1. Sets, Relations and Functions

1.2 Sets: A set is a well-defined collection of objects.

$a \in S$ "a is an element of S"

$a \notin S'$ "a is not an element of S"

In the set There are two Methods of representing a set:

(i) Roster or tabular form (ii) Set-builder form.

(i) In roster form: all the elements of a set are listed, the elements are being separated by commas and are enclosed within braces { }.

Example: 1. Write the following in roster form

(i) $\{x \in N : x^2 < 121$ and x is a prime$\}$

Solution: let $A = \{x \in N : x^2 < 121$ and x is a prime$\}$

$x^2 < 121$

$x < \sqrt{121}$

$x < 11$

$A = \{2, 3, 5, 7\}$

(ii) the set of all positive roots of the equation $(x-1)(x+1)(x^2-1) = 0$

Solution: let $B = \{$ the set of all positive roots of the eqn $(x-1)(x+1)(x^2-1) = 0\}$

$\Rightarrow x = 1, -1$

$B = \{1\}$

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(iii) $\{x \in N : 4x + 9 < 52\}$

Solution: let $C = \{x \in N : 4x + 9 < 52\}$

$C = \{x \in N : 4x < 43\}$

$C = \{x \in N : x < 10.75\}$

$C = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$

(iv) $\left\{x : \frac{x-4}{x+2} = 3, \ x \in R - \{-2\}\right\}$.

Solution: let $S = \left\{x : \frac{x-4}{x+2} = 3, \ x \in R - \{-2\}\right\}$

$S = \{x : x - 4 = 3x + 6 \quad x \in R \setminus \{-2\}\}$

$S = \{x : x = -5, \ x \in R \setminus \{-2\}\}$

$S = \{-5\}$

(ii) Set - builder form:

If P denotes a certain property of elements, then $\{x : P\}$ stands for the set of all elements x for which the property P is meaningful and true.

Example: 1. Write the set $\{-1, 1\}$ in set builder form.

Solution: let $A = \{-1, 1\}$

$\Rightarrow A = \{x \in \mathbb{R} : x^2 = 1\}$

2. Write the set $A = \{1, 4, 9, 16, 25, \ldots\}$ in set builder form

Solution: let $A = \{1, 4, 9, 16, 25, \ldots\}$

$\Rightarrow A = \{x : x \text{ is the square of a natural number}\}$

$\Rightarrow A = \{x : x = n^2, \text{ where } n \in \mathbb{N}\}$

Empty set (or) Void set:

A set which does not contain any element is called the empty set or the null set or the void set.

It is usually denoted by ϕ or $\{\}$.

The null set is the set having no elements; it is a subset of every set.

Sub set: The set A will be said to be a subset of the set B if every element in A is an element of B

ie, if $a \in A \Rightarrow a \in B$ we write $A \subset B$

If A is not a subset of B we write $A \not\subset B$

Example: $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$

Trivially, every set is a subset of itself. $(A \subseteq A)$

Superset:- If A is a subset of B then B is a superset of A we write $B \supseteq A$

Equal set: For any two sets A and B if $A \subseteq B$ and $B \subseteq A$ then the two sets are equal.

$A = B$ if both $A \subset B$ and $B \subset A$

Trivial subsets: For any set A, The empty set ϕ and the set A are always subsets of A. These two subsets are called trivial subsets.

Proper subset: A subset A of B will be called a proper subset of B. If $A \subset B$ but $A \neq B$ (A is not equal to B)

improper subset: every element of A is an element of A, we have $A \subseteq A$. Thus, any set is a subset of itself. This subset is called an improper subset.

Operations on sets: let A and B are two sets.

(i) Union: $A \cup B = \{x : x \in A \text{ or } x \in B\}$

(ii) Intersection: $A \cap B = \{x : x \in A \text{ and } x \in B\}$

(iii) Set ~~D~~ Difference: $A \setminus B = \{a : a \in A \text{ and } a \notin B\}$

(iv) Symmetric Difference: $A \Delta B = (A-B) \cup (B-A)$

$\Rightarrow \quad A \Delta B = (A \cup B) - (A \cap B)$

(v) Complement: ~~let~~ If $A \subseteq U$, where U is a Universal set, Then $U \setminus A$ is called the complement of A with respect to U is denoted as A' or A^c and defined as

$$A' = \{x : x \in U \text{ and } x \notin A\}$$

(vi) Disjoint set: Two sets A and B are disjoint if they do not have any common element.

ie, A and B are disjoint if $A \cap B = \phi$

Power set: Given a set A, let $P(A)$ denote the collection of all subsets of A. The set $P(A)$ is called the power set.

ie, $P(A) = \{ B : B \subseteq A \}$

The number of elements in $P(A)$ is 2^n where n is the number of elements in A

$n(P(A)) = 2^{n(A)}$ $\quad n(A) = n$

Example: $A = \{a, b, c\}$ $\quad n(A) = 3$

then $P(A) = \{ \phi, \{a\}, \{b\}, \{c\}, \{a,c\}, \{b,c\}, \{a,b,c\} \}$

$n(P(A)) = 2^3 = 8$

Notations: $\sum_{i=1}^{n} a_i = a_1 + a_2 + \cdots + a_n$

$\bigcup_{i=1}^{n} A_i = A_1 \cup A_2 \cup \cdots \cup A_n = \{ x : x \in A_i \text{ for some } i \}$

$\bigcap_{i=1}^{n} A_i = A_1 \cap A_2 \cap \cdots \cap A_n = \{ x : x \in A_i \text{ for each } i \}$

Results:

(i) $U - A = A'$ (ii) $A - A = \phi$ (iii) $\phi - A = \phi$ (iv) $A - \phi = A$ (v) $A - U = \phi$

1.2.1 Properties of set Operations:

Commutative (i) $A \cup B = B \cup A$ (ii) $A \cap B = B \cap A$

Associative: (i) $(A \cup B) \cup C = A \cup (B \cup C)$ (ii) $(A \cap B) \cap C = A \cap (B \cap C)$

Distributive:

(i) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ (ii) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

Identity: (i) $A \cup \phi = A$ (ii) $A \cup U = A$

Idempotent: (i) $A \cup A = A$ (ii) $A \cap A = A$

Absorption: (i) $A \cup (A \cup B) = A$ (ii) $A \cap (A \cup B) = A$

De Morgan Laws: (i) $(A \cup B)' = A' \cap B'$ (ii) $(A \cap B)' = A' \cup B'$

(iii) $A - (B \cup C) = (A - B) \cap (A - C)$ (iv) $A - (B \cap C) = (A - B) \cup (A - C)$

On Symmetric Difference: (i) $A \Delta B = B \Delta A$ (ii) $(A \Delta B) \Delta C = A \Delta (B \Delta C)$

(iii) $A \cap (B \Delta C) = (A \cap B) \Delta (A \cap C)$

On Empty set and Universal set: (i) $\phi' = U$ (ii) $U' = \phi$

(iii) $A \cup A' = U$ (iv) $A \cap A' = \phi$ (v) $A \cup U = U$ (vi) $A \cap U = A$

on Cardinality:

(i) For any two finite sets A and B, $n(A\cup B)=n(A)+n(B)-n(A\cap B)$

(ii) A and B are disjoint finite sets, then $n(A\cup B)=n(A)+n(B)$

(iii) For any theree finite sets A, B and C

$n(A\cup B\cup C)=n(A)+n(B)+n(C)-n(A\cap B)-n(B\cap C)-n(C\cap A)+n(A\cap B\cap C)$

Finite set: A set is said to be a finite set if it contains only a finite number of elements in it.

A set X is said to be a finite set if it has K elements for some $K\in W$

cardinality of a set X is denoted by $n(X)=K$.

Infinite Set: A set which is not finite is called an infinite set.

For an infinite set A, the cardinality is infinity (∞)

Note: (i) If $n(A)=1$, then it is called a singleton set.

(ii) $n(\phi)=0$ and $n(\{\phi\})=1$

Exercise 1.1

3. State whether the following sets are finite or infinite.

(i) $\{x\in N : x$ is an even prime number$\}$

Solution: let $A=\{x\in N : x$ is an even prime number$\}$

$\Rightarrow A=\{2\}$

$\Rightarrow$ A is a finite set.

(ii) $\{x\in N : x$ is an ~~odd~~ prime number$\}$

Solution: let $B=\{x\in N : x$ is an odd prime number$\}$

$\Rightarrow B=\{3,5,7,11,\dots\}$ is an infinite set.

