Chapter4 Inverse Trigonometric Functions

Example 4.1 Find the principal value of

$sin^{-1}\left(-\frac{1}{2}\right)$(in radians and degrees).

Solution:

We know $If\ sin^{-1}(x) = \theta$

$then\ \sin\theta = x$

and $sin(-\theta) = -\sin\theta$

Let $sin^{-1}\left(-\frac{1}{2}\right) = x$

then $\sin x = -\frac{1}{2}$

The range of the principal value of

$$sin^{-1}(x) = \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

Clearly $-\frac{\pi}{6} \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

and $sin\left(-\frac{\pi}{6}\right) = -\sin\left(\frac{\pi}{6}\right)$

$= -\frac{1}{2}$

The principal value of

$sin^{-1}\left(-\frac{1}{2}\right) = -\frac{\pi}{6}$ (in radians)

$= -30^0$(in degrees)

Example 4.2 Find the principal value of

$sin^{-1}(2)$, if it exists.

Solution:

The domain of $sin^{-1}(x) = [-1,1]$

Since $2 \notin [-1,1]$

So, $sin^{-1}(2)$ does not exists.

Example 4.3 Find the principal value of

(i) $sin^{-1}\left(\frac{1}{\sqrt{2}}\right)$

Let $sin^{-1}\left(\frac{1}{\sqrt{2}}\right) = x$

then $\sin x = \frac{1}{\sqrt{2}}$

The range of the principal value of

$$sin^{-1}(x) = \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

We know, $\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$

Clearly $\frac{\pi}{4} \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

(ii) $sin^{-1}\left[sin\left(-\frac{\pi}{3}\right)\right]$

Solution: We know $sin^{-1}[sin(\theta)] = \theta$

So, $sin^{-1}\left[sin\left(-\frac{\pi}{3}\right)\right] = -\frac{\pi}{3}$

The range of the principal value of

$$sin^{-1}(x) = \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

Clearly $-\frac{\pi}{3} \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

(iii) $sin^{-1}\left[sin\left(\frac{5\pi}{6}\right)\right]$

Solution: We know $sin^{-1}[sin(\theta)] = \theta$

and $sin(\pi - \theta) = \sin\theta$

So, $sin^{-1}\left[sin\left(\frac{5\pi}{6}\right)\right]$

$= sin^{-1}\left[sin\left(\frac{6\pi}{6} - \frac{\pi}{6}\right)\right]$

$= sin^{-1}\left[sin\left(\pi - \frac{\pi}{6}\right)\right]$

$= sin^{-1}\left[sin\left(\frac{\pi}{6}\right)\right]$

$= \frac{\pi}{6}$

Clearly $\frac{\pi}{6} \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

Example 4.4 Find the domain of $sin^{-1}(2 - 3x^2)$

Solution: The domain of $sin^{-1}(x) = [-1,1]$

ie., $-1 \le x \le 1$

So, the domain of $sin^{-1}(2 - 3x^2)$

$= -1 \le 2 - 3x^2 \le 1$

Adding -2, $-1 - 2 \le 2 - 2 - 3x^2 \le 1 - 2$

$-3 \le -3x^2 \le -1$

Hence, $-3 \le -3x^2$ gives $\Rightarrow x^2 \le 1$

and, $-3x^2 \le -1$ gives $\Rightarrow x^2 \ge \frac{1}{3}$

We get $\frac{1}{3} \le x^2 \le 1$

$\therefore \frac{1}{\sqrt{3}} \le |x| \le 1$, which gives

$x \in \left[-1, -\frac{1}{\sqrt{3}}\right] \cup \left[\frac{1}{\sqrt{3}}, 1\right]$

since $a \le |x| \le b$, implies

$x \in [-b, -a] \cup [a, b]$

EXERCISE 4.1

1. Find all the values of x such that

(i) $-10\pi \le x \le 10\pi$ and $\sin x = 0$

Solution: $\sin x = 0$ gives

$x = n\pi$ [because $\sin n\pi = 0$]

where n = 0, ±1, ±2, … … ± 10

(ii) $-8\pi \le x \le 8\pi$ and $\sin x = -1$

Solution: $\sin x = -1$ gives

$\sin x = \sin\left(-\frac{\pi}{2}\right)$

$x = -\frac{\pi}{2}, \frac{3\pi}{2}, \frac{7\pi}{2}, \ldots$

$x = (4n-1)\frac{\pi}{2}, n \in Z$

where n = 0, ±1, ±2, ±3, 4

2. Find the period and amplitude of

(i) $y = \sin 7x$

Solution: $y = \sin 7x$

Period of the function $\sin x$ is 2π

∴ Period of the function $\sin 7x$ is $\frac{2\pi}{7}$

since $\sin 7\left(\frac{2\pi}{7}\right) = \sin 2\pi$

Amplitude of $\sin x$ is 1

∵ [$Max\ ht.$ sin $curve$ is 1]

∴ Amplitude of $\sin 7x$ is also 1

(ii) $y = -\sin\left(\frac{1}{3}x\right)$

Solution: $y = -\sin\left(\frac{1}{3}x\right)$

Period of the function $\sin x$ is 2π

∴ Period of the function $\sin\left(\frac{1}{3}x\right)$ is 6π

since $\sin\left(\frac{1}{3}6\pi\right) = \sin 2\pi$

Amplitude of $\sin\left(\frac{1}{3}x\right)$ is 1

(iii) $y = 4\sin(-2x)$

Solution: $y = 4\sin(-2x) = -4\sin(2x)$

Period of the function $\sin x$ is 2π

∴ Period of the function $\sin(2x)$ is π

since $\sin 2(\pi) = \sin 2\pi$

Amplitude of $\sin(2x)$ is 4

3. Sketch the graph of

$y = \sin\left(\frac{1}{3}x\right)$ for $0 \le x \le 6\pi$

Solution: $y = \sin\left(\frac{1}{3}x\right)$

Period of the function $\sin x$ is 2π

∴ Period of the function $\sin\left(\frac{1}{3}x\right)$ is 6π

since $\sin\left(\frac{1}{3}6\pi\right) = \sin 2\pi$

Amplitude of $\sin\left(\frac{1}{3}x\right)$ is 1

Taking $\frac{1}{3}x = y$

Then, $x = 0, y = 0, x = \pi, y = \frac{\pi}{3}$

$x = 2\pi, y = \frac{2\pi}{3}, x = 3\pi, y = \frac{3\pi}{3} = \pi$

$x = 4\pi, y = \frac{4\pi}{3}, x = 5\pi, y = \frac{5\pi}{3}$

$x = 6\pi, y = \frac{6\pi}{3} = 2\pi$

By plotting the points the graph is

4. Find the value of

(i) $\sin^{-1}\left[\sin\left(\frac{2\pi}{3}\right)\right]$

Solution:

The principal value of $\sin^{-1}(x) = \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

But $\frac{2\pi}{3} \notin \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

$\sin\left(\frac{2\pi}{3}\right) = \sin\left(\frac{3\pi}{3} - \frac{\pi}{3}\right)$

$= \sin\left(\pi - \frac{\pi}{3}\right)$

$= \sin\left(\frac{\pi}{3}\right) \qquad \frac{\pi}{3} \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

$[\because sin(\pi-\theta)=\sin\theta]$

$sin\left(\frac{2\pi}{3}\right)= sin\left(\frac{\pi}{3}\right)$

So, $sin^{-1}\left[sin\left(\frac{2\pi}{3}\right)\right]=sin^{-1}\left[sin\left(\frac{\pi}{3}\right)\right]$

$=\frac{\pi}{3}$

$[\because\ sin^{-1}[sin(\theta)]=\theta]$

(ii) $sin^{-1}\left[sin\left(\frac{5\pi}{4}\right)\right]$

Solution: $sin\left(\frac{5\pi}{4}\right)= sin\left(\frac{4\pi}{4}+\frac{\pi}{4}\right)$

$= sin\left(\pi+\frac{\pi}{4}\right)$

$=- sin\left(\frac{\pi}{4}\right)$

$= sin\left(-\frac{\pi}{4}\right)$

$[\because\ sin(\pi+\theta)=-\sin\theta]$ and

$[\because\ sin(-\theta)=-\sin\theta]$

$sin\left(\frac{5\pi}{4}\right)= sin\left(-\frac{\pi}{4}\right)$

So, $sin^{-1}\left[sin\left(\frac{5\pi}{4}\right)\right]=sin^{-1}\left[sin\left(-\frac{\pi}{4}\right)\right]$

$=-\frac{\pi}{4}$

$[\because\ sin^{-1}[sin(\theta)]=\theta]$

5. For what value of x does $sin\ x=sin^{-1}x$?

Solution: $sin\ x=sin^{-1}x$ is possible only

when $x\ =\ 0,\because x\ \in R$

6. Find the domain of the following

(i) $f(x)=sin^{-1}\left(\frac{x^2+1}{2x}\right)$

Solution:

The domain of $sin^{-1}(x)=[-1,1]$

ie., $-1\le x\le 1$

So, the domain of $sin^{-1}\left(\frac{x^2+1}{2x}\right)$

$=-1\le\left(\frac{x^2+1}{2x}\right)\le 1$

So, $\left(\frac{x^2+1}{2x}\right)\ge -1$

$x^2\ +\ 1\ge -2x$

$x^2+2x+\ 1\ge 0$

$(x+1)^2\ge 0$

$-1\le x\le 1$

and, $\left(\frac{x^2+1}{2x}\right)\le 1$

$x^2\ +\ 1\le 2x$

$x^2-2x+\ 1\le 0$

$(x-1)^2\le 0$ which is impossible.

