - z_0| = \mathrm{r}$

centre is $(2 - 4i)$ and radius is $\frac{8}{3}$

5. Obtain the Cartesian form of the locus of $z = x + iy$ in each of the following cases.

(i) $|Z - 4| = 16$

Solution: $z = x + iy$

$|(x + yi) - 4| = 16$

$|x + yi - 4| = 16$

$$|(x-4)+yi| = 16$$
$$\sqrt{(x-4)^2+y^2} = 16$$

Squaring,

$$(x-4)^2+y^2 = (16)^2$$
$$x^2+16-8x+y^2 = 256$$
$$x^2+16-8x+y^2-256 = 0$$
$$x^2+y^2-8x-240 = 0,$$

Is the required Cartesian equation.

(ii) $|z-4|^2-|z-1|^2 = 16.$

Solution: $z = x+iy$

$$|z-4|^2-|z-1|^2 = 16$$
$$|(x+yi)-4|^2-|(x+yi)-1|^2 = 16$$
$$|x+yi-4|^2-|x+yi-1|^2 = 16$$
$$|(x-4)+yi|^2-|(x-1)+yi|^2 = 16$$
$$\left[\sqrt{(x-4)^2+y^2}\right]^2-\left[\sqrt{(x-1)^2+y^2}\right]^2 = 16$$
$$(x-4)^2+y^2-[(x-1)^2+y^2] = 16$$
$$x^2+16-8x+y^2-[x^2+1-2x+y^2] = 16$$
$$x^2+16-8x+y^2-x^2-1+2x-y^2 = 16$$
$$-6x+15-16 = 0$$
$$-6x-1 = 0$$
$$6x+1 = 0$$

Is the required Cartesian equation.

Example 2.22 Find the modulus and principal argument of the following complex numbers.

$(i)\ \sqrt{3}+i$

$$x+yi = \sqrt{3}+i$$
$$\text{Modulus} = \sqrt{x^2+y^2} = \sqrt{\left(\sqrt{3}\right)^2+1^2}$$
$$= \sqrt{3+1}$$
$$= \sqrt{4}$$
$$r = 2$$
$$\tan\alpha = \left|\frac{y}{x}\right|$$
$$= \frac{1}{\sqrt{3}}$$
$$\alpha = \frac{\pi}{6}$$

$\sqrt{3}+i$, lies in I quadrant, has the principal value $\theta = \alpha \Rightarrow \theta = \frac{\pi}{6}$

Therefore, the modulus and principal argument of $\sqrt{3}+i$, are 2 and $\frac{\pi}{6}$

$(ii)\ -\sqrt{3}+i$

$$x+yi = -\sqrt{3}+i$$
$$\text{Modulus} = \sqrt{x^2+y^2} = \sqrt{\left(-\sqrt{3}\right)^2+1^2}$$
$$= \sqrt{3+1}$$
$$= \sqrt{4}$$
$$r = 2$$
$$\tan\alpha = \left|\frac{y}{x}\right|$$
$$= \frac{1}{\sqrt{3}}$$
$$\alpha = \frac{\pi}{6}$$

$-\sqrt{3}+i$, lies in II quadrant, has the principal value $\theta = \pi-\alpha \Rightarrow$

$$\theta = \pi-\frac{\pi}{6}$$
$$= \frac{6\pi-\pi}{6}$$
$$= \frac{5\pi}{6}$$

Therefore, the modulus and principal argument of $-\sqrt{3}+i$, are 2 and $\frac{5\pi}{6}$

$(iii)\ -\sqrt{3}-i$

$$x+yi = -\sqrt{3}-i$$
$$\text{Modulus} = \sqrt{x^2+y^2} = \sqrt{\left(-\sqrt{3}\right)^2+(-1)^2}$$
$$= \sqrt{3+1}$$
$$= \sqrt{4}$$
$$r = 2$$
$$\tan\alpha = \left|\frac{y}{x}\right|$$

$$= \frac{1}{\sqrt{3}}$$

$$\alpha = \frac{\pi}{6}$$

$-\sqrt{3} - i$, lies in III quadrant, has the principal value $\theta = -\pi + \alpha \Rightarrow$

$$\theta = -\pi + \frac{\pi}{6}$$

$$= \frac{-6\pi + \pi}{6}$$

$$= -\frac{5\pi}{6}$$

Therefore, the modulus and principal argument of $-\sqrt{3} - i$, are 2 and $-\frac{5\pi}{6}$

(iv) $\sqrt{3} - i$

$$x + yi = \sqrt{3} - i$$

$$\text{Modulus} = \sqrt{x^2 + y^2} = \sqrt{\left(\sqrt{3}\right)^2 + (-1)^2}$$

$$= \sqrt{3 + 1}$$

$$= \sqrt{4}$$

$$r = 2$$

$$\tan\alpha = \left|\frac{y}{x}\right|$$

$$= \frac{1}{\sqrt{3}}$$

$$\alpha = \frac{\pi}{6}$$

$\sqrt{3} - i$, lies in IV quadrant, has the principal value $\theta = -\alpha \Rightarrow$

$$\theta = -\frac{\pi}{6}$$

$$= -\frac{\pi}{6}$$

Therefore, the modulus and principal argument of $\sqrt{3} - i$, are 2 and $-\frac{\pi}{6}$

Example 2.23 Represent the complex number (i) $-1 - i$ (ii) $1 + i\sqrt{3}$ in polar form.

(i) $-1 - i$

Solution:

Let $-1 - i = r(\cos\theta + i\sin\theta)$

$$x + yi = -1 - i$$

$$\text{Modulus} = \sqrt{x^2 + y^2} = \sqrt{(-1)^2 + (-1)^2}$$

$$= \sqrt{1 + 1}$$

$$= \sqrt{2}$$

$$r = \sqrt{2}$$

$$\tan\alpha = \left|\frac{y}{x}\right|$$

$$= \frac{1}{1}$$

$$= 1$$

$$\alpha = \frac{\pi}{4}$$

$-1 - i$, lies in III quadrant, has the principal value $\theta = -\pi + \alpha \Rightarrow$

$$\theta = -\pi + \frac{\pi}{4}$$

$$= \frac{-4\pi + \pi}{4}$$

$$= -\frac{3\pi}{4}$$

$$-1 - i = r(\cos\theta + i\sin\theta)$$

$$= \sqrt{2}\left[\cos\left(-\frac{3\pi}{4}\right) + i\sin\left(-\frac{3\pi}{4}\right)\right]$$

$$= \sqrt{2}\left[\cos\left(\frac{3\pi}{4}\right) - i\sin\left(\frac{3\pi}{4}\right)\right] =$$

$$\sqrt{2}\left[\cos\left(\frac{3\pi}{4} + 2k\pi\right) - i\sin\left(\frac{3\pi}{4} + 2k\pi\right)\right], k \in Z$$

(ii) $1 + i\sqrt{3}$

Solution:

Let $1 + \sqrt{3}i = r(\cos\theta + i\sin\theta)$

$$x + yi = 1 + \sqrt{3}i$$

$$\text{Modulus} = \sqrt{x^2 + y^2} = \sqrt{(1)^2 + \left(\sqrt{3}\right)^2}$$

$$= \sqrt{1 + 3}$$

$$= \sqrt{4}$$

$$r = 2$$

$$\tan\alpha = \left|\frac{y}{x}\right|$$

$$= \frac{1}{\sqrt{3}}$$

$$\alpha = \frac{\pi}{6}$$

$1 + \sqrt{3}i$, lies in I quadrant, has the principal value $\theta = \alpha \Rightarrow$

$$\theta = \frac{\pi}{6}$$

$$1 + \sqrt{3}i = r(\cos\theta + i\sin\theta)$$

$$= 2\left[\cos\left(\frac{\pi}{6}\right) + i\sin\left(\frac{\pi}{6}\right)\right]$$

$$= 2\left[\cos\left(\frac{\pi}{6} + 2k\pi\right) + i\sin\left(\frac{\pi}{6} + 2k\pi\right)\right], k \in Z$$

Example 2.24 Find the principal argument Arg Z, when $Z = \frac{-2}{1 + i\sqrt{3}}$

Solution: $Z = \frac{-2}{1 + i\sqrt{3}}$

$$\text{Arg } Z = \arg\frac{-2}{1 + i\sqrt{3}}$$

$$= \arg(-2) - \arg\left(1 + i\sqrt{3}\right)$$

$$= \left[\pi - tan^{-1}\left(\frac{0}{2}\right)\right] - tan^{-1}\left(\frac{\sqrt{3}}{1}\right)$$

$$= \pi - \frac{\pi}{3}$$

$$= \frac{3\pi - \pi}{3}$$

$$= \frac{2\pi}{3}$$

This implies that one of the values of $\arg z$ is $\frac{2\pi}{3}$. Since $\frac{2\pi}{3}$ lies between $-\pi$ and π, the principal argument Arg z is $\frac{2\pi}{3}$.

Example 2.25 Find the product of

$$\frac{3}{2}\left(\cos\frac{\pi}{3} + i\,sin\,\frac{\pi}{3}\right).6\left(\cos\frac{5\pi}{6} + i\,sin\,\frac{5\pi}{6}\right)$$

in rectangular form.

Solution:

$$\frac{3}{2}\left(\cos\frac{\pi}{3} + i\,sin\,\frac{\pi}{3}\right).6\left(\cos\frac{5\pi}{6} + i\,sin\,\frac{5\pi}{6}\right) =$$

$$= \left(\frac{3}{2}\right)(6)\left(\cos\frac{\pi}{3} + i\,sin\,\frac{\pi}{3}\right)\left(\cos\frac{5\pi}{6} + i\,sin\,\frac{5\pi}{6}\right)$$

$$= 9\left[\cos\left(\frac{\pi}{3} + \frac{5\pi}{6}\right) + i\,sin\,\left(\frac{\pi}{3} + \frac{5\pi}{6}\right)\right]$$

$$= 9\left[\cos\left(\frac{2\pi + 5\pi}{6}\right) + i\,sin\,\left(\frac{2\pi + 5\pi}{6}\right)\right]$$

$$= 9\left[\cos\left(\frac{7\pi}{6}\right) + i\,sin\,\left(\frac{7\pi}{6}\right)\right]$$

$$= 9\left[\cos\left(\frac{6\pi + \pi}{6}\right) + i\,sin\,\left(\frac{6\pi + \pi}{6}\right)\right]$$

$$= 9\left[\cos\left(\frac{6\pi}{6} + \frac{\pi}{6}\right) + i\,sin\,\left(\frac{6\pi}{6} + \frac{\pi}{6}\right)\right]$$

$$= 9\left[\cos\left(\pi + \frac{\pi}{6}\right) + i\,sin\,\left(\pi + \frac{\pi}{6}\right)\right]$$

$$= 9\left[-\cos\left(\frac{\pi}{6}\right) - i\,sin\,\left(\frac{\pi}{6}\right)\right]$$

$$= 9\left[-\frac{\sqrt{3}}{2} - i\,\frac{1}{2}\right]$$

$$= -\frac{9\sqrt{3}}{2} - \frac{9}{2}i$$

Example 2.26 Find the quotient $\frac{2\left(\cos\frac{9\pi}{4} + i\,sin\,\frac{9\pi}{4}\right)}{4\left[\cos\left(\frac{-3\pi}{2}\right) + isin\left(\frac{-3\pi}{2}\right)\right]}$ in rectangular form.

Solution:

$$\frac{2\left(\cos\frac{9\pi}{4} + i\,sin\,\frac{9\pi}{4}\right)}{4\left[\cos\left(\frac{-3\pi}{2}\right) + isin\left(\frac{-3\pi}{2}\right)\right]} = \frac{2}{4}\frac{\left(\cos\frac{9\pi}{4} + i\,sin\,\frac{9\pi}{4}\right)}{\left[\cos\left(\frac{-3\pi}{2}\right) + isin\left(\frac{-3\pi}{2}\right)\right]}$$

$$= \frac{1}{2}\left[\text{cis}\left(\left(\frac{9\pi}{4}\right) - \left(\frac{-3\pi}{2}\right)\right)\right]$$

$$= \frac{1}{2}\left[\cos\left(\frac{9\pi}{4} + \frac{3\pi}{2}\right) + i\,sin\,\left(\frac{9\pi}{4} + \frac{3\pi}{2}\right)\right]$$

$$= \frac{1}{2}\left[\cos\left(\frac{9\pi + 6\pi}{4}\right) + i\,sin\,\left(\frac{9\pi + 6\pi}{4}\right)\right]$$

$$= \frac{1}{2}\left[\cos\left(\frac{15\pi}{4}\right) + i\,sin\,\left(\frac{15\pi}{4}\right)\right]$$

$$= \frac{1}{2}\left[\cos\left(\frac{16\pi - \pi}{4}\right) + i\,sin\,\left(\frac{16\pi - \pi}{4}\right)\right]$$

$$= \frac{1}{2}\left[\cos\left(\frac{16\pi}{4} - \frac{\pi}{4}\right) + i\,sin\,\left(\frac{16\pi}{4} - \frac{\pi}{4}\right)\right]$$

$$= \frac{1}{2}\left[\cos\left(4\pi - \frac{\pi}{4}\right) + i\,sin\,\left(4\pi - \frac{\pi}{4}\right)\right]$$

$$= \frac{1}{2}\left[\cos\left(-\frac{\pi}{4}\right) + i\,sin\,\left(-\frac{\pi}{4}\right)\right]$$

$$= \frac{1}{2}\left[\cos\left(\frac{\pi}{4}\right) - i\,sin\,\left(\frac{\pi}{4}\right)\right]$$

$$= \frac{1}{2}\left[\frac{1}{\sqrt{2}} - i\,\frac{1}{\sqrt{2}}\right]$$

$$= \frac{1}{2\sqrt{2}} - \frac{1}{2\sqrt{2}}i$$

Example 2.27 If $z = x + iy$ and $\arg\left(\frac{z-1}{z+1}\right) = \frac{\pi}{2}$, then show that $x^2 + y^2 = 1$.

Solution: Given $z = x + iy$

$$\frac{z-1}{z+1} = \frac{(x+iy)-1}{(x+iy)+1}$$

$$= \frac{x+iy-1}{x+iy+1}$$

$$= \frac{(x-1)+iy}{(x+1)+iy}$$

If $Z = \frac{a+bi}{c+di}$ then

$$\frac{a+bi}{c+di} \times \frac{c-di}{c-di} = \frac{(ac+bd)+(bc-ad)i}{c^2+d^2}$$

$$\text{Re}(Z) = \frac{ac+bd}{c^2+d^2}, \ \text{Im}(Z) = \frac{bc-ad}{c^2+d^2}$$

$$\arg(Z) = \frac{bc-ad}{ac+bd}$$

$a = x - 1, b = y, c = x + 1$ and $d = y$

$$\arg(Z) = \frac{bc-ad}{ac+bd}$$

$$\arg\left(\frac{z-1}{z+1}\right) = \frac{\pi}{2}$$

$$\frac{bc-ad}{ac+bd} = \tan\left(\frac{\pi}{2}\right)$$

$$ac + bd = 0$$

$$x^2 + y^2 - 1 = 0$$

$x^2 + y^2 = 1$, is proved.

EXERCISE 2.7

1. Write in polar form of the following complex numbers.

(i) $2 + i2\sqrt{3}$

Solution:

Let $2 + i2\sqrt{3} = r(\cos\theta + i\sin\theta)$

$$x + yi = 2 + 2\sqrt{3}i$$

$$\text{Modulus} = \sqrt{x^2+y^2} = \sqrt{(2)^2 + (2\sqrt{3})^2}$$

$$= \sqrt{4+12}$$

$$= \sqrt{16}$$

$$r = 4$$

$$\tan\alpha = \left|\frac{y}{x}\right|$$

$$= \frac{2\sqrt{3}}{2}$$

$$= \sqrt{3}$$

$$\alpha = \frac{\pi}{3}$$

$2 + 2\sqrt{3}i$, lies in I quadrant, has the principal value $\theta = \alpha \Rightarrow$

$$\theta = \frac{\pi}{3}$$

$$2 + 2\sqrt{3}i = r(\cos\theta + i\sin\theta)$$

$$= 4\left[\cos\left(\frac{\pi}{3}\right) + i\sin\left(\frac{\pi}{3}\right)\right]$$

$$= 4\left[\cos\left(\frac{\pi}{3} + 2k\pi\right) + i\sin\left(\frac{\pi}{3} + 2k\pi\right)\right], k \in Z$$

(ii) $3 - i\sqrt{3}$

Solution:

Let $3 - i\sqrt{3} = r(\cos\theta + i\sin\theta)$

$$x + yi = 3 - \sqrt{3}i$$

$$\text{Modulus} = \sqrt{x^2+y^2} = \sqrt{(3)^2 + (-\sqrt{3})^2}$$

$$= \sqrt{9+3}$$

$$= \sqrt{12}$$

$$= \sqrt{4 \times 3}$$

$$r = 2\sqrt{3}$$

$$\tan\alpha = \left|\frac{y}{x}\right|$$

$$= \frac{\sqrt{3}}{3}$$

$$= \frac{\sqrt{3}}{\sqrt{3}\sqrt{3}}$$

$$= \frac{1}{\sqrt{3}}$$

$$\alpha = \frac{\pi}{6}$$

$3 - i\sqrt{3}$, lies in IV quadrant, has the principal value $\theta = -\alpha \Rightarrow$

$$\theta = -\frac{\pi}{6}$$

$$3-\sqrt{3}i = r(\cos\theta + i\sin\theta)$$

$$= 2\sqrt{3}\left[\cos\left(-\frac{\pi}{6}\right) + i\sin\left(-\frac{\pi}{6}\right)\right]$$

$$= 2\sqrt{3}\left[\cos\left(-\frac{\pi}{6}+2k\pi\right) + i\sin\left(-\frac{\pi}{6}+2k\pi\right)\right], k \in Z$$

(iii) $-2-2i$

Solution:

Let $-2-2i = r(\cos\theta + i\sin\theta)$

$$x+yi = -2-2i$$

$$\text{Modulus} = \sqrt{x^2+y^2} = \sqrt{(-2)^2+(-2)^2}$$

$$= \sqrt{4+4}$$

$$= \sqrt{8}$$

$$= \sqrt{4\times 2}$$

$$r = 2\sqrt{2}$$

$$\tan\alpha = \left|\frac{y}{x}\right|$$

$$= \frac{2}{2}$$

$$= 1$$

$$\alpha = \frac{\pi}{4}$$

$-2-2i$, lies in III quadrant, has the principal value $\theta = -\pi+\alpha \Rightarrow$

$$\theta = -\pi + \frac{\pi}{4}$$

$$= \frac{-4\pi+\pi}{4}$$

$$= -\frac{3\pi}{4}$$

$$-2-2i = r(\cos\theta + i\sin\theta)$$

$$= 2\sqrt{2}\left[\cos\left(-\frac{3\pi}{4}\right) + i\sin\left(-\frac{3\pi}{4}\right)\right]$$

$$= 2\sqrt{3}\left[\cos\left(-\frac{3\pi}{4}+2k\pi\right) + i\sin\left(-\frac{3\pi}{4}+2k\pi\right)\right], k \in Z$$

(iv) $\dfrac{i-1}{\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}}$

$$i-1 = -1+i$$

$$\text{Modulus} = \sqrt{x^2+y^2} = \sqrt{(-1)^2+(1)^2}$$

$$= \sqrt{1+1}$$

$$r = \sqrt{2}$$

$$\tan\alpha = \left|\frac{y}{x}\right|$$

$$= \frac{1}{1}$$

$$= 1$$

$$\alpha = \frac{\pi}{4}$$

$-1+i$, lies in II quadrant, has the principal value $\theta = \pi - \alpha \Rightarrow$

$$\theta = \pi - \frac{\pi}{4}$$

$$= \frac{4\pi-\pi}{4}$$

$$= \frac{3\pi}{4}$$

$$-1+i = r(\cos\theta + i\sin\theta)$$

$$= \sqrt{2}\left[\cos\left(\frac{3\pi}{4}\right) + i\sin\left(\frac{3\pi}{4}\right)\right]$$

$$\frac{-1+i}{\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}} = \frac{\sqrt{2}\left[\cos\left(\frac{3\pi}{4}\right)+i\sin\left(\frac{3\pi}{4}\right)\right]}{\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}}$$

$$= \sqrt{2}\left[\cos\left(\frac{3\pi}{4}-\frac{\pi}{3}\right) + i\sin\left(\frac{3\pi}{4}-\frac{\pi}{3}\right)\right]$$

$$= \sqrt{2}\left[\cos\left(\frac{9\pi-4\pi}{12}\right) + i\sin\left(\frac{9\pi-4\pi}{12}\right)\right]$$

$$= \sqrt{2}\left[\cos\left(\frac{5\pi}{12}\right) + i\sin\left(\frac{5\pi}{12}\right)\right]$$

$$= \sqrt{2}\left[\cos\left(\frac{5\pi}{12}+2k\pi\right) + i\sin\left(\frac{5\pi}{12}+2k\pi\right)\right], k \in Z$$

2. Find the rectangular form of the complex numbers

(i) $\left(\cos\frac{\pi}{6}+i\,sin\,\frac{\pi}{6}\right)\left(\cos\frac{\pi}{12}+i\,sin\,\frac{\pi}{12}\right)$

$$= \left[\cos\left(\frac{\pi}{6}+\frac{\pi}{12}\right) + i\,sin\,\left(\frac{\pi}{6}+\frac{\pi}{12}\right)\right]$$

$$= \left[\cos\left(\frac{2\pi+\pi}{12}\right) + i\,sin\,\left(\frac{2\pi+\pi}{12}\right)\right]$$

$$= \left[\cos\left(\frac{3\pi}{12}\right) + i\,sin\,\left(\frac{3\pi}{12}\right)\right]$$

$$= \left[\cos\left(\frac{\pi}{4}\right) + i\,sin\,\left(\frac{\pi}{4}\right)\right]$$

$$= \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i$$

$$(ii)\ \frac{\cos\frac{\pi}{6}-i\sin\frac{\pi}{6}}{2\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)}$$

$$=\frac{1}{2}\frac{\cos\frac{\pi}{6}-i\sin\frac{\pi}{6}}{\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)}$$

$$=\frac{1}{2}\frac{\cos\left(-\frac{\pi}{6}\right)+i\sin\left(-\frac{\pi}{6}\right)}{\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)}$$

$$=\frac{1}{2}\left[\cos\left(-\frac{\pi}{6}-\frac{\pi}{3}\right)+i\sin\left(-\frac{\pi}{6}-\frac{\pi}{3}\right)\right]$$

$$=\frac{1}{2}\left[\cos\left(\frac{-\pi-2\pi}{6}\right)+i\sin\left(\frac{-\pi-2\pi}{6}\right)\right]$$

$$=\frac{1}{2}\left[\cos\left(\frac{-3\pi}{6}\right)+i\sin\left(\frac{-3\pi}{6}\right)\right]$$

$$=\frac{1}{2}\left[\cos\left(\frac{-\pi}{2}\right)+i\sin\left(\frac{-\pi}{2}\right)\right]$$

$$=\frac{1}{2}\left[\cos\left(\frac{\pi}{2}\right)-i\sin\left(\frac{\pi}{2}\right)\right]$$

$$=\frac{1}{2}[0-i(1)]$$

$$=-\frac{i}{2}$$

3. If $(x_1+iy_1)(x_2+iy_2)(x_3+iy_3)\ldots\ldots(x_n+iy_n)=a+ib$,

show that

(i) $(x_1^2+y_1^2),(x_2^2+y_2^2)(x_3^2+y_3^2)\ldots\ldots(x_n^2+y_n^2)=a^2+b^2$

Solution:

Given

$$(x_1+iy_1)(x_2+iy_2)(x_3+iy_3)\ldots\ldots(x_n+iy_n)=a+ib$$

Taking modulus on both sides,

$$|(x_1+iy_1)(x_2+iy_2)(x_3+iy_3)\ldots\ldots(x_n+iy_n)|=|a+ib|$$

$$|(x_1+iy_1)|\,|(x_2+iy_2)|\,|(x_3+iy_3)|\ldots\ldots|(x_n+iy_n)|=|(a+ib)|$$

$$\sqrt{x_1{}^2+y_1{}^2}\sqrt{x_2{}^2+y_2{}^2}\sqrt{x_3{}^2+y_3{}^2}\ldots\ldots\sqrt{x_n{}^2+y_n{}^2}=\sqrt{a^2+b^2}$$

Squaring,

$$(x_1^2+y_1^2),(x_2^2+y_2^2)(x_3^2+y_3^2)\ldots\ldots(x_n^2+y_n^2)=a^2+b^2$$

(ii) $\sum_{r=1}^{n}tan^{-1}\left(\frac{y_r}{x_r}\right)=tan^{-1}\left(\frac{b}{a}\right)+2k\pi, k\in Z$

Solution:

Given

$$(x_1+iy_1)(x_2+iy_2)(x_3+iy_3)\ldots\ldots(x_n+iy_n)=a+ib$$

Taking argument on both sides,

$$\text{Arg}\,[(x_1+iy_1)(x_2+iy_2)\ldots(x_n+iy_n)]=Arg(a+ib)$$

$$\arg(x_1+iy_1)+\arg(x_2+iy_2)\ldots\ldots\ldots+\arg(x_n+iy_n)=arg(a+ib)$$

$$tan^{-1}\left(\frac{y_1}{x_1}\right)+tan^{-1}\left(\frac{y_2}{x_2}\right)+\ldots+tan^{-1}\left(\frac{y_n}{x_n}\right)=tan^{-1}\left(\frac{b}{a}\right)+2k\pi, k\in Z$$

Hence

$$\sum_{r=1}^{n}tan^{-1}\left(\frac{y_r}{x_r}\right)=tan^{-1}\left(\frac{b}{a}\right)+2k\pi, k\in Z$$

is proved.