(iii) $\{x\in Z : x$ is even and less than $10\}$

Solution: let $C=\{x\in Z : x$ is even and less than $10\}$

$\Rightarrow C=\{2,4,6,8\}$ is a finite set

(iv) $\{x\in Z : x$ is a rational number$\}$

Solution: let $D=\{x\in R : x$ is a rational number$\}=\{$set of all rational number$\}$

$\Rightarrow$ D is an infinite set

(v) $\{x\in N : x$ is a rational number$\}$

Solution: let $N=\{x\in N : x$ is a rational number$\}=\{\frac{1}{1},\frac{2}{1},\frac{3}{1},\frac{4}{1}\dots\infty\}$ is an infinite set

1.3 Cartesian product:

Given sets A and B, we can define a new set $A \times B$, called the Cartesian product of A and B, as a set of ordered pairs. ie, $A \times B = \{(a,b) : a \in A \text{ and } b \in B\}$

Ex: If $A = \{1,2,3\}$ and $B = \{2,4,6\}$ then

$$A \times B = \{(1,2),(1,4),(1,6),(2,2),(2,4),(2,6),(3,2),(3,4),(3,6)\}$$

Here $A \times B$ is a subset of $\mathbb{R} \times \mathbb{R}$

Similarly, the Cartesian product $A \times B \times C$ is defined by

$$A \times B \times C = \{(a,b,c) : a \in A, b \in B, c \in C\}$$

Note: (i) For non-empty sets $A \times B \neq B \times A$

(ii) $A \times B = B \times A$ if $A = B$

We know that $\mathbb{R}$ denotes the set of real numbers and

$$\mathbb{R}^2 = \mathbb{R} \times \mathbb{R} = \{(x,y) : x,y \in \mathbb{R}\}$$

$$\mathbb{R}^3 = \mathbb{R} \times \mathbb{R} \times \mathbb{R} = \{(x,y,z) : x,y,z \in \mathbb{R}\}$$

Exercise - 1.1

3. By taking suitable sets A, B, C verify the following results:

(i) $A \times (B \cap C) = (A \times B) \cap (A \times C)$

Solution: Let $A = \{1,2,3\}$ $B = \{3,4,5\}$ $C = \{5,6,7\}$ and $U = \{1,2,3,4,5,6,7\}$

L.H.S $B \cap C = \{5\}$

$$A \times (B \cap C) = \{(1,5),(2,5),(3,5)\}$$

R.H.S $A \times B = \{(1,3),(1,4),(1,5),(2,3),(2,4),(2,5),(3,3),(3,4),(3,5)\}$

$$A \times C = \{(1,5),(1,6),(1,7),(2,5),(2,6),(2,7),(3,5),(3,6),(3,7)\}$$

$$\therefore (A \times B) \cap (A \times C) = \{(1,5),(2,5),(3,5)\}$$

$$A \times (B \cap C) = (A \times B) \cap (A \times C)$$

Given $n(P(A)) = 1024 = 2^{10}$ $\quad n(P(B)) = 32 = 2^5$

$\Rightarrow n(A) = 10$ $\quad \Rightarrow n(B) = 5$

W.K.T $n(A\cup B) = n(A) + n(B) - n(A\cap B)$

$15 = 10 + 5 - n(A\cap B)$

$\Rightarrow$

$\Rightarrow n(A\cap B) = 0$

7. If $n(A\cap B) = 3$ and $n(A\cup B) = 10$ then find $n(P(A\Delta B))$

Solution: $A\Delta B = (A\cup B) - (A\cap B)$

$n(A\Delta B) = n(A\cup B) - n(A\cap B)$

$\Rightarrow n(A\Delta B) = 10 - 3 = 7$

$\therefore n[P(A\Delta B)] = 2^7 = 128$

8. For a set A, $A\times A$ contains 16 elements and two of its elements are $(1,3)$ and $(0,2)$. Find the elements of A.

Solution: $n(A\times A) = 16$ then A must have 4 element

$\Rightarrow n(A) = 4$ $\quad \because n(A\times B) = n(A)\cdot n(B)$

The elements of $A\times A$ are $(1,3)$ and $(0,2)$

$\therefore$ The possibilities of elements of A are $\{0, 1, 2, 3\}$

9. Let A and B be two sets such that $n(A) = 3$ and $n(B) = 2$

If $(x,1), (y,2), (z,1)$ are in $A\times B$ find A and B where x, y, z are distinct elem

Solution:
Given If $(x,1)(y,2)(z,1)$ are in $A\times B$

$n(A) = 3$, $n(B) = 2$

$\Rightarrow n(A\times B) = 6$

$\therefore A\times B = \{(x,1)(x,2), (y,1), (y,2), (z,1), (z,2)\}$

$A = \{x, y, z\}$ and $B = \{1, 2\}$

10. If $A\times A$ has 16 elements, $S' = \{(a,b)\in A\times A : a<b\}$; $(-1,2)$ and $(0,1)$ are two elements of S'. Then find the remaining elements of S.

Solution:

$n(A\times A) = 16$ $\quad \Rightarrow n(A) = 4$

Given: $(-1,2)$ and $(0,1)$ are two elements of S'

$A = \{-1, 0, 1, 2\}$

$A\times A = \{(-1,-1), (-1,0), (-1,1), (-1,2), (0,-1), (0,0), (0,1), (0,2), (1,-1), (1,0), (1,1), (1,2), (2,-1), (2,0), (2,1), (2,2)\}$

Now $S' = \{(a,b)\in A\times A : a<b\} = \{(-1,0), (-1,1), (-1,2), (0,1), (0,2)(1,2)\}$

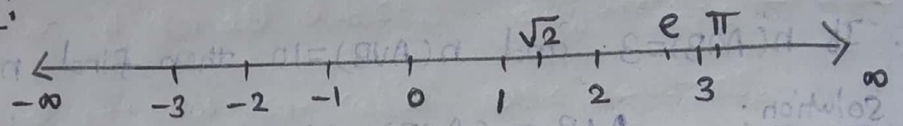
$\therefore$ The remaining elements of S' are $(-1,0)\ (-1,1)\ (0,2)(1,2)$

1.4.1 Constants and Variables:

examples: (i) $A = lb$ (ii) $A = \pi r^2$ (iii) $V = lbh$

A and V are dependent variable; b, h, l, r are independent variables and π is a constant.

Real line:



If x lies to the left of y on the real line then $x < y$.

As there is no gap in a line, we have infinitely many real numbers between any two real numbers.

Definition 1.1: A subset I of $\mathbb{R}$ is said to be an interval if

(i) I contains at least two elements and

(ii) $a, b \in I$ and $a < c < b$ then $c \in I$

Type of Intervals: The set of all points between a and b is called an interval.

Let $a, b \in \mathbb{R}$ s.t $a < b$.

(i) The open interval $(a, b) = \{x : a < x < b\}$

(ii) The closed interval $[a, b] = \{x : a \le x \le b\}$

(iii) The half-open intervals $(a, b] = \{x : a < x \le b\}$

$[a, b) = \{x : a \le x < b\}$

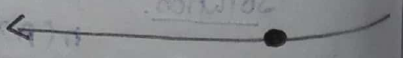
Infinite intervals are defined as follows:

$(a, +\infty) = \{x : x > a\}$

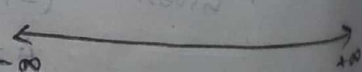
$[a, +\infty) = \{x : x \ge a\}$

$(-\infty, a) = \{x : x < a\}$

$(-\infty, a] = \{x : x \le a\}$



$\mathbb{R}$ or $(-\infty, \infty) = \{x : -\infty < x < \infty\}$

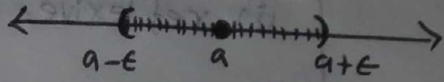


or the set of real numbers.

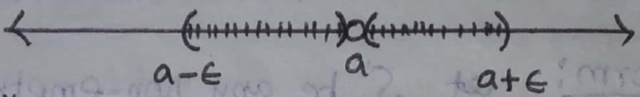
Neighbourhood

ϵ-neighborhood: let $a \in \mathbb{R}$ and ϵ-neighborhood of a is the set $N(x;\epsilon) = \{ x \in \mathbb{R} : |x-a| < \epsilon \}$

$-\epsilon < x-a < \epsilon \iff a-\epsilon < x < a+\epsilon$



Deleted neighborhood: The set $(a-\epsilon, a+\epsilon) - \{a\}$ is called deleted neighorhood of 'a' and it is denoted as $0 < |x-a| < \epsilon$



$(a-\epsilon, a) \cup (a \;\; a+\epsilon)$

NOTICE: $N^*(x;\epsilon) = N(x;\epsilon) \setminus \{x\}$

1.5 Relations

Definition 1.2 let A and B be any two non-empty sets. A relation R from A to B is a non-empty subset of $A \times B$.

That is, $R \subseteq A \times B$

let R be a relation from a set A into a set B.

If $(x,y) \in R$, we write xRy or $R(x) = y$.

Domain of $R = \{ x \in A \mid (x,y) \in R$ for some $y \in B \}$

Range of $R = \{ y \in B \mid (x,y) \in R$ for some $x \in A \}$

Example: let $A = \{4,5,7,8,9\}$ and $B = \{16,18,20,22\}$.

Define $R \subseteq A \times B$ by $R = \{(4,16),(4,20),(5,20),(8,16),(9,18)\}$

Then R is a relation from A into B.