(ii) $g(x)=2sin^{-1}(2x-1)-\frac{\pi}{4}$

Solution:

The domain of $sin^{-1}(x)=[-1,1]$

ie., $-1\le x\le 1$

So, the domain of $sin^{-1}(2x-1)$

$=-1\le(2x-1)\le 1$

Adding 1, $0\le 2x\le 2$

$0\le\ x\ \le 1$

7. Find the value of

$sin^{-1}\left(sin\frac{5\pi}{9}\cos\frac{\pi}{9}+cos\frac{5\pi}{9}\sin\frac{\pi}{9}\right)$

Solution:

We know

$\sin(A+B)=\sin A\cos B+\cos A\sin B$

$\therefore\left(sin\frac{5\pi}{9}\cos\frac{\pi}{9}+cos\frac{5\pi}{9}\sin\frac{\pi}{9}\right)=\sin\left(\frac{5\pi}{9}+\frac{\pi}{9}\right)$

$=\sin\left(\frac{6\pi}{9}\right)$

$=\sin\left(\frac{2\pi}{3}\right)$

But $\frac{2\pi}{3}\notin\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$

$sin\left(\frac{2\pi}{3}\right)= sin\left(\frac{3\pi}{3}-\frac{\pi}{3}\right)$

$= sin\left(\pi-\frac{\pi}{3}\right)$

$= sin\left(\frac{\pi}{3}\right)$

$[\because \ sin(\pi - \theta) = \sin\theta]$

$\therefore sin^{-1}\left(sin\frac{5\pi}{9}\cos\frac{\pi}{9} + cos\frac{5\pi}{9}\sin\frac{\pi}{9}\right)$

$= sin^{-1}\left(sin\left(\frac{\pi}{3}\right)\right)$

$= \frac{\pi}{3}, \quad \frac{\pi}{3} \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

$[\because \ sin^{-1}[sin(\theta)] = \theta]$

Example 4.5

Find the principal value of $cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$

Solution: Let $cos^{-1}\left(\frac{\sqrt{3}}{2}\right) = y,$

then $\cos y = \frac{\sqrt{3}}{2}$

The range of the principal value of $cos^{-1}(x)$ is $[0, \pi]$. Hence the value of y $\in [0, \pi]$

We know, $\cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$ and $\frac{\pi}{6} \in [0, \pi]$

So, the principal value of $cos^{-1}\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{6}$

Example 4.6 Find

(i) $cos^{-1}\left(-\frac{1}{\sqrt{2}}\right)$

Solution: Let $cos^{-1}\left(-\frac{1}{\sqrt{2}}\right) = y$

Then $\cos y = -\frac{1}{\sqrt{2}}$

The range of the principal value of $cos^{-1}(x)$ is $[0, \pi]$. Hence the value of y $\in [0, \pi]$

We know, $\cos\frac{3\pi}{4} = -\frac{1}{\sqrt{2}}$ and $\frac{3\pi}{4} \in [0, \pi]$

So, the principal value of $cos^{-1}\left(-\frac{1}{\sqrt{2}}\right) = \frac{3\pi}{4}$

Because, $\cos\frac{3\pi}{4} = \cos\left(\frac{4\pi}{4} - \frac{\pi}{4}\right)$

$= \cos\left(\pi - \frac{\pi}{4}\right)$

$= -\cos\left(\frac{\pi}{4}\right)$

$[\because cos(\pi - \theta) = -\cos\theta]$

$\cos\frac{3\pi}{4} = -\frac{1}{\sqrt{2}}$

(ii) $cos^{-1}\left[cos\left(-\frac{\pi}{3}\right)\right]$

Solution: $cos\left(-\frac{\pi}{3}\right) = cos\left(\frac{\pi}{3}\right)$

Because, $cos(-\theta) = \cos\theta$

The range of the principal value of $cos^{-1}(x)$ is $[0, \pi]$

Hence the value of y $\in [0, \pi]$ since $\frac{\pi}{3} \in [0, \pi]$

$cos^{-1}\left[cos\left(-\frac{\pi}{3}\right)\right] = cos^{-1}\left[cos\left(\frac{\pi}{3}\right)\right]$

$= \frac{\pi}{3}$

$[\because \ cos^{-1}[cos(\theta)] = \theta]$

(iii) $cos^{-1}\left[cos\left(\frac{7\pi}{6}\right)\right]$

Solution: The range of the principal value of $cos^{-1}(x)$ is $[0, \pi]$

Hence the value of y $\in [0, \pi]$ since $\frac{7\pi}{6} \notin [0, \pi]$

$cos\left(\frac{7\pi}{6}\right) = cos\left(\frac{12\pi}{6} - \frac{5\pi}{6}\right)$

$= cos\left(2\pi - \frac{5\pi}{6}\right)$

$= cos\left(-\frac{5\pi}{6}\right)$

$= cos\left(\frac{5\pi}{6}\right)$

$[\because cos(2\pi - \theta) = cos(-\theta) = \cos\theta]$

$cos\left(\frac{7\pi}{6}\right) = cos\left(\frac{5\pi}{6}\right), \quad \frac{5\pi}{6} \in [0, \pi]$

$cos^{-1}\left[cos\left(\frac{7\pi}{6}\right)\right] = cos^{-1}\left[cos\left(\frac{5\pi}{6}\right)\right]$

$= \frac{5\pi}{6}$

$[\because \ cos^{-1}[cos(\theta)] = \theta]$

Example 4.7 Find the domain of $cos^{-1}\left(\frac{2 + \sin x}{3}\right)$

Solution:

The domain of $cos^{-1}(x)$ is $-1 \le x \le 1$ or $|x| \le 1$

$-1 \le \frac{2 + \sin x}{3} \le 1$

Multiply by 3, $-3 \le 2 + \sin x \le 3$

Adding -2, $-5 \le \sin x \le 1$

Reduces to $-1 \le \sin x \le 1$

$\therefore \ sin^{-1}(-1) \le x \le sin^{-1}(1)$ or

$$-\frac{\pi}{2} \le x \le \frac{\pi}{2}$$

So, domain of $cos^{-1}\left(\frac{2+\sin x}{3}\right)$ is $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$

EXERCISE 4.2

1. Find all values of x such that

(i) $-6\pi \le x \le 6\pi$ and $cos\ x = 0$

Solution: $cos\ x = 0$

$$\Rightarrow x = (2n+1) \pm \frac{\pi}{2}$$

$$n = 0, \pm1, \pm2, \pm3, \pm4, \pm5$$

(ii) $-5\pi \le x \le 5\pi$ and $cos\ x = 1$

Solution: $cos\ x = 1$

$$\Rightarrow x = (2n+1)\pi$$

$$n = 0, \pm1, \pm2, -3$$

2. State the reason for $cos^{-1}\left[cos\left(-\frac{\pi}{6}\right)\right] \ne -\frac{\pi}{6}$.

Solution: $cos\left(-\frac{\pi}{6}\right) = cos\left(\frac{\pi}{6}\right)$

Since $cos(-\theta) = cos(\theta)$

$$cos^{-1}\left[cos\left(-\frac{\pi}{6}\right)\right] = cos^{-1}\left[cos\left(\frac{\pi}{6}\right)\right]$$

$$= \frac{\pi}{6}$$

Since $cos^{-1}[cos(\theta)] = \theta$

$$\therefore cos^{-1}\left[cos\left(-\frac{\pi}{6}\right)\right] \ne -\frac{\pi}{6}$$

3. Is $cos^{-1}(-x) = \pi - cos^{-1}(x)$ true?

Justify your answer.

Solution: Let $\theta = cos^{-1}(-x)$

Then, $\cos\theta = -x$

$$\cos(\pi-\theta) = -x$$

$$\pi - \theta = cos^{-1}(x)$$

$$\Rightarrow \theta = \pi - cos^{-1}(x)$$

ie., $cos^{-1}(-x) = \pi - cos^{-1}(x)$

So, it is true.