4. If $\frac{1+Z}{1-Z}=\cos 2\theta+i\sin 2\theta$,

show that $Z=i\tan\theta$

Solution:

Given $\frac{1+Z}{1-Z}=\cos 2\theta+i\sin 2\theta$

$$\frac{1+Z}{1-Z}=\frac{1+(x+yi)}{1-(x+yi)}$$

$$=\frac{(1+x)+yi}{(1-x)-yi}\times\frac{+yi}{(1-x)+yi}$$

$$=\frac{(1+x)(1-x)+(1+x)yi+(1-x)yi+y^2i^2}{(1-x)^2+y^2}$$

$$=\frac{(1+x)(1-x)+(1+x)yi+(1-x)yi+y^2(-1)}{(1-x)^2+y^2}$$

$$=\frac{(1-x^2-y^2)+[(1+x)y+(1-x)y]i}{(1-x)^2+y^2}$$

$$=\frac{(1-x^2-y^2)+[y+xy+y-xy]i}{(1-x)^2+y^2}$$

$$= \frac{(1-x^2-y^2)+(2y)i}{(1-x)^2+y^2}$$

$$\frac{1+Z}{1-Z} = \frac{(1-x^2-y^2)}{(1-x)^2+y^2} + \frac{(2y)i}{(1-x)^2+y^2}$$

Given $\frac{1+Z}{1-Z} = \cos 2\theta + i\sin 2\theta$

$$\cos 2\theta = \frac{(1-x^2-y^2)}{(1-x)^2+y^2}$$

$$\sin 2\theta = \frac{2y}{(1-x)^2+y^2}$$

$$\tan 2\theta = {\sin 2\theta}/{\cos 2\theta}$$

$$= \frac{2y}{(1-x)^2+y^2} \Big/ \frac{(1-x^2-y^2)}{(1-x)^2+y^2}$$

$$\tan 2\theta = \frac{2y}{(1-x^2-y^2)}$$

If $Z = i\tan\theta$

$x + yi = i\tan\theta$

Then $x = 0, and\ y = \tan\theta$

Substituting $x = 0, and\ y = \tan\theta$ in

$$\tan 2\theta = \frac{2y}{(1-x^2-y^2)}$$

$$= \frac{2\tan\theta}{(1-tan^2\theta)}$$

$$= \tan 2\theta$$

5. If $\cos\alpha + \cos\beta + \cos\gamma = 0 = \sin\alpha + \sin\beta + \sin\gamma$,

prove that

(i) $\cos 3\alpha + \cos 3\beta + \cos 3\gamma = 3\cos(\alpha+\beta+\gamma)$

(ii) $\sin 3\alpha + \sin 3\beta + \sin 3\gamma = 3\sin(\alpha+\beta+\gamma)$

Solution: Given

$\cos\alpha + \cos\beta + \cos\gamma = 0 = \sin\alpha + \sin\beta + \sin\gamma$

$(\cos\alpha + \cos\beta + \cos\gamma)$
$+i(\sin\alpha + \sin\beta + \sin\gamma) = 0$

$(\cos\alpha + i\sin\alpha) + (\cos\beta + i\sin\beta)$
$+ (\cos\gamma + i\sin\gamma) = 0$

If $a + b + c = 0, then\ a^3 + b^3 + c^3 = 3abc$

where $a = \cos\alpha + i\sin\alpha$, $b = \cos\beta + i\sin\beta$

and $c = \cos\gamma + i\sin\gamma$

$\therefore (\cos\alpha + i\sin\alpha)^3 + (\cos\beta + i\sin\beta)^3$
$+(\cos\gamma + i\sin\gamma)^3$

$= 3(\cos\alpha + i\sin\alpha)(\cos\alpha + i\sin\alpha)(\cos\alpha + i\sin\alpha)$

$(\cos 3\alpha + i\sin 3\alpha) + (\cos 3\beta + i\sin 3\beta)$
$+(\cos 3\gamma + i\sin 3\gamma)$

$=3[\cos(\alpha+\beta+\gamma) + i\sin(\alpha+\beta+\gamma)]$

$(\cos 3\alpha + \cos 3\beta + \cos 3\gamma)$
$+i(\sin 3\alpha + \sin 3\beta + \sin 3\gamma)$

$= [3\cos(\alpha+\beta+\gamma) + i\,3\sin(\alpha+\beta+\gamma)]$

Equating real and imaginary parts,

(i) $\cos 3\alpha + \cos 3\beta + \cos 3\gamma = 3\cos(\alpha+\beta+\gamma)$

(ii) $\sin 3\alpha + \sin 3\beta + \sin 3\gamma = 3\sin(\alpha+\beta+\gamma)$

is proved.

6. If $Z = x + iy$ and $\arg\left(\frac{Z-i}{Z+2}\right) = \frac{\pi}{4}$, then show that $x^2 + y^2 + 3x - 3y + 2 = 0$.

Solution: Given $Z = x + iy$

$$\frac{Z-i}{Z+2} = \frac{x+iy-i}{x+iy+2}$$

$$= \frac{x+(y-1)i}{(x+2)+iy}$$

$$\arg(Z) = \frac{bc-ad}{ac+bd}$$

$a = x$, $b = y - 1$, $c = x + 2$ and $d = y$

$$\arg\left(\frac{Z-1}{Z+1}\right) = \frac{\pi}{4}$$

$$\frac{bc-ad}{ac+bd} = tan\left(\frac{\pi}{4}\right)$$

$$\frac{bc-ad}{ac+bd} = 1$$

$$bc - ad = ac + bd$$

$$(y-1)(x+2) - xy = x(x+2) + (y-1)y$$

$$xy + 2y - x - 2 - xy = x^2 + 2x + y^2 - y$$

$$2y - x - 2 = x^2 + 2x + y^2 - y$$

$$x^2 + 2x + y^2 - y - 2y + x + 2 = 0$$

$x^2 + y^2 + 3x - 3y + 2 = 0$, is proved.

Example 2.28

If $Z = (\cos\theta + i\sin\theta\ )$, show that

(i) $z^n + \frac{1}{z^n} = 2\cos\theta$ (ii) $z^n - \frac{1}{z^n} = 2i\sin\theta$.

Solution:

$$\text{Given } Z = (\cos\theta + i\sin\theta\ )$$

$$\therefore Z^n = (\cos\theta + i\sin\theta\ )^n$$

$$= (\cos n\theta + i\sin n\theta\ )$$

$$\frac{1}{z^n} = z^{-n}$$

$$= (\cos n\theta - i\sin n\theta\ )$$

(i) $z^n + \frac{1}{z^n}$

$$= (\cos n\theta + i\sin n\theta\ ) + (\cos n\theta - i\sin n\theta\ )$$

$$= \cos n\theta + i\sin n\theta + \cos n\theta - i\sin n\theta$$

$$= 2\cos n\theta$$

(ii) $z^n - \frac{1}{z^n}$

$$= (\cos n\theta + i\sin n\theta\ ) - (\cos n\theta - i\sin n\theta\ )$$

$$= \cos n\theta + i\sin n\theta - \cos n\theta + i\sin n\theta$$

$$= 2i\ sin\ n\theta$$

Example 2.29 Simplify $\left(\sin\frac{\pi}{6} + i\cos\frac{\pi}{6}\right)^{18}$

Solution:

$$\sin\frac{\pi}{6} + i\cos\frac{\pi}{6} = \sin\left(\frac{\pi}{2} - \frac{\pi}{3}\right) + i\cos\left(\frac{\pi}{2} - \frac{\pi}{3}\right)$$

$$= \cos\frac{\pi}{3} + i\sin\frac{\pi}{3}$$

$$\therefore \left(\sin\frac{\pi}{6} + i\cos\frac{\pi}{6}\right)^{18} = \left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)^{18}$$

$$= \cos 18\left(\frac{\pi}{3}\right) + i\sin 18\left(\frac{\pi}{3}\right)$$

$$= \cos 6\pi + i\sin 6\pi$$

$$= 1 + i0$$

$$\therefore \left(\sin\frac{\pi}{6} + i\cos\frac{\pi}{6}\right)^{18} = 1$$

Example 2.30 Simplify $\left(\frac{1+\cos 2\theta + i\sin 2\theta}{1+\cos 2\theta - \mathrm{i}\sin 2\theta}\right)^{30}$

Solution: Let $(\cos 2\theta + i\sin 2\theta) = \mathrm{Z}$

$$\therefore \frac{1}{Z} = (\cos 2\theta - i\sin 2\theta)$$

$$\therefore \left(\frac{1+\cos 2\theta + i\sin 2\theta}{1+\cos 2\theta - \mathrm{i}\sin 2\theta}\right) = \left(\frac{1+Z}{1+\frac{1}{Z}}\right)$$

$$= \left(\frac{1+Z}{\frac{Z+1}{Z}}\right)$$

$$= \left(\frac{1+Z}{1+Z} \times z\right)$$

$$= \mathrm{Z}$$

$$\left(\frac{1+\cos 2\theta + i\sin 2\theta}{1+\cos 2\theta - \mathrm{i}\sin 2\theta}\right)^{30} = z^{30}$$

$$= (\cos 2\theta + i\sin 2\theta)^{30}$$

$$= (\cos 60\theta + i\sin 60\theta)$$

Example 2.31 Simplify

$(i)\ (1+i)^{18}$

Solution:

$$\text{Let } 1 + i\ = r(\cos\theta + i\sin\theta)$$

$$x + yi = 1 + i$$

$$\text{Modulus} = \sqrt{x^2 + y^2} = \sqrt{(1)^2 + (1)^2}$$

$$= \sqrt{1+1}$$

$$= \sqrt{2}$$

$$r = \sqrt{2}$$

$$\tan\alpha = \left|\frac{y}{x}\right|$$

$$= \frac{1}{1}$$

$$= 1$$

$$\alpha = \frac{\pi}{4}$$

$1 + i$, lies in I quadrant, has the principal value $\theta = \alpha \Rightarrow$

$$\theta = \frac{\pi}{4}$$

$$1 + i = r(\cos\theta + i\sin\theta)$$

$$= \sqrt{2}\left[\cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right)\right]$$

$$\therefore (1+i)^{18} = \sqrt{2}^{18}\left[\cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right)\right]^{18}$$

$$= 2^9\left[\cos 18\left(\frac{\pi}{4}\right) + i\sin 18\left(\frac{\pi}{4}\right)\right]$$

$$= 512\left[\cos\left(\frac{9\pi}{2}\right) + i\sin\left(\frac{9\pi}{2}\right)\right]$$

$$= 512\left[\cos\left(\frac{8\pi}{2} + \frac{\pi}{2}\right) + i\sin\left(\frac{8\pi}{2} + \frac{\pi}{2}\right)\right]$$

$$= 512\left[\cos\left(4\pi+\frac{\pi}{2}\right)+i\sin\left(4\pi+\frac{\pi}{2}\right)\right]$$
$$= 512\left[\cos\left(\frac{\pi}{2}\right)+i\sin\left(\frac{\pi}{2}\right)\right]$$
$$= 512[0+i(1)]$$
$$= 512i$$

(ii) $\left(-\sqrt{3}+3i\right)^{31}$

Solution:

Let $-\sqrt{3}+3i = r(\cos\theta + i\sin\theta)$

$$x+yi = -\sqrt{3}+3i$$

$$\text{Modulus} = \sqrt{x^2+y^2} = \sqrt{\left(-\sqrt{3}\right)^2+(3)^2}$$
$$= \sqrt{3+9}$$
$$= \sqrt{12}$$
$$= \sqrt{4\times 3}$$
$$r = 2\sqrt{3}$$
$$\tan\alpha = \left|\frac{y}{x}\right|$$
$$= \frac{3}{\sqrt{3}}$$
$$= \frac{\sqrt{3}\sqrt{3}}{\sqrt{3}}$$
$$= \sqrt{3}$$
$$\alpha = \frac{\pi}{3}$$

$-\sqrt{3}+3i$, lies in II quadrant, has the principal value $\theta = \pi - \alpha \Rightarrow$

$$\theta = \pi - \frac{\pi}{3}$$
$$= \frac{3\pi-\pi}{3}$$
$$\theta = \frac{2\pi}{3}$$
$$-\sqrt{3}+3i = r(\cos\theta + i\sin\theta)$$
$$= 2\sqrt{3}\left[\cos\left(\frac{2\pi}{3}\right)+i\sin\left(\frac{2\pi}{3}\right)\right]$$
$$\left(-\sqrt{3}+3i\right)^{31} = \left(2\sqrt{3}\right)^{31}\left[\cos\left(\frac{2\pi}{3}\right)+i\sin\left(\frac{2\pi}{3}\right)\right]^{31}$$
$$= \left(2\sqrt{3}\right)^{31}\left[\cos 31\left(\frac{2\pi}{3}\right)+i\sin 31\left(\frac{2\pi}{3}\right)\right]$$
$$= \left(2\sqrt{3}\right)^{31}\left[\cos\left(\frac{62\pi}{3}\right)+i\sin\left(\frac{62\pi}{3}\right)\right]$$
$$= \left(2\sqrt{3}\right)^{31}\left[\cos\left(\frac{60\pi}{3}+\frac{2\pi}{3}\right)+i\sin\left(\frac{60\pi}{3}+\frac{2\pi}{3}\right)\right]$$
$$= \left(2\sqrt{3}\right)^{31}\left[\cos\left(20\pi+\frac{2\pi}{3}\right)+i\sin\left(20\pi+\frac{2\pi}{3}\right)\right]$$
$$= \left(2\sqrt{3}\right)^{31}\left[\cos\left(\frac{2\pi}{3}\right)+i\sin\left(\frac{2\pi}{3}\right)\right]$$
$$= \left(2\sqrt{3}\right)^{31}\left[\cos\left(\frac{3\pi-\pi}{3}\right)+i\sin\left(\frac{3\pi-\pi}{3}\right)\right]$$
$$= \left(2\sqrt{3}\right)^{31}\left[\cos\left(\frac{3\pi}{3}-\frac{\pi}{3}\right)+i\sin\left(\frac{3\pi}{3}-\frac{\pi}{3}\right)\right]$$
$$= \left(2\sqrt{3}\right)^{31}\left[\cos\left(\pi-\frac{\pi}{3}\right)+i\sin\left(\pi-\frac{\pi}{3}\right)\right]$$
$$= \left(2\sqrt{3}\right)^{31}\left[-\cos\left(\frac{\pi}{3}\right)+i\sin\left(\frac{\pi}{3}\right)\right]$$
$$= \left(2\sqrt{3}\right)^{31}\left[-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right]$$
$$\left(-\sqrt{3}+3i\right)^{31} = \left(2\sqrt{3}\right)^{31}\left[-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right]$$

Example 2.32 Find the cube roots of unity.

Solution: Let $x = (1)^{\frac{1}{3}}$

$$= (\cos 0 + i\sin 0)^{\frac{1}{3}}$$
$$= (\cos 2k\pi + i\sin 2k\pi)^{\frac{1}{3}}$$
$$= \left(\cos\frac{1}{3}(2k\pi) + i\sin\frac{1}{3}(2k\pi)\right)$$

Substituting k=0, 1, 2 we get 3 values

k	Values
0	$cis\ 0$
1	$cis\,\frac{2\pi}{3}$
2	$cis\,\frac{4\pi}{3}$

Example 2.33 Find the fourth roots of unity.

Solution: Let $x = (1)^{\frac{1}{4}}$

$$= (\cos 0 + i\sin 0)^{\frac{1}{4}}$$
$$= (\cos 2k\pi + i\sin 2k\pi)^{\frac{1}{4}}$$
$$= \left(\cos\frac{1}{4}(2k\pi) + i\sin\frac{1}{4}(2k\pi)\right)$$

Substituting k=0, 1, 2, 3 we get 4 values

k	Values
0	$cis\ 0$
1	$cis\frac{\pi}{2}$
2	$cis\ \pi$
3	$cis\frac{3\pi}{2}$

Example 2.34 Solve the equation $Z^3 + 8i = 0$, where $Z \in \mathcal{C}$.

Solution: $Z^3 + 8i = 0$

$$Z^3 = -8i$$

$$Z^3 = 8(-i)$$

$$= 8cis\left(-\frac{\pi}{2}\right)$$

$$Z = \left[8cis\left(-\frac{\pi}{2}\right)\right]^{\frac{1}{3}}$$

$$= 8^{\frac{1}{3}}cis\left(-\frac{\pi}{2}\right)^{\frac{1}{3}}$$

$$= 2cis\left(2k\pi - \frac{\pi}{2}\right)^{\frac{1}{3}}$$

$$= 2cis\ \frac{1}{3}\left(2k\pi - \frac{\pi}{2}\right)$$

$$= 2cis\ \frac{\pi}{3}\left(2k - \frac{1}{2}\right)$$

$$= 2cis\ \frac{\pi}{3}\left(\frac{4k-1}{2}\right)$$

Substituting k = 0, 1, 2 we get 3 values

k	Values
0	$2cis\left(-\frac{\pi}{6}\right)$
1	$2cis\left(\frac{\pi}{2}\right)$
2	$2cis\left(\frac{7\pi}{6}\right)$

Example 2.35 Find all cube roots of $\sqrt{3} + i$.

Solution:

Let $\sqrt{3} + i = r(\cos\theta + i\sin\theta)$

$$x + yi = \sqrt{3} + i$$

$$\text{Modulus} = \sqrt{x^2 + y^2} = \sqrt{\left(\sqrt{3}\right)^2 + (1)^2}$$

$$= \sqrt{3+1}$$

$$= \sqrt{4}$$

$$r = 2$$

$$\tan\alpha = \left|\frac{y}{x}\right|$$

$$= \frac{1}{\sqrt{3}}$$

$$\alpha = \frac{\pi}{6}$$

$\sqrt{3} + i$, lies in I quadrant, has the principal value $\theta = \alpha \Rightarrow$

$$\theta = \frac{\pi}{6}$$

$$\sqrt{3} + i = r(\cos\theta + i\sin\theta)$$

$$= 2\left[\cos\left(\frac{\pi}{6}\right) + i\sin\left(\frac{\pi}{6}\right)\right]$$

cube roots of $\sqrt{3} + i = \left(\sqrt{3} + i\right)^{\frac{1}{3}}$

$$\left(\sqrt{3} + i\right)^{\frac{1}{3}} = (2)^{\frac{1}{3}}\left[\cos\left(\frac{\pi}{6}\right) + i\sin\left(\frac{\pi}{6}\right)\right]^{\frac{1}{3}}$$

$$= (2)^{\frac{1}{3}}\left[\text{cis}\left(2k\pi + \frac{\pi}{6}\right)\right]^{\frac{1}{3}}$$

$$= (2)^{\frac{1}{3}}\left[\text{cis}\frac{1}{3}\left(2k\pi + \frac{\pi}{6}\right)\right]$$

$$= (2)^{\frac{1}{3}}\left[\text{cis}\frac{1}{3}\left(\frac{12k\pi + \pi}{6}\right)\right]$$

$$= (2)^{\frac{1}{3}}\left[\text{cis}\frac{\pi}{3}\left(\frac{12k + 1}{6}\right)\right]$$

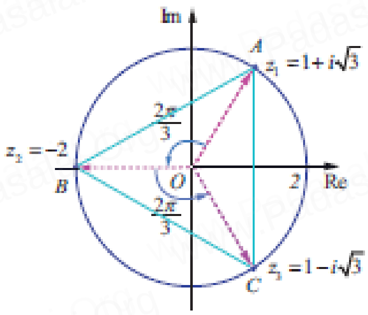
Substituting k = 0, 1, 2 we get 3 values

k	Values
0	$(2)^{\frac{1}{3}}cis\left(\frac{\pi}{18}\right)$
1	$(2)^{\frac{1}{3}}cis\left(\frac{13\pi}{18}\right)$
2	$(2)^{\frac{1}{3}}cis\left(\frac{25\pi}{18}\right)$

Example 2.36 Suppose $Z_1, Z_2, and\ Z_3$ are the vertices of an equilateral triangle inscribed in the circle $|Z| = 2$. If $Z_1 = 1 + i\sqrt{3}$, then find Z_2 and Z_3.

Solution: $|Z| = 2$ represents the circle with centre (0, 0) and radius 2. Let *A, B,* and *C* be the vertices of the given triangle. Since the vertices $Z_1, Z_2, and\ Z_3$ form an equilateral triangle inscribed in the circle $|Z| = 2$, the sides of this triangle *AB, BC,* and *CA* subtend $\frac{2\pi}{3}$ radians (120 degree) at the origin (circumcenter of the triangle).