Here, $(a,b) \in R \iff a$ divides b where $a \in A$ and $b \in B$

Domain of R, we have $D(R) = \{4,5,8,9\}$ and

Range of R. we have $I(R) = \{16,18,20\}$

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1.5.1 Type of Relations:

Definition 1.3: Let S be any non-empty set. Let R be a relation on S. Then

(i) reflexive if for all $a \in S$, aRa

(ii) Symmetric if for all $a, b \in S$, $aRb \Rightarrow bRa$

(iii) transitive if for all $a, b, c \in S$ aRb and $bRc \Rightarrow aRc$

These three relations are called basic relations.

Different form: Let S be any non-empty set. Let R be a relation on S. Then R is

* reflexive if $(a,a) \in R$ for all $a \in S$
* Symmetric if $(a,b) \in R \Rightarrow (b,a) \in R$
* transitive if $(a,b),(b,c) \in R \Rightarrow (a,c) \in R$

Definition 1.4: Let S be any set. A relation S is said to be an equivalence relation if it is reflexive, Symmetric, and transitive.

Exercise - 1.2

1. Discuss the following relations for reflexivity, symmetricity and transitivity:

(i) The relation R defined on the set of all positive integers by " mRn if m divides n

Solution $R = \{(m,n) : m \text{ divides } n\}$

Reflexivity: mRm

$\Rightarrow m|m$ for all positive integers m

$\therefore R$ is reflexive

Symmetricity: $mRn \Rightarrow nRm$

$m|n \Rightarrow n\nmid m$ for all positive integers m and n

R is not Symmetric

Transitive: mRn and $nRp \Rightarrow mRn$

$\Rightarrow m|n$ and $n|p \Rightarrow m|p$

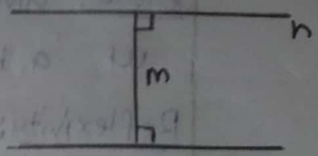
$\therefore R$ is transitive.

R is reflexive, not Symmetric and transitive.

(ii) Let P denote the set of all straight lines in a plane. The relation R defined by "lRm if l is perpendicular to m"

Solution: $R = \{(l, m) : l \text{ is } \perp^r \text{ to } m\}$

Let $l, m, n \in P$



Reflexivity:

R is not reflexive, as a line l can not be $\perp^r$ to itself.

ie, $(l, l) \notin R$

Symmetry: $(l, m) \in R$

$\Rightarrow$ l is $\perp^r$ to m

$\Rightarrow$ m is $\perp^r$ to l

$\Rightarrow$ $(m, l) \in R$

R is symmetric

Transitivity: $(l, m) \in R$ and $(m, n) \in R$

$\Rightarrow$ l is $\perp^r$ to m and m is $\perp^r$ to n then

$\Rightarrow$ l can never be $\perp^r$ to n (l is $\parallel^{le}$ to n)

$\Rightarrow$ $(l, n) \notin R$

ie, $(l, m) \in R$, $(m, n) \in R$ but $(l, n) \notin R$

R is not transitive.

(iii) Let A be the set consisting of all the members of family. The relation R defined by "aRb if a is not a sister of b"

Solution: Given relation is "aRb if a is not a sister of b" and $a, b, c \in A$

Reflexivity: $aRa \Rightarrow a$ is not a sister of a

$\therefore$ R is reflexive.

Symmetricity: aRb

$\Rightarrow$ a is not a sister of b

$\Rightarrow$ b is not a sister of a

$\Rightarrow$ bRa

$\therefore$ R is symmetric

Transitivity: aRb and $bRc \Rightarrow aRc$

a is not a sister of b and b is not a sister of c

$\Rightarrow$ a is not a sister of c

R is transitive.

$\therefore$ R is reflexive, symmetric and transitive.

(iv) Let A be the set consisting of all the female members of a family. The relation R defined by "aRb if a is not a sister of b"

<u>Solution:</u> Given relation is "aRb if a is not a sister of b"

Let $a, b, c \in A$

<u>Reflexivity:</u> $aRa \Rightarrow$ a is not a sister of a

$\therefore$ R is reflexive.

<u>Symmetricity:</u> $aRb \Rightarrow bRa$

a is not a sister of b $\Rightarrow$ b is not a sister of a

$\therefore$ R is Symmetric

<u>Transitivity:</u>

aRb and $bRc \Rightarrow aRc$

a is not a sister of b, b is not a sister of c

$\Rightarrow$ a is a sister of c

$\therefore$ R is not transitive.

$\therefore$ R is reflexive, symmetric and but not transitive.

(v) On the set of natural numbers the relation R defined by "xRy if $x+2y=1$"

<u>Solution:</u> $R = \{(x,y) : x+2y=1, x,y \in N\} = \{\ \}$ or ϕ

$R(x)=y$
$R(1)=0 \notin N$
$R(2)=-\frac{1}{2} \notin N$
$R(3)=-1 \notin N$
etc

$N = \{1, 2, 3, 4, 5, \ldots\}$ and $R = \{\ \}$ or ϕ where ϕ is the empty set

As $(1,1) \notin R$, $(2,2) \notin R$ it is not reflexive.

As We Cannot find a pair (x,y) in R such that $(y,x) \notin R$ the relation is not 'not symmetric'; so it is Symmetric

Similarly, it is transitive.

⑤ On the set of natural numbers let R be the relation defined by aRb if $2a+3b=30$. Write down the relation by listing all the pairs. Check whether it is

(i) reflexive (ii) symmetric (iii) transitive (iv) equivalence

Solution: $N=\{1,2,3 \ldots\}$ and $R=\{(a,b): 2a+3b=30, a,b\in N\}$

$R=\{(3,8),(6,6),(9,4),(12,2)\}$

Reflexivity: $(3,3),(9,9),(12,12)\notin R \Rightarrow R$ is not reflexive.

Symmetricity: $(3,8)\in R \Rightarrow (8,3)\notin R$

$(9,4)\in R \Rightarrow (4,9)\notin R$ $\quad (12,2)\in R \Rightarrow (2,12)\notin R$.

$\therefore R$ is not symmetric

Transitivity: clearly R is not transitive

R is not an equivalence relation.

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(7) $N=\{1,2,3 \ldots\}$ and $R=\{(a,b): a+b\le 6, a,b\in N\}$

Solution: $R=\{(1,1),(1,2),(1,3),(1,4),(1,5),(2,1),(2,2),(2,3),(2,4),(3,1),$
$(3,2),(3,3),(4,1),(4,2),(5,1)\}$

Reflexivity: $(4,4),(5,5)\notin R \Rightarrow R$ is not reflexive.

Symmetric: $(1,2)\in R \Rightarrow (2,1)\in R$, $(1,5)\in R \Rightarrow (5,1)\in R$.

$(a,b)\in R \Rightarrow (b,a)\in R$

$\therefore R$ is symmetric.

Transitivity: $(4,2)\in R, (2,3)\in R \Rightarrow (4,3)\notin R$

$(a,b),(b,c)\in R \Rightarrow (a,c)\notin R$

R is not transitive.

$\therefore$ R is not an equivalence relation.

---

(8) Let $A=\{a,b,c\}$. What is the equivalence relation of smallest cardinality on A? What is the equivalence relation of largest cardinality on A?

Solution: Given $A=\{a,b,c\}$

$A\times A=\{(a,a),(a,b),(a,c),(b,a),(b,b),(b,c),(c,a),(c,b),(c,c)\}$

(i) Let $R=\{(a,a),(b,b),(c,c)\}$ $\quad n(R)=3$

R is reflexive, symmetric and transitive.

$\Rightarrow$ R is an equivalence relation

(ii) Let $R=\{(a,a),(a,b),(a,c),(b,a),(b,b),(b,c),(c,a),(c,b),(c,c)\}$

R is reflexive since $(a,a),(b,b)$ and $(c,c)\in R$

R is symmetric since $(a,b)\in R \Rightarrow (b,a)\in R$, $(b,c)\in R \Rightarrow (c,b)\in R$
$(c,a)\in R \Rightarrow (a,c)\in R$

R is also transitive since $(a,b),(b,c)\in R \Rightarrow (a,c)\in R$

$\therefore n(R)=9$

Example 1.13 In the set Z of integers, define mRn if $m-n$ is a multiple of 12. prove that R is an equivalence relation.

Solution:

(i) For all $m \in Z$, $m-m=0=0\times 12$

Hence, for all $m \in Z$, mRm

$\therefore R$ is reflexive.

(ii) let $m,n \in Z$

let mRn Then $m-n=12K$ for some integer K

$\Rightarrow nRm \Rightarrow n-m=12(-K)$

$\Rightarrow nRm$ $\therefore R$ is Symmetric.

(iii) let mRn and nRp; then

there exist $K, l \in Z$ such that

$m-n=12K$ and $n-p=12l$

$\Rightarrow m-p=12(K+l)$ $(K+l) \in Z$

$\Rightarrow mRp$

R is transitive.