4. Find the principal value of $cos^{-1}\left(\frac{1}{2}\right)$.

Solution: Let $cos^{-1}\left(\frac{1}{2}\right) = \theta$

Then, $\cos\theta = \frac{1}{2} = \cos\frac{\pi}{3}$

$$\theta = \frac{\pi}{3} \in [0,\pi]$$

$$\therefore cos^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3}$$

5. Find the value of

(i) $2cos^{-1}\left(\frac{1}{2}\right) + sin^{-1}\left(\frac{1}{2}\right)$

Solution: Let $cos^{-1}\left(\frac{1}{2}\right) = \theta$

Then, $\cos\theta = \frac{1}{2} = \cos\frac{\pi}{3}$

$$\theta = \frac{\pi}{3} \in [0,\pi]$$

$$\therefore cos^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3}$$

If $sin^{-1}\left(\frac{1}{2}\right) = \theta$

Then, $\sin\theta = \frac{1}{2} = \sin\frac{\pi}{6}$

$$\theta = \frac{\pi}{6}$$

$$\therefore sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}$$

$$2cos^{-1}\left(\frac{1}{2}\right) + sin^{-1}\left(\frac{1}{2}\right) = 2\left(\frac{\pi}{3}\right) + \frac{\pi}{6}$$

$$= \frac{2\pi}{3} + \frac{\pi}{6}$$

$$= \frac{4\pi+\pi}{6}$$

$$= \frac{5\pi}{6}$$

(ii) $cos^{-1}\left(\frac{1}{2}\right) + sin^{-1}(-1)$

Solution: Let $cos^{-1}\left(\frac{1}{2}\right) = \theta$

Then, $\cos\theta = \frac{1}{2} = \cos\frac{\pi}{3}$

$$\theta = \frac{\pi}{3} \in [0,\pi]$$

$$\therefore cos^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3}$$

If $sin^{-1}(-1) = \theta$

Then, $\sin\theta = -1 = \sin\left(-\frac{\pi}{2}\right)$

$$\theta = -\frac{\pi}{2}$$

$$\therefore sin^{-1}(-1) = -\frac{\pi}{2}$$

$$cos^{-1}\left(\frac{1}{2}\right) + sin^{-1}(-1) = \frac{\pi}{3} - \frac{\pi}{2}$$

$$= \frac{2\pi - 3\pi}{6}$$

$$= -\frac{\pi}{6}$$

(iii) $cos^{-1}\left(cos\frac{\pi}{7}\cos\frac{\pi}{17} - sin\frac{\pi}{7}\sin\frac{\pi}{17}\right)$

Solution:

We know

$\cos(A+B) = \cos A\cos B - \sin A\sin B$

$$cos\frac{\pi}{7}\cos\frac{\pi}{17} - sin\frac{\pi}{7}\sin\frac{\pi}{17} = \cos\left(\frac{\pi}{7} + \frac{\pi}{17}\right)$$

$$= \cos\left(\frac{17\pi+7\pi}{119}\right)$$

$$= \cos\left(\frac{24\pi}{119}\right)$$

$$\therefore cos^{-1}\left(cos\frac{\pi}{7}\cos\frac{\pi}{17} - sin\frac{\pi}{7}\sin\frac{\pi}{17}\right)$$

$$= cos^{-1}\left(\cos\left(\frac{24\pi}{119}\right)\right)$$

$$= \frac{24\pi}{119}$$

Since $cos^{-1}[cos(\theta)] = \theta$

6. Find the domain of

(i) $f(x) = sin^{-1}\left(\frac{|x|-2}{3}\right) + cos^{-1}\left(\frac{1-|x|}{4}\right)$

Solution:

$$f(x) = sin^{-1}\left(\frac{|x|-2}{3}\right) + cos^{-1}\left(\frac{1-|x|}{4}\right)$$

Let $g(x) = sin^{-1}\left(\frac{|x|-2}{3}\right)$

$$-1 \le \frac{|x|-2}{3} \le 1$$

$$-3 \le |x| - 2 \le 3$$

$$-1 \le |x| \le 5 \quad ......(1)$$

Let $h(x) = cos^{-1}\left(\frac{1-|x|}{4}\right)$

$$-1 \le \left(\frac{1-|x|}{4}\right) \le 1$$

$$-4 \le 1 - |x| \le 4$$

$$-5 \le |x| \le 3 \quad ......(2)$$

From (1) and (2)

$$f(x) = sin^{-1}\left(\frac{|x|-2}{3}\right) + cos^{-1}\left(\frac{1-|x|}{4}\right)$$

$x \le 5$, This gives

$$-5 \le x \le 5$$

(ii) $g(x) = sin^{-1}x + cos^{-1}x$

Solution: $g(x) = sin^{-1}x + cos^{-1}x$

$$-1 \le x \le 1$$

7. For what value of x , the inequality $\frac{\pi}{2} < cos^{-1}(3x-1) < \pi$ holds?

Solution: $\frac{\pi}{2} < cos^{-1}(3x-1) < \pi$

$$cos\frac{\pi}{2} < (3x-1) < \cos\pi$$

$$0 < 3x - 1 < -1$$

$$1 < 3x < 0$$

$$\frac{1}{3} < x < 0$$

This inequality holds only if $x < 0$ or $x > \frac{1}{3}$

8. Find the value of

(i) $cos\left[cos^{-1}\left(\frac{4}{5}\right) + sin^{-1}\left(\frac{4}{5}\right)\right]$

Solution: We know that,

$$cos^{-1}(x) + sin^{-1}(x) = \frac{\pi}{2}$$

$$\therefore cos^{-1}\left(\frac{4}{5}\right) + sin^{-1}\left(\frac{4}{5}\right) = \frac{\pi}{2}$$

$$cos\left[cos^{-1}\left(\frac{4}{5}\right) + sin^{-1}\left(\frac{4}{5}\right)\right] = cos\left[\frac{\pi}{2}\right]$$

$$= 0$$

(ii) $cos^{-1}\left[cos\left(\frac{4\pi}{3}\right)\right] + cos^{-1}\left[cos\left(\frac{5\pi}{4}\right)\right]$

Solution: $cos\left(\frac{4\pi}{3}\right) = cos\left(\frac{3\pi}{3} + \frac{\pi}{3}\right)$

$$= cos\left(\pi + \frac{\pi}{3}\right)$$

Since $cos(\pi + \theta) = -\cos\theta$

$$cos\left(\pi+\frac{\pi}{3}\right)=-\,cos\left(\frac{\pi}{3}\right)$$

$$\therefore\ cos\left(\frac{4\pi}{3}\right)=-\,cos\left(\frac{\pi}{3}\right)$$

$$cos^{-1}\left[cos\left(\frac{4\pi}{3}\right)\right]=cos^{-1}\left[-\,cos\left(\frac{\pi}{3}\right)\right]$$

$$=-\,cos^{-1}\left[cos\left(\frac{\pi}{3}\right)\right]$$

$$=-\frac{\pi}{3}$$

and $cos\left(\frac{5\pi}{4}\right)=cos\left(\frac{4\pi}{4}+\frac{\pi}{4}\right)$

$$=cos\left(\pi+\frac{\pi}{4}\right)$$

Since $cos(\pi+\theta)=-\cos\theta$

$$cos\left(\pi+\frac{\pi}{4}\right)=-\,cos\left(\frac{\pi}{4}\right)$$

$$\therefore\ cos\left(\frac{5\pi}{4}\right)=-\,cos\left(\frac{\pi}{4}\right)$$

$$cos^{-1}\left[cos\left(\frac{5\pi}{4}\right)\right]=cos^{-1}\left[-\,cos\left(\frac{\pi}{4}\right)\right]$$

$$=-\,cos^{-1}\left[cos\left(\frac{\pi}{4}\right)\right]$$

$$=-\frac{\pi}{4}$$

$$\therefore\ cos^{-1}\left[cos\left(\frac{4\pi}{3}\right)\right]+cos^{-1}\left[cos\left(\frac{5\pi}{4}\right)\right]$$

$$=-\frac{\pi}{3}-\frac{\pi}{4}$$

$$=-\left(\frac{\pi}{3}+\frac{\pi}{4}\right)$$

$$=-\left(\frac{4\pi+3\pi}{12}\right)$$

$$=-\left(\frac{7\pi}{12}\right)$$

Example 4.8

Find the principal value of $tan^{-1}(\sqrt{3})$

Solution: Let $tan^{-1}(\sqrt{3})=y$

Then, $\tan y=\sqrt{3}$

We know $\tan\frac{\pi}{3}=\sqrt{3}$

Thus $y=\frac{\pi}{3}\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$

Hence, $tan^{-1}(\sqrt{3})=\frac{\pi}{3}$

Example 4.9 Find

(i) $tan^{-1}(-\sqrt{3})$

Solution:

We know $\tan(-\theta)=-\tan\theta$

So, $\tan\left(-\frac{\pi}{3}\right)=-\tan\frac{\pi}{3}$

$$=-\sqrt{3}$$

$$\therefore\ tan^{-1}(-\sqrt{3})=tan^{-1}\left(\tan\left(-\frac{\pi}{3}\right)\right)$$

$$=-\frac{\pi}{3},\ \text{since } -\frac{\pi}{3}\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$$