(The complex number $Ze^{i\theta}$ is a rotation of Z by θ radians in the counter clockwise direction about the origin.)



Therefore, we can obtain $Z_2, and\ Z_3$ by the rotation of Z_1 by $\frac{2\pi}{3}$ and $\frac{4\pi}{3}$ respectively.

Given that $\overline{OA} = Z_1 = 1 + i\sqrt{3}$

$$Z_2 = \overline{OB} = Z_1 e^{i\frac{2\pi}{3}}$$

$$= (1 + i\sqrt{3})e^{i\frac{2\pi}{3}}$$

$$= (1 + i\sqrt{3})\left[\cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right)\right]$$

$$= (1 + i\sqrt{3})\left[-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right]$$

$$= -\frac{1}{2} + i\frac{\sqrt{3}}{2} - i\frac{\sqrt{3}}{2} + \frac{3}{2}(i)^2$$

$$= -\frac{1}{2} - \frac{3}{2}$$

$$= \frac{-1-3}{2}$$

$$= \frac{-4}{2}$$

$$= -2$$

$$Z_3 = \overline{OC} = Z_1 e^{i\frac{4\pi}{3}}$$

$$= Z_2 e^{i\frac{2\pi}{3}}$$

$$= (-2)e^{i\frac{2\pi}{3}}$$

$$= (-2)\left[\cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right)\right]$$

$$= (-2)\left[-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right]$$

$$= 1 - \sqrt{3}i$$

Hence $Z_2 = -2$ and $Z_3 = 1 - \sqrt{3}i$

EXERCISE 2.8

1. If $\omega \neq 1$ is a cube root of unity, then show that $\frac{a+b\omega+c\omega^2}{b+c\omega+a\omega^2} + \frac{a+b\omega+c\omega^2}{c+a\omega+b\omega^2} = -1$

Solution: $\omega \neq 1$ is a cube root of unity,

$1 + \omega + \omega^2 = 0, and\ \omega^3 = 1$

$$LHS = \frac{a+b\omega+c\omega^2}{b+c\omega+a\omega^2} + \frac{a+b\omega+c\omega^2}{c+a\omega+b\omega^2}$$

$$= \frac{a\omega^3+b\omega+c\omega^2}{b+c\omega+a\omega^2} + \frac{a\omega^3+b\omega+c\omega^2}{c+a\omega+b\omega^2}$$

$$= \frac{\omega(a\omega^2+b+c\omega)}{b+c\omega+a\omega^2} + \frac{\omega^2(a\omega+b\omega^2+c)}{c+a\omega+b\omega^2}$$

$$= \omega + \omega^2$$

$$= -1$$

$$= RHS$$

2. Show that $\left(\frac{\sqrt{3}}{2} + \frac{i}{2}\right)^5 + \left(\frac{\sqrt{3}}{2} - \frac{i}{2}\right)^5 = -\sqrt{3}$

Solution:

Let $\frac{\sqrt{3}}{2} + \frac{i}{2} = r(\cos\theta + i\sin\theta)$

$$x + yi = \frac{\sqrt{3}}{2} + \frac{i}{2}$$

$$\text{Modulus} = \sqrt{x^2 + y^2} = \sqrt{\left(\frac{\sqrt{3}}{2}\right)^2 + \left(\frac{1}{2}\right)^2}$$

$$= \sqrt{\frac{3}{4} + \frac{1}{4}}$$

$$= \sqrt{\frac{3+1}{4}}$$

$$= \sqrt{\frac{4}{4}}$$

$$r = 1$$

$$\tan\alpha = \left|\frac{y}{x}\right|$$

$$= \frac{1}{2} \Big/ \frac{\sqrt{3}}{2}$$

$$= \frac{1}{2} \times \frac{2}{\sqrt{3}}$$

$$= \frac{1}{\sqrt{3}}$$

$$\alpha = \frac{\pi}{6}$$

$\frac{\sqrt{3}}{2} + \frac{i}{2}$, lies in I quadrant, has the principal value $\theta = \alpha \Rightarrow$

$$\theta = \frac{\pi}{6}$$

$$\frac{\sqrt{3}}{2} + \frac{i}{2} = r(\cos\theta + i\sin\theta)$$

$$= 1\left[\cos\left(\frac{\pi}{6}\right) + i\sin\left(\frac{\pi}{6}\right)\right]$$

$$\left(\frac{\sqrt{3}}{2} + \frac{i}{2}\right)^5 = \left[\cos\left(\frac{\pi}{6}\right) + i\sin\left(\frac{\pi}{6}\right)\right]^5$$

$$= \cos 5\left(\frac{\pi}{6}\right) + i\sin 5\left(\frac{\pi}{6}\right)$$

$$\left(\frac{\sqrt{3}}{2} + \frac{i}{2}\right)^5 = \cos\left(\frac{5\pi}{6}\right) + i\sin\left(\frac{5\pi}{6}\right)$$

Similarly

$$\left(\frac{\sqrt{3}}{2} - \frac{i}{2}\right)^5 = \cos\left(\frac{5\pi}{6}\right) - i\sin\left(\frac{5\pi}{6}\right)$$

Hence

$$\left(\frac{\sqrt{3}}{2} + \frac{i}{2}\right)^5 + \left(\frac{\sqrt{3}}{2} - \frac{i}{2}\right)^5$$

$$= \cos\left(\frac{5\pi}{6}\right) + i\sin\left(\frac{5\pi}{6}\right) + \cos\left(\frac{5\pi}{6}\right) - i\sin\left(\frac{5\pi}{6}\right)$$

$$= 2\cos\left(\frac{5\pi}{6}\right)$$

$$= 2\cos\left(\frac{6\pi - \pi}{6}\right)$$

$$= 2\cos\left(\frac{6\pi}{6} - \frac{\pi}{6}\right)$$

$$= 2\cos\left(\pi - \frac{\pi}{6}\right) \qquad [\because \cos(180 - \theta) = -\cos\theta]$$

$$= -2\cos\left(\frac{\pi}{6}\right)$$

$$= -2\left(\frac{\sqrt{3}}{2}\right)$$

$$= -\sqrt{3}, \textit{Hence proved.}$$

3. Find the value of $\left(\frac{1 + \sin\frac{\pi}{10} + i\cos\frac{\pi}{10}}{1 + \sin\frac{\pi}{10} - i\cos\frac{\pi}{10}}\right)^{10}$

Solution: Let $\left(\sin\frac{\pi}{10} + i\cos\frac{\pi}{10}\right) = Z$

$$\therefore \frac{1}{Z} = \left(\sin\frac{\pi}{10} - i\cos\frac{\pi}{10}\right)$$

$$\therefore \left(\frac{1 + \sin\frac{\pi}{10} + i\cos\frac{\pi}{10}}{1 + \sin\frac{\pi}{10} - i\cos\frac{\pi}{10}}\right) = \left(\frac{1 + Z}{1 + \frac{1}{Z}}\right)$$

$$= \left(\frac{1 + Z}{\frac{Z+1}{Z}}\right)$$

$$= \left(\frac{1+Z}{1+Z} \times z\right)$$

$$= Z$$

$$\left(\frac{1 + \sin\frac{\pi}{10} + i\cos\frac{\pi}{10}}{1 + \sin\frac{\pi}{10} - i\cos\frac{\pi}{10}}\right)^{10} = z^{10}$$

$$= \left(\sin\frac{\pi}{10} + i\cos\frac{\pi}{10}\right)^{10}$$

$$= i^{10}\left(\cos\frac{\pi}{10} - i\sin\frac{\pi}{10}\right)^{10}$$

$$= i^8 i^2\left(\cos\frac{\pi}{10} - i\sin\frac{\pi}{10}\right)^{10}$$

$$= i^2\left(\cos\frac{\pi}{10} - i\sin\frac{\pi}{10}\right)^{10}$$

$$= (-1)\left[\cos 10\left(\frac{\pi}{10}\right) - i\sin 10\left(\frac{\pi}{10}\right)\right]$$

$$= (-1)(\cos\pi + i\sin\pi)$$

$$= (-1)(-1)$$

$$= 1$$

4. If $2\cos\alpha = x + \frac{1}{x}$ and If $2\cos\beta = y + \frac{1}{y}$

then show that

$(i)\ \frac{x}{y} + \frac{y}{x} = 2\cos(\alpha - \beta)$

$(ii)\ xy - \frac{1}{xy} = 2i\sin(\alpha + \beta)$

$(iii)\ \frac{x^m}{y^n} - \frac{y^n}{x^m} = 2i\sin(m\alpha - n\beta)$

$$(iv) x^m y^n + \frac{1}{x^m y^n} = 2\cos(m\alpha + n\beta)$$

Solution: Given $2\cos\alpha = x + \frac{1}{x}$

$$x + \frac{1}{x} = 2\cos\alpha$$

$$\frac{x^2+1}{x} = 2\cos\alpha$$

$$x^2 + 1 = 2x\cos\alpha$$

$$x^2 - 2x\cos\alpha + 1 = 0$$

$$a = 1, b = -2\cos\alpha \text{ and } c = 1$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{2\cos\alpha \pm \sqrt{(-2\cos\alpha)^2 - 4(1)(1)}}{2(1)}$$

$$= \frac{2\cos\alpha \pm \sqrt{4\cos^2\alpha - 4}}{2}$$

$$= \frac{2\cos\alpha \pm \sqrt{-4(1-\cos^2\alpha)}}{2}$$

$$= \frac{2\cos\alpha \pm \sqrt{-4\sin^2\alpha}}{2}$$

$$= \frac{2\cos\alpha \pm i\,2\sin\alpha}{2}$$

$$= \frac{2(\cos\alpha \pm i\sin\alpha)}{2}$$

$$= (\cos\alpha \pm i\sin\alpha)$$

Let $x = (\cos\alpha + i\sin\alpha)$

Similarly $y = (\cos\beta + i\sin\beta)$

$$\therefore \frac{x}{y} = \frac{(\cos\alpha + i\sin\alpha)}{(\cos\beta + i\sin\beta)}$$

$$= [\cos(\alpha-\beta) + i\sin(\alpha-\beta)]$$

hence $\frac{y}{x} = [\cos(\alpha-\beta) - i\sin(\alpha-\beta)]$

(i) $\therefore \frac{x}{y} + \frac{y}{x} = \cos(\alpha-\beta) + i\sin(\alpha-\beta)$

$$+ \cos(\alpha-\beta) - i\sin(\alpha-\beta)$$

$$= 2\cos(\alpha-\beta)$$

(ii) $xy = (\cos\alpha + i\sin\alpha)(\cos\beta + i\sin\beta)$

$$= [\cos(\alpha+\beta) + i\sin(\alpha+\beta)]$$

hence $\frac{1}{xy} = [\cos(\alpha+\beta) - i\sin(\alpha+\beta)]$

$$\therefore xy - \frac{1}{xy} = \cos(\alpha+\beta) + i\sin(\alpha+\beta)$$

$$- \cos(\alpha+\beta) + i\sin(\alpha+\beta)$$

$$= 2i\sin(\alpha+\beta)$$

(iii) $x^m = (\cos\alpha + i\sin\alpha)^m$

$$= (\cos m\alpha + i\sin m\alpha)$$

$$y^n = (\cos n\beta + i\sin n\beta)$$

$$\therefore \frac{x^m}{y^n} = \frac{(\cos m\alpha + i\sin m\alpha)}{(\cos n\beta + i\sin n\beta)}$$

$$= [\cos(m\alpha - n\beta) + i\sin(m\alpha - n\beta)]$$

hence $\frac{y^n}{x^m} = [\cos(m\alpha - n\beta) - i\sin(m\alpha - n\beta)]$

$$\therefore \frac{x^m}{y^n} - \frac{y^n}{x^m}$$

$$= \cos(m\alpha - n\beta) + i\sin(m\alpha - n\beta)$$

$$- \cos(m\alpha - n\beta) + i\sin(m\alpha - n\beta)$$

$$\therefore \frac{x^m}{y^n} - \frac{y^n}{x^m} = 2i\sin(m\alpha - n\beta)$$

(iv) $x^m y^n$

$$= (\cos m\alpha + i\sin m\alpha)(\cos n\beta + i\sin n\beta)$$

$$= [\cos(m\alpha + n\beta) + i\sin(m\alpha + n\beta)]$$

$$\therefore \frac{1}{x^m y^n} = [\cos(m\alpha + n\beta) - i\sin(m\alpha + n\beta)]$$

Hence, $x^m y^n + \frac{1}{x^m y^n}$

$$= \cos(m\alpha + n\beta) + i\sin(m\alpha + n\beta)$$

$$+ \cos(m\alpha + n\beta) - i\sin(m\alpha + n\beta)$$

$$\therefore x^m y^n + \frac{1}{x^m y^n} = 2\cos(m\alpha + n\beta)$$

5. Solve $Z^3 + 27 = 0$.

Solution: Given $Z^3 + 27 = 0$

$$Z^3 = -27$$

$$Z^3 = 27(-1)$$

$$Z^3 = 3^3(-1)$$

$$Z = 3(-1)^{\frac{1}{3}}$$

$$= 3(cis\,\pi)^{\frac{1}{3}}$$

$$= 3[cis\,(2k\pi + \pi)]^{\frac{1}{3}}$$

$$= 3\left[cis\ \tfrac{1}{3}(2k\pi + \pi)\right]$$

$$Z = 3\left[cis\ \tfrac{\pi}{3}(2k+1)\right]$$

Substituting k = 0, 1, 2 we get 3 values

k	Values
0	$3cis\left(\frac{\pi}{3}\right)$
1	$3cis\left(\frac{3\pi}{3}\right) = 3cis(\pi) = -3$
2	$3cis\left(\frac{5\pi}{3}\right)$

6. If $\omega \neq 1$ is a cube root of unity, then show that the roots of the equation $(Z-1)^3 + 8 = 0$ are $-1, 1-2\omega, 1-2\omega^2$.

Solution: $(Z-1)^3 + 8 = 0$

$$(Z-1)^3 = -8$$

$$= 8(-1)$$

$$= 2^3(-1)$$

$$(Z-1) = 2(-1)^{\frac{1}{3}}$$

$$= 2(cis\ \pi)^{\frac{1}{3}}$$

$$= 2[cis\ (2k\pi + \pi)]^{\frac{1}{3}}$$

$$= 2\left[cis\ \tfrac{1}{3}(2k\pi + \pi)\right]$$

$$(Z-1) = 2\left[cis\ \tfrac{\pi}{3}(2k+1)\right]$$

Substituting k = 0, 1, 2 we get 3 values

(i) k = 0

$$Z - 1 = 2\ cis\left(\frac{\pi}{3}\right)$$

$$= 2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$$

$$= 2\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)$$

$$Z - 1 = 1 + i\sqrt{3}$$

We know that, $\left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) = \omega$ and

$$\left(-\frac{1}{2} - i\frac{\sqrt{3}}{2}\right) = \omega^2$$

So, $-\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) = \omega^2$

$$\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) = -\omega^2$$

$$\left(\frac{1+i\sqrt{3}}{2}\right) = -\omega^2$$

$$(1 + i\sqrt{3}) = -2\omega^2$$

$$Z - 1 = 1 + i\sqrt{3}$$

$$= -2\omega^2$$

$$Z = 1 - 2\omega^2$$

(ii) k = 1

$$Z - 1 = 2\ cis\left(\frac{3\pi}{3}\right)$$

$$= 2\ cis(\pi)$$

$$= 2(-1)$$

$$= -2$$

$$Z = -2 + 1$$

$$Z = -1$$

(iii) k = 2

$$Z - 1 = 2\ cis\left(\frac{5\pi}{3}\right)$$

$$= 2\ cis\left(\frac{6\pi}{3} - \frac{\pi}{3}\right)$$

$$= 2\ cis\left(2\pi - \frac{\pi}{3}\right)$$

$$= 2\ cis\left(-\frac{\pi}{3}\right)$$

$$= 2\left(\cos\frac{\pi}{3} - i\sin\frac{\pi}{3}\right)$$

$$= 2\left(\frac{1}{2} - i\frac{\sqrt{3}}{2}\right)$$

$$Z - 1 = 1 - i\sqrt{3}$$

We know $\left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) = \omega$

$$\left(\frac{-1+i\sqrt{3}}{2}\right) = \omega$$

$$-1 + i\sqrt{3} = 2\omega$$

$$\therefore -(1 - i\sqrt{3}) = -2\omega$$

Hence $Z - 1 = -2\omega$

$$Z = -2\omega + 1 \Longrightarrow Z = 1 - 2\omega$$

<u>Another Method</u>

Solution: $(Z-1)^3 + 8 = 0$

Let $(Z-1)=A$

Then $A^3+8=0$

$$A^3+2^3=0$$

$$(A+2)(A^2-2A+4)=0$$

$$[a^3+b^3=(a+b)(a^2-ab+b^2)]$$

So, $(A+2)=0$ or $(A^2-2A+4)=0$

(i) If $(A+2)=0$

$$A=-2$$

$$Z-1=-2$$

$$Z=-2+1$$

$$Z=-1$$

(ii) If $A^2-2A+4=0$

$$a=1, b=\ -2 \ and\ c=4$$

$$A=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

$$=\frac{2\pm\sqrt{(-2)^2-4(1)(4)}}{2(1)}$$

$$=\frac{2\pm\sqrt{4-16}}{2}$$

$$=\frac{2\pm\sqrt{-12}}{2}$$

$$=\frac{2\pm\sqrt{-(4)(3)}}{2}$$

$$=\frac{2\pm2\sqrt{3}i}{2}$$

$$=\frac{2(1\pm\sqrt{3}i)}{2}$$

$$A=1\ \pm\sqrt{3}i$$

$$A=1+\sqrt{3}i\ and\ A=1-\sqrt{3}i$$

We know that, $\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)=\omega$ and

$$\left(-\frac{1}{2}-i\frac{\sqrt{3}}{2}\right)=\omega^2$$

So, $-\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)=\omega^2$

$$\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)=-\ \omega^2$$

$$\left(\frac{1+i\sqrt{3}}{2}\right)=-\ \omega^2$$

$$(1\ +\ i\sqrt{3})=-2\omega^2$$

$$A=Z-1\ =1+i\sqrt{3}$$

$$=-2\omega^2$$

$$\therefore\ Z=1-2\omega^2$$

If $A=Z-1\ =1-i\sqrt{3}$

We know $\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)=\omega$

$$\left(\frac{-1+i\sqrt{3}}{2}\right)=\omega$$

$$-1+i\sqrt{3}=2\omega$$

$$\therefore -(1-i\sqrt{3})=-2\omega$$

Hence $A=Z-1\ =-2\omega$

$$Z=-2\omega+1\Longrightarrow$$

$$Z=1-2\omega$$

Solutions are $Z=-1, 1-2\omega^2$ and $1-2\omega$

7. Find the value of $\sum_{k=1}^{8}\left(\cos\frac{2k\pi}{9}+i\sin\frac{2k\pi}{9}\right)$

Solution:

$\sum_{k=1}^{8}\left(\cos\frac{2k\pi}{9}+i\sin\frac{2k\pi}{9}\right)$

$$=cis\left(\frac{2\pi}{9}\right)+cis\left(\frac{4\pi}{9}\right)+cis\left(\frac{6\pi}{9}\right)+\cdots+cis\left(\frac{16\pi}{9}\right)$$

$$=cis\left(\frac{2\pi}{9}\right)(1+2+3+4+5+6+7+8)$$

$$=cis\left(\frac{2\pi}{9}\right)(36)$$

$$=cis(2\pi)(4)$$

$$=cis(8\pi)$$

$$=1$$

8. If $\omega\neq1$ is a cube root of unity, then show that

(i) $(1-\omega+\omega^2)^6+(1+\omega-\omega^2)^6=128$

$$(1-\omega+\omega^2)^6+(1+\omega-\omega^2)^6$$

$$=(-\omega-\omega)^6+(-\omega^2-\omega^2)^6$$

$$=(-2\omega)^6+(-2\omega^2)^6$$

$$=(-2)^6(\omega)^6+(-2)^6(\omega^2)^6$$

$$=64\omega^6+64\omega^{12}$$

$$=64(1)+64(1)$$

$$=128$$

(ii) $(1+\omega)\,(1+\omega^2)(1+\omega^4)(1+\omega^8)$

$$\ldots\ldots(1+\omega^{2^{11}})=1$$

Solution:

$$(1+\omega)\,(1+\omega^2)(1+\omega^4)(1+\omega^8)\ldots\ldots(1+\omega^{2^{11}})$$

$$=(1+\omega)\,(1+\omega^2)(1+\omega)\,(1+\omega^2)\ldots\ldots(1+\omega)\,(1+\omega^2)\text{ 12 terms}$$

$$=[(1+\omega)\,(1+\omega^2)]^6$$

$$=[(-\omega^2)\,(-\omega)]^6$$

$$=[\omega^3]^6$$

$$=1^6$$

$=1$ Hence proved.