Thus R is an equivalence relation.

9. In the set Z of integers, define mRn if ~~mRn~~ if $m-n$ is divisible by 7. prove that R is an equivalence relation.

$7|0$
7 divides 0
7 divides m-n
$7|m-n$
$\Rightarrow m-n=7K$

Solution: (i) As $m-m=0$

$m-m=0$ is divisible by 7 $\Rightarrow mRm$

$\therefore R$ is reflexive.

(ii) let mRn then $m-n=7K$ for some integer K

$\Rightarrow n-m=7(-K)$

$\Rightarrow nRm$

$\therefore R$ is symmetric.

(iii) let mRn and nRp

$\Rightarrow m-n=7K$ and $n-p=7l$ $K, l \in Z$

$\Rightarrow m-p=7(K+l)$ $K+l \in Z$

$\Rightarrow mRp$

$\therefore R$ is transitive.

Thus, R is an equivalence relation.

* Let n be a fixed positive integer in Z. Define the relation $\equiv_n$ on Z by for all $x, y \in Z$. $x \equiv y \mod n \iff n \mid (x-y)$, ie $x-y = nk$ for some $k \in Z$. We now show that $\equiv_n$ is an equivalence relation on Z.

Solution: (i) For all $x \in Z$, $x - x = 0 = 0n$

Hence, for all $x \in Z$, $x \equiv x \mod n$

$\Rightarrow \equiv_n$ is reflexive.

(ii) Let $x, y \in Z$. Suppose $x \equiv y \mod n$ then there exists $q \in Z$ such that $qn = x - y$

$\Rightarrow (-q)n = y - x$

$\Rightarrow y \equiv x \mod n$

Hence, $\equiv_n$ is symmetric.

(iii) Let $x, y, z \in Z$. Suppose $x \equiv y \mod n$ and $y \equiv z \mod n$ Then $\exists\, q, r \in Z$ such that $x - y = qn$ and $y - z = rn$

$\Rightarrow x - z = (q+r)n$ and $q + r \in Z$

$\Rightarrow x \equiv z \mod n$

Hence, $\equiv_n$ is transitive.

$\equiv_n$ is an equivalence relation on Z.

1.6 Functions :

Definition 1.6: Let A and B be any two non-empty sets. A function from A to B is a relation.

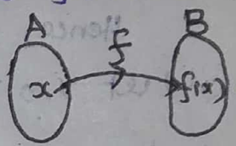
$f \subseteq A \times B$ such that the following hold :

(i) for all $a \in A$ there is only one $b \in B$ such that $(a,b) \in f$

(ii) if $(a,b) \in f$ and $(a,c) \in f$ then $b = c$.

ie, a function is a relation in which each element in the domain is mapped to exactly one element in the range.

A function from A to B is denoted by $f : A \to B$ and if $(x,y) \in f$, then we write $y = f(x)$



f maps A into B

A is called the domain of f and

B is called the co-domain of f

If (a,b) is in f then we write $f(a) = b$;

the element b is called the image of a and

and $f^{-1}(b) = a$; the element a is called the pre-image of b

The range of f is the set $\{ b \in B \mid b = f(a) \text{ for some } b \}$

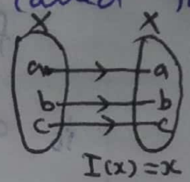
That is, the rang of f is the subset of B consisting of all images of elements of A.

Real-valued function: The range of a given function f is contained in the set of all real numbers. If $f : A \to \mathbb{R}$ we call f a real-valued function.

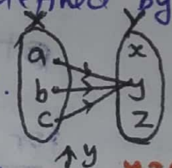
Two functions f and g are equal iff f and g have the same domain A and $f(a) = g(a)$ $(a \in A)$.

1.6.2 Some Elementary Functions:

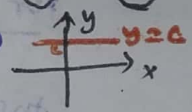
(i) **Identity function:** let X be any non-empty set. $f: X \to X$ defined by $f(x) = x$ for all $x \in X$ is called the identity function on X. It is denoted by I_X or I



(ii) **constant function:** let X and Y be two sets. let c be a fixed element of Y. $f: X \to Y$ defined by $f(x) = c$ for all $x \in X$ is called a constant function.



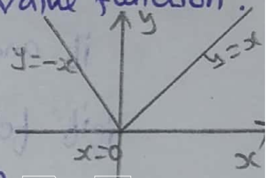
Note: $f(x) = 0$ for all $x \in X$ is called the zero function



(iii) Modulus function or absolute value function:

$f: \mathbb{R} \to \mathbb{R}$, $f(x) = |x|$ where $|x|$ is the modulus or absolute value of x is called the modulus function or absolute value function.

$$|x| = \begin{cases} -x & \text{if } x < 0 \\ 0 & \text{if } x = 0 \\ x & \text{if } x > 0 \end{cases} \quad \text{or } |x| = \begin{cases} -x & \text{if } x \le 0 \\ x & \text{if } x > 0 \end{cases} \quad \text{or } |x| = \begin{cases} -x & \text{if } x < 0 \\ x & \text{if } x \ge 0 \end{cases}$$



(iv) $f: \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \begin{cases} \frac{x}{|x|} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$ is called the Signum function. Domain $= \mathbb{R}$ Range : $\{-1, 0, 1\}$

(v) **Step functions:**

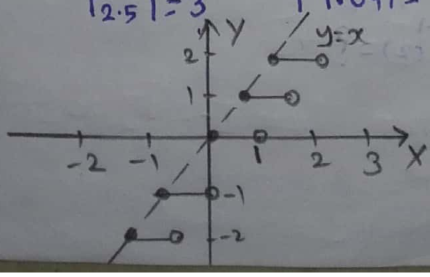
(a) **Greatest integer function:** $f: \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \lfloor x \rfloor$ is called the greatest integer function or the floor function.

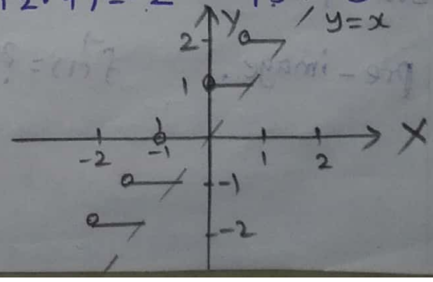
(b) **Least integer function:** $f: \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \lceil x \rceil$ is called the smallest integer function or the ceil function.

Note:

$\lfloor 1\frac{1}{5} \rfloor = 1$ $\lfloor 2.5 \rfloor = 2$, $\lfloor 3.9 \rfloor = 3$, $\lfloor -2.1 \rfloor = -3$

$\lceil 2.5 \rceil = 3$ $\lceil 1.09 \rceil = 2$ $\lceil 2.9 \rceil = -2$ $\lceil 3 \rceil = 3$





1.6.3 Types of Functions:

Definition: A function $f : A \to B$ is said to be one-to-one (or injective)

$f(a_1) = f(a_2) \Rightarrow a_1 = a_2$ $(a_1, a_2 \in A)$ (or)

if $a_1, a_2 \in A$, $a_1 \neq a_2 \Rightarrow f(a_1) \neq f(a_2)$

$\Rightarrow$ if f is 1-1 and $b = f(a_1)$ then $b \neq f(a_2)$ for any $a_2 \in A$ distinct from a_1

Otherwise, f is called many-one.

For example: $f(x) = x^2$ $(-\infty \leq x < \infty)$ is not 1-1 but the function $g(x) = x^2$ $(0 \leq x < \infty)$ is 1-1

Definition: A function $f : X \to Y$ is said to be onto (or surjective) if and only if B = range of f.

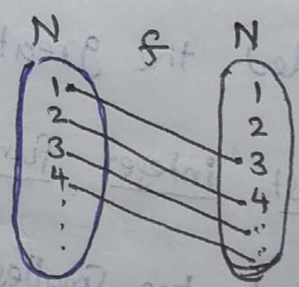
i.e., if for each $b \in B$ there exists at least one element $a \in A$ such that $f(a) = b$

Definition: A function $f : X \to Y$ is said to be 1-1 and onto (or bijective), if f is both one-one and onto.

Example 1.14 Check whether the following function are 1-1 and onto.

(i) $f : N \to N$ defined by $f(n) = n+2$

Solution:



$f(1) = 3$

$f(2) = 4$

If $f(n) = f(m)$ then

$n+2 = m+2$

$\Rightarrow n = m$

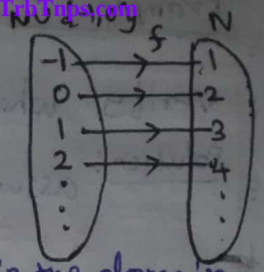
$\Rightarrow f$ is one-to-one

$1 \neq 2 \Rightarrow f(1) \neq f(2)$

f is 1-1

But f is not onto, As 1, 2 has no pre-image. $f^{-1}(1) = ?$, $f^{-1}(2) = ?$

(ii) $f : \mathbb{N} \cup \{-1, 0\} \to \mathbb{N}$ defined by $f(n) = n+2$



$-1 \neq 0 \Rightarrow f(-1) \neq f(0)$

f is one-to-one.