(ii) $tan^{-1}\left(\tan\frac{3\pi}{5}\right)$

Solution:

Since the tanget function has period π,

$$\tan\frac{3\pi}{5}=\tan\left(\frac{3\pi}{5}-\pi\right)$$

$$=\tan\left(\frac{3\pi-5\pi}{5}\right)$$

$$=\tan\left(-\frac{2\pi}{5}\right)$$

$$\therefore\ tan^{-1}\left(\tan\frac{3\pi}{5}\right)=tan^{-1}\left(\tan\left(-\frac{2\pi}{5}\right)\right)$$

$$=-\frac{2\pi}{5}\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$$

(iii) $\tan[\,tan^{-1}(2019)]$

Solution:

We know, $\tan[\,tan^{-1}(x)]=x, x\in R$,

$\tan[\,tan^{-1}(2019)]=2019$

Example 4.10 Find the value of

$tan^{-1}(-1)+cos^{-1}\left(\frac{1}{2}\right)+sin^{-1}\left(-\frac{1}{2}\right)$

Solution: Let $tan^{-1}(-1)=y$

Then, $\tan y=-1$

$$\tan y=-\tan\frac{\pi}{4}$$

$$=\tan\left(-\frac{\pi}{4}\right)$$

$$tan^{-1}(-1)=-\frac{\pi}{4}\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$$

Let $cos^{-1}\left(\frac{1}{2}\right)=y$

Then, $\cos y=\frac{1}{2}$

$$\cos y = \cos\frac{\pi}{3}$$

$$cos^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3} \in [0, \pi]$$

Now $sin^{-1}\left(-\frac{1}{2}\right) = y$

Then, $\sin y = -\frac{1}{2}$

$$\sin y = -\sin\frac{\pi}{6}$$

$$= \sin\left(-\frac{\pi}{6}\right)$$

$$sin^{-1}\left(-\frac{1}{2}\right) = -\frac{\pi}{6} \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

$$\therefore\ tan^{-1}(-1) + cos^{-1}\left(\frac{1}{2}\right) + sin^{-1}\left(-\frac{1}{2}\right)$$

$$= -\frac{\pi}{4} + \frac{\pi}{3} - \frac{\pi}{6}$$

$$= \frac{-3\pi + 4\pi - 2\pi}{12}$$

$$= \frac{-5\pi + 4\pi}{12}$$

$$= -\frac{\pi}{12}$$

Example 4.11

Prove that $tan(sin^{-1}x) = \frac{x}{\sqrt{1-x^2}}, -1 < x < 1$

Solution: If $x = 0$, then both sides are equal to 0

Assume that, $0 < x < 1$

Then $\theta = sin^{-1}x$ gives $0 < \theta < \frac{\pi}{2}$

Now, $\sin\theta = \frac{x}{1}$ gives $\tan\theta = \frac{x}{\sqrt{1-x^2}}$

Hence $tan(sin^{-1}x) = \frac{x}{\sqrt{1-x^2}}$ ......(1)

Assume that, $-1 < x < 0$

Then $\theta = sin^{-1}x$ gives $-\frac{\pi}{2} < \theta < 0$

Now, $\sin\theta = \frac{x}{1}$ gives $\tan\theta = \frac{x}{\sqrt{1-x^2}}$

So, that $tan(sin^{-1}x) = \frac{x}{\sqrt{1-x^2}}$ ......(2)

From the above results

$$tan(sin^{-1}x) = \frac{x}{\sqrt{1-x^2}}, -1 < x < 1$$

EXERCISE 4.3

1. Find the domain of the following functions:

$(i) tan^{-1}\left(\sqrt{9-x^2}\right)$

Solution: Let $f(x) = tan^{-1}\left(\sqrt{9-x^2}\right)$

$$\sqrt{9-x^2} \in R$$

But $\sqrt{9-x^2} \geq 0$

$$\therefore 9 - x^2 \geq 0$$

Hence $x^2 - 9 \leq 0$

$$(x+3)(x-3) \leq 0$$

$\therefore$ Domain is $[-3,3]$

$(ii) \frac{1}{2}tan^{-1}(1-x^2) - \frac{\pi}{4}$

Solution: $tan^{-1}(x)$ is function with $(-\infty, \infty)$ as domain.

$\therefore \frac{1}{2}tan^{-1}(1-x^2) - \frac{\pi}{4}$ is R.

2. Find the value of

$(i)\ tan^{-1}\left(tan\frac{5\pi}{4}\right)$

Solution: $\frac{5\pi}{4} \notin \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

$$tan\left(\frac{5\pi}{4}\right) = tan\left(\frac{4\pi}{4} + \frac{\pi}{4}\right)$$

$$= tan\left(\pi + \frac{\pi}{4}\right)$$

$$= tan\left(\frac{\pi}{4}\right)$$

Since $= tan(\pi + \theta) = \tan\theta$

$$\therefore\ tan^{-1}\left(tan\frac{5\pi}{4}\right) = tan^{-1}\left(tan\frac{\pi}{4}\right)$$

$$= \frac{\pi}{4} \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

$(ii)\ tan^{-1}\left(tan\left(-\frac{\pi}{6}\right)\right)$

Solution: $tan^{-1}\left(tan\left(-\frac{\pi}{6}\right)\right)$

$$= -\frac{\pi}{6},\ \text{as}\ -\frac{\pi}{6} \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

3. Find the value of

(i) $tan^{-1}\left(tan\frac{7\pi}{4}\right)$

Solution:

We know, $\tan[\,tan^{-1}(x)] = x, x \in R$,

$\tan\left[\,tan^{-1}\left(\frac{7\pi}{4}\right)\right] = \frac{7\pi}{4}$

(ii) $\tan[\,tan^{-1}(1947)]$

Solution:

We know, $\tan[\,tan^{-1}(x)] = x, x \in R$,

$\tan[\,tan^{-1}(1947)] = 1947$

(iii) $\tan[\,tan^{-1}(-0.2021)]$

Solution:

We know, $\tan[\,tan^{-1}(x)] = x, x \in R$,

$\tan[\,tan^{-1}(-0.2021)] = -0.2021$

4. Find the value of

(i) $tan\left[cos^{-1}\left(\frac{1}{2}\right) - sin^{-1}\left(-\frac{1}{2}\right)\right]$

Solution:

Let $cos^{-1}\left(\frac{1}{2}\right) = y$

Then, $\cos y = \frac{1}{2}$

$\cos y = \cos\frac{\pi}{3}$

$cos^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3} \in [0, \pi]$

Now $sin^{-1}\left(-\frac{1}{2}\right) = y$

Then, $\sin y = -\frac{1}{2}$

$\sin y = -\sin\frac{\pi}{6}$

$= \sin\left(-\frac{\pi}{6}\right)$

$sin^{-1}\left(-\frac{1}{2}\right) = -\frac{\pi}{6} \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

$\therefore\ cos^{-1}\left(\frac{1}{2}\right) - sin^{-1}\left(-\frac{1}{2}\right) = \frac{\pi}{3} - \left(-\frac{\pi}{6}\right)$

$= \frac{\pi}{3} + \frac{\pi}{6}$

$= \frac{2\pi + \pi}{6}$

$= \frac{3\pi}{6} = \frac{\pi}{2}$

$\therefore\ tan\left[cos^{-1}\left(\frac{1}{2}\right) - sin^{-1}\left(-\frac{1}{2}\right)\right] = \tan\left(\frac{\pi}{2}\right)$

$= \infty$

(ii) $sin\left[tan^{-1}\left(\frac{1}{2}\right) - cos^{-1}\left(\frac{4}{5}\right)\right]$

Solution: Let $tan^{-1}\left(\frac{1}{2}\right) = A$

So, $\tan A = \frac{1}{2}$

$\frac{\text{opp}}{\text{adj}} = \frac{1}{2}$

By using Pythagoras Theorem,

$hyp^2 = opp^2 + adj^2$

$= 1^2 + 2^2$

$= 1 + 4$

$= 5$

$hyp = \sqrt{5}$

$\sin A = \frac{\text{opp}}{\text{hyp}}$

$= \frac{1}{\sqrt{5}}$

and $\cos A = \frac{\text{adj}}{\text{hyp}}$

$= \frac{2}{\sqrt{5}}$

Similarly, Let $cos^{-1}\left(\frac{4}{5}\right) = B$

So, $\cos B = \frac{4}{5}$

$\frac{\text{adj}}{\text{hyp}} = \frac{4}{5}$

By using Pythagoras Theorem,

$hyp^2 = opp^2 + adj^2$

$5^2 = opp^2 + 4^2$

$25 = opp^2 + 16$

$opp^2 = 25 - 16$

$= 9$

$opp = \sqrt{9} = 3$

$\sin B = \frac{\text{opp}}{\text{hyp}}$

$= \frac{3}{5}\therefore$

$sin\left[tan^{-1}\left(\frac{1}{2}\right) - cos^{-1}\left(\frac{4}{5}\right)\right] = \sin(A - B)$