9. If $Z=2-2i$, find the rotation of Z by θ radians in the counter clockwise direction about the origin when

(i) $\theta=\frac{\pi}{3}$

Solution:

$$Z=2-2i$$

$$\text{Let } 2-2i=r(\cos\theta+i\sin\theta)$$

$$x+yi=2-2i$$

$$\text{Modulus}=\sqrt{x^2+y^2}=\sqrt{(2)^2+(-2)^2}$$

$$=\sqrt{4+4}$$

$$=\sqrt{8}$$

$$=\sqrt{4\times 2}$$

$$r=2\sqrt{2}$$

$$\tan\alpha=\left|\frac{y}{x}\right|$$

$$=\frac{2}{2}$$

$$=1$$

$$\alpha=\frac{\pi}{4}$$

$2-2i$, lies in IV quadrant, has the principal value $\theta=-\alpha\Rightarrow$

$$\theta=-\frac{\pi}{4}$$

$$2-2i=r(\cos\theta+i\sin\theta)$$

$$=2\sqrt{2}\left[\cos\left(-\frac{\pi}{4}\right)+i\sin\left(-\frac{\pi}{4}\right)\right]$$

$$=2\sqrt{2}e^{-\frac{i\pi}{4}}$$

The rotation of Z by θ radians in the counter clockwise direction about the origin in $Ze^{i\theta}$

(i) $\theta=\frac{\pi}{3}$

$\therefore$ Rotation of Z is $Ze^{i\frac{\pi}{3}}$

$$=2\sqrt{2}e^{-\frac{i\pi}{4}}e^{i\frac{\pi}{3}}$$

$$=2\sqrt{2}e^{-\frac{i\pi}{4}+i\frac{\pi}{3}}$$

$$=2\sqrt{2}e^{\frac{-i3\pi+i4\pi}{12}}$$

$$=2\sqrt{2}e^{\frac{i\pi}{12}}$$

(ii) $\theta=\frac{2\pi}{3}$

$\therefore$ Rotation of Z is $Ze^{i\frac{2\pi}{3}}$

$$=2\sqrt{2}e^{-\frac{i\pi}{4}}e^{i\frac{2\pi}{3}}$$

$$=2\sqrt{2}e^{-\frac{i\pi}{4}+i\frac{2\pi}{3}}$$

$$=2\sqrt{2}e^{\frac{-i3\pi+i8\pi}{12}}$$

$$=2\sqrt{2}e^{\frac{i5\pi}{12}}$$

(iii) $\theta=\frac{3\pi}{2}$

$\therefore$ Rotation of Z is $Ze^{i\frac{3\pi}{2}}$

$$=2\sqrt{2}e^{-\frac{i\pi}{4}}e^{i\frac{3\pi}{2}}$$

$$=2\sqrt{2}e^{-\frac{i\pi}{4}+i\frac{3\pi}{2}}$$

$$=2\sqrt{2}e^{\frac{-i\pi+i6\pi}{4}}$$

$$=2\sqrt{2}e^{\frac{i5\pi}{4}}$$

10. Prove that the values of

$\sqrt[4]{-1}$ are $\pm\frac{1}{\sqrt{2}}(1\pm i)$

Solution:

$$(-1) = cis(\pi)$$
$$= cis(2k\pi + \pi)$$
$$\sqrt[4]{-1} = [cis(2k\pi + \pi)]^{\frac{1}{4}}$$
$$= \left[cis\frac{1}{4}(2k\pi + \pi)\right]$$
$$= \left[cis\frac{\pi}{4}(2k + 1)\right]$$

Substituting k = 0, 1, 2 and 3 we get 4 values

(i) *When* $k = 0$

$$= cis\frac{\pi}{4}$$
$$= \left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$$
$$= \frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}$$
$$= \frac{1}{\sqrt{2}}(1 + i)$$

(ii) *When* $k = 1$

$$= cis\frac{3\pi}{4}$$
$$= cis\left(\frac{4\pi}{4} - \frac{\pi}{4}\right)$$
$$= cis\left(\pi - \frac{\pi}{4}\right)$$
$$= \left[\cos\left(\pi - \frac{\pi}{4}\right) + i\sin\left(\pi - \frac{\pi}{4}\right)\right]$$
$$= \left[-\cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right)\right]$$
$$= -\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}$$
$$= \frac{1}{\sqrt{2}}(-1 + i)$$

(iii) *When* $k = 2$

$$= cis\frac{5\pi}{4}$$
$$= cis\left(\frac{4\pi}{4} + \frac{\pi}{4}\right)$$
$$= cis\left(\pi + \frac{\pi}{4}\right)$$
$$= \left[\cos\left(\pi + \frac{\pi}{4}\right) + i\sin\left(\pi + \frac{\pi}{4}\right)\right]$$
$$= \left[-\cos\left(\frac{\pi}{4}\right) - i\sin\left(\frac{\pi}{4}\right)\right]$$
$$= -\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}$$
$$= \frac{1}{\sqrt{2}}(-1 - i)$$

(iv) *When* $k = 3$

$$= cis\frac{7\pi}{4}$$
$$= cis\left(\frac{8\pi}{4} - \frac{\pi}{4}\right)$$
$$= cis\left(2\pi - \frac{\pi}{4}\right)$$
$$= \left[\cos\left(2\pi - \frac{\pi}{4}\right) + i\sin\left(2\pi - \frac{\pi}{4}\right)\right]$$
$$= \left[\cos\left(\frac{\pi}{4}\right) - i\sin\left(\frac{\pi}{4}\right)\right]$$
$$= \frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}$$
$$= \frac{1}{\sqrt{2}}(1 - i)$$

Hence the four roots are

$$\frac{1}{\sqrt{2}}(1 + i),\ \frac{1}{\sqrt{2}}(-1 + i),\ \frac{1}{\sqrt{2}}(-1 - i),\ \frac{1}{\sqrt{2}}(1 - i)$$

$\therefore$ *The roots are* $\pm\frac{1}{\sqrt{2}}(1 \pm i)$

EXERCISE 2.9

Choose the correct or the most suitable answer from the given four alternatives:

1. $i^n + i^{n+1} + i^{n+2} + i^{n+3}$ is

(1) 0 (2) 1 (3) -1 (4) i

2. The value of $\sum_{i=1}^{13}\left(i^n + i^{n-1}\right)$ is

(1) 1+ i (2) i (3) 1 (4) 0

3. The area of the triangle formed by the complex numbers $z, iz,$ *and* $z + iz$, in the Argand's diagram is

(1) $\mathbf{\frac{1}{2}|Z|^2}$ (2) $|Z|^2$

(3) $\frac{3}{2}|Z|^2$ (4) $2|Z|^2$

4. The conjugate of a complex number is $\frac{1}{i-2}$. Then, the complex number is

(1) $\frac{1}{i+2}$ **(2)** $\mathbf{\frac{-1}{i+2}}$

(3) $\frac{-1}{i-2}$ (4) $\frac{1}{i-2}$

5. If $Z = \frac{(\sqrt{3}+i)^3(3i+4)^2}{(8+6i)^2}$, then | z | is equal to

(1) 0 (2) 1 **(3) 2** (4) 3

6. If z is a non zero complex number,

such that $2iz^2 = \bar{z}$ then | z | is

(1) $\frac{1}{2}$ (2) 1 (3) 2 (4) 3

7. If $|z - 2 + i| \leq 2$, then the greatest value of | z | is

(1) $\sqrt{3} - 2$ (2) $\sqrt{3} + 2$

(3) $\sqrt{5} - 2$ **(4) $\sqrt{5} + 2$**

8. If $\left|z - \frac{3}{z}\right| = 2$, then the least value of | z | is

(1) 1 (2) 2 (3) 3 (4) 5

9. If | z | =1, then the value of $\frac{1+z}{1+\bar{z}}$ is

(1) z (2) $\bar{z}$ (3) $\frac{1}{z}$ (4) 1

10. The solution of the equation | z |− z =1+ 2 i is

(1) $\frac{3}{2} - 2i$ (2) $-\frac{3}{2} + 2i$

(3) $2 - \frac{3}{2}i$ (4) $2 + \frac{3}{2}i$

11. If $|Z_1| = 1, |Z_2| = 2, |Z_3| = 3$ *and* $|9z_1z_2 + 4z_1z_3 + z_2z_3| = 6$ the value of $|Z_1 + Z_2 + Z_3|$ is

(1) 1 **(2) 2** (3) 3 (4) 4

12. If z is a complex number such that $z \in C \backslash R$ and $z + \frac{1}{z} \in R$, then | z | is

(1) 0 **(2) 1** (3) 2 (4) 3

13. If Z_1, Z_2 and Z_3 are three complex numbers such that $|Z_1 + Z_2 + Z_3| = 1$ and $|Z_1| = |Z_2| = |Z_3| = 1$then $Z_1^2 + Z_2^2 + Z_3^2$ is

(1) 3 (2) 2 (3) 1 **(4) 0**

14. If $\frac{z-1}{z+1}$ is purely imaginary, then | z | is

(1) $\frac{1}{2}$ (2) 1 **(3) 2** (4) 3

15. If $z = x + iy$ is a complex number such that | z + 2| = | z − 2 |, then the locus of z is

(1) real axis (2) imaginary axis

(3) ellipse (4) circle

16. The principal argument of $\frac{3}{-1+i}$ is

(1) $\frac{-5\pi}{6}$ (2) $\frac{-2\pi}{3}$

(3) $\frac{-3\pi}{4}$ (4) $\frac{-\pi}{2}$

17. The principal argument of $(\sin 40^0 + i \cos 40^0)$ is

(1) −110° (2) −70°

(3) 70° (4) 110°

18. If $(1 + i)(1 + 2i)(1 + 3i) \ldots\ldots (1 + ni) = (x + iy)$, then $2.5.10 \ldots (1 + n^2)$is

(1) 1 (2) i

(3) $x^2 + y^2$ (4) $1 + n^2$

19. If $\omega \neq 1$is a cubic root of unity and $(1 + \omega)^7 = A + B\omega$, then (A , B) equals

(1) (1 , 0) (2) (−1, 1)

(3) (0 , 4) **(4) (1 , 1)**

20. The principal argument of the complex number $\frac{(1+i\sqrt{3})^2}{4i(1-i\sqrt{3})}$ is

(1) $\frac{2\pi}{3}$ (2) $\frac{\pi}{6}$ (3) $\frac{5\pi}{6}$ **(4) $\frac{\pi}{2}$**

21. If α and β are the roots of $x^2 + x + 1 = 0$, then $\alpha^{2020} + \beta^{2020}$ is

(1) −2 **(2) −1** (3) 1 (4) 2

22. The product of all four values of

$\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)^{\frac{3}{4}}$ is

(1) -2 (2) -1 **(3) 1** (4) 2

23. If $\omega \neq 1$is a cubic root of unity and

$\begin{vmatrix} 1 & 1 & 1 \\ 1 & -\omega^2 - 1 & \omega^2 \\ 1 & \omega^2 & \omega^7 \end{vmatrix} = 3k,$

then k is equal to

(1) 1 (2) -1 (3) $\sqrt{3}$i **(4) $-\sqrt{3}$i**

24. The value of $\left(\frac{1+\sqrt{3}i}{1-\sqrt{3}i}\right)^{10}$ is

(1) $cis\ \frac{2\pi}{3}$ (2) $cis\ \frac{4\pi}{3}$

(3) $-cis\ \frac{2\pi}{3}$ (4) $-cis\ \frac{4\pi}{3}$

25. If $= cis\ \frac{2\pi}{3}$, then the number of distinct roots of $\begin{vmatrix} z+1 & \omega & \omega^2 \\ \omega & z+\omega^2 & 1 \\ \omega^2 & 1 & z+\omega \end{vmatrix} = 0$

(1) 1 (2) 2 (3) 3 (4) 4

<u>Formulae:</u>

1. If $z = \frac{a+ib}{c+id}$ then

Re (z) $= \frac{ac+bd}{c^2+d^2}$

2. Im (z) $= \frac{bc-ad}{c^2+d^2}$

3. If $z = \frac{a+ib}{c+id}$ then

Arg (z) $= \frac{bc-ad}{ac+bd}$

4. If arg $(z) = \frac{\pi}{2}$ then

Re $(z) = 0$

5. $i = \sqrt{-1}$, $i^2 = -1$ and $i^4 = 1$

6.

7. Principle values of θ

II Quadrant (−, +)	I Quadrant (+, +)
$\theta = \pi - \alpha$	$\theta = \alpha$
III Quadrant (−, −)	IV Quadrant (+, −)
$\theta = -\pi + \alpha$	$\theta = -\alpha$

8. Working rules to find the n values.

i) Write in Polar form

ii) Add $2k\pi$ with θ

iii) Apply De Moivre's Theorem.

iv) Substitute k = 0, 1, 2...

9. Polar form of

$1 = \cos 0 + i\sin 0$

$-1 = \cos\pi + i\sin\pi$

$i = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2}$

$-i = \cos\frac{\pi}{2} - i\sin\frac{\pi}{2}$

10. Z is purely real if and if $z = \bar{z}$

11. Z is imaginary if and if $z = -\bar{z}$

12. Triangular law of inequality

$(i)\ |Z_1 + Z_2| \leq ||Z_1| + |Z_2||$

$(ii)\ |Z_1 + Z_2| \geq ||Z_1| - |Z_2||$

13. The form $|z - z_0| = $ r, represents a circle with centre z_0 and radius r.

14. The polar form of $x + yi = r(\cos\theta + i\sin\theta)$

Modulus $= r = \sqrt{x^2+y^2}$

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
tan θ	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	∞

argument

$\tan\alpha = \left|\frac{y}{x}\right|$

15. If $a+b+c=0, then\ a^3+b^3+c^3=3abc$

16. De Moivre's Theorem states that

$(\cos\theta + i\sin\theta)^n = (\cos n\theta + i\sin n\theta)$

17. If ω is the cube roots of unity then

$(1+\omega+\omega^2)=0$, and $\omega^3=1$

$(1+\omega+\omega^2)=0$ gives,

$1+\omega = -\omega^2$

$1+\omega^2 = -\omega$

$\omega+\omega^2 = -1$

where $\omega = \left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)$

and $\omega^2 = \left(-\frac{1}{2}-i\frac{\sqrt{3}}{2}\right)$

18. The fourth roots of unity are $\pm 1, \pm i$

19. The rotation of Z by θ radians in the counter clockwise direction about the origin in $Ze^{i\theta}$

20. If $Z=a+ib$ then

$\sqrt{a+ib} = \pm\sqrt{\frac{|Z|+a}{2}} + i\frac{b}{|b|}\sqrt{\frac{|Z|-a}{2}}$

DEPARTMENT OF MATHEMATICS

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UNIT - 2

COMPLEX NUMBERS

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UNIT - 3

Theory of Equations

Chapter3: Theory of Equations

Example 3.1 If α and β are the roots of the quadratic equation $17x^2 + 43x - 73 = 0$, construct a quadratic equation whose roots are $\alpha + 2$and $\beta + 2$.

Solution: If α, β are the roots of equation $ax^2 + bx + c = 0, then\ \alpha + \beta = \frac{-b}{a}$ and $\alpha\beta = \frac{c}{a}$

Given: $17x^2 + 43x - 73 = 0$

a = 17, b= 43 and c = –73

$\alpha + \beta = \frac{-b}{a} \Longrightarrow \frac{-43}{17}$ and $\alpha\beta = \frac{c}{a} \Rightarrow \frac{-73}{17}$

To construct a quadratic equation whose roots are $\alpha + 2$and $\beta + 2$.

Sum of the roots $= (\alpha + 2) + (\beta + 2)$

$$= \alpha + \beta + 4$$

$$= \frac{-43}{17} + 4$$

$$= \frac{-43 + 68}{17}$$

$$= \frac{25}{17}$$

Product of the roots $= (\alpha + 2) \times (\beta + 2)$

$$= \alpha\beta + 2\alpha + 2\beta + 4$$

$$= \alpha\beta + 2(\alpha + \beta) + 4$$

$$= \frac{-73}{17} + 2\left(\frac{-43}{17}\right) + 4$$

$$= \frac{-73}{17} - \frac{86}{17} + 4$$

$$= \frac{-159}{17} + 4$$

$$= \frac{-159 + 68}{17}$$

$$= -\frac{91}{17}$$

Required equation is $x^2 - (sum)x + prod = 0$

$$x^2 - \left(\frac{25}{17}\right)x - \frac{91}{17} = 0$$

Multiplying by 17,

∴ Required Equation is $17x^2 - 25x - 91 = 0$.

Example 3.2 If α and β are the roots of the quadratic equation $2x^2 - 7x + 13 = 0$, construct a quadratic equation whose roots are $\alpha^2$and β^2.

Solution: If α, β are the roots of equation $ax^2 + bx + c = 0, then\ \alpha + \beta = \frac{-b}{a}$ and $\alpha\beta = \frac{c}{a}$

Given: $2x^2 - 7x + 13 = 0$

a = 2, b= – 7 and c = 13

$\alpha + \beta = \frac{-b}{a} \Longrightarrow \frac{7}{2}$ and $\alpha\beta = \frac{c}{a} \Rightarrow \frac{13}{2}$

To construct a quadratic equation whose roots are $\alpha^2$and β^2.

Sum of the roots = $\alpha^2 + \beta^2$

$= (\alpha + \beta)^2 - 2\alpha\beta$

$= \left(\frac{7}{2}\right)^2 - 2\left(\frac{13}{2}\right)$

$= \frac{49}{4} - 13$

$= \frac{49 - 52}{4}$

$= \frac{-3}{4}$

Product of the roots = $\alpha^2 \times \beta^2$

$= (\alpha\beta)^2$

$= \left(\frac{13}{2}\right)^2$

$= \frac{169}{4}$

Required equation is $x^2 - (sum)x + prod = 0$

$$x^2 - \left(\frac{-3}{4}\right)x + \frac{169}{4} = 0$$

Multiplying by 4,

∴ Required Equation is $4x^2 + 3x + 169 = 0$.

Example 3.3 If α , β , and γ are the roots of the equation $x^3 + px^2 + qx + r = 0$, find the value of $\sum \frac{1}{\beta\gamma}$ in terms of the coefficients.

Solution:

If α , β , and γ are the roots of the equation

$x^3 + px^2 + qx + r = 0$

$\sum_1 = (\alpha + \beta + \gamma) = -p$ and

$\sum_3 = (\alpha\beta\gamma) = -r$

To find the value of $\sum \frac{1}{\beta\gamma} = \frac{1}{\beta\gamma} + \frac{1}{\gamma\alpha} + \frac{1}{\alpha\beta}$

$= \frac{\alpha + \beta + \gamma}{\alpha\beta\gamma}$

$= \frac{-p}{-r}$

$= \frac{p}{r}$

Example 3.4 Find the sum of the squares of the roots of $ax^4 + bx^3 + cx^2 + dx + e = 0$.

Solution: If $\alpha, \beta, \gamma, and\ \delta\ are\ the\ roots\ of$ $ax^4 + bx^3 + cx^2 + dx + e = 0$, then

$\sum_1 = (\alpha + \beta + \gamma + \delta) = -\frac{b}{a}$

$\sum_2 = (\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta) = \frac{c}{a}$

$\sum_3 = (\alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta) = -\frac{d}{a}$

$\sum_4 = (\alpha\beta\gamma\delta) = \frac{e}{a}$

We have to find $(\alpha^2 + \beta^2 + \gamma^2 + \delta^2)$.

Applying the algebraic identity

$(a + b + c + d)^2 =$

$a^2 + b^2 + c^2 + d^2 + 2(ab + ac + ad + bc + bd + cd)$

we get

$a^2 + b^2 + c^2 + d^2 =$

$(a + b + c + d)^2 - 2(ab + ac + ad + bc + bd + cd)$

Hence,

$\alpha^2 + \beta^2 + \gamma^2 + \delta^2 =$

$(\alpha + \beta + \gamma + \delta)^2 - 2(\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta)$

$= \left(-\frac{b}{a}\right)^2 - 2\left(\frac{c}{a}\right)$

$= \frac{b^2}{a^2} - \frac{2c}{a}$

$\alpha^2+\beta^2+\gamma^2+\delta^2 = \frac{b^2-2ac}{a^2}$

Example 3.5 Find the condition that the roots of $x^3+ax^2+bx+c=0$ are in the ratio $p:q:r$.

Solution: The roots of $x^3+ax^2+bx+c=0$ are in the ratio $p:q:r$.

So let the roots are $p\lambda, q\lambda$ *and* $r\lambda$ then

$\sum_1 = (p\lambda+q\lambda+r\lambda) = -a$

$\sum_2 = (p\lambda)(q\lambda)+(p\lambda)(r\lambda)+(q\lambda)(r\lambda) = b$

$\sum_3 = (p\lambda)(q\lambda)(r\lambda) = -c$

$$p\lambda+q\lambda+r\lambda = -a$$

$$(p+q+r)\lambda = -a$$

$$\lambda = \frac{-a}{p+q+r}$$

and $(p\lambda)(q\lambda)(r\lambda) = -c$

$$pqr\lambda^3 = -c$$

So, $\lambda^3 = -\frac{c}{pqr}$

$$\left(\frac{-a}{p+q+r}\right)^3 = -\frac{c}{pqr}$$

$$\frac{-a^3}{(p+q+r)^3} = -\frac{c}{pqr}$$

$$\frac{a^3}{(p+q+r)^3} = \frac{c}{pqr}$$

$$a^3pqr = c(p+q+r)^3$$

Example 3.6 Form the equation whose roots are the squares of the roots of the cubic equation $x^3+ax^2+bx+c=0$.

Solution: If $\alpha, \beta, and\ \gamma\ are\ the\ roots\ of$ $x^3+ax^2+bx+c=0$, then

$\sum_1 = (\alpha+\beta+\gamma) = -a$

$\sum_2 = (\alpha\beta+\alpha\gamma+\beta\gamma) = b$

$\sum_3 = (\alpha\beta\gamma) = -c$

To construct a quadratic equation whose roots are α^2, β^2 *and* γ^2.

$\sum_1 = \alpha^2+\beta^2+\gamma^2$

$= (\alpha+\beta+\gamma)^2 - 2(\alpha\beta+\alpha\gamma+\beta\gamma)$

$= (-a)^2 - 2(b)$

$= a^2 - 2b$

$\sum_2 = \alpha^2\beta^2+\alpha^2\gamma^2+\beta^2\gamma^2$

$= (\alpha\beta+\alpha\gamma+\beta\gamma)^2$

$-2[(\alpha\beta)(\alpha\gamma)+(\alpha\beta)(\beta\gamma)+(\alpha\gamma)(\beta\gamma)]$

$= (\alpha\beta+\alpha\gamma+\beta\gamma)^2 - 2(\alpha\beta\gamma)(\beta+\gamma+\alpha)$

$= (b)^2 - 2(-c)(b)$

$= b^2 + 2bc$

$\sum_3 = \alpha^2\beta^2\gamma^2$

$= (\alpha\beta\gamma)^2$

$= (-c)^2$

$= c^2$

Hence, the required equation is

$x^3 - (\alpha^2+\beta^2+\gamma^2)x^2$

$+(\alpha^2\beta^2+\alpha^2\gamma^2+\beta^2\gamma^2)x - (\alpha^2\beta^2\gamma^2) = 0$

$x^3 - (a^2-2b)x^2 + (b^2+2bc)x - c^2 = 0$

Example 3.7 If p is real, discuss the nature of the roots of the equation $4x^2+4px+p+2=0$, in terms of p.

Solution: $4x^2+4px+(p+2)=0$

a = 4, b = 4p and c = p + 2

Discriminant $\Delta = b^2-4ac$

$= (4p)^2 - 4(4)(p+2)$

$= 16p^2 - 16(p+2)$

$= 16p^2 - 16p - 32$

$= 16(p^2-p-2)$

$= 16(p+1)(p-2)$

If $\Delta < 0\ then -1 < p < 2$

If $\Delta = 0\ then\ p = -1 and\ p = 2$

If $\Delta > 0$ *then* $-\infty < p < -1$ *or* $2 < p < \infty$

Thus the given polynomial has

imaginary roots if $-1 < p < 2$

equal real roots if $p = -1$ or $p = 2$;

distinct real roots if $-\infty < p < -1$ *or* $2 < p < \infty$

EXERCISE 3.1

1. If the sides of a cubic box are increased by 1, 2, 3 units respectively to form a cuboid, then the volume is increased by 52 cubic units. Find the volume of the cuboid.

Solution:

Let x be the side and V be the volume of the cube, then volume of the cube

$V = x \times x \times x = x^3$

If the sides are increased by 1, 2, 3 units, Volume of the cuboid is equal to volume of the cube increased by 52

$V + 52 = (x+1) \times (x+2) \times (x+3)$

If $\alpha, \beta,$ *and* γ *are the roots*, then If

$\alpha = -1,\ \beta = -2$ *and* $\gamma = -3$.