Now, If m is in the co-domain, then $m-2$ is in the domain.

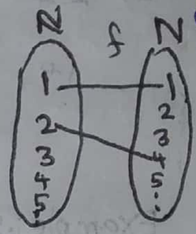
$m \in \mathbb{N}$, $f(m-2) = (m-2)+2 = m$

$\Rightarrow$ m has a pre-image.

$\therefore$ f is onto.

<u>Example 1.15</u> Check the following function for one-to-oneness and ontoness.

(i) $f : \mathbb{N} \to \mathbb{N}$ defined by $f(n) = n^2$



$f(1) = 1$
$f(2) = 4$

<u>Solution</u> $f(m) = f(n)$

$\Rightarrow m^2 = n^2$

$\Rightarrow m = n$ Since $m, n \in \mathbb{N}$

f is one-to-one.

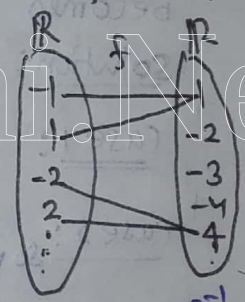
$1 \neq 2 \Rightarrow f(1) \neq f(2)$
f is 1-1

But, non-perfect square elements in the co-domain do not have pre-images

$f^{-1}(2) = ?$
$f^{-1}(3) = ?$

$\Rightarrow$ f is not onto.

(ii) $f : \mathbb{R} \to \mathbb{R}$ defined by $f(n) = n^2$



<u>Solution:</u> Since $f(-1) = 1$, $f(1) = 1$

$-1 \neq 1 \Rightarrow f(-1) = f(1)$

f is not one-to-one.

$f^{-1}(-2) = ?$

Also, the element -2 in the co-domain $\mathbb{R}$ is not image of any element x in the domain $\mathbb{R}$.

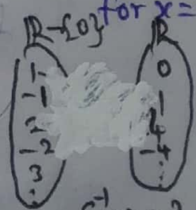
$\therefore$ f is not onto.

<u>Example 1.16</u> Check whether the following for one-to-oneness and ontoness.

(i) $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \frac{1}{x}$

<u>Solution:</u> This is not at all a function because $f(x)$ is not defined for $x = 0$

(ii) $f : \mathbb{R} - \{0\} \to \mathbb{R}$ defined by $f(x) = \frac{1}{x}$



<u>Solution</u> $f(1) = \frac{1}{1}$ $f(-1) = -1$

$1 \neq -1 \Rightarrow f(1) \neq f(-1)$ f is one-to-one.

but not onto because 0 has no pre-image.

$f^{-1}(0) = ?$

<u>Note:</u> $f : \mathbb{R} \setminus \{0\} \to \mathbb{R} \setminus \{0\}$ $f(x) = 1/x$

f is 1-1 & onto. f is bijection.

Example 1.19: If $f : \mathbb{R} - \{-1, 1\} \to \mathbb{R}$ is defined by $f(x) = \frac{x}{x^2-1}$ verify whether f is one-to-one or not.

Solution: assumption $f(x) = f(y)$ Then

$$\frac{x}{x^2-1} = \frac{y}{y^2-1}$$

$$\Rightarrow (y-x)(xy+1) = 0$$

$$\Rightarrow y = x \text{ or } xy = -1$$

$x = 2,\ y = -\frac{1}{2}$ $(2, -\frac{1}{2})$

$2 \neq -\frac{1}{2} \Rightarrow f(2) = f(-\frac{1}{2}) = \frac{2}{3}$

$$\Rightarrow 2 \neq -\frac{1}{2} \Rightarrow f(2) = f(-\tfrac{1}{2}) = \frac{2}{3}$$

ie, $f(x) = f(y) \not\Rightarrow x = y$

f is not one-to-one.

Example 1.20: Check whether the function $f(x) = x|x|$ defined on $[-2, 2]$ is one-to-one or not, If it is one-to-one, find a suitable co-domain so that the function becomes a bijection.

Solution: Let $x, y \in [-2, 2]$ such that $f(x) = f(y)$.

Case 1: If $y = 0$ then $x = 0 \Rightarrow f(x) = f(y)$

Case 2: So let $y \neq 0$ and hence $x \neq 0$.

Now, $f(x) = f(y)$
$$x|x| = y|y|$$

$$\Rightarrow \frac{x}{y} = \frac{|y|}{|x|} \quad \text{Since } \frac{|y|}{|x|} > 0,\ \frac{x}{y} > 0;\ x \text{ and } y \text{ are both +ve or -ve}$$

$$\Rightarrow x^2 = y^2$$

This is possible only if $x = y$

Thus f is one-to-one

$$f(x) = \begin{cases} -x^2 & x < 0 \\ x^2 & x \geq 0 \end{cases}$$

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So the range is $[-4, 4]$

So f becomes a bijection from $[-2, 2]$ to $[-4, 4]$.

DOMAIN AND RANGE:

You can think of a function as a Machine:

* We input Something (like a number) into the Machine.
* The machine processes the input, based on the rule.
* The machine produces an output (a number).

DOMAI AND RANGE: The domain of a function gives us a set of "allowable" inputs.

The range of a function gives us and a set of achieved outputs.

Top tips: Domain: (i) can't divide by zero.

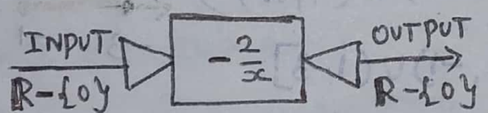
(ii) can't take Square root of neg. number.

Range: (iii) square roots can't be negative.

(iv) $f(x) = \frac{1}{g(x)} \neq 0$

Example: Show that the relation $xy=-2$ is a function for a suitable domain. Find the domain and the range of the function.

Solution: given $xy = -2 \Rightarrow y = -\frac{2}{x}$



Now $f(a_1) = f(a_2)$

$\Rightarrow -\frac{2}{a_1} = -\frac{2}{a_2}$

$\Rightarrow a_1 = a_2$

Domain $= \mathbb{R}-\{0\}$ and Range $= \mathbb{R}-\{0\}$ $\therefore f$ is a 1-1 function

How to Find the Domain Of Any Function:

① POLYNOMIAL only

$f(x) = x^2+3x+1$

$\Rightarrow$ DOMAIN = ALL REAL NUMBERS $\{\mathbb{R}\}$ or $(-\infty, \infty)$

② FRACTION only

$f(x) = \frac{2x+1}{x^2+5x+6}$ or $f(x) = \frac{2}{x^2+3}$

$\Rightarrow$ DOMAIN = SET BOTTOM $\neq 0$

③ SQUARE ROOT only

$f(x) = \sqrt{x+1}$ or

$f(x) = \sqrt{x^2+5x+6}$

$\Rightarrow$ DOMAIN = SET PART UNDER ROOT ≥ 0

④ SQUARE ROOT ON BOTTOM

$f(x) = \frac{x^2+2x+3}{\sqrt{x+1}}$

$\Rightarrow$ DOMAIN = SET PART UNDER ROOT > 0

⑤ SQUARE ROOT ON TOP

$f(x) = \frac{\sqrt{x+1}}{x^2+4}$

$\Rightarrow$ DOMAIN = BOTTOM $\neq 0$

UNDER ROOT ≥ 0

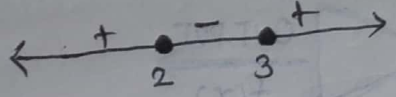
TAKE INTERSECTION

Example 1.21: Find the largest possible domain for the real valued function f defined by $f(x)=\sqrt{x^2-5x+6}$

Solution:

$x^2-5x+6 \geq 0$

$(x-2)(x-3)\geq 0$



$x\leq 2 \cup x\geq 3$

DOMAIN

$\mathbb{R},\ x\leq 2,\ x\geq 3$

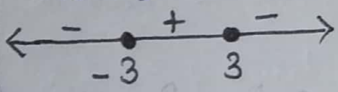
$(-\infty, 2] \cup [3,\infty)$

Example 1.24 Find the largest possible domain for the real valued function given by $f(x)=\frac{\sqrt{9-x^2}}{\sqrt{x^2-1}}$