We know $\sin(A-B) = \sin A\cos B - \cos A\sin B$

$$= \frac{1}{\sqrt{5}}\left(\frac{4}{5}\right) - \frac{2}{\sqrt{5}}\left(\frac{3}{5}\right)$$

$$= \frac{4}{5\sqrt{5}} - \frac{6}{5\sqrt{5}}$$

$$\sin(A-B) = -\frac{2}{5\sqrt{5}}$$

$$\therefore sin\left[tan^{-1}\left(\frac{1}{2}\right) - cos^{-1}\left(\frac{4}{5}\right)\right] = -\frac{2}{5\sqrt{5}}$$

(iii) $cos\left[sin^{-1}\left(\frac{4}{5}\right) - tan^{-1}\left(\frac{3}{4}\right)\right]$

Solution: Let $sin^{-1}\left(\frac{4}{5}\right) = A$

So, $\sin A = \frac{4}{5}$

$$\frac{\text{opp}}{\text{hyp}} = \frac{4}{5}$$

By using Pythagoras Theorem,

$$hyp^2 = opp^2 + adj^2$$

$$5^2 = 4^2 + adj^2$$

$$25 = 16 + adj^2$$

$$adj^2 = 25 - 16$$

$$= 9$$

$$adj = \sqrt{9} = 3$$

$$\cos A = \frac{\text{adj}}{\text{hyp}}$$

$$= \frac{3}{5}$$

Let $tan^{-1}\left(\frac{3}{4}\right) = B$

So, $\tan B = \frac{3}{4}$

$$\frac{\text{opp}}{\text{adj}} = \frac{3}{4}$$

By using Pythagoras Theorem,

$$hyp^2 = opp^2 + adj^2$$

$$= 3^2 + 4^2$$

$$= 9 + 16$$

$$= 25$$

$$hyp = 5$$

$$\sin B = \frac{\text{opp}}{\text{hyp}}$$

$$= \frac{3}{5}$$

and $\cos B = \frac{\text{adj}}{\text{hyp}}$

$$= \frac{4}{5}$$

$$\therefore cos\left[sin^{-1}\left(\frac{4}{5}\right) - tan^{-1}\left(\frac{3}{4}\right)\right] = \cos(A-B)$$

We know $\cos(A-B) = \cos A\cos B + \sin A\sin B$

$$= \frac{3}{5}\left(\frac{4}{5}\right) + \frac{4}{5}\left(\frac{3}{5}\right)$$

$$= \frac{12}{25} + \frac{12}{25}$$

$$= \frac{24}{25}$$

$$\cos(A-B) = \frac{24}{25}$$

$$\therefore cos\left[sin^{-1}\left(\frac{4}{5}\right) - tan^{-1}\left(\frac{3}{4}\right)\right] = \frac{24}{25}$$

Example 4.12

Find the principal value of

(i) $cosec^{-1}(-1)$

Solution: Let $cosec^{-1}(-1) = y$

Then, $cosec y = -1$

Since the range of the principal value of

$y = cosec^{-1}x$ is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \setminus \{0\}$ and

$cosec\left(-\frac{\pi}{2}\right) = -1$ we have $y = -\frac{\pi}{2}$

and $-\frac{\pi}{2} \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \setminus \{0\}$

The principal value of $cosec^{-1}(-1) = -\frac{\pi}{2}$

(ii) $sec^{-1}(-2)$

Solution: Let $sec^{-1}(-2) = y$

Then, $sec y = -2$

Since the range of the principal value of

$y = sec^{-1}x$ is $[0, \pi] \setminus \left\{\frac{\pi}{2}\right\}$ and

$sec y = -2 \Rightarrow \cos y = -\frac{1}{2}$

We have $\cos y = -\frac{1}{2} \Rightarrow \cos\left(-\frac{\pi}{3}\right)$

and $-\frac{\pi}{3} \notin [0, \pi] \setminus \left\{\frac{\pi}{2}\right\}$

But $\cos\left(-\frac{\pi}{3}\right) = \cos\left(\pi - \frac{\pi}{3}\right)$ because

$\cos(-\theta) = \cos(\pi - \theta)$

$\therefore \cos\left(-\frac{\pi}{3}\right) = \cos\left(\frac{3\pi - \pi}{3}\right)$

$= \cos\left(\frac{2\pi}{3}\right)$ *and* $\frac{2\pi}{3} \in [0, \pi] \setminus \left\{\frac{\pi}{2}\right\}$

The principal value of $sec^{-1}(-2) = \frac{2\pi}{3}$

Example 4.13 Find the value of $sec^{-1}\left(-\frac{2\sqrt{3}}{3}\right)$

Solution: Let $sec^{-1}\left(-\frac{2\sqrt{3}}{3}\right) = \theta$

Then, $\sec\theta = \left(-\frac{2\sqrt{3}}{3}\right)$ *and* $\theta \in [0, \pi] \setminus \left\{\frac{\pi}{2}\right\}$

$\cos\theta = -\frac{3}{2\sqrt{3}} \Rightarrow -\frac{\sqrt{3}\sqrt{3}}{2\sqrt{3}}$

$\cos\theta = -\frac{\sqrt{3}}{2}$

We have $\cos y = -\frac{\sqrt{3}}{2} \Rightarrow \cos\left(-\frac{\pi}{6}\right)$

and $-\frac{\pi}{6} \notin [0, \pi] \setminus \left\{\frac{\pi}{2}\right\}$

But $\cos\left(-\frac{\pi}{6}\right) = \cos\left(\pi - \frac{\pi}{6}\right)$ because

$\cos(-\theta) = \cos(\pi - \theta)$

$\therefore \cos\left(-\frac{\pi}{6}\right) = \cos\left(\frac{6\pi - \pi}{6}\right)$

$= \cos\left(\frac{5\pi}{6}\right)$ *and* $\frac{5\pi}{6} \in [0, \pi] \setminus \left\{\frac{\pi}{2}\right\}$

The value of $sec^{-1}\left(-\frac{2\sqrt{3}}{3}\right) = \frac{5\pi}{6}$

Example 4.14

If $cot^{-1}\left(\frac{1}{7}\right) = \theta$, find the value of $cos\theta$.

Solution: Given $cot^{-1}\left(\frac{1}{7}\right) = \theta$

Hence, $\cot\theta = \frac{1}{7}$

$\therefore \tan\theta = \frac{7}{1}$

$\frac{opp}{adj} = \frac{7}{1}$

By using Pythagoras Theorem,

$hyp^2 = opp^2 + adj^2$

$= 7^2 + 1^2$

$= 49 + 1$

$= 50$

$= 25 \times 2$

$hyp = 5\sqrt{2}$

and $\cos\theta = \frac{\text{adj}}{\text{hyp}}$

$= \frac{1}{5\sqrt{2}}$

Example 4.15

Show that $cot^{-1}\left(\frac{1}{\sqrt{x^2-1}}\right) = sec^{-1}x$, $|x| > 1$.

Solution: Let $cot^{-1}\left(\frac{1}{\sqrt{x^2-1}}\right) = \theta$

Then, $\cot\theta = \left(\frac{1}{\sqrt{x^2-1}}\right)$

$\therefore \tan\theta = \frac{\sqrt{x^2-1}}{1}$

$\frac{opp}{adj} = \frac{\sqrt{x^2-1}}{1}$

By using Pythagoras Theorem,

$hyp^2 = opp^2 + adj^2$

$= x^2 - 1 + 1$

$= x^2$

$hyp = x$

and $\cos\theta = \frac{\text{adj}}{\text{hyp}}$

$= \frac{1}{x}$

gives $\sec\theta = \frac{x}{1} = x$

$\therefore sec^{-1}x = \theta$

$= cot^{-1}\left(\frac{1}{\sqrt{x^2-1}}\right)$

EXERCISE 4.4

1. Find the principal value of

(i) $sec^{-1}\left(\frac{2}{\sqrt{3}}\right)$

Solution: Let $sec^{-1}\left(\frac{2}{\sqrt{3}}\right) = \theta$

Then, $\sec\theta = \left(\frac{2}{\sqrt{3}}\right)$ *and* $\theta \in [0,\pi] \setminus \left\{\frac{\pi}{2}\right\}$