$\Sigma_1 = (\alpha + \beta + \gamma)$

$= -1 - 2 - 3$

$= -6$

$\Sigma_2 = (\alpha\beta + \alpha\gamma + \beta\gamma)$

$= (-1)(-2) + (-1)(-3) + (-2)(-3)$

$= 2 + 3 + 6$

$= 11$

$\Sigma_3 = (\alpha\beta\gamma)$

$= (-1)(-2)(-3)$

$= -6$

Hence, the volume of the cuboid equation is

$x^3 - \Sigma_1 x^2 + \Sigma_2 x - \Sigma 3$

$= x^3 - (-6)x^2 + (11)x - (-6)$

$= x^3 + 6x^2 + 11x + 6$

$\therefore V + 52 = x^3 + 6x^2 + 11x + 6$

$x^3 + 52 = x^3 + 6x^2 + 11x + 6$

$x^3 + 6x^2 + 11x + 6 - x^3 - 52 = 0$

$6x^2 + 11x - 46 = 0$

$(6x + 23)(x - 2) = 0$

$6x + 23 = 0$ or $x - 2 = 0$

$6x + 23 = 0$ gives,

$6x = -23$

$x = -\frac{23}{6}$, is impossible

So $x - 2 = 0$ gives,

$x = 2$

Substituting $x = 2$, volume of the cuboid

$V = (2+1) \times (2+2) \times (2+3)$

$= (3) \times (4) \times (5)$

$= 60$ cubic units.

2. Construct a cubic equation with roots

(i) 1, 2 and 3

Solution: Given

$\alpha = 1,\ \beta = 2$ *and* $\gamma = 3$.

$\Sigma_1 = (\alpha + \beta + \gamma)$

$= 1 + 2 + 3$

$= 6$

$\Sigma_2 = (\alpha\beta + \alpha\gamma + \beta\gamma)$

$= (1)(2) + (1)(3) + (2)(3)$

$= 2 + 3 + 6$

$= 11$

$\Sigma_3 = (\alpha\beta\gamma)$

$= (1)(2)(3)$

$= 6$

Hence, the required equation is

$x^3 - \Sigma_1 x^2 + \Sigma_2 x - \Sigma_3 = 0$

$x^3 - (6)x^2 + (11)x - (6) = 0$

$x^3 - 6x^2 + 11x - 6 = 0$

(ii) 1, 1 and −2

Solution: Given

$\alpha = 1,\ \beta = 1\ and\ \gamma = -2.$

$\sum_1 = (\alpha + \beta + \gamma)$

$= 1 + 1 - 2$

$= 2 - 2$

$= 0$

$\sum_2 = (\alpha\beta + \alpha\gamma + \beta\gamma)$

$= (1)(1) + (1)(-2) + (1)(-2)$

$= 1 - 2 - 2$

$= 1 - 4$

$= -3$

$\sum_3 = (\alpha\beta\gamma)$

$= (1)(1)(-2)$

$= -2$

Hence, the required equation is

$x^3 - \sum_1 x^2 + \sum_2 x - \sum_3 = 0$

$x^3 - (0)x^2 + (-3)x - (-2) = 0$

$x^3 - 3x + 2 = 0$

(iii) 2 , −2 , and 4 .

Solution: Given

$\alpha = 2,\ \beta = -2\ and\ \gamma = 4.$

$\sum_1 = (\alpha + \beta + \gamma)$

$= 2 - 2 + 4$

$= 4$

$\sum_2 = (\alpha\beta + \alpha\gamma + \beta\gamma)$

$= (2)(-2) + (2)(4) + (-2)(4)$

$= -4 + 8 - 8$

$= -4$

$\sum_3 = (\alpha\beta\gamma)$

$= (2)(-2)(4)$

$= -16$

Hence, the required equation is

$x^3 - \sum_1 x^2 + \sum_2 x - \sum_3 = 0$

$x^3 - (4)x^2 + (-4)x - (-16) = 0$

$x^3 - 4x^2 - 4x + 16 = 0$

(iv) 2, $\frac{1}{2}$, and1.

Solution: Given

$\alpha = 2,\ \beta = \frac{1}{2}\ and\ \gamma = 1.$

$\sum_1 = (\alpha + \beta + \gamma)$

$= 2 + \frac{1}{2} + 1$

$= 3 + \frac{1}{2}$

$= \frac{6+1}{2}$

$= \frac{7}{2}$

$\sum_2 = (\alpha\beta + \alpha\gamma + \beta\gamma)$

$= (2)\left(\frac{1}{2}\right) + (2)(1) + \left(\frac{1}{2}\right)(1)$

$= 1 + 2 + \frac{1}{2}$

$= 3 + \frac{1}{2}$

$= \frac{6+1}{2}$

$= \frac{7}{2}$

$\sum_3 = (\alpha\beta\gamma)$

$= (2)\left(\frac{1}{2}\right)(1)$

$= 1$

Hence, the required equation is

$x^3 - \sum_1 x^2 + \sum_2 x - \sum_3 = 0$

$x^3 - \left(\frac{7}{2}\right)x^2 + \left(\frac{7}{2}\right)x - (1) = 0$

$x^3 - \frac{7}{2}x^2 + \frac{7}{2}x - 1 = 0$

Multiplying by 2

$2x^3 - 7x^2 + 7x - 2 = 0$

3. If α , β and γ are the roots of the cubic equation $x^3 + 2x^2 + 3x + 4 = 0$, form a cubic equation whose roots are

(i) $2\alpha, 2\beta, 2\gamma$

<u>Solution</u>: Given $x^3 + 2x^2 + 3x + 4 = 0$

Comparing with $x^3 - \sum_1 x^2 + \sum_2 x - \sum_3 = 0$

$\sum_1 = (\alpha + \beta + \gamma) = -2$

$\sum_2 = (\alpha\beta + \alpha\gamma + \beta\gamma) = 3$

$\sum_3 = (\alpha\beta\gamma) = -4$

To construct a quadratic equation whose roots are $2\alpha, 2\beta, 2\gamma$.

$\sum_1 = 2\alpha + 2\beta + 2\gamma$

$= 2(\alpha + \beta + \gamma)$

$= 2(-2)$

$= -4$

$\sum_2 = (2\alpha)(2\beta) + (2\alpha)(2\gamma) + (2\beta)(2\gamma)$

$= 4\alpha\beta + 4\alpha\gamma + 4\beta\gamma$

$= 4(\alpha\beta + \alpha\gamma + \beta\gamma)$

$= 4(3)$

$= 12$

$\sum_3 = (2\alpha)(2\beta)(2\gamma)$

$= 8(\alpha\beta\gamma)$

$= 8(-4)$

$= -32$

Hence, the required equation is

$$x^3 - \sum_1 x^2 + \sum_2 x - \sum_3 = 0$$

$$x^3 - (-4)x^2 + (12)x - (-32) = 0$$

$$x^3 + 4x^2 + 12x + 32 = 0$$

(ii) $\frac{1}{\alpha}, \frac{1}{\beta}, \frac{1}{\gamma}$

To construct a quadratic equation whose roots are $\frac{1}{\alpha}, \frac{1}{\beta}, \frac{1}{\gamma}$.

$\sum_1 = \frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}$

$= \frac{\beta\gamma + \alpha\gamma + \alpha\beta}{\alpha\beta\gamma}$

$= \frac{3}{-4} = -\frac{3}{4}$

$\sum_2 = \left(\frac{1}{\alpha}\right)\left(\frac{1}{\beta}\right) + \left(\frac{1}{\alpha}\right)\left(\frac{1}{\gamma}\right) + \left(\frac{1}{\beta}\right)\left(\frac{1}{\gamma}\right)$

$= \frac{1}{\alpha\beta} + \frac{1}{\alpha\gamma} + \frac{1}{\beta\gamma}$

$= \frac{\gamma + \beta + \alpha}{\alpha\beta\gamma}$

$= \frac{-2}{-4}$

$= \frac{2}{4}$

$= \frac{1}{2}$

$\sum_3 = \left(\frac{1}{\alpha}\right)\left(\frac{1}{\beta}\right)\left(\frac{1}{\gamma}\right)$

$= \left(\frac{1}{\alpha\beta\gamma}\right)$

$= \left(\frac{1}{-4}\right)$

$= -\frac{1}{4}$

Hence, the required equation is

$$x^3 - \sum_1 x^2 + \sum_2 x - \sum_3 = 0$$

$$x^3 - \left(-\frac{3}{4}\right)x^2 + \left(\frac{1}{2}\right)x - \left(-\frac{1}{4}\right) = 0$$

$$x^3 + \frac{3}{4}x^2 + \frac{1}{2}x + \frac{1}{4} = 0$$

Multiplying by 4

$$4x^3 + 3x^2 + 2x + 1 = 0$$

(iii) $-\alpha, -\beta, -\gamma$

To construct a quadratic equation whose roots are $-\alpha, -\beta, -\gamma$

$\sum_1 = -\alpha - \beta - \gamma$

$= -(\alpha + \beta + \gamma)$

$= -(-2)$

$= 2$

$\sum_2 = (-\alpha)(-\beta) + (-\alpha)(-\gamma) + (-\beta)(-\gamma)$

$= \alpha\beta + \alpha\gamma + \beta\gamma$

$= 3$

$\sum_3 = (-\alpha)(-\beta)(-\gamma)$

$= -(\alpha\beta\gamma)$

$= -(-4)$

$= 4$

Hence, the required equation is

$$x^3 - \sum_1 x^2 + \sum_2 x - \sum_3 = 0$$

$$x^3 - (2)x^2 + (3)x - (4) = 0$$

$$x^3 - 2x^2 + 3x - 4 = 0$$

4. Solve the equation $3x^3 - 16x^2 + 23x - 6 = 0$ if the product of two roots is 1.

<u>Solution</u>: Given $3x^3 - 16x^2 + 23x - 6 = 0$

$\div 3, x^3 - \frac{16}{3}x^2 + \frac{23}{3}x - 2 = 0$

Comparing with $x^3 - \sum_1 x^2 + \sum_2 x - \sum_3 = 0$

$\sum_1 = (\alpha + \beta + \gamma) = \frac{16}{3}$

$\sum_2 = (\alpha\beta + \alpha\gamma + \beta\gamma) = \frac{23}{3}$

$\sum_3 = (\alpha\beta\gamma) = 2$

Given the product of two roots is 1

So let, $\beta = \frac{1}{\alpha}$

$\sum_1 = \left(\alpha + \frac{1}{\alpha} + \gamma\right) = \frac{16}{3}$

$\sum_2 = \left(\alpha\frac{1}{\alpha} + \alpha\gamma + \frac{1}{\alpha}\gamma\right) = \frac{23}{3}$

$\sum_3 = \left(\alpha\frac{1}{\alpha}\gamma\right) = 2$

$\therefore \gamma = 2$

Substituting $\gamma = 2$ in

$1 + \alpha\gamma + \frac{1}{\alpha}\gamma = \frac{23}{3}$ we get,

$$1 + 2\alpha + \frac{2}{\alpha} = \frac{23}{3}$$

$$2\alpha + \frac{2}{\alpha} = \frac{23}{3} - 1$$

$$\frac{2\alpha^2 + 2}{\alpha} = \frac{23-3}{3}$$

$$\frac{2\alpha^2 + 2}{\alpha} = \frac{20}{3}$$

$$3(2\alpha^2 + 2) = 20\alpha$$

$$6\alpha^2 + 6 = 20\alpha$$

$$6\alpha^2 - 20\alpha + 6 = 0$$

Dividing by 2,

$$3\alpha^2 - 10\alpha + 3 = 0$$

$$(3\alpha - 1)(\alpha - 3) = 0$$

$3\alpha - 1 = 0$ or $\alpha - 3 = 0$

If $3\alpha - 1 = 0$ gives

$$3\alpha = 1$$

$\alpha = \frac{1}{3} \Rightarrow \beta = 3$ and

If $\alpha - 3 = 0$

$$\alpha = 3 \Rightarrow \beta = \frac{1}{3}$$

Hence the roots are $3, \frac{1}{3}, 2$

5. Find the sum of squares of roots of the equation $2x^4 - 8x^3 + 6x^2 - 3 = 0$.

<u>Solution</u>: Given $2x^4 - 8x^3 + 6x^2 - 3 = 0$

$\div 2, \quad x^4 - 4x^3 + 3x^2 - \frac{3}{2} = 0$

Comparing with

$$x^4 - \sum_1 x^3 + \sum_2 x^2 - \sum_3 x + \sum_4 = 0$$

If $\alpha, \beta, \gamma, and\ \delta\ are\ the\ roots$

$\sum_1 = (\alpha + \beta + \gamma + \delta) = 4$

$\sum_2 = (\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta) = 3$

$\sum_3 = (\alpha\beta\gamma) + (\alpha\beta\delta) + (\beta\gamma\delta) = 0$

$\sum_4 = (\alpha\beta\gamma\delta) = -\frac{3}{2}$

We have to find $(\alpha^2 + \beta^2 + \gamma^2 + \delta^2)$.

Applying the algebraic identity

$(a + b + c + d)^2 =$

$a^2 + b^2 + c^2 + d^2 + 2(ab + ac + ad + bc + bd + cd)$

we get

$a^2 + b^2 + c^2 + d^2 =$

$(a + b + c + d)^2 - 2(ab + ac + ad + bc + bd + cd)$

Hence,

$\alpha^2 + \beta^2 + \gamma^2 + \delta^2 =$

$(\alpha + \beta + \gamma + \delta)^2 - 2(\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta)$

$$= (4)^2 - 2(3)$$

$= 16 - 6$

$= 10$

6. Solve the equation $x^3 - 9x^2 + 14x + 24 = 0$, if it is given that two of its roots are in the ratio 3 : 2 .

Solution: Given $x^3 - 9x^2 + 14x + 24 = 0$

Two of its roots are in the ratio 3 : 2

Let $\alpha : \beta = 3 : 2$

$\frac{\alpha}{\beta} = \frac{3}{2}$

$\therefore 2\alpha = 3\beta$

Comparing with $x^3 - \sum_1 x^2 + \sum_2 x - \sum_3 = 0$

$\sum_1 = (\alpha + \beta + \gamma) = 9$

$\sum_2 = (\alpha\beta + \alpha\gamma + \beta\gamma) = 14$

$\sum_3 = (\alpha\beta\gamma) = -24$

From $\alpha + \beta + \gamma = 9$

$2\alpha + 2\beta + 2\gamma = 18$

Substituting $2\alpha = 3\beta$

$3\beta + 2\beta + 2\gamma = 18$

$5\beta + 2\gamma = 18$

$2\gamma = 18 - 5\beta$ .....(1)

From $\alpha\beta\gamma = -24$

$2\alpha\beta\gamma = -48$

Substituting $2\alpha = 3\beta$

$(3\beta)\beta\gamma = -48$

$3\beta^2\gamma = -48$

$\beta^2\gamma = -16$

Multiplying by 2, $\beta^2 2\gamma = -32$

Substituting $2\gamma = 18 - 5\beta$

$\beta^2(18 - 5\beta) = -32$

$18\beta^2 - 5\beta^3 = -32$

$5\beta^3 - 18\beta^2 - 32 = 0$

$(\beta - 4)(5\beta^2 + 2\beta + 8) = 0$

$\beta - 4 = 0$, gives $\beta = 4$

Substituting $\beta = 4$ *in* $2\alpha = 3\beta$

$2\alpha = 3(4)$

$2\alpha = 12$

$\therefore \alpha = 6$

Substituting $\beta = 4$ in $2\gamma = 18 - 5\beta$

$2\gamma = 18 - 5(4)$

$2\gamma = 18 - 20$

$2\gamma = -2$

$\therefore \gamma = -1$

Hence the roots are 6, 4 and -1

7. If α , β , and γ are the roots of the polynomial equation $ax^3 + bx^2 + cx + d = 0$, find the value of $\sum \frac{\alpha}{\beta\gamma}$ in terms of the coefficients.

Solution: If $\alpha, \beta,$ *and* γ *are the roots of* $ax^3 + bx^2 + cx + d = 0$, then

$\sum_1 = (\alpha + \beta + \gamma) = -\frac{b}{a}$

$\sum_2 = (\alpha\beta + \alpha\gamma + \beta\gamma) = \frac{c}{a}$

$\sum_3 = (\alpha\beta\gamma) = -\frac{d}{a}$

To find the value of $\sum \frac{\alpha}{\beta\gamma}$

$$\sum \frac{\alpha}{\beta\gamma} = \frac{\alpha}{\beta\gamma} + \frac{\beta}{\alpha\gamma} + \frac{\gamma}{\alpha\beta}$$

$$= \frac{\alpha^2 + \beta^2 + \gamma^2}{\alpha\beta\gamma}$$

We know

$$\alpha^2 + \beta^2 + \gamma^2 = (\alpha + \beta + \gamma)^2 - 2(\alpha\beta + \alpha\gamma + \beta\gamma)$$

$$= \left(-\frac{b}{a}\right)^2 - 2\left(\frac{c}{a}\right)$$

$$= \frac{b^2}{a^2} - \frac{2c}{a}$$

$$= \frac{b^2 - 2ac}{a^2}$$

$$\therefore \sum \frac{\alpha}{\beta\gamma} = \frac{\alpha^2 + \beta^2 + \gamma^2}{\alpha\beta\gamma}$$

$$= \frac{\frac{b^2-2ac}{a^2}}{-\frac{d}{a}}$$

$$= \frac{b^2-2ac}{a^2} \times -\frac{a}{d}$$

$$= -\frac{(b^2-2ac)}{ad}$$

$$\therefore \sum \frac{\alpha}{\beta\gamma} = \frac{(2ac-b^2)}{ad}$$

8. If α , β , γ and δ are the roots of the equation $2x^4 + 5x^3 - 7x^2 + 8 = 0$, find a quadratic equation with integer coefficients whose roots are $\alpha+\beta+\gamma+\delta$ and $\alpha\beta\gamma\delta$.

Solution: Given $2x^4 + 5x^3 - 7x^2 + 8 = 0$

$\div 2,\quad x^4 + \frac{5}{2}x^3 - \frac{7}{2}x^2 + 4 = 0$

Comparing with

$$x^4 - \sum_1 x^3 + \sum_2 x^2 - \sum_3 x + \sum_4 = 0$$

Given $\alpha, \beta, \gamma, and\ \delta\ are\ the\ roots$

$$\sum_1 = (\alpha+\beta+\gamma+\delta) = -\frac{5}{2}$$

$$\sum_4 = (\alpha\beta\gamma\delta) = 4$$

To construct a quadratic equation whose roots are $\alpha+\beta+\gamma+\delta$ and $\alpha\beta\gamma\delta$

$$\sum_1 = (\alpha+\beta+\gamma+\delta) + (\alpha\beta\gamma\delta)$$

$$= -\frac{5}{2} + 4$$

$$= \frac{-5+8}{2}$$

$$= \frac{3}{2}$$

$$\sum_2 = (\alpha+\beta+\gamma+\delta)(\alpha\beta\gamma\delta)$$

$$= -\left(\frac{5}{2}\right) \times (4)$$

$$= -10$$

Hence, the required equation is

$$x^2 - \sum_1 x + \sum_2 = 0$$

$$x^2 - \left(\frac{3}{2}\right)x + (-10) = 0$$

$$x^2 - \frac{3}{2}x - 10 = 0$$

Multiplying by 2

$$2x^2 - 3x - 20 = 0$$

9. If p and q are the roots of the equation $lx^2 + nx + n = 0$, show that $\sqrt{\frac{p}{q}} + \sqrt{\frac{q}{p}} + \sqrt{\frac{n}{l}} = 0.$

Solution:

p and q are the roots of the equation

$$lx^2 + nx + n = 0$$

$\div l,\quad x^2 + \frac{n}{l}x + \frac{n}{l} = 0$

Comparing with

$$x^2 - \sum_1 x + \sum_2 = 0$$

$$\sum_1 = (p+q) = -\frac{n}{l}$$

$$\sum_2 = (pq) = \frac{n}{l}$$

To prove: $\sqrt{\frac{p}{q}} + \sqrt{\frac{q}{p}} + \sqrt{\frac{n}{l}} = 0$

$$\text{LHS} = \sqrt{\frac{p}{q}} + \sqrt{\frac{q}{p}} + \sqrt{\frac{n}{l}}$$

$$= \frac{\sqrt{p}\sqrt{p} + \sqrt{q}\sqrt{q}}{\sqrt{p}\sqrt{q}} + \sqrt{\frac{n}{l}}$$

$$= \frac{p+q}{\sqrt{pq}} + \sqrt{\frac{n}{l}}$$

$$= \frac{-\frac{n}{l}}{\sqrt{\frac{n}{l}}} + \sqrt{\frac{n}{l}}$$

$$= \left(-\sqrt{\frac{n}{l}}\sqrt{\frac{n}{l}} \times \sqrt{\frac{l}{n}}\right) + \sqrt{\frac{n}{l}}$$

$$= -\sqrt{\frac{n}{l}} + \sqrt{\frac{n}{l}}$$

$= 0$ RHS

Hence proved.

10. If the equations $x^2 + px + q = 0$ and $x^2 + p'x + q' = 0$ have a common root, show that it must be equal to $\frac{pq' - p'q}{q - q'}$ or $\frac{q - q'}{p' - p}$.

Solution: Let α be the common root of the given equations. Then,

$\alpha^2 + p\alpha + q = 0 \quad .....(1)$

$\alpha^2 + p'\alpha + q' = 0 \quad .....(2)$

Solving the equations by cross multiplication method,

$$\begin{array}{cccc} p & q & 1 & p \\ p' & q' & 1 & p' \end{array}$$

$$\frac{\alpha^2}{pq' - p'q} = \frac{\alpha}{q - q'} = \frac{1}{p' - p}$$

From $\frac{\alpha^2}{pq' - p'q} = \frac{\alpha}{q - q'}$

$$\frac{\alpha^2}{\alpha} = \frac{pq' - p'q}{q - q'}$$

$$\alpha = \frac{pq' - p'q}{q - q'}$$

and from $\frac{\alpha}{q - q'} = \frac{1}{p' - p}$

$$\alpha = \frac{q - q'}{p' - p}$$

Hence, $\alpha = \frac{pq' - p'q}{q - q'}$ or $\alpha = \frac{q - q'}{p' - p}$ is proved.

11. Formulate into a mathematical problem to find a number such that when its cube root is add to it, the result is 6 .

Solution: Let x be the number. Then,

$$\sqrt[3]{x} + x = 6$$

$$\sqrt[3]{x} = 6 - x$$

Raising to the power of 3 on both sides,

$$x = (6 - x)^3$$

$$= 6^3 - 3(6)^2(x) + 3(6)(x)^2 - (x)^3$$

$$= 216 - 3(36)x + 18x^2 - x^3$$

$$= 216 - 108x + 18x^2 - x^3$$

$$x^3 - 18x^2 + 108x + x - 216 = 0$$

$$\therefore\ x^3 - 18x^2 + 109x - 216 = 0$$

12. A 12 metre tall tree was broken into two parts. It was found that the height of the part which was left standing was the cube root of the length of the part that was cut away. Formulate this into a mathematical problem to find the height of the part which was cut away.

Solution: Height of the tree is 12 meters.