Solution: Given $f(x)=\frac{\sqrt{9-x^2}}{\sqrt{x^2-1}}$

$9-x^2 \geq 0$

$(3-x)(3+x)\geq 0$



$x \in [-3,3]$

$x^2-1>0$

$(x-1)(x+1)>0$

$x\neq -1,\ x\neq 1$

$(-\infty,-1)\cup(1,\infty)$

$x^2>1$

$\sqrt{x^2}>\sqrt{1}$

$x>1$ & $x<-1$

TAKE INTERSECTION

$[-3,3]\cap((-\infty,-1)\cup(1,\infty))$

DOMAIN $=[-3,-1)\cup(1,3]$

7. Find the largest possible domain of the real valued fun $f(x)=\frac{\sqrt{4-x^2}}{\sqrt{x^2-9}}$

Solution: $f(x)=\frac{\sqrt{4-x^2}}{\sqrt{x^2-9}}$

$4-x^2\geq 0$

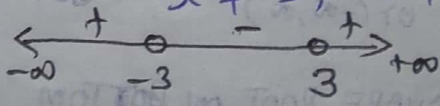
$(2+x)(2-x)\geq 0$

$[-2,2]$

$\sqrt{x^2-9}\neq 0$, so $x^2-9\neq 0$

so $(x-3)(x+3)\neq 0$

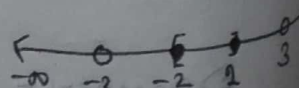
$x\neq -3,\ x\neq 3$



$(-\infty,-3)\cup(3,\infty)$

Combining

Domain of $f=[-2,2]\cap((-\infty,-3)\cup(3,\infty))$



DOMAIN $=\{\ \}$ or ϕ

Solution: $1-2\cos x \neq 0$

The function is defined for all $x \in R$ except $1-2\cos x = 0$

ie, except $\cos x = \frac{1}{2}$

ie except $x = 2n\pi \pm \frac{\pi}{3}$, $n \in Z$

Hence, the domain is $R - \{2n\pi \pm \frac{\pi}{3}\}$, $n \in Z$

6 Find the domain of $f(x) = \frac{1}{1-2\sin x}$

Solution: $1-2\sin x \neq 0$

The function is defined for all $x \in R$ except $1-2\sin x = 0$

ie, except $x = \sin^{-1}(\frac{1}{2})$

ie, except $x = n\pi + (-1)^n \frac{\pi}{6}$, $n \in Z$

Domain of $f(x)$ is $R - (n\pi + (-1)^n \frac{\pi}{6})$ $n \in Z$

Example 1.23 Find the range of the function $f(x) = \frac{1}{1-3\cos x}$

Solution: $-1 \leq \cos x \leq 1$

$\Rightarrow 3 \geq -3\cos x \geq -3$

$\Rightarrow -3 \leq -3\cos x \leq 3$

$\Rightarrow 1-3 \leq 1-3\cos x \leq 1+3$

$-2 \leq 1-3\cos x$ and $1-3\cos x \leq 4$

$\frac{1}{1-3\cos x} \leq -\frac{1}{2}$ and $\frac{1}{1-3\cos x} \geq \frac{1}{4}$

Range of $f(x)$ is $(-\infty, -\frac{1}{2}] \cup [\frac{1}{4}, \infty)$

8 Find the range of the function $\frac{1}{2\cos x - 1}$

Solution: $-1 \leq \cos x \leq 1$

$\Rightarrow -2 \leq 2\cos x \leq 2$

$\Rightarrow -2-1 \leq 2\cos x - 1 \leq 2-1$

$\Rightarrow -3 \leq 2\cos x - 1 \leq 1$

$-3 \leq 2\cos x - 1$ and $2\cos x - 1 \leq 1$

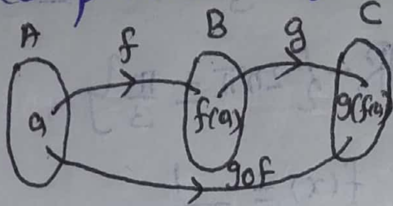
$\frac{1}{2\cos x - 1} \leq -\frac{1}{3}$ and $\frac{1}{2\cos x - 1} \geq 1$

Range of $f(x)$ is $(-\infty, -\frac{1}{3}] \cup [1, \infty)$

1.6.4 Operation on Functions.

Definition: (THE COMPOSITION OF FUNCTIONS):

Let $f: X \to Y$ and $g: Y \to Z$ be two function. Then the function $h: X \to Z$ defined as $h(x) = gof(x) = g(f(x))$ for every x is called the composition $\circ$ of f with g.



Note: (i) If $f: X \to Y_1$, $g: Y_2 \to Z$ and $Y_1 \subseteq Y_2$ then also we define gof

So we can define $gof \iff$ the range of f is contained in the domain of g

we can define $fog \iff$ the range of g is contained in the domain of f

Example 1.25: Let $f = \{(1,2), (3,4), (2,2)\}$ and $g = \{(2,1), (3,1), (4,2)\}$ Find gof and fog.

Solution: Domain of $f = \{1, 2, 3\}$ Domain of $g = \{2, 3, 4\}$

Range of $f = \{2, 4\}$ Range of $g = \{1, 2\}$

Range of $f = \{2, 4\} \subset$ Domain of $g = \{2, 3, 4\}$ we can define gof.

$$gof(x) = g(f(x))$$

$gof(1) = g(f(1)) = g(2) = 1$

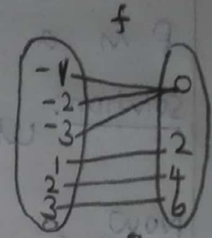
$gof(2) = g(f(2)) = g(2) = 1$

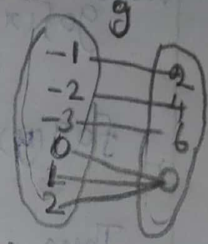
$gof(3) = g(f(3)) = g(4) = 2$

so $gof = \{(1,1), (2,1), (3,2)\}$

similarly $fog = \{(2,2), (3,2), (4,2)\}$

10 If $f, g : \mathbb{R} \to \mathbb{R}$ are defined by $f(x) = |x| + x$ and $g(x) = |x| - x$ find $g \circ f$ and $f \circ g$?





Solution:

$$f(x) = \begin{cases} 0 & x \le 0 \\ 2x & x > 0 \end{cases} \qquad g(x) = \begin{cases} -2x & x \le 0 \\ 0 & x > 0 \end{cases}$$

Let $x \le 0$ Then $(g \circ f)(x) = g(f(x)) = g(0) = -2(0) = 0$

Let $x > 0$ Then $(g \circ f)(x) = g(f(x)) = g(2x) = 0$

$g \circ f(-1) = g(f(-1)) = g(0) =$

$\Rightarrow (g \circ f)(x) = 0$ for all x.

$g \circ f(-2) = g(f(-2)) = g(0) = 0$

Let $x > 0$ then $(f \circ g)(x) = 0$, Let $x \le 0$ then $(f \circ g)(x) = f(g(x)) = f(-2x) = 2(-2x) = -4x$.

$g \circ f(1) = g(f(1)) = g(2) = 0$

1.6.5 Inverse of a Function

Definition 1.9: Let $f : X \to Y$ be a bijection. The function $g : Y \to X$ defined by $g(y) = x$ if $f(x) = y$ is called the inverse of f and is denoted by f^{-1}

Definition 1.10: A function $f : X \to Y$ is said to be invertible if there exists a function $g : Y \to X$ such that $g \circ f = I_X$ and $f \circ g = I_Y$ where I_X and I_Y are identity function on X and Y. In this case, g is called the inverse of f and g is denoted by f^{-1}

$f : X \to Y$ and $g : Y \to X$ such that $g \circ f = I_X$ and $f \circ g = I_Y$ then both f and g are bijections and inverse to each other. $f^{-1} = g$ and $g = f^{-1}$

For example,

1) $f(x) = \dfrac{x+2}{x}$ $(x \ne 0)$

$y = \dfrac{x+2}{x}$

$x = \dfrac{y+2}{y}$

$f^{-1}(x) = \dfrac{2}{x-1}$

(2) $f(x) = \{(1,2)\ (3,-4), (5,-8)\}$

$f^{-1}(x) = \{(2,1), (-4,3), (-8,-5)\}$

$f(x) = \dfrac{x-3}{5}$

$y = \dfrac{x-3}{5}$

$x = \dfrac{y-3}{5}$

$f^{-1}(x) = \cancel{y} = 5x+3$

12. If $f:\mathbb{R}\to\mathbb{R}$ is defined by $f(x)=3x-5$ prove that f is a bijection and find its inverse.

Solution: Let $f(x)=3x-5$. Let $g(y)=\frac{y+5}{3}$

Now
$$g\circ f(x)=g(f(x))=g(3x-5)=\frac{(3x-5)+5}{3}=x$$

$$f\circ g(y)=f(g(y))=f\left(\frac{y+5}{3}\right)=3\left(\frac{y+5}{3}\right)-5=y$$

Thus, $g\circ f=I_X$ and $f\circ g=I_Y$

$\Rightarrow$ f and g are bijections and $f^{-1}=g$, $g=f^{-1}$

Hence f is a bijection and $f^{-1}(y)=\frac{y+5}{3}$

Replacing y by x $\quad f^{-1}(x)=\frac{x+5}{3}$

13

1.7 Graphing Functions using Transformations

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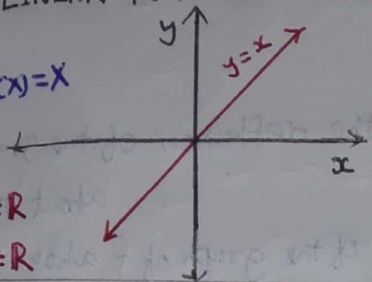
Graphs of Basic Functions