$\cos\theta = \frac{\sqrt{3}}{2}$

We have $\cos y = \frac{\sqrt{3}}{2} \Rightarrow \cos\left(\frac{\pi}{6}\right)$

and $\frac{\pi}{6} \in [0,\pi] \setminus \left\{\frac{\pi}{2}\right\}$

The principal value of $sec^{-1}\left(\frac{2}{\sqrt{3}}\right) = \frac{\pi}{6}$

(ii) $cot^{-1}(\sqrt{3})$

Solution: Let $cot^{-1}(\sqrt{3}) = \theta$

Then, $\cot\theta = \sqrt{3}$

$\therefore \tan\theta = \frac{1}{\sqrt{3}}$

as $\tan\frac{\pi}{6} = \frac{1}{\sqrt{3}}$

and $\frac{\pi}{6} \in [0,\pi]$, $cot^{-1}(\sqrt{3}) = \frac{\pi}{6}$

(iii) $cosec^{-1}(-\sqrt{2})$

Solution: Let $cosec^{-1}(-\sqrt{2}) = y$

Then, $cosec y = -\sqrt{2}$

Since the range of the principal value of

$y = cosec^{-1}x$ is $\left[-\frac{\pi}{2},\frac{\pi}{2}\right] \setminus \{0\}$ and

$cosec y = -\sqrt{2}$ gives $\sin y = -\frac{1}{\sqrt{2}}$

we have $y = -\frac{\pi}{4}$

and $-\frac{\pi}{4} \in \left[-\frac{\pi}{2},\frac{\pi}{2}\right] \setminus \{0\}$

The principal value of $cosec^{-1}(-\sqrt{2}) = -\frac{\pi}{4}$

2. Find the value of

(i) $tan^{-1}(\sqrt{3}) - sec^{-1}(-2)$

Solution: Let $tan^{-1}(\sqrt{3}) = \theta$

Then, $\tan\theta = (\sqrt{3})$

as $\tan\frac{\pi}{3} = \sqrt{3}$

and $\frac{\pi}{3} \in [0,\pi]$, $tan^{-1}(\sqrt{3}) = \frac{\pi}{3}$

Let $sec^{-1}(-2) = y$

Then, $sec y = -2$

Since the range of the principal value of

$y = sec^{-1}x$ is $[0,\pi] \setminus \left\{\frac{\pi}{2}\right\}$ and

$sec y = -2 \Rightarrow \cos y = -\frac{1}{2}$

We have $\cos y = -\frac{1}{2} \Rightarrow \cos\left(-\frac{\pi}{3}\right)$

and $-\frac{\pi}{3} \notin [0,\pi] \setminus \left\{\frac{\pi}{2}\right\}$

But $\cos\left(-\frac{\pi}{3}\right) = \cos\left(\pi - \frac{\pi}{3}\right)$ because

$\cos(-\theta) = \cos(\pi - \theta)$

$\therefore \cos\left(-\frac{\pi}{3}\right) = \cos\left(\frac{3\pi - \pi}{3}\right)$

$= \cos\left(\frac{2\pi}{3}\right)$ *and* $\frac{2\pi}{3} \in [0,\pi] \setminus \left\{\frac{\pi}{2}\right\}$

The principal value of $sec^{-1}(-2) = \frac{2\pi}{3}$

$\therefore tan^{-1}(\sqrt{3}) - sec^{-1}(-2) = \frac{\pi}{3} - \frac{2\pi}{3}$

$= \frac{\pi - 2\pi}{3}$

$= -\frac{\pi}{3}$

(ii) $sin^{-1}(-1) + cos^{-1}\left(\frac{1}{2}\right) + cot^{-1}(2)$

Solution: Now $sin^{-1}(-1) = y$

Then, $\sin y = -1$

$\sin y = -\sin\frac{\pi}{2}$

$= \sin\left(-\frac{\pi}{2}\right)$

$sin^{-1}(-1) = -\frac{\pi}{2} \in \left[-\frac{\pi}{2},\frac{\pi}{2}\right]$

Let $cos^{-1}\left(\frac{1}{2}\right) = y$

Then, $\cos y = \frac{1}{2}$

$\cos y = \cos\frac{\pi}{3}$

$cos^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3} \in [0,\pi]$

$\therefore sin^{-1}(-1) + cos^{-1}\left(\frac{1}{2}\right) + cot^{-1}(2)$

$= -\frac{\pi}{2} + \frac{\pi}{3} + cot^{-1}(2)$

$= \frac{-3\pi + 2\pi}{6} + cot^{-1}(2)$

$= -\frac{\pi}{6} + cot^{-1}(2)$

$(iii)\ cot^{-1}(1) + sin^{-1}\left(-\frac{\sqrt{3}}{2}\right) - sec^{-1}(-\sqrt{2})$

Solution: Let $cot^{-1}(1) = \theta$

Then, $\cot\theta = 1$

$\therefore \tan\theta = 1$

as $\tan\frac{\pi}{4} = 1$

and $\frac{\pi}{4} \in [0, \pi]$, $cot^{-1}(1) = \frac{\pi}{4}$

Now $sin^{-1}\left(-\frac{\sqrt{3}}{2}\right) = y$

Then, $\sin y = -\frac{\sqrt{3}}{2}$

$\sin y = -\sin\frac{\pi}{3}$

$= \sin\left(-\frac{\pi}{3}\right)$

$sin^{-1}\left(-\frac{\sqrt{3}}{2}\right) = -\frac{\pi}{3} \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

Let $sec^{-1}(-\sqrt{2}) = y$

Then, $secy = -\sqrt{2}$

Since the range of the principal value of

$y = sec^{-1}x$ is $[0, \pi] \setminus \left\{\frac{\pi}{2}\right\}$ and

$secy = -\sqrt{2} \Rightarrow \cos y = -\frac{1}{\sqrt{2}}$

We have $\cos y = -\frac{1}{\sqrt{2}} \Rightarrow \cos\left(-\frac{\pi}{4}\right)$

and $-\frac{\pi}{4} \notin [0, \pi] \setminus \left\{\frac{\pi}{2}\right\}$

But $\cos\left(-\frac{\pi}{4}\right) = \cos\left(\pi - \frac{\pi}{4}\right)$ because

$\cos(-\theta) = \cos(\pi - \theta)$

$\therefore \cos\left(-\frac{\pi}{4}\right) = \cos\left(\frac{4\pi - \pi}{4}\right)$

$= \cos\left(\frac{3\pi}{4}\right)$ *and* $\frac{3\pi}{4} \in [0, \pi] \setminus \left\{\frac{\pi}{2}\right\}$

The principal value of $sec^{-1}(-\sqrt{2}) = \frac{3\pi}{4}$

$\therefore cot^{-1}(1) + sin^{-1}\left(-\frac{\sqrt{3}}{2}\right) - sec^{-1}(-\sqrt{2})$

$= \frac{\pi}{4} - \frac{\pi}{3} - \frac{3\pi}{4}$

$= \frac{3\pi - 4\pi - 9\pi}{12}$

$= \frac{3\pi - 13\pi}{12}$

$= -\frac{10\pi}{12}$

$= -\frac{5\pi}{6}$

Example 4.16 Prove that

$\frac{\pi}{2} \le sin^{-1}x + 2cos^{-1}x \le \frac{3\pi}{2}$

Solution:

We know that $sin^{-1}x + cos^{-1}x = \frac{\pi}{2}$

$sin^{-1}x + 2cos^{-1}x = sin^{-1}x + cos^{-1}x + cos^{-1}x$

$= \frac{\pi}{2} + cos^{-1}x$

Since the range of $cos^{-1}x$ is $[0, \pi]$

We have, $0 \le cos^{-1}x \le \pi$

Adding $\frac{\pi}{2}$,

$\frac{\pi}{2} + 0 \le \frac{\pi}{2} + cos^{-1}x \le \frac{\pi}{2} + \pi$

$\frac{\pi}{2} \le sin^{-1}x + cos^{-1}x + cos^{-1}x \le \frac{\pi + 2\pi}{2}$

$\therefore \frac{\pi}{2} \le sin^{-1}x + 2cos^{-1}x \le \frac{3\pi}{2}$ is proved.