Let the cut part be $= x$

Then uncut part $= 12 - x$

Given, $\sqrt[3]{(12 - x)} = x$

Raising to the power of 3 on both sides,

$$(12 - x) = x^3$$

$\therefore\ x^3 + x - 12 = 0$, is the required equation.

Example 3.8 Find the monic polynomial equation of minimum degree with real coefficients having $2 - \sqrt{3}i$ as a root.

Solution: Given $2 - \sqrt{3}i$ is a root

$\therefore 2 + \sqrt{3}i$ is another root.

Hence sum of the roots = $2 - \sqrt{3}i + 2 + \sqrt{3}i$

= 4

Product of the roots = $(2 - \sqrt{3}i)(2 + \sqrt{3}i)$

= 4 + 3

= 7

Hence the required equation is

$$x^2 - (sum\ of\ roots)x + \ Prod.of\ roots = 0$$

$$x^2 - 4x + \ 7 = 0$$

Example 3.9 Find a polynomial equation of minimum degree with rational coefficients, having $2 - \sqrt{3}$ as a root.

Solution: Given $2 - \sqrt{3}$ is a root

$\therefore 2 + \sqrt{3}$ is another root.

Hence sum of the roots = $2 - \sqrt{3} + 2 + \sqrt{3}$

= 4

Product of the roots = $\left(2 - \sqrt{3}\right)\left(2 + \sqrt{3}\right)$

= 4 − 3

= 1

Hence the required equation is

$$x^2 - (sum\ of\ roots)x + \ Prod.of\ roots = 0$$

$$x^2 - 4x + \ 1 = 0$$

Example 3.10 Form a polynomial equation with integer coefficients with $\sqrt{\frac{\sqrt{2}}{\sqrt{3}}}$ as a root.

Solution: Given $\sqrt{\frac{\sqrt{2}}{\sqrt{3}}}$ is a root.

So, $\left(x - \sqrt{\frac{\sqrt{2}}{\sqrt{3}}}\right)$ is a factor.

$\therefore \left(x + \sqrt{\frac{\sqrt{2}}{\sqrt{3}}}\right)$ is another factor.

Their product = $\left(x - \sqrt{\frac{\sqrt{2}}{\sqrt{3}}}\right)\left(x + \sqrt{\frac{\sqrt{2}}{\sqrt{3}}}\right)$

$= \left(x^2 - \frac{\sqrt{2}}{\sqrt{3}}\right)$

To remove square root we include $\left(x^2 + \frac{\sqrt{2}}{\sqrt{3}}\right)$

$\therefore$ Required equation is

$$\left(x^2 - \frac{\sqrt{2}}{\sqrt{3}}\right)\left(x^2 + \frac{\sqrt{2}}{\sqrt{3}}\right) = 0$$

$$x^4 - \frac{2}{3} = 0$$

$$3x^4 - 2 = 0$$

Example 3.11 Show that the equation $2x^2 - 6x + 7 = 0$ cannot be satisfied by any real values of x.

Solution: $2x^2 - 6x + 7 = 0$

$a = 2, b = -6\ and\ c = 7$

$\Delta = \ b^2 - 4ac$

$= \ (-6)^2 - 4(2)(7)$

$= \ 36 - 56$

$= \ -20 < 0$

Hence the roots are imaginary.

Example 3.12 If $x^2 + 2(k+2)x + 9k = 0$ has equal roots, find k.

Solution: $x^2 + 2(k+2)x + 9k = 0$

$a = 1, b = 2(k+2)\ and\ c = 9k$

$\Delta = \ b^2 - 4ac$

Given the roots are equal,

$$b^2 - 4ac = 0$$

$$[2(k+2)]^2 - 4(1)(9k) = 0$$

$$4(k+2)^2 - 36k = 0$$

$\div 4 \qquad (k+2)^2 - 9k = 0$

$$k^2 + 4k + 4 - 9k = 0$$

$$k^2 - 5k + 4 = 0$$

$$(k-1)(k-4) = 0$$

$$k - 1 = \ 0 \quad or \quad k - 4 = 0$$

$$k - 1 = \ 0 \Rightarrow k = 1$$

$$k - 4 = \ 0 \Rightarrow k = 4$$

Example 3.13 Show that, if p, q, r are rational, the roots of the equation $x^2 - 2px + p^2 - q^2 + 2qr - r^2 = 0$ are rational.

Solution: $x^2 - 2px + (p^2 - q^2 + 2qr - r^2) = 0$

$a = 1, b = -2p\ and\ c = (p^2 - q^2 + 2qr - r^2)$

$\Delta = b^2 - 4ac$

$= (-2p)^2 - 4(1)(p^2 - q^2 + 2qr - r^2)$

$= 4p^2 - 4(p^2 - q^2 + 2qr - r^2)$

$= 4p^2 - 4p^2 + 4q^2 - 8qr + 4r^2$

$= 4q^2 - 8qr + 4r^2$

$= 4(q^2 - 2qr + r^2)$

$= 4(q - r)^2$ which is a perfect square.

Hence the roots are rational.

EXERCISE 3.2

1. If k is real, discuss the nature of the roots of the polynomial equation $2x^2 + kx + k = 0$, in terms of k.

Solution: $2x^2 + kx + k = 0$

$a = 2, b = k \; and \; c = k$

$\Delta = b^2 - 4ac$

$= k^2 - 4(2)(k)$

$= k^2 - 8k$

$= k(k - 8)$

(i) When $k < 0$, the polynomial has real roots.

(ii) When $k = 0$ or $k = 8$, the roots are real and equal.

(iii) When $0 < k < 8$, roots are imaginary.

(iv) When $k > 8$, roots are real and distinct.

2. Find a polynomial equation of minimum degree with rational coefficients, having $2 + \sqrt{3}i$ as a root.

Solution: Given $2 + \sqrt{3}i$ is a root

$\therefore 2 - \sqrt{3}i$ is another root.

Hence sum of the roots = $2 + \sqrt{3}i + 2 - \sqrt{3}i$

= 4

Product of the roots = $(2 + \sqrt{3}i)(2 - \sqrt{3}i)$

= 4 + 3

= 7

Hence the required equation is

$x^2 - (sum\ of\ roots)x + Prod.of\ roots = 0$

$x^2 - 4x + 7 = 0$

3. Find a polynomial equation of minimum degree with rational coefficients, having $2i + 3$ as a root.

Solution: Given $2i + 3 \Rightarrow 3 + 2i$ is a root

$\therefore 3 - 2i$ is another root.

Hence sum of the roots = $3 + 2i + 3 - 2i$

= 6

Product of the roots = $(3 + 2i)(3 - 2i)$

= 9 + 4

= 13

Hence the required equation is

$x^2 - (sum\ of\ roots)x + Prod.of\ roots = 0$

$x^2 - 6x + 13 = 0$

4. Find a polynomial equation of minimum degree with rational coefficients, having $\sqrt{5} - \sqrt{3}$ as a root.

Solution: Given $\sqrt{5} - \sqrt{3}$ is a root

$\therefore \sqrt{5} + \sqrt{3}$ is another root.

Hence sum of the roots = $\sqrt{5} - \sqrt{3} + \sqrt{5} + \sqrt{3}$

= $2\sqrt{5}$

Product of the roots = $(\sqrt{5} - \sqrt{3})(\sqrt{5} - \sqrt{3})$

= $5 - 3$

= 2

Hence one of the factor is

$= x^2 - (sum\ of\ roots)x + Prod.of\ roots$

$= x^2 - 2\sqrt{5}x + 2$

To remove square root we include the other factor $(x^2 + 2\sqrt{5}x + 2)$

The required equation is,

$$\left(x^2-2\sqrt{5}x+2\right)\left(x^2+2\sqrt{5}x+2\right)=0$$
$$\left(x^2+2-2\sqrt{5}x\right)\left(x^2+2+2\sqrt{5}x\right)=0$$
$$(x^2+2)^2-\left(2\sqrt{5}x\right)^2=0$$
$$x^4+4x^2+4-4(5)x^2=0$$
$$x^4+4x^2+4-20x^2=0$$
$$x^4-16x^2+4=0$$

5. Prove that a straight line and parabola cannot intersect at more than two points.

Solution:

Let the equation of the line be $y=mx+c$

Equation of parabola is $y^2=4ax$ ....(1)

Substituting the value of $y=mx+c$ in....(1)

$$(mx+c)^2=4ax$$
$$m^2x^2+2mcx+c^2=4ax$$

This is a second degree equation in x the straight line and parabola intersect at two points

Example 3.15

If $2+i$ and $3-\sqrt{2}$ are roots of the equation $x^6-13x^5+62x^4-126x^3+65x^2+127x-140=0$, find all roots.

Solution:

Since the coefficient of the equations are all rational numbers, and $2+i$ and $3-\sqrt{2}$ are roots, we get $2-i$ and $3+\sqrt{2}$ are also roots of the given equation. Thus, $[x-(2+i)]$, $[x-(2-i)]$, $\left[x-\left(3-\sqrt{2}\right)\right]$ and $\left[x-\left(3+\sqrt{2}\right)\right]$ are the factors.

Their products,

$$(x-2-i)(x-2+i)\left(x-3+\sqrt{2}\right)\left(x-3-\sqrt{2}\right)$$

is a factor of the polynomial equation.

$$[(x-2)^2+1][(x-3)^2-2]$$
$$[(x^2-4x+4)+1][(x^2-6x+9)-2]$$
$$(x^2-4x+5)(x^2-6x+7)$$
$$x^4-6x^3+7x^2-4x^3+24x^2-28x+5x^2-30x+35$$

$(x^4-10x^3+36x^2-58x+35)$ is a factor.

Hence,

$$(x^6-13x^5+62x^4-126x^3+65x^2+127x-140)$$
$$=(x^4-10x^3+36x^2-58x+35)(x^2+px-4)$$

Equating the x term,

$$127=232+35p$$
$$127-232=35p$$
$$-105=35p$$
$$p=\frac{-105}{35}$$
$$p=-3$$

$x^2-3x-4=0$ is another factor.

$$(x-4)(x+1)=0$$
$$x-4=0 \quad or \quad x+1=0$$
$$x-4=0\Rightarrow x=4$$
$$x+1=0\Rightarrow x=-1$$

The roots of the given polynomial equation are $2+i, 2-i, 3-\sqrt{2}, 3+\sqrt{2}, 4$ and -1.

Example 3.16

Solve the equation $x^4-9x^2+20=0$.

Solution: $x^4-9x^2+20=0$.

Let $x^2=A$

Then, $A^2-9A+20=0$.

$$(A-4)(A-5)=0$$
$$(i)\ When, A-4=0$$
$$A=4$$
$$ie.,\ x^2=4$$
$$x=\pm 2$$
$$(ii)\ When, A-5=0$$
$$A=5$$
$$ie.,\ x^2=5$$
$$x=\pm\sqrt{5}$$

The solutions are, $\pm 2, \pm\sqrt{5}$.

Example 3.17

Solve the equation $x^3 - 3x^2 - 33x + 35 = 0$.

Solution:

The sum of the coefficients of the equations is 0

Hence 1 is a root of the polynomial.

$(x-1)$ is a factor.

$x^3 - 3x^2 - 33x + 35 = (x-1)(x^2 + px - 35)$

Equating the x term,

$$-33 = -35 - p$$
$$-33 + 35 = -p$$
$$2 = -p$$
$$p = -2$$

$x^2 - 2x - 35 = 0$ is a factor.

$$(x-7)(x+5) = 0$$
$$x - 7 = 0 \quad or \quad x + 5 = 0$$
$$x - 7 = 0 \Rightarrow x = 7$$
$$x + 5 = 0 \Rightarrow x = -5$$

The roots are $1, 7, -5$

Example 3.18

Solve the equation $2x^3 + 11x^2 - 9x - 18 = 0$.

Solution:

We observe that the sum of the coefficients of the odd powers and that of the even powers are equal. Hence -1 is a root.

$(x+1)$ is a factor.

$$2x^3 + 11x^2 - 9x - 18$$
$$= (x+1)(2x^2 + px - 18)$$

Equating the x term,

$$-9 = -18 + p$$
$$-9 + 18 = p$$
$$9 = p$$

$2x^2 + 9x - 18 = 0$ is another factor.

$$2x^2 + 12x - 3x - 18 = 0$$
$$2x(x+6) - 3(x+6) = 0$$
$$(2x-3)(x+6) = 0$$
$$2x - 3 = 0 \quad or \quad x + 6 = 0$$
$$2x - 3 = 0 \Rightarrow 2x = 3 \Rightarrow x = \frac{3}{2}$$
$$x + 6 = 0 \Rightarrow x = -6$$

The roots are $-1, \frac{3}{2}, -6$

Example 3.19 Obtain the condition that the roots of $x^3 + px^2 + qx + r = 0$ are in A.P.

Solution:

The roots are in A.P. Then, we can assume them in the form $\alpha - d, \alpha, \alpha + d$.

Applying the Vieta's formula

$$\alpha - d + \alpha + \alpha + d = -\frac{p}{1} = -p$$
$$\Rightarrow 3\alpha = -p$$
$$\Rightarrow \alpha = -\frac{p}{3}.$$

But, we note that α is a root of the given equation. Therefore, we get

$$\left(-\frac{p}{3}\right)^3 + p\left(-\frac{p}{3}\right)^2 + q\left(-\frac{p}{3}\right) + r = 0$$
$$-\frac{p^3}{27} + p\left(\frac{p^2}{9}\right) - \frac{qp}{3} + r = 0$$
$$-\frac{p^3}{27} + \frac{p^3}{9} - \frac{qp}{3} + r = 0$$

Multiplying by 27,

$$-p^3 + 3p^3 - 9pq + 27r = 0$$
$$2p^3 - 9pq + 27r = 0$$
$$2p^3 + 27r = 9pq$$

Example 3.20 Find the condition that the roots of $ax^3 + bx^2 + cx + d = 0$ are in geometric progression. Assume $a, b, c, d \neq 0$

Solution:

The roots are in G.P. Then, we can assume them in the form $\frac{\alpha}{r}, \alpha, \alpha r$.

Applying the Vieta's formula

$$\Sigma_1 = \left(\frac{\alpha}{r} + \alpha + \alpha r\right) = -\frac{b}{a}$$

$$\alpha\left(\frac{1}{r}+1+r\right)=-\frac{b}{a}\ldots\ldots(1)$$

$$\Sigma_2=\left(\frac{\alpha}{r}\alpha+\frac{\alpha}{r}\alpha r+\alpha\alpha r\right)=\frac{c}{a}$$

$$\left(\frac{\alpha^2}{r}+\alpha^2+\alpha^2 r\right)=\frac{c}{a}$$

$$\alpha^2\left(\frac{1}{r}+1+r\right)=\frac{c}{a}\ldots\ldots(2)$$

$$\frac{(2)}{(1)}\Rightarrow\frac{\alpha^2\left(\frac{1}{r}+1+r\right)}{\alpha\left(\frac{1}{r}+1+r\right)}=\frac{\frac{c}{a}}{-\frac{b}{a}}$$

$$\frac{(2)}{(1)}\Rightarrow\alpha=\frac{c}{a}\times\left(-\frac{a}{b}\right)$$

$$\therefore\alpha=-\frac{c}{b}$$

$$\Sigma_3\left(\frac{\alpha}{r}\right)(\alpha)(\alpha r)=-\frac{d}{a}$$

$$\alpha^3=-\frac{d}{a}$$

Substituting $\alpha=-\frac{c}{b}$ in $\alpha^3=-\frac{d}{a}$

$$\left(-\frac{c}{b}\right)^3=-\frac{d}{a}$$

$$-\frac{c^3}{b^3}=-\frac{d}{a}$$

$$\frac{c^3}{b^3}=\frac{d}{a}$$

$$ac^3=b^3d$$

Example 3.21

If the roots of $x^3+px^2+qx+r=0$ are in H.P. prove that $9pqr=27r^3+2p$.

Solution:

The roots of $x^3+px^2+qx+r=0$ are in H.P.

Then the reciprocal roots are in AP.

$$\frac{1}{x^3}+p\frac{1}{x^2}+q\frac{1}{x}+r=0$$

$$\frac{1+px+qx^2+rx^3}{x^3}=0$$

$$1+px+qx^2+rx^3=0$$

$rx^3+qx^2+px+1=0\ldots..(1)$ is the equation has AP roots

Let $a-d, a\ a+d$ be the roots.

$$\Sigma_1=(a-d+a+\ a+d)=-\frac{q}{r}$$

$$3a=-\frac{q}{r}$$

$$a=-\frac{q}{3r}$$

Since a is the root of the equation(1)

$$r\left(-\frac{q}{3r}\right)^3+q\left(-\frac{q}{3r}\right)^2+p\left(-\frac{q}{3r}\right)+1=0$$

$$r\left(-\frac{q^3}{27r^3}\right)+q\left(\frac{q^2}{9r^2}\right)-\frac{pq}{3r}+1=0$$

$$\left(-\frac{q^3}{27r^2}\right)+q\left(\frac{q^2}{9r^2}\right)-\frac{pq}{3r}+1=0$$

Multiplying by$27r^2$,

$$-q^3+3q^3-pq9r+27r^2=0$$

$$2q^3-pq9r+27r^2=0$$

$$2q^3+27r^2=9pqr$$

Example 3.22 It is known that the roots of the equation $x^3-6x^2-4x+24=0$ are in arithmetic progression. Find its roots.

Solution: $x^3-6x^2-4x+24=0$has AP roots.

The roots are in A.P. Then, we can assume them in the form $\alpha-d,\alpha,\alpha+d$.

Applying the Vieta's formula

$$\alpha-d+\alpha+\ \alpha+d=-\frac{(-6)}{1}=6$$

$$\Rightarrow 3\alpha=6$$

$$\Rightarrow\alpha=\frac{6}{3}=2$$

$$(\alpha-d)(\alpha)\,(\alpha+d)=-\frac{24}{1}=-24$$

Substituting $\alpha=2$,

$$(2-d)(2)\,(2+d)=-24$$

$$(2-d)\,(2+d)=-12$$

$$4-d^2=-12$$

$$-d^2=-12-4$$

$$d^2=16$$

$$d=\pm 4$$

(i) $When\ \alpha = 2\ and\ d = 4$

The roots $a-d, a\ a+d$ becomes,

$2-4, 2, 2+4 \Rightarrow -2, 2, 6$

(ii) $When\ \alpha = 2\ and\ d = -4$

The roots $a-d, a\ a+d$ becomes,

$2+4, 2, 2-4 \Rightarrow 6, 2, -2$

EXERCISE 3.3

1. Solve the cubic equation:

$2x^3 - x^2 - 18x + 9 = 0$ if sum of two of its roots vanishes.

Solution:

The sum of the two roots is zero.

So, $\alpha, -\alpha, \beta$ be the roots.

$\sum_1(\alpha - \alpha + \beta) = -\frac{b}{a}$

$= -\frac{(-1)}{2}$

$\beta = \frac{1}{2}$

$\sum_3(\alpha)(-\alpha)(\beta) = -\frac{d}{a}$

$(\alpha)(-\alpha)\left(\frac{1}{2}\right) = -\frac{9}{2}$

$\frac{-\alpha^2}{2} = -\frac{9}{2}$

$\alpha^2 = 9$

$\alpha = \pm 3$

Hence $3, -3, \frac{1}{2}$ are the roots.

2. Solve the equation

$9x^3 - 36x^2 + 44x - 16 = 0$ if the roots form an arithmetic progression.

Solution:

The roots are in A.P. Then, we can assume them in the form $\alpha - d, \alpha, \alpha + d$.

Applying the Vieta's formula

$\alpha - d + \alpha + \alpha + d = -\frac{b}{a}$

$\Rightarrow 3\alpha = -\frac{(-36)}{9} = \frac{36}{9}$

$\Rightarrow 3\alpha = 4$

$\alpha = \frac{4}{3}$

$(\alpha - d)(\alpha)(\alpha + d) = -\frac{d}{a}$

$(\alpha - d)\left(\frac{4}{3}\right)(\alpha + d) = -\frac{(-16)}{9}$

$(\alpha - d)(\alpha + d) = \frac{16}{9}\left(\frac{3}{4}\right)$

$\alpha^2 - d^2 = \frac{4}{3}$

$\left(\frac{4}{3}\right)^2 - d^2 = \frac{4}{3}$

$\frac{16}{9} - d^2 = \frac{4}{3}$

$\frac{16}{9} - \frac{4}{3} = d^2$

$d^2 = \frac{16-12}{9}$

$= \frac{4}{9}$

$d = \pm\frac{2}{3}$

(i) $When\ \alpha = \frac{4}{3}\ and\ d = \frac{2}{3}$

The roots $a-d, a\ a+d$ becomes,

$\frac{4}{3} - \frac{2}{3}, \frac{4}{3}, \frac{4}{3} + \frac{2}{3} \Rightarrow \frac{2}{3}, \frac{4}{3}, 2$

(ii) $When\ \alpha = \frac{4}{3}\ and\ d = -\frac{2}{3}$

The roots $a-d, a\ a+d$ becomes,

$\frac{4}{3} + \frac{2}{3}, \frac{4}{3}, \frac{4}{3} - \frac{2}{3} \Rightarrow 2, \frac{4}{3} \frac{2}{3}$

3. Solve the equation

$3x^3 - 26x^2 + 52x - 24 = 0$ if its roots form a geometric progression.

Solution:

The roots are in G.P. Then, we can assume them in the form $\frac{\alpha}{r}, \alpha, \alpha r$.

Applying the Vieta's formula

$\sum_1 = \left(\frac{\alpha}{r} + \alpha + \alpha r\right) = -\frac{b}{a}$

$\frac{\alpha}{r} + \alpha + \alpha r = -\frac{(-26)}{3} = \frac{26}{3}$

$\sum_3 = \left(\frac{\alpha}{r}\right)(\alpha)(\alpha r) = -\frac{d}{a}$

$\left(\frac{\alpha}{r}\right)(\alpha)(\alpha r) = -\frac{(-24)}{3} = \frac{24}{3}$

$\alpha^3 = 8$

$\alpha = 2$

Substituting $\alpha = 2$, in

$\frac{\alpha}{r} + \alpha + \alpha r = \frac{26}{3}$

$\frac{2}{r} + 2 + 2r = \frac{26}{3}$

$\frac{2}{r} + 2r = \frac{26}{3} - 2$

$\frac{2}{r} + 2r = \frac{26-6}{3}$

$\frac{2+2r^2}{r} = \frac{20}{3}$

$3(2 + 2r^2) = 20r$

$6 + 6r^2 = 20r$

$6r^2 - 20r + 6 = 0$

$\div 2, \quad 3r^2 - 10r + 3 = 0$

$(3r - 1)(r - 3) = 0$

$3r - 1 = 0 \quad or \quad r - 3 = 0$

$3r - 1 = 0 \Rightarrow 3r = 1 \Rightarrow r = \frac{1}{3}$

$r - 3 = 0 \Rightarrow r = 3$

(i) *When* $\alpha = 2$ *and* $r = \frac{1}{3}$

The roots $\frac{\alpha}{r}, \alpha, \alpha r$ becomes $6, 2, \frac{1}{3}$

(ii) *When* $\alpha = 2$ *and* $r = 3$

The roots $\frac{\alpha}{r}, \alpha, \alpha r$ becomes $\frac{1}{3}, 2, 6$

4. Determine k and solve the equation

$2x^3 - 6x^2 + 3x + k = 0$ if one of its roots is twice the sum of the other two roots.