1) LINEAR FUNCTION

$f(x) = x$

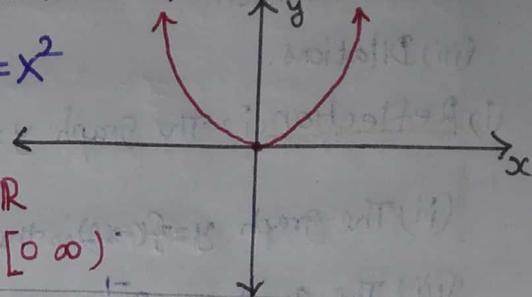


$D = \mathbb{R}$

$R = \mathbb{R}$

(2) QUADRATIC FUNCTION

$f(x) = x^2$

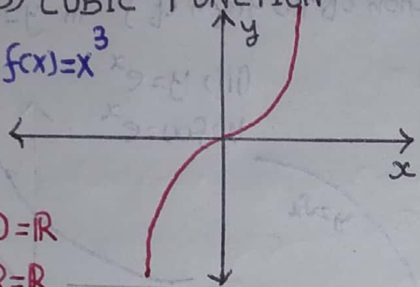


$D = \mathbb{R}$

$R = [0\ \infty)$

(3) CUBIC FUNCTION

$f(x) = x^3$



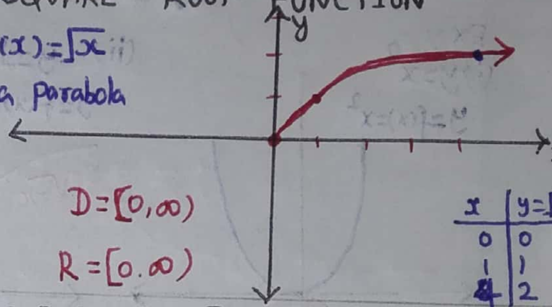
$D = \mathbb{R}$

$R = \mathbb{R}$

(4) SQUARE ROOT FUNCTION

$f(x) = \sqrt{x}$

half a parabola



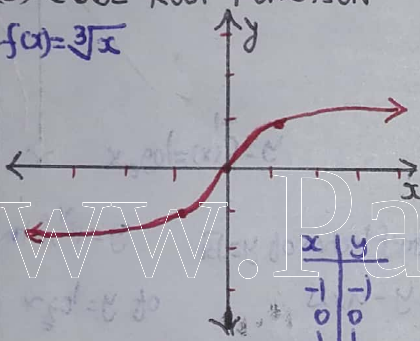
$D = [0, \infty)$

$R = [0. \infty)$

x	$y = \sqrt{x}$
0	0
1	1
4	2

(5) CUBE ROOT FUNCTION

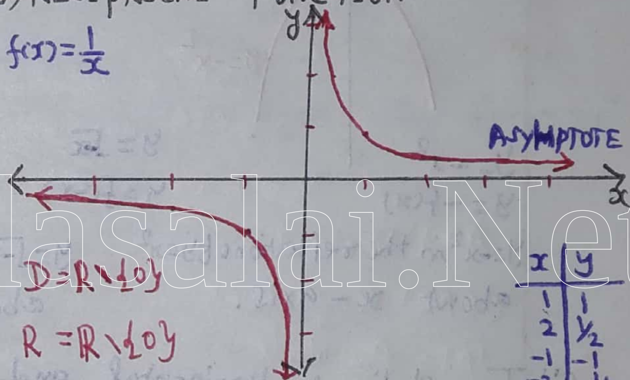
$f(x) = \sqrt[3]{x}$



x	y
-1	-1
0	0
1	1
8	2

(6) RECIPROCAL FUNCTION

$f(x) = \frac{1}{x}$



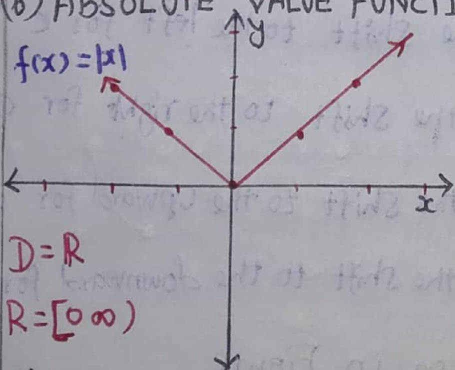
$D = \mathbb{R} \setminus \{0\}$

$R = \mathbb{R} \setminus \{0\}$

x	y
1	1
2	½
-1	-1
-2	-½

(6) ABSOLUTE VALUE FUNCTION

$f(x) = |x|$

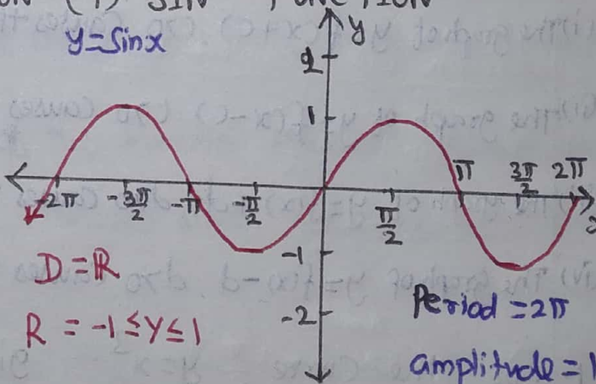


$D = \mathbb{R}$

$R = [0\ \infty)$

(7) SIN FUNCTION

$y = \sin x$



$D = \mathbb{R}$

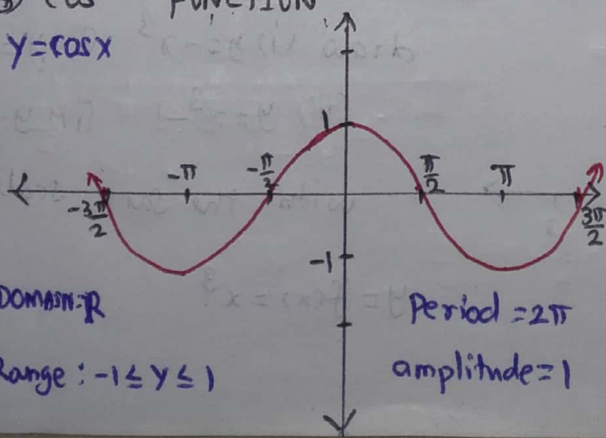
$R = -1 \le y \le 1$

Period $= 2\pi$

amplitude $= 1$

(8) COS FUNCTION

$y = \cos x$



DOMAIN: $\mathbb{R}$

Range : $-1 \le y \le 1$

Period $= 2\pi$

amplitude $= 1$

1.7 Graphing Functions using Transformation.

(i) Reflection

(ii) Translation.

(iii) Dilations.

(i) Reflection: (i) The graph $y=-f(x)$ is the reflection of the graph of f about the x-axis.

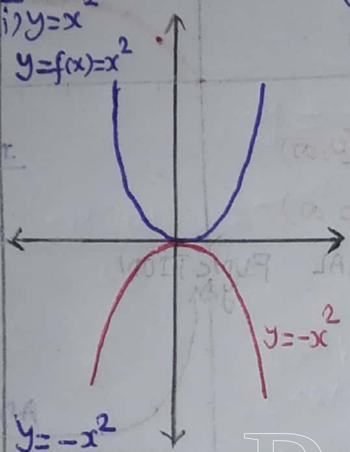
(ii) The graph $y=f(-x)$ is the reflection of the graph of f about the y-axis.

(iii) The graph $y=f^{-1}(x)$ is the reflection of the graph of f in $y=x$.

Ex:

(i) $y=x^2$

$y=f(x)=x^2$



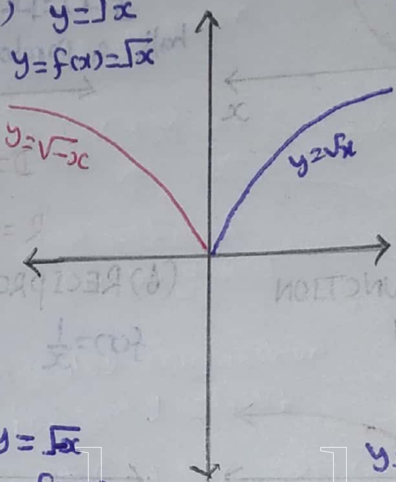
$y=-x^2$

$y=-f(x)$

$y=-x^2$ is the reflection of $y=x^2$ about x-axis.