Example 4.17 Simplify

$(i)\ cos^{-1}\left[cos\left(\frac{13\pi}{3}\right)\right]$

Solution: Since the range of $cos^{-1}x$ is $[0, \pi]$

and $\frac{13\pi}{3} \notin [0, \pi]$, we have

$cos\left(\frac{13\pi}{3}\right) = cos\left(\frac{12\pi}{3} + \frac{\pi}{3}\right)$

$= cos\left(4\pi + \frac{\pi}{3}\right)$

$= cos\left(\frac{\pi}{3}\right)$ and $\frac{\pi}{3} \in [0, \pi]$

$\therefore cos^{-1}\left[cos\left(\frac{13\pi}{3}\right)\right] = \frac{\pi}{3}$

(ii) $tan^{-1}\left[tan\left(\frac{3\pi}{4}\right)\right]$

Solution: Range of $tan^{-1}x$ is $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$

and $\frac{3\pi}{4} \notin \left[-\frac{\pi}{2},\frac{\pi}{2}\right]$, we have

$$tan\left(\frac{3\pi}{4}\right) = tan\left(\pi - \frac{\pi}{4}\right)$$

$$= -\,tan\left(\frac{\pi}{4}\right)$$

$$= tan\left(-\frac{\pi}{4}\right) \text{ and } -\frac{\pi}{4} \in \left[-\frac{\pi}{2},\frac{\pi}{2}\right]$$

$$\therefore tan^{-1}\left[tan\left(\frac{3\pi}{4}\right)\right] = -\frac{\pi}{4}$$

(iii) $sec^{-1}\left[sec\left(\frac{5\pi}{3}\right)\right]$

Solution: Range of $sec^{-1}x$ is $[0,\pi] \setminus \left\{\frac{\pi}{2}\right\}$

and $\frac{5\pi}{3}$ does not in the interval.

$$sec\left(\frac{5\pi}{3}\right) = sec\left(\frac{6\pi}{3} - \frac{\pi}{3}\right)$$

$$= sec\left(2\pi - \frac{\pi}{3}\right)$$

$$= sec\left(\frac{\pi}{3}\right) \text{ and } \frac{\pi}{3} \in [0,\pi] \setminus \left\{\frac{\pi}{2}\right\}$$

$$\therefore sec^{-1}\left[sec\left(\frac{5\pi}{3}\right)\right] = \frac{\pi}{3}$$

(iv) $sin^{-1}[sin 10]$

Solution: $sin^{-1}[sin\theta] = \theta$, as $\theta \in \left[-\frac{\pi}{2},\frac{\pi}{2}\right]$

So, 10 is not in the interval, but

$10 - 3\pi \in \left[-\frac{\pi}{2},\frac{\pi}{2}\right]$

$$sin 10 = sin(3\pi + (10 - 3\pi))$$

$$= -\,sin(10 - 3\pi)$$

$$= sin(3\pi - 10)$$

$$\therefore sin^{-1}[sin 10] = sin^{-1}[sin(3\pi - 10)]$$

$$= 3\pi - 10$$

Example 4.18 Find the value of

(i) $sin\left[\frac{\pi}{3} - sin^{-1}\left(-\frac{1}{2}\right)\right]$

Solution: Let $sin^{-1}\left(-\frac{1}{2}\right) = y$

Then, $\sin y = -\frac{1}{2}$

$$\sin y = -\sin\frac{\pi}{6}$$

$$= \sin\left(-\frac{\pi}{6}\right)$$

$$sin^{-1}\left(-\frac{1}{2}\right) = -\frac{\pi}{6} \in \left[-\frac{\pi}{2},\frac{\pi}{2}\right]$$

$$\therefore sin\left[\frac{\pi}{3} - sin^{-1}\left(-\frac{1}{2}\right)\right] = sin\left[\frac{\pi}{3} - \left(-\frac{\pi}{6}\right)\right]$$

$$= sin\left[\frac{\pi}{3} + \frac{\pi}{6}\right]$$

$$= sin\left[\frac{2\pi+\pi}{6}\right]$$

$$= sin\left[\frac{3\pi}{6}\right]$$

$$= sin\left[\frac{\pi}{2}\right]$$

$$= 1$$

(ii) $cos\left[\frac{1}{2}cos^{-1}\left(\frac{1}{8}\right)\right]$

Solution: Let $cos^{-1}\left(\frac{1}{8}\right) = \theta$

Then, $\cos\theta = \left(\frac{1}{8}\right)$

We know $2cos^2\theta - 1 = \cos 2\theta$

$$\cos\theta = 2cos^2\frac{\theta}{2} - 1$$

$$2cos^2\frac{\theta}{2} - 1 = \frac{1}{8}$$

$$2cos^2\frac{\theta}{2} = \frac{1}{8} + 1$$

$$= \frac{1+8}{8}$$

$$= \frac{9}{8}$$

$$\therefore cos^2\frac{\theta}{2} = \frac{9}{16}$$

Hence $\cos\frac{\theta}{2} = \frac{3}{4}$

$$\therefore cos\left[\frac{1}{2}cos^{-1}\left(\frac{1}{8}\right)\right] = cos\left[\frac{1}{2}\theta\right]$$

$$= cos\left[\frac{\theta}{2}\right]$$

$$= \frac{3}{4}$$

(iii) $tan\left[\frac{1}{2}sin^{-1}\left(\frac{2a}{1+a^2}\right) + \frac{1}{2}cos^{-1}\left(\frac{1-a^2}{1+a^2}\right)\right]$

Solution: We know

$$\sin 2\theta = \frac{2\tan\theta}{1+tan^2\theta} \quad \text{and}$$

$$\cos 2\theta = \frac{1-tan^2\theta}{1+tan^2\theta}$$

$$\therefore tan\left[\frac{1}{2}sin^{-1}\left(\frac{2a}{1+a^2}\right)+\frac{1}{2}cos^{-1}\left(\frac{1-a^2}{1+a^2}\right)\right]$$

$$= tan\left[\frac{1}{2}sin^{-1}(\sin 2\theta)+\frac{1}{2}cos^{-1}(\cos 2\theta)\right]$$

$$= tan\left[\frac{1}{2}(2\theta)+\frac{1}{2}(2\theta)\right]$$

$$= tan[\theta+\theta]$$

$$= tan[2\theta]$$

$$= \frac{2\tan\theta}{1-tan^2\theta}$$

Example 4.19 Prove that

$tan\,(sin^{-1}x) = \frac{x}{\sqrt{1-x^2}}$ $for\ |x| < 1.$

Solution: Let $sin^{-1}x = \theta$

Then, $\sin\theta = x$

$$\frac{opp}{hyp} = \frac{x}{1}$$

By using Pythagoras Theorem,

$$hyp^2 = opp^2 + adj^2$$

$$1^2 = x^2 + adj^2$$

$$1 = x^2 + adj^2$$

$$adj^2 = 1 - x^2$$

$$adj = \sqrt{1-x^2}$$

$$\tan\theta = \frac{\text{opp}}{\text{adj}}$$

$$= \frac{x}{\sqrt{1-x^2}}$$

$$\therefore \tan(sin^{-1}x) = tan\,(\theta)$$

$$= \frac{x}{\sqrt{1-x^2}}$$ Proved.

Example 4.20 Evaluate

$sin\left[sin^{-1}\left(\frac{3}{5}\right)+sec^{-1}\left(\frac{5}{4}\right)\right]$

Solution: Let $sin^{-1}\left(\frac{3}{5}\right) = A$

So, $\sin A = \frac{3}{5}$

$$\frac{\text{opp}}{\text{hyp}} = \frac{3}{5}$$

By using Pythagoras Theorem,

$$hyp^2 = opp^2 + adj^2$$

$$5^2 = 3^2 + adj^2$$

$$25 = 9 + adj^2$$

$$adj^2 = 25 - 9$$

$$= 16$$

$$adj = \sqrt{16} = 4$$

$$\cos A = \frac{\text{adj}}{\text{hyp}}$$

$$= \frac{4}{5}$$

Let $sec^{-1}\left(\frac{5}{4}\right) = B$

So, $\sec B = \frac{5}{4}$

$$\cos B = \frac{4}{5} = \frac{adj}{hyp}$$

By using Pythagoras Theorem,

$$hyp^2 = opp^2 + adj^2$$

$$5^2 = opp^2 + 4^2$$

$$opp^2 = 5^2 - 4^2$$

$$= 25 - 16$$

$$= 9$$

$$opp = 3$$

$$\sin B = \frac{\text{opp}}{\text{hyp}}$$

$$= \frac{3}{5}$$

$$\therefore sin\left[sin^{-1}\left(\frac{3}{5}\right)+sec^{-1}\left(\frac{5}{4}\right)\right] = sin\,[A+B]$$

We know $\sin(A+B) = \sin A\cos B + \cos A\sin B$

$$= \frac{3}{5}\left(\frac{4}{5}\right)+\frac{4}{5}\left(\frac{3}{5}\right)$$

$$= \frac{12}{25}+\frac{12}{25}$$

$$= \frac{24}{25}$$

$$\sin(A+B) = \frac{24}{25}$$

$$\therefore sin\left[tan^{-1}\left(\frac{1}{2}\right)-cos^{-1}\left(\frac{4}{5}\right)\right] = \frac{24}{25} \in [-1,1]$$