Solution:

Given $2x^3 - 6x^2 + 3x + k = 0$

Let α, β, γ be the roots.

$\alpha = 2(\beta + \gamma) \Rightarrow \alpha = 2\beta + 2\gamma$

$\Sigma_1 = (\alpha + \beta + \gamma) = -\frac{b}{a}$

$\alpha + \beta + \gamma = -\frac{(-6)}{2} = \frac{6}{2} = 3$

$\therefore 2\alpha + 2\beta + 2\gamma = 6$

$2\alpha + \alpha = 6$

$3\alpha = 6$

$\alpha = 2$

$\Sigma_2 = (\alpha\beta + \alpha\gamma + \beta\gamma) = \frac{c}{a}$

$\alpha\beta + \alpha\gamma + \beta\gamma = \frac{3}{2}$

Substituting $\alpha = 2$

$2\beta + 2\gamma + \beta\gamma = \frac{3}{2}$

$\alpha + \beta\gamma = \frac{3}{2}$

$2 + \beta\gamma = \frac{3}{2}$

$\beta\gamma = \frac{3}{2} - 2$

$= \frac{3-4}{2}$

$\beta\gamma = -\frac{1}{2}$

$\Sigma_3 = (\alpha\beta\gamma) = -\frac{d}{a}$

$2\beta\gamma = -\frac{k}{2}$

$\therefore \beta\gamma = -\frac{k}{4}$

Hence $-\frac{1}{2} = -\frac{k}{4}$

$\frac{1}{2} = \frac{k}{4}$

$k = \frac{4}{2}$

$k = 2$

From, $\beta\gamma = -\frac{1}{2}$

$\gamma = -\frac{1}{2\beta}$

Substituting $\alpha = 2$ and $\gamma = -\frac{1}{2\beta}$ in

$\alpha + \beta + \gamma = 3$

$2 + \beta - \frac{1}{2\beta} = 3$

$\beta - \frac{1}{2\beta} = 3 - 2$

$$\beta - \frac{1}{2\beta} = 1$$

$$\frac{2\beta^2 - 1}{2\beta} = 1$$

$$2\beta^2 - 1 = 2\beta$$

$$2\beta^2 - 2\beta - 1 = 0$$

$$\beta = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{2 \pm \sqrt{4 - 4(2)(-1)}}{2(2)}$$

$$= \frac{2 \pm \sqrt{4+8}}{4}$$

$$= \frac{2 \pm \sqrt{12}}{4}$$

$$= \frac{2 \pm \sqrt{4 \times 3}}{4}$$

$$= \frac{2 \pm 2\sqrt{3}}{4}$$

$$= \frac{2(1 \pm \sqrt{3})}{4}$$

$$= \frac{(1 \pm \sqrt{3})}{2}$$

$k = 2$, and the roots are $2, \frac{(1 \pm \sqrt{3})}{2}$.

5. Find all zeros of the polynomial $x^6 - 3x^5 - 5x^4 + 22x^3 - 39x^2 - 39x + 135$, if it known that $1+2i$ and $\sqrt{3}$ are two of its zeros.

Solution:

Since $1+2i$ and $\sqrt{3}$ are two of polynomial zeros, we get $1 - 2i$ and $-\sqrt{3}$ are also zeros of the given equation. Thus, $[x - (1 + 2i)]$, $[x - (1 - 2i)]$, $[x - (\sqrt{3})]$ and $[x - (-\sqrt{3})]$ are the factors.

Their products,

$(x - 1 - 2i)(x - 1 + 2i)(x - \sqrt{3})(x + \sqrt{3})$

is a factor of the polynomial equation.

$$[(x-1)^2 + 4][(x)^2 - 3]$$

$$[(x^2 - 2x + 1) + 4][(x^2 - 3)]$$

$$(x^2 - 2x + 5)(x^2 - 3)$$

$$x^4 - 3x^2 - 2x^3 + 6x + 5x^2 - 15$$

$(x^4 - 2x^3 + 2x^2 + 6x - 15)$ is a factor.

Hence,

$$x^6 - 3x^5 - 5x^4 + 22x^3 - 39x^2 - 39x + 135$$

$$= (x^4 - 2x^3 + 2x^2 + 6x - 15)(x^2 + px - 9)$$

Equating the x term,

$$-39 = -54 - 15p$$

$$-39 + 54 = -15p$$

$$15 = -15p$$

$$p = \frac{-15}{15}$$

$$p = -1$$

$x^2 - x - 9 = 0$ is another factor.

$$x^2 - x - 9 = 0$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{1 \pm \sqrt{1 - 4(1)(-9)}}{2(1)}$$

$$= \frac{1 \pm \sqrt{1+36}}{2}$$

$$= \frac{1 \pm \sqrt{37}}{2}$$

The zeros of the given polynomial equation are

$1 + 2i, 1 - 2i, \sqrt{3}, -\sqrt{3}, \frac{1 + \sqrt{37}}{2}$ and $\frac{1 - \sqrt{37}}{2}$

6. Solve the cubic equations:

(i) $2x^3 - 9x^2 + 10x = 3$

Solution:

$$2x^3 - 9x^2 + 10x = 3$$

$$2x^3 - 9x^2 + 10x - 3 = 0$$

The sum of the coefficients of the equations is 0

Hence 1 is a root of the polynomial.

$(x - 1)$ is a factor.

$$2x^3 - 9x^2 + 10x - 3 = (x - 1)(2x^2 + px + 3)$$

Equating the x term,

$$10 = 3 - p$$

$$10 - 3 = -p$$

$$7 = -p$$

$$p = -7$$

$2x^2 - 7x + 3 = 0$ is a factor.

$$2x^2 - 6x - x + 3 = 0$$

$$2x(x-3) - 1(x-3) = 0$$

$$(2x-1)(x-3) = 0$$

$$2x - 1 = 0 \quad or \quad x - 3 = 0$$

$$2x - 1 = 0 \Rightarrow x = \frac{1}{2}$$

$$x - 3 = 0 \Rightarrow x = 3$$

The roots are 1, 3, $\frac{1}{2}$

(ii) $8x^3 - 2x^2 - 7x + 3 = 0$

Solution:

We observe that the sum of the coefficients of the odd powers and that of the even powers are equal. Hence -1 is a root.

$(x+1)$ is a factor.

$8x^3 - 2x^2 - 7x + 3$

$$= (x+1)(8x^2 + px + 3)$$

Equating the x term,

$$-7 = 3 + p$$

$$-7 - 3 = p$$

$$-10 = p$$

$8x^2 - 10x + 3 = 0$ is another factor.

$$8x^2 - 6x - 4x + 3 = 0$$

$$2x(4x-3) - 1(4x-3) = 0$$

$$(2x-1)(4x-3) = 0$$

$$2x - 1 = 0 \quad or \quad 4x - 3 = 0$$

$$2x - 1 = 0 \Rightarrow x = \frac{1}{2}$$

$$4x - 3 = 0 \Rightarrow x = \frac{3}{4}$$

The roots are 1, $\frac{1}{2}$, $\frac{3}{4}$

7. Solve the equation: $x^4 - 14x^2 + 45 = 0.$

Solution: $x^4 - 14x^2 + 45 = 0.$

Let $x^2 = A$

Then, $A^2 - 14A + 45 = 0.$

$$(A-9)(A-5) = 0$$

$$(i)\ When, A - 9 = 0$$

$$A = 9$$

$$ie., \ x^2 = 9$$

$$x = \pm 3$$

$$(ii)\ When, A - 5 = 0$$

$$A = 5$$

$$ie., \ x^2 = 5$$

$$x = \pm\sqrt{5}$$

The solutions are, ± 3 , $\pm\sqrt{5}$.

Example 3.23 Solve the equation

$(x-2)(x-7)(x-3)(x+2) + 19 = 0$

Solution:

$$(x-2)(x-7)(x-3)(x+2) + 19 = 0$$

$$(x-2)(x-3)(x-7)(x+2) + 19 = 0$$

$$(x^2 - 3x - 2x + 6)(x^2 + 2x - 7x - 14) + 19 = 0$$

$$(x^2 - 5x + 6)(x^2 - 5x - 14) + 19 = 0$$

Let $x^2 - 5x = A$

Then, $(A+6)(A-14) + 19 = 0$

$$A^2 - 14A + 6A - 84 + 19 = 0$$

$$A^2 - 8A - 65 = 0$$

$$A = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{8 \pm \sqrt{64 - 4(1)(-65)}}{2(1)}$$

$$= \frac{8 \pm \sqrt{64 + 260}}{2}$$

$$= \frac{8 \pm \sqrt{324}}{2}$$

$$= \frac{8 \pm 18}{2}$$

$A = \frac{8+18}{2}$ and $A = \frac{8-18}{2}$

(i) When $A = \frac{8+18}{2}$

$= \frac{26}{2}$

$= 13$

Hence, Let $x^2 - 5x = 13$

$x^2 - 5x - 13 = 0$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{5 \pm \sqrt{25 - 4(1)(-13)}}{2(1)}$$

$$= \frac{5 \pm \sqrt{25+52}}{2}$$

$$= \frac{5 \pm \sqrt{77}}{2}$$

(ii) When $A = \frac{8-18}{2}$

$= -\frac{10}{2}$

$= -5$

Hence, Let $x^2 - 5x = -5$

$x^2 - 5x + 5 = 0$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{5 \pm \sqrt{25 - 4(1)(5)}}{2(1)}$$

$$= \frac{5 \pm \sqrt{25-20}}{2}$$

$$= \frac{5 \pm \sqrt{5}}{2}$$

The solutions are $\frac{5 \pm \sqrt{77}}{2}, \frac{5 \pm \sqrt{5}}{2}$

Example 3.24 Solve the equation

$(2x-3)(6x-1)(3x-2)(x-12) - 7 = 0$

<u>Solution:</u>

$(2x-3)(6x-1)(3x-2)(x-12) - 7 = 0$

$(2x-3)(3x-2)(6x-1)(x-12) - 7 = 0$

$(6x^2 - 4x - 9x + 6)(6x^2 - 12x - x + 12) - 7 = 0$

$(6x^2 - 13x + 6)(6x^2 - 13x + 12) - 7 = 0$

Let $6x^2 - 13x = A$

Then, $(A+6)(A+12) - 7 = 0$

$A^2 + 12A + 6A + 72 - 7 = 0$

$A^2 + 18A + 65 = 0$

$A^2 + 13A + 5A + 65 = 0$

$A(A+13) + 5(A+13) = 0$

$(A+13)(A+5) = 0$

$(A+13) = 0 \Rightarrow A = -13$

and $(A+5) = 0 \Rightarrow A = -5$

(i) When $A = -13$

Hence, $6x^2 - 13x = -13$

$6x^2 - 13x + 13 = 0$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{13 \pm \sqrt{169 - 4(6)(13)}}{2(6)}$$

$$= \frac{13 \pm \sqrt{169 - 312}}{12}$$

$$= \frac{13 \pm \sqrt{-143}}{12}$$

$$= \frac{13 \pm \sqrt{143}i}{12}$$

(ii) When $A = -5$

Hence, $6x^2 - 13x = -5$

$6x^2 - 13x + 5 = 0$

$6x^2 - 10x - 3x + 5 = 0$

$2x(3x-5) - 1(3x-5) = 0$

$(2x-1)(3x-5) = 0$

$2x - 1 = 0$ *or* $4x - 3 = 0$

$2x - 1 = 0 \Rightarrow x = \frac{1}{2}$

$3x - 5 = 0 \Rightarrow x = \frac{5}{3}$

The roots are $\frac{13 \pm \sqrt{143}i}{12}, \frac{1}{2}, \frac{5}{3}$

EXERCISE 3.4

1. Solve : (i) $(x-5)(x-7)(x+6)(x+4)=504$

Solution:

$$(x-5)(x-7)(x+6)(x+4)=504$$
$$(x-5)(x+4)(x-7)(x+6)=504$$
$$(x^2+4x-5x-20)(x^2+6x-7x-42)=504$$
$$(x^2-x-20)(x^2-x-42)-504=0$$

Let $x^2-x=A$

Then, $(A-20)(A-42)-504=0$

$$A^2-42A-20A+840-504=0$$
$$A^2-62A+336=0$$
$$A^2-56A-6A+336=0$$
$$A(A-56)-6(A-56)=0$$
$$(A-56)(A-6)=0$$
$$(A-56)=0\Rightarrow A=56$$

and $(A-6)=0\Rightarrow A=6$

(i) When $A=56$

Hence, $x^2-x=56$

$$x^2-x-56=0$$
$$(x-8)(x+7)=0$$
$$x-8=0 \quad or \quad x+7=0$$
$$x-8=0\Rightarrow x=8$$
$$x+7=0\Rightarrow x=-7$$

(i) When $A=6$

Hence, $x^2-x=6$

$$x^2-x-6=0$$
$$(x-3)(x+2)=0$$
$$x-3=0 \quad or \quad x+2=0$$
$$x-3=0\Rightarrow x=3$$
$$x+2=0\Rightarrow x=-2$$

The roots are $x=8,-7,3,-2$

(ii) $(x-4)(x-7)(x-2)(x+1)=16.$

Solution:

$$(x-4)(x-7)(x-2)(x+1)=16$$
$$(x-4)(x-2)(x-7)(x+1)=16$$
$$(x^2-2x-4x+8)(x^2+x-7x-7)=16$$
$$(x^2-6x+8)(x^2-6x-7)-16=0$$

Let $x^2-6x=A$

Then, $(A+8)(A-7)-16=0$

$$(A^2-7A+8A-56)-16=0$$
$$A^2+A-72=0$$
$$(A+9)(A-8)=0$$
$$(A+9)=0\Rightarrow A=-9$$

and $(A-8)=0\Rightarrow A=8$

(i) When $A=-9$

$$x^2-6x=-9$$
$$x^2-6x+9=0$$
$$(x-3)(x-3)=0$$
$$x=3,3$$

(ii) When $A=8$

$$x^2-6x=8$$
$$x^2-6x-8=0$$
$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$
$$=\frac{6\pm\sqrt{36-4(1)(-8)}}{2(1)}$$
$$=\frac{6\pm\sqrt{36+32}}{2}$$
$$=\frac{6\pm\sqrt{68}}{2}$$
$$=\frac{6\pm\sqrt{4\times17}}{2}$$

$$= \frac{6\pm2\sqrt{17}}{2}$$

$$= \frac{2(3\pm\sqrt{17})}{2}$$

$$= (3 \pm \sqrt{17})$$

The roots are $x = 3, 3, 3 + \sqrt{17}, 3 - \sqrt{17}$

2. Solve : $(2x-1)(x+3)(x-2)(2x+3) + 20 = 0$

<u>Solution:</u>

$$(2x-1)(x+3)(x-2)(2x+3) + 20 = 0$$

$$(2x-1)(2x+3)(x+3)(x-2) + 20 = 0$$

$$(4x^2+6x-2x-3)(x^2-2x+3x-6) + 20 = 0$$

$$(4x^2+4x-3)(x^2+x-6) + 20 = 0$$

$$[4(x^2+x)-3](x^2+x-6) + 20 = 0$$

Let $x^2 + x = A$

Then, $(4A-3)(A-6) + 20 = 0$

$$4A^2 - 24A - 3A + 18 + 20 = 0$$

$$4A^2 - 27A + 38 = 0$$

$$4A^2 - 8A - 19A + 38 = 0$$

$$4A(A-2) - 19(A-2) = 0$$

$$(A-2)(4A-19) = 0$$

$$(A-2) = 0 \Rightarrow A = 2$$

and $\quad (4A-19) = 0 \Rightarrow A = \frac{19}{4}$

(i) When $A = 2$

$$x^2 + x = 2$$

$$x^2 + x - 2 = 0$$

$$(x+2)(x-1) = 0$$

$$(x+2) = 0 \Rightarrow x = -2$$

and $\quad (x-1) = 0 \Rightarrow x = 1$

(ii) When $A = \frac{19}{4}$

$$x^2 + x = \frac{19}{4}$$

$$x^2 + x - \frac{19}{4} = 0$$

Multiplying by 4

$$4x^2 + 4x - 19 = 0$$

$$x = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

$$= \frac{-4\pm\sqrt{16-4(4)(-19)}}{2(4)}$$

$$= \frac{-4\pm\sqrt{16+304}}{8}$$

$$= \frac{-4\pm\sqrt{320}}{8}$$

$$= \frac{-4\pm\sqrt{64\times5}}{8}$$

$$= \frac{-4\pm8\sqrt{5}}{8}$$

$$= \frac{4(-1\pm2\sqrt{5})}{8}$$

$$= \frac{(-1\pm2\sqrt{5})}{2}$$

The roots are $x = 1, -2, \frac{(-1-2\sqrt{5})}{2}, \frac{(-1+2\sqrt{5})}{2}$

Example 3.25

Solve the equation: $x^3 - 5x^2 - 4x + 20 = 0$

<u>Solution:</u> $x^3 - 5x^2 - 4x + 20 = 0$

2 is a solution hence $(x-2)$ is a factor.

$$x^3 - 5x^2 - 4x + 20 = (x-2)(x^2+px-10)$$

Equating the x term,

$$-4 = -10 - 2p$$

$$-4 + 10 = -2p$$

$$6 = -2p$$

$$p = -3$$

$x^2 - 3x - 10 = 0$ is another factor.

$$(x-5)(x+2)=0$$
$$x-5=0 \quad or \quad x+2=0$$
$$x-5=0 \Rightarrow x=5$$
$$x+2=0 \Rightarrow x=-2$$

The roots are 2, 5 and−2.

Example 3.26

Find the roots of $2x^3+3x^2+2x+3$

Solution: $2x^3+3x^2+2x+3$

$t_1=a_n=2, and\ last\ term\ t_l=a_0=3$

the root $\frac{p}{q}=\frac{a_0}{a_n}=\frac{3}{2}$

as $(p,q)=1$ p must divide 3 and q must divide 2. Clearly, the possible values of p are $-1,1,-3,3$ and the possible values of q are $-1,1,-2,2$

We find $-\frac{3}{2}$ is the root.

$ie.,(2x+3)$ is a factor.

$$2x^3+3x^2+2x+3=(2x+3)(x^2+px+1)$$

Equating the x term,

$$2=2+3p$$
$$2-2=3p$$
$$0=3p$$
$$p=0$$

$x^2+1=0$ is another factor.

$x^2+1=0$ gives,

$$x^2=-1$$
$$x=\pm i$$

$-\frac{3}{2}, -i, i$ are the roots.

Example 3.27

Solve the equation $7x^3-43x^2=43x-7$

Solution:

$$7x^3-43x^2=43x-7$$
$$7x^3-43x^2-43x+7=0$$

This is an odd degree reciprocal equation of Type I. Thus -1 is a solution and hence $(x+1)$ is a factor.

$$7x^3-43x^2-43x+7=(x+1)(7x^2+px+7)$$

Equating the x term,

$$-43=7+p$$
$$-43-7=p$$
$$-50=p$$
$$p=-50$$

$7x^2-50x+7=0$ is another factor.

$$7x^2-49x-x+7=0$$
$$7x(x-7)-1(x-7)=0$$
$$(x-7)(7x-1)=0$$
$$(x-7)=0 \Rightarrow x=7$$
$$(7x-1)=0 \Rightarrow x=\frac{1}{7}$$

Hence, $x=-1,\frac{1}{7}$ *and* 7 *solution.*

Example 3.28 Solve the following equation:

$x^4-10x^3+26x^2-10x+1=0$

Solution: $x^4-10x^3+26x^2-10x+1=0$

This equation is Type I even degree reciprocal equation. Hence it can be rewritten as

$$x^2\left[x^2-10x+26-\frac{10}{x}+\frac{1}{x^2}\right]=0$$
$$\left[x^2+\frac{1}{x^2}-10x-\frac{10}{x}+26\right]=0$$
$$\left[\left(x+\frac{1}{x}\right)^2-2-10\left(x+\frac{1}{x}\right)+26\right]=0$$
$$\left[\left(x+\frac{1}{x}\right)^2-10\left(x+\frac{1}{x}\right)+24\right]=0$$

Let $x+\frac{1}{x}=A$

then, $A^2-10A+24=0$

$$(A-6)(A-4)=0$$
$$(A-6)=0 \Rightarrow A=6$$

and $(A-4)=0 \Rightarrow A=4$

(i) When $A = 6$

$$x + \frac{1}{x} = 6$$

$$\frac{x^2+1}{x} = 6$$

$$x^2 + 1 = 6x$$

$$x^2 - 6x + 1 = 0$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{6 \pm \sqrt{36 - 4(1)(1)}}{2(1)}$$

$$= \frac{6 \pm \sqrt{36 - 4}}{2}$$

$$= \frac{6 \pm \sqrt{32}}{2}$$

$$= \frac{6 \pm \sqrt{16 \times 2}}{2}$$

$$= \frac{6 \pm 4\sqrt{2}}{2}$$

$$= \frac{2(3 \pm 2\sqrt{2})}{2}$$

$$= 3 \pm 2\sqrt{2}$$

(ii) When $A = 4$

$$x + \frac{1}{x} = 4$$

$$\frac{x^2+1}{x} = 4$$

$$x^2 + 1 = 4x$$

$$x^2 - 4x + 1 = 0$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{4 \pm \sqrt{16 - 4(1)(1)}}{2(1)}$$

$$= \frac{4 \pm \sqrt{16 - 4}}{2}$$

$$= \frac{4 \pm \sqrt{12}}{2}$$

$$= \frac{4 \pm \sqrt{4 \times 3}}{2}$$

$$= \frac{4 \pm 2\sqrt{3}}{2}$$

$$= \frac{2(2 \pm \sqrt{3})}{2}$$

$$= 2 \pm \sqrt{3}$$

Hence, $x = (3 \pm 2\sqrt{2})\, and\, (2 \pm \sqrt{3})$

Example 3.29 Find solution, if any, of the equation $2cos^2 x - 9\cos x + 4 = 0$

Solution:

Substitute $cos\ x = A$, then

$$2A^2 - 9A + 4 = 0$$

$$2A^2 - 8A - A + 4 = 0$$

$$2A(A - 4) - 1(A - 4) = 0$$

$$(A - 4)(2A - 1) = 0$$

$$(A - 4) = 0 \Rightarrow A = 4$$

and $(2A - 1) = 0 \Rightarrow A = \frac{1}{2}$

(i) When $A = 4$

$cos\ x = A$ gives,

$cos\ x = 4$

There is no possible solution for $cos\ x = 4$

(ii) When $A = \frac{1}{2}$

$cos\ x = A$ gives,

$cos\ x = \frac{1}{2}$

for all $n \in Z, x = 2n\pi \pm \frac{\pi}{3}$ are solutions, but repeat the steps by taking the given equation, we find that there is no solution.