(ii) $y=\sqrt{x}$

$y=f(x)=\sqrt{x}$



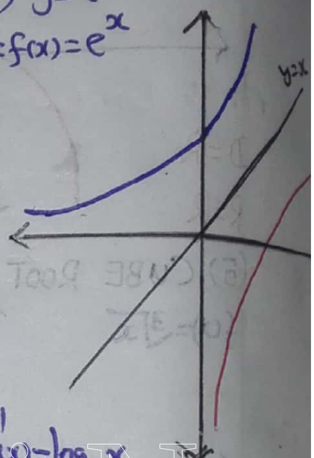
$y=\sqrt{-x}$

$y=f(-x)$

$y=\sqrt{-x}$ is the reflection of $y=\sqrt{x}$ about y-axis.

(iii) $y=e^x$

$y=f(x)=e^x$



$y=f^{-1}(x)=\log_e x$

$y=e^x$ is the reflection of $y=\log_e x$ about $y=x$

(ii) Translation: Horizontal and Vertical Translations.

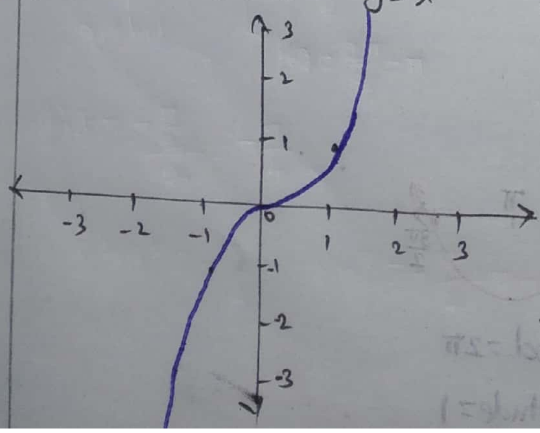
(i) The graph of $y=f(x+c)$, $c>0$ causes the shift to the left for c units.

(ii) The graph of $y=f(x-c)$ $c>0$ causes the shift to the right for c units.

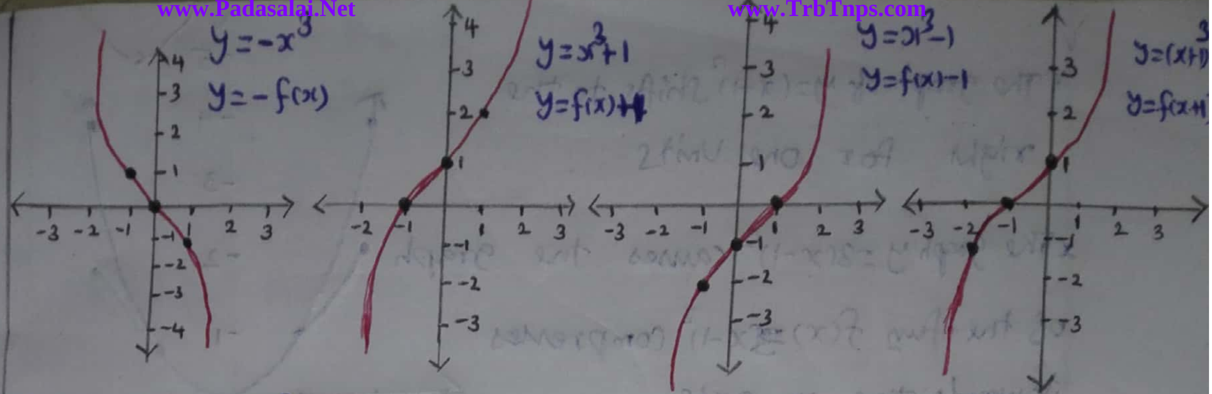
(iii) The graph of $y=f(x)+d$, $d>0$ causes the shift to the upward for d units.

(iv) The graph of $y=f(x)-d$, $d>0$ causes the shift to the downward for d units.

1. For the curve $y=x^3$ given in Figure draw (i) $y=-x^3$ (ii) $y=x^3+1$ (iii) $y=x^3-1$ (iv) $y=(x+1)^3$ with the same scale.



$y=f(x)=x^3$



(i) The graph $y=-x^3$ is the reflection of the graph of f about the x-axis

(ii) The graph of $y=x^3+1$ causes the shift to the upward for 1 units.

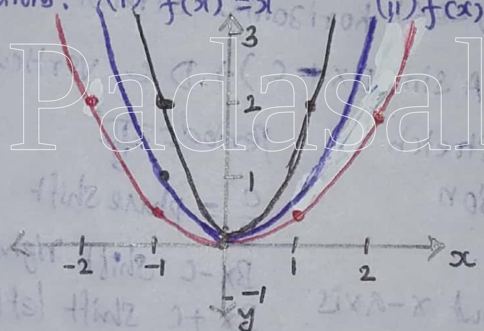
(iii) $y=x^3-1$ causes the graph of the function $f(x)=x^3$ shifts to downward for one units

(iv) $y=(x+1)^3$ causes the graph of the function $y=x^3$ shifts to the left for one unit.

(iii) <u>Dilation</u>: Dilation is also a transformation which causes the curve stretches (expands) or compresses (contracts).

* $a>1$ the graph moves away from the x-axis.
* $0<a<1$ the graph moves towards the x-axis.

<u>Ex</u>: Consider the functions: (i) $f(x)=x^2$ (ii) $f(x)=\frac{1}{2}x^2$ (iii) $f(x)=2x^2$



$f(x)=\frac{1}{2}x^2$ causes the graph of the function $f(x)=x^2$ stretches towards the x-axis.

$f(x)=2x^2$ causes the graph of the function $f(x)=x^2$ compresses towards the y-axis that is, moves away from the x-axis.

<u>Exercise</u>

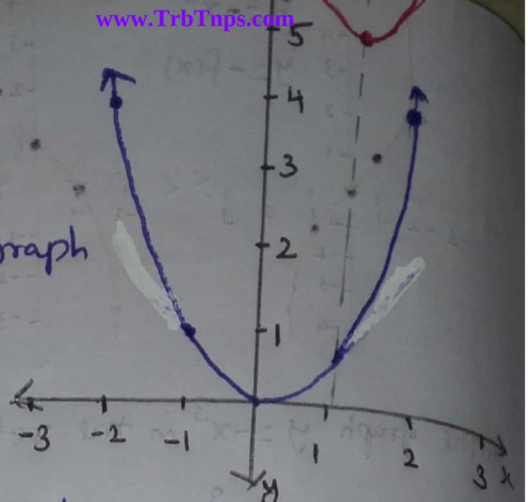
4. Write the steps to obtain the graph of the fun $y=3(x-1)^2+5$ from the graph $y=x^2$

<u>Sol</u>

$f(x)=a(x-h)^2+K$

vertex $=(h,K)$ axis of Symmetry $x=h$

vertex $=(1,5)$ axis of Symmetry $x=1$



* The graph of $y=(x-1)^2$ shifts to the right for one units

* The graph $y=3(x-1)^2$ causes the graph of the fun $f(x)=(x-1)^2$ compresses towards the y-axis

* $f(x)=3(x-1)^2+5$, causes the graph of the function $f(x)=3(x-1)^2$ shifts to the upward for 5 units

Note:

(i)

a>1 stretch
a<1 compression

$$y = \pm a\, f(\pm x \pm c) \pm d \leftarrow \text{vertical shift.}$$

-ve reflection over x-axis

-ve reflection over the y-axis

$x+c \leftarrow$

$x-c \rightarrow$

(ii)

B>1 horizontal compression
0<B<1 horizontal stretch

$$y = A\sin(Bx + C) + D \leftarrow \text{vertical shift.}$$

A>1 vertical stretch

0<A<1 compression

|A| = Amplitude.

-A reflection about x-axis

Period $= \frac{2\pi}{B}$

$\frac{C}{B}$ = phase shift

Bx-C shift right

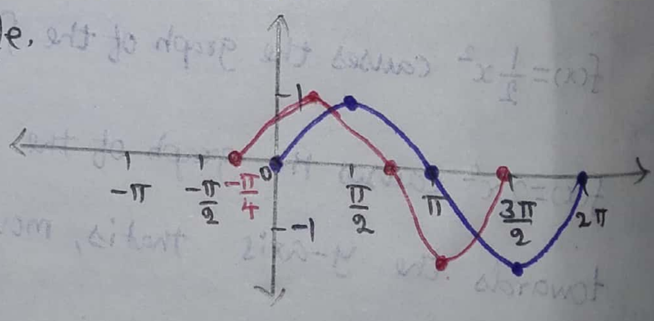
Bx+C shift left

Graph one complete cycle.

1. $y = \sin\left(x+\frac{\pi}{4}\right)$

$P.S = -\frac{\pi}{4}$

$x+\frac{\pi}{4}=0$



2. $y = -1\cos\left(2x+\frac{\pi}{2}\right)$

Amp = 1

reflect

Per $= \frac{2\pi}{2} = \pi$

$P.S = -\frac{\frac{\pi}{2}}{2} = -\frac{\pi}{4}$

Start $= -\frac{\pi}{4}$ End $= -\frac{\pi}{4}+\pi = \frac{3\pi}{4}$