EXERCISE 3.5

1. Solve the following equations

(i) $sin^2 x - 5\sin x + 4 = 0$

Solution:

Substitute $sin\, x = A$, then

$$A^2 - 5A + 4 = 0$$

$$(A-4)(A-1) = 0$$

$$(A-4) = 0 \Rightarrow A = 4$$

and $(A-1) = 0 \Rightarrow A = 1$

(i) When $A = 4$

$sin\, x = A$ gives,

$sin\, x = 4$

There is no possible solution

for $sin\, x = 4$ since the range is $[-1,1]$

(ii) When $A = 1$

$sin\, x = A$ gives,

$sin\, x = 1$

for all $n \in Z, x = 2n\pi \pm \frac{\pi}{2}$ is solution.

(ii) $12x^3 + 8x = 29x^2 - 4$

Solution: $12x^3 + 8x = 29x^2 - 4$

$$12x^3 - 29x^2 + 8x + 4 = 0$$

2 is a solution hence $(x-2)$ is a factor.

$$12x^3 - 29x^2 + 8x + 4 = (x-2)(12x^2 + px - 2)$$

Equating the x term,

$$8 = -2 - 2p$$

$$8 + 2 = -2p$$

$$10 = -2p$$

$$p = -5$$

$12x^2 - 5x - 2 = 0$ is another factor.

$$12x^2 - 5x - 2 = 0$$

$$12x^2 - 8x + 3x - 2 = 0$$

$$4x(3x-2) + 1(3x-2) = 0$$

$$(3x-2)(4x+1) = 0$$

$$(3x-2) = 0 \Rightarrow x = \frac{2}{3}$$

and $(4x+1) = 0 \Rightarrow x = -\frac{1}{4}$

The roots are 2, $\frac{2}{3}, -\frac{1}{4}$

2. Examine for the rational roots of

(i) $2x^3 - x^2 - 1$

Solution: $2x^3 - x^2 - 1$

The sum of the coefficients of the

equations is 0

Hence 1 is a root of the polynomial.

$(x-1)$ is a factor.

$$2x^3 - x^2 - 1 \quad = (x-1)(2x^2 + px + 1)$$

Equating the x term,

$$0 = 1 - p$$

$$1 - p = 0$$

$$p = 1$$

$2x^2 + x + 1$ is a factor.

$$2x^2 + x + 1 = 0$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-1 \pm \sqrt{1 - 4(2)(1)}}{2(2)}$$

$$= \frac{-1 \pm \sqrt{1 - 8}}{4}$$

$$= \frac{-1 \pm \sqrt{-7}}{4}$$

$$= \frac{-1 \pm \sqrt{7}\, i}{4}$$

$\therefore x = 1$, is a rational root.

(ii) $x^8 - 3x + 1 = 0$.

Solution: $x^8 - 3x + 1 = 0$

$t_1 = a_n = 1$, *and last term* $t_l = a_0 = 1$

the root $\frac{p}{q} = \frac{1}{1} = 1$, then as $(p\,, q\,) = 1$

p is a factor of $a_0 = 1$

and q is a factor of $a_n = 1$

Since 1 has no factor, the given equation has no rational roots.

3. Solve: $8x^{\frac{3}{2n}} - 8x^{\frac{-3}{2n}} = 63$

Solution:

$$8x^{\frac{3}{2n}} - 8x^{\frac{-3}{2n}} = 63$$

$$8\left(x^{\frac{3}{2n}} - x^{\frac{-3}{2n}}\right) = 63$$

$$8\left[\left(x^{\frac{1}{2n}}\right)^3 - \left(x^{\frac{-1}{2n}}\right)^3\right] = 63$$

Let $x^{\frac{1}{2n}} = A$, then $x^{\frac{-1}{2n}} = \frac{1}{A}$

$$8\left[(A)^3 - \left(\frac{1}{A}\right)^3\right] = 63$$

$$8\left[A^3 - \frac{1}{A^3}\right] = 63$$

$$A^3 - \frac{1}{A^3} = \frac{63}{8}$$

$$\frac{A^6 - 1}{A^3} = \frac{63}{8}$$

$$8\,(A^6 - 1) = 63A^3$$

$$8A^6 - 8 = 63A^3$$

$$8A^6 - 63A^3 - 8 = 0$$

$$8A^6 - 64A^3 + A^3 - 8 = 0$$

$$8A^3(A^3 - 8) + 1(A^3 - 8) = 0$$

$$(A^3 - 8)(8A^3 + 1) = 0$$

$$A^3 - 8 = 0 \; and \; 8A^3 + 1 = 0$$

$$A^3 - 8 = 0$$

$$\Rightarrow A^3 = 8$$

$$\Rightarrow A = 2$$

and $8A^3 + 1 = 0$

$$\Rightarrow 8A^3 = -1$$

$$\Rightarrow A^3 = -\frac{1}{8}$$

$$\Rightarrow A = -\frac{1}{2}$$

(i) When A = 2

$$x^{\frac{1}{2n}} = A, gives$$

$$x^{\frac{1}{2n}} = 2$$

Raising to the power of $2n$

$$x = 2^{2n}$$

$$= (2^2)^n$$

$$\therefore x = 4^n$$

(ii) When $A = -\frac{1}{2}$

$$x^{\frac{1}{2n}} = A, gives$$

$$x^{\frac{1}{2n}} = -\frac{1}{2}$$

Raising to the power of $2n$

$$x = -\frac{1}{2^{2n}}$$

$$= -\frac{1}{(4)^n}$$

$$\therefore x = -\frac{1}{4^n}$$

4. Solve: $2\sqrt{\frac{x}{a}} + 3\sqrt{\frac{a}{x}} = \frac{b}{a} + \frac{6a}{b}$

Solution: $2\sqrt{\frac{x}{a}} + 3\sqrt{\frac{a}{x}} = \frac{b}{a} + \frac{6a}{b}$

Let $\sqrt{\frac{x}{a}} = A$

$$2A + 3\frac{1}{A} = \frac{b}{a} + \frac{6a}{b}$$

$$2A + \frac{3}{A} = \frac{b}{a} + \frac{6a}{b}$$

$$\frac{2A^2 + 3}{A} = \frac{b^2 + 6a^2}{ab}$$

$$ab(2A^2 + 3) = A(b^2 + 6a^2)$$

$$2A^2ab + 3ab = Ab^2 + 6Aa^2$$

$$2A^2ab - Ab^2 - 6Aa^2 + 3ab = 0$$

$$bA(2Aa - b) - 3a(2Aa - b) = 0$$

$$(2Aa - b)(bA - 3a) = 0$$

$$(2Aa - b) = 0, and\ (bA - 3a) = 0$$

$$2Aa - b = 0, gives$$

$$2Aa = b$$

$$A = \frac{b}{2a}$$

$$(bA - 3a) = 0, gives$$

$$bA - 3a = 0$$

$$bA = 3a$$

$$A = \frac{3a}{b}$$

(i) When $A = \frac{b}{2a}$

$$\sqrt{\frac{x}{a}} = A, \text{ becomes}$$

$$\sqrt{\frac{x}{a}} = \frac{b}{2a}$$

Squaring,

$$\frac{x}{a} = \frac{b^2}{4a^2}$$

$$x = \frac{b^2}{4a}$$

(ii) When $A = \frac{3a}{b}$

$$\sqrt{\frac{x}{a}} = A, \text{ becomes}$$

$$\sqrt{\frac{x}{a}} = \frac{3a}{b}$$

Squaring,

$$\frac{x}{a} = \frac{9a^2}{b^2}$$

$$x = \frac{9a^3}{b^2}$$

$\therefore$ *The roots are* $\frac{b^2}{4a}, \frac{9a^3}{b^2}$

5. Solve the equation:

(i) $6x^4 - 35x^3 + 62x^2 - 35x + 6 = 0$

Solution: $6x^4 - 35x^3 + 62x^2 - 35x + 6 = 0$

This equation is Type I even degree reciprocal equation. Hence it can be rewritten as

$$x^2\left[6x^2 - 35x + 62 - \frac{35}{x} + \frac{6}{x^2}\right] = 0$$

$$\left[6x^2 + \frac{6}{x^2} - 35x - \frac{35}{x} + 62\right] = 0$$

$$\left[6\left(x + \frac{1}{x}\right)^2 - 12 - 35\left(x + \frac{1}{x}\right) + 62\right] = 0$$

$$\left[6\left(x + \frac{1}{x}\right)^2 - 35\left(x + \frac{1}{x}\right) + 50\right] = 0$$

Let $x + \frac{1}{x} = A$

then, $6A^2 - 35A + 50 = 0$

$$6A^2 - 20A - 15A + 50 = 0$$

$$2A(3A - 10) - 5(3A - 10) = 0$$

$$(3A - 10)(2A - 5) = 0$$

$$3A - 10 = 0, and\ 2A - 5 = 0$$

$$(3A - 10) = 0 \Rightarrow A = \frac{10}{3}$$

and $(2A - 5) = 0 \Rightarrow A = \frac{5}{2}$

(i) When $A = \frac{10}{3}$

$$x + \frac{1}{x} = \frac{10}{3}$$

$$\frac{x^2+1}{x} = \frac{10}{3}$$

$$3(x^2 + 1) = 10x$$

$$3x^2 - 10x + 3 = 0$$

$$3x^2 - 9x - x + 3 = 0$$

$$3x(x - 3) - 1(x - 3) = 0$$

$$(3x - 1)(x - 3) = 0$$

$$3x - 1 = 0, and\ x - 3 = 0$$

$$(3x - 1) = 0 \Rightarrow x = \frac{1}{3}$$

and $(x - 3) = 0 \Rightarrow x = 3$

(ii) When $A = \frac{5}{2}$

$$x + \frac{1}{x} = \frac{5}{2}$$

$$\frac{x^2+1}{x} = \frac{5}{2}$$

$$2(x^2 + 1) = 5x$$

$$2x^2 + 2 = 5x$$

$$2x^2 - 5x + 2 = 0$$

$$2x^2 - 4x - x + 2 = 0$$

$$2x(x - 2) - 1(x - 2) = 0$$

$$(2x-1)(x-2)=0$$

$$2x-1=0, and\ x-2=0$$

$$(2x-1)=0 \Rightarrow x=\frac{1}{2}$$

and $(x-2)=0 \Rightarrow x=2$

Hence, $x=\frac{1}{3}, 3, \frac{1}{2}, 2$

(ii) $x^4+3x^3-3x-1=0$

Solution: $x^4+3x^3-3x-1=0$

The sum of the coefficients of the equations is 0

Hence 1 is a root of the polynomial.

$(x-1)$ is a factor.

By synthetic division we find -1 is a root of the polynomial.

$(x+1)$ is also another factor.

$(x-1)(x+1)=(x^2-1)$is a factor.

$\therefore x^4+3x^3-3x-1=(x^2-1)(x^2+px+1)$

Equating the x term,

$$-3=-p$$

$$p=3$$

x^2+3x+1 is a factor.

$$x^2+3x+1=0$$

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

$$=\frac{-3\pm\sqrt{9-4(1)(1)}}{2(1)}$$

$$=\frac{-3\pm\sqrt{9-4}}{2}$$

$$=\frac{-3\pm\sqrt{5}}{2}$$

$$=\frac{-3\pm\sqrt{5}}{2}$$

$\therefore x=-1, 1, \frac{-3\pm\sqrt{5}}{2}$ are roots

6. Find all real numbers satisfying

$$4^x-3(2^{x+2})+2^5=0$$

Solution: $4^x-3(2^{x+2})+2^5=0$

$$4^x-3(2^x)(2^2)+2^5=0$$

$$4^x-3(2^x)4+2^5=0$$

$$4^x-12(2^x)+2^5=0$$

$$(2^2)^x-12(2^x)+2^5=0$$

$$(2^x)^2-12(2^x)+2^5=0$$

Let $2^x=A$

$$A^2-12A+32=0$$

$$(A-8)(A-4)=0$$

$$A-8=0, and\ A-4=0$$

$$A-8=0 \Rightarrow A=8$$

and $A-4=0 \Rightarrow A=4$

(i) When $A=8$

$$2^x=A, gives$$

$$2^x=8$$

$$2^x=2^3$$

$$x=3$$

(ii) When $A=4$

$$2^x=A, gives$$

$$2^x=4$$

$$2^x=2^2$$

$$x=\pm 2$$

$\therefore x=-2, 2, 3$ are roots

7. Solve the equation

$6x^4-5x^3-38x^2-5x+6=0$ if it is known that $\frac{1}{3}$ is a solution.

Solution: $6x^4-5x^3-38x^2-5x+6=0$

This equation is Type I even degree reciprocal equation. Hence it can be rewritten as

$$x^2\left[6x^2-5x-38-\frac{5}{x}+\frac{6}{x^2}\right]=0$$

$$6x^2 - 5x - 38 - \frac{5}{x} + \frac{6}{x^2} = 0$$

$$\left[6x^2 + \frac{6}{x^2} - 5x - \frac{5}{x} - 38\right] = 0$$

$$\left[6\left(x + \frac{1}{x}\right)^2 - 12 - 5\left(x + \frac{1}{x}\right) - 38\right] = 0$$

$$\left[6\left(x + \frac{1}{x}\right)^2 - 5\left(x + \frac{1}{x}\right) - 50\right] = 0$$

Let $x + \frac{1}{x} = A$

then, $6A^2 - 5A - 50 = 0$

$$6A^2 - 20A - 15A - 50 = 0$$

$$2A(3A - 10) + 5(3A - 10) = 0$$

$$(3A - 10)(2A + 5) = 0$$

$$3A - 10 = 0, and\ 2A + 5 = 0$$

$$(3A - 10) = 0 \Rightarrow A = \frac{10}{3}$$

and $(2A + 5) = 0 \Rightarrow A = -\frac{5}{2}$

(i) When $A = \frac{10}{3}$

$$x + \frac{1}{x} = \frac{10}{3}$$

$$\frac{x^2+1}{x} = \frac{10}{3}$$

$$3(x^2 + 1) = 10x$$

$$3x^2 - 10x + 3 = 0$$

$$3x^2 - 9x - x + 3 = 0$$

$$3x(x - 3) - 1(x - 3) = 0$$

$$(3x - 1)(x - 3) = 0$$

$$3x - 1 = 0, and\ x - 3 = 0$$

$$(3x - 1) = 0 \Rightarrow x = \frac{1}{3}$$

and $(x - 3) = 0 \Rightarrow x = 3$

(ii) When $A = -\frac{5}{2}$

$$x + \frac{1}{x} = -\frac{5}{2}$$

$$\frac{x^2+1}{x} = -\frac{5}{2}$$

$$2(x^2 + 1) = -5x$$

$$2x^2 + 2 = -5x$$

$$2x^2 + 5x + 2 = 0$$

$$2x^2 + 4x + x + 2 = 0$$

$$2x(x + 2) + 1(x + 2) = 0$$

$$(2x + 1)(x + 2) = 0$$

$$2x + 1 = 0, and\ x + 2 = 0$$

$$(2x + 1) = 0 \Rightarrow x = -\frac{1}{2}$$

and $(x + 2) = 0 \Rightarrow x = -2$

Hence, $x = \frac{1}{3}, 3, -\frac{1}{2}, -2$

Example 3.30 Show that the polynomial $9x^9 + 2x^5 - x^4 - 7x^2 + 2 = 0$ has at least six imaginary roots.

Solution:

$$P(x) = 9x^9 + 2x^5 - x^4 - 7x^2 + 2$$

There are 2 sign changes for $P(x)$

Hence it has at most 2 positive roots.

$$P(-x) = 9(-x)^9 + 2(-x)^5 - (-x)^4 - 7(-x)^2 + 2$$
$$= -9x^9 - 2x^5 - x^4 - 7x^2 + 2$$

There is 1 sign changes for $P(-x)$

Hence it has at most 1 negative root.

Since $P(x)$s of degree 9, the $P(x)$ has at least 6 imaginary roots.

Example 3.31 Discuss the nature of the roots of the following polynomials:

(i) $x^{2018} + 1947x^{1950} + 15x^8 + 26x^6 + 2019$

Solution:

$$P(x) = x^{2018} + 1947x^{1950} + 15x^8 + 26x^6 + 2019$$

There are no sign changes for $P(x)$

Hence it has no positive roots.

$$P(-x) = x^{2018} + 1947x^{1950} + 15x^8 + 26x^6 + 2019$$
$$= x^{2018} + 1947x^{1950} + 15x^8 + 26x^6 + 2019$$

There are no sign changes for $P(-x)$

Hence $P(x)$ has no negative root.

Since $P(x)$s of degree 9, the $P(x)$ has at least 9 imaginary roots.

(ii) $x^5 - 19x^4 + 2x^3 + 5x^2 + 11$

Solution:

$P(x) = x^5 - 19x^4 + 2x^3 + 5x^2 + 11$

There are 2 sign changes for $P(x)$

Hence it has at most 2 positive roots.

$P(-x) = (-x)^5 - 19(-x)^4 + 2(-x)^3 + 5(-x)^2 + 11$

$= -x^5 - 19x^4 - 2x^3 + 5x^2 + 11$

There is 1 sign changes for $P(-x)$

Hence $P(x)$ has at most 1 negative root.

Since $P(x)$s of degree 5, the $P(x)$ has at least 2 imaginary roots.

EXERCISE 3.6

1. Discuss the maximum possible number of positive and negative roots of polynomial equation $9x^9 - 4x^8 + 4x^7 - 3x^6 + 2x^5 + x^3 + 7x^2 + 7x + 2 = 0$.

Solution:

$P(x) = 9x^9 - 4x^8 + 4x^7 - 3x^6 + 2x^5 + x^3 + 7x^2 + 7x + 2$

There are 4 sign changes for $P(x)$

Hence it has at most 4 positive roots.

$P(-x) = 9(-x)^9 - 4(-x)^8 + 4(-x)^7 - 3(-x)^6 + 2(-x)^5 + (-x)^3 + 7(-x)^2 + 7(-x) + 2$

$= -9x^9 - 4x^8 - 4x^7 - 3x^6 - 2x^5 - x^3 + 7x^2 - 7x + 2$

There are 3 sign changes for $P(-x)$

Hence $P(x)$ has at most 3 negative roots.

Since the difference between the number of sign changes in co- efficient $P(-x)$ and number of negative roots of the polynomial $P(x)$ is even.

The number of imaginary roots at most 2.

2. Discuss the maximum possible number of positive and negative roots of polynomial equations $x^2 - 5x + 6$ and $x^2 - 5x + 16$ Also draw rough sketch of the graphs.

Solution:

Let $P(x) = x^2 - 5x + 6$

There are 2 sign changes for $P(x)$

Hence it has at most 2 positive roots.

$P(-x) = (-x)^2 - 5(-x) + 6$

$= x^2 + 5x + 6$

There are no sign changes for $P(-x)$

Hence $P(x)$has no negative root.

Let $Q(x) = x^2 - 5x + 16$

There are 2 sign changes for $Q(x)$

Hence it has at most 2 positive roots.

$Q(-x) = (-x)^2 - 5(-x) + 16$

$= x^2 + 5x + 16$

There are no sign changes for $Q(-x)$

Hence $Q(x)$ has no negative root.

3. Show that the equation $x^9 - 5x^5 + 4x^4 + 2x^2 + 1 = 0$ has at least 6 imaginary solutions.

Solution:

Let $P(x) = x^9 - 5x^5 + 4x^4 + 2x^2 + 1$

There are 2 sign changes for $P(x)$

Hence it has at most 2 positive roots.

$P(-x) = (-x)^9 - 5(-x)^5 + 4(-x)^4 + 2(-x)^2 + 1$

$= -x^9 + 5x^5 + 4x^4 + 2x^2 + 1$

There is 1 sign changes for $P(-x)$

Hence $P(x)$ has one negative root.

Since the difference between the number of sign changes in co- efficient $P(-x)$ and number of negative zeros is even.

$P(x)$ has one negative root.

0 is not the zero of the polynomial $P(x)$.

So the number of real roots is at most 3.

The number of imaginary roots at least 6.

4. Determine the number of positive and negative roots of the equation $x^9 - 5x^8 - 14x^7 = 0$.

Solution:

Let $P(x) = x^9 - 5x^8 - 14x^7$

There is 1 sign change for $P(x)$

Hence it has at most 1 positive root.

$P(-x) = (-x)^9 - 5(-x)^8 - 14(-x)^7$

$= -x^9 - 5x^8 + 14x^7$

There is 1 sign change for $P(-x)$

Hence $P(x)$ has at most 1 negative root.

Clearly 0 is a root of $P(x)$.

The number of imaginary roots at least 6.

5. Find the exact number of real roots and imaginary of the equn. $x^9 + 9x^7 + 5x^3 + 3x$.

Solution:

Let $P(x) = x^9 + 9x^7 + 5x^3 + 3x$

There is no sign change for $P(x)$

Hence it has no positive root.

$P(-x) = (-x)^9 + 9(-x)^7 + 5(-x)^3 + 3(-x)$

$= -x^9 - 9x^7 - 5x^3 - 3x$

There is no sign change for $P(-x)$

Hence $P(x)$ has no negative root.

Clearly 0 is a root of $P(x)$.

EXERCISE 3.7

Choose the most suitable answer.

1. A zero of $x^3 + 64$ is . . . . .
(1) 0 (2) 4 (3) $4i$ **(4) – 4**

2. If f and g are polynomials of degrees m and n respectively, and if $h(x) = (f o g)(x)$, then the degree of h is
(1)mn (2) m + n (3) m^n (4) n^m

3. A polynomial equation in x of degree n always has
(1) n distinct roots (2) n real roots
(3) n imaginary roots (4) at most one root.

4. If α , β , and γ are the roots of the equation $x^3 + px^2 + qx + r = 0$, then $\sum \frac{1}{\alpha}$ is
(1) $-\frac{q}{r}$ **(2) $-\frac{p}{r}$** (3) $\frac{q}{r}$ (4) $-\frac{q}{p}$

5. According to the rational root theorem, which number is not possible rational root of $4x^7 + 2x^4 - 10x^3 - 5$?
(1) −1 **(2) $\frac{5}{4}$** (3) $\frac{4}{5}$ (4) 5

6. The polynomial $x^3 - kx^2 + 9x$ has three real roots if and only if, k satisfies
(1) $|k| \le 6$ (2) $k = 0$
(3) $|k| > 6$ **(4) $|k| \ge 6$**

7. The number of real numbers in $[0, 2\pi]$ satisfying $sin^4 x - 2sin^2 x + 1$ is
(1) 2 (2) 4 (3)1 (4) ∞

8. If $x^3 + 12x^2 + 10ax + 1999$ definitely has a positive root, if and only if
(1) $a \ge 0$ (2) $a > 0$ **(3) $a < 0$** (4) $a \le 0$

9. The polynomial $x^3 + 2x + 3$ has

(1) one negative and two real roots

(2) one positive and two imaginary roots

(3) three real roots

(4) no solution

10. The number of positive roots of the polynomial $\sum_{j=0}^{n} n_{C_r}(-1)^r x^j$ is

(1) 0 **(2) n** (3) $< n$ (4) r

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