Example 4.21 Prove that

$(i) tan^{-1}\frac{1}{2} + tan^{-1}\frac{1}{3} = \frac{\pi}{4}$

Solution: $tan^{-1}A + tan^{-1}B = tan^{-1}\left[\frac{A+B}{1-AB}\right]$

$$\therefore tan^{-1}\frac{1}{2} + tan^{-1}\frac{1}{3} = tan^{-1}\left[\frac{\frac{1}{2}+\frac{1}{3}}{1-\left(\frac{1}{2}\right)\left(\frac{1}{3}\right)}\right]$$

$$= tan^{-1}\left[\frac{\frac{3+2}{6}}{1-\left(\frac{1}{6}\right)}\right]$$

$$= tan^{-1}\left[\frac{\frac{5}{6}}{\frac{6-1}{6}}\right]$$

$$= tan^{-1}\left[\frac{\frac{5}{6}}{\frac{5}{6}}\right]$$

$$= tan^{-1}[1]$$

$$= \frac{\pi}{4},$$

Since $tan\frac{\pi}{4} = 1 \Rightarrow tan^{-1}(1) = \frac{\pi}{4}$

(ii) $2tan^{-1}\frac{1}{2} + tan^{-1}\frac{1}{7} = tan^{-1}\frac{31}{17}$

Solution: $tan^{-1}A + tan^{-1}B = tan^{-1}\left[\frac{A+B}{1-AB}\right]$

$$\therefore tan^{-1}\frac{1}{2} + tan^{-1}\frac{1}{7} = tan^{-1}\left[\frac{\frac{1}{2}+\frac{1}{7}}{1-\left(\frac{1}{2}\right)\left(\frac{1}{7}\right)}\right]$$

$$= tan^{-1}\left[\frac{\frac{7+2}{14}}{1-\left(\frac{1}{14}\right)}\right]$$

$$= tan^{-1}\left[\frac{\frac{9}{14}}{\frac{14-1}{14}}\right]$$

$$= tan^{-1}\left[\frac{\frac{9}{14}}{\frac{13}{14}}\right]$$

$$= tan^{-1}\left[\frac{9}{14}\times\frac{14}{13}\right]$$

$$= tan^{-1}\left[\frac{9}{13}\right]$$

$$\therefore 2tan^{-1}\frac{1}{2} + tan^{-1}\frac{1}{7}$$

$$= tan^{-1}\frac{1}{2} + tan^{-1}\frac{1}{2} + tan^{-1}\frac{1}{7}$$

$$= tan^{-1}\frac{1}{2} + tan^{-1}\left[\frac{9}{13}\right]$$

$$= tan^{-1}\left[\frac{\frac{1}{2}+\frac{9}{13}}{1-\left(\frac{1}{2}\right)\left(\frac{9}{13}\right)}\right]$$

$$= tan^{-1}\left[\frac{\frac{13+18}{26}}{1-\left(\frac{9}{26}\right)}\right]$$

$$= tan^{-1}\left[\frac{\frac{31}{26}}{\frac{26-9}{26}}\right]$$

$$= tan^{-1}\left[\frac{\frac{31}{26}}{\frac{17}{26}}\right]$$

$$= tan^{-1}\left[\frac{31}{26}\times\frac{26}{17}\right]$$

$$= tan^{-1}\left[\frac{31}{17}\right]$$

$\therefore 2tan^{-1}\frac{1}{2} + tan^{-1}\frac{1}{7} = tan^{-1}\frac{31}{17}$ Proved.

Example 4.22 If $cos^{-1}x + cos^{-1}y + cos^{-1}z = \pi$ and $0 < x, y, z < 1$ show that $x^2 + y^2 + z^2 + 2xyz = 1$

Solution:

Let $cos^{-1}x = \alpha \Rightarrow \cos\alpha = x$

$cos^{-1}y = \beta \Rightarrow \cos\beta = y$

$cos^{-1}z = \gamma \Rightarrow \cos\gamma = z$

Given $cos^{-1}x + cos^{-1}y + cos^{-1}z = \pi$

$\alpha + \beta + cos^{-1}z = \pi$

$\alpha + \beta = \pi - cos^{-1}z$

$cos(\alpha+\beta) = cos(\pi - cos^{-1}z)$

Now $cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\cos\beta$

Since $\cos\alpha = x$

we get $cos^2\alpha = x^2$

$sin^2\alpha = 1 - cos^2\alpha = 1 - x^2$

$\therefore \sin\alpha = \sqrt{1-x^2}$ similarly

$\therefore \sin\beta = \sqrt{1-y^2}$

$\therefore cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\cos\beta$

$= xy - \sqrt{1-x^2}\sqrt{1-y^2}$

Now $cos(\pi - cos^{-1}z) = -cos(cos^{-1}z)$

$= -z$

Since $cos(\alpha+\beta) = cos(\pi - cos^{-1}z)$

$xy - \sqrt{1-x^2}\sqrt{1-y^2} = -z$

$xy + z = \sqrt{1-x^2}\sqrt{1-y^2}$

Squaring,

$(xy+z)^2 = (1-x^2)(1-y^2)$

$x^2y^2 + z^2 + 2xyz = 1 - y^2 - x^2 + x^2y^2$

$$z^2 + 2xyz = 1 - y^2 - x^2$$

$\therefore x^2 + y^2 + z^2 + 2xyz = 1$ Hence Proved.

Example 4.23 If $a_1, a_2, a_3, \dots a_n$ is an arithmetic prog. with common difference d, prove that

$$\tan\left[tan^{-1}\left(\frac{d}{1+a_1a_2}\right) + tan^{-1}\left(\frac{d}{1+a_2a_3}\right) + \cdots + tan^{-1}\left(\frac{d}{1+a_na_{n-1}}\right)\right] = \frac{a_n - a_1}{1 + a_1a_n}$$

Solution: We know that

$$tan^{-1}A - tan^{-1}B = tan^{-1}\left(\frac{A-B}{1+AB}\right)$$

$$tan^{-1}\left(\frac{d}{1+a_1a_2}\right) = tan^{-1}\left(\frac{a_2-a_1}{1+a_1a_2}\right)$$

$$= tan^{-1}a_2 - tan^{-1}a_1$$

$$tan^{-1}\left(\frac{d}{1+a_2a_3}\right) = tan^{-1}\left(\frac{a_3-a_2}{1+a_2a_3}\right)$$

$$= tan^{-1}a_3 - tan^{-1}a_2$$

$$tan^{-1}\left(\frac{d}{1+a_na_{n-1}}\right) = tan^{-1}\left(\frac{a_n-a_{n-1}}{1+a_na_{n-1}}\right)$$

$$= tan^{-1}a_n - tan^{-1}a_{n-1}$$

$$\therefore tan^{-1}\left(\frac{d}{1+a_1a_2}\right) + tan^{-1}\left(\frac{d}{1+a_2a_3}\right) + \cdots + tan^{-1}\left(\frac{d}{1+a_na_{n-1}}\right)$$

$$= tan^{-1}a_2 - tan^{-1}a_n$$

$$= tan^{-1}\left(\frac{a_n - a_1}{1 + a_1a_n}\right)$$

$$\text{LHS} = \tan\left[tan^{-1}\left(\frac{a_n - a_1}{1 + a_1a_n}\right)\right]$$

$$= \frac{a_n - a_1}{1 + a_1a_n}$$

= RHS Hence Proved.

Example 4.24 Solve

$$tan^{-1}\left(\frac{1-x}{1+x}\right) = \frac{1}{2}tan^{-1}x \; for \; x > 0$$

Solution: We know that

$$tan^{-1}A - tan^{-1}B = tan^{-1}\left(\frac{A-B}{1+AB}\right)$$

$$\therefore tan^{-1}1 - tan^{-1}x = tan^{-1}\left(\frac{1-x}{1+x}\right)$$

Given $tan^{-1}\left(\frac{1-x}{1+x}\right) = \frac{1}{2}tan^{-1}x$

$$\therefore tan^{-1}1 - tan^{-1}x = \frac{1}{2}tan^{-1}x$$

$$tan^{-1}1 = \frac{1}{2}tan^{-1}x + tan^{-1}x$$

$$tan^{-1}1 = tan^{-1}x\left(\frac{1}{2} + 1\right)$$

$$= tan^{-1}x\left(\frac{1+2}{2}\right)$$

$$= tan^{-1}x\left(\frac{3}{2}\right)$$

$$\frac{\pi}{4} = tan^{-1}x\left(\frac{3}{2}\right)$$

$$\frac{\pi}{4}\left(\frac{2}{3}\right) = tan^{-1}x$$

$$\frac{\pi}{6} = tan^{-1}x$$

So, $\tan\frac{\pi}{6} = x$

$$\frac{1}{\sqrt{3}} = x$$

Hence the solution is $x = \frac{1}{\sqrt{3}}$

Example 4.25 Solve $sin^{-1}x > cos^{-1}x$

Solution:

We know that $sin^{-1}x + cos^{-1}x = \frac{\pi}{2}$

Given $sin^{-1}x > cos^{-1}x$

Adding $sin^{-1}x$ on both sides, we get

$$sin^{-1}x + sin^{-1}x > sin^{-1}x + cos^{-1}x$$

$$2\, sin^{-1}x > \frac{\pi}{2}$$

$$sin^{-1}x > \frac{\pi}{4}$$

Sine function increases in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

$$x > sin\frac{\pi}{4}$$

$$x > \frac{1}{\sqrt{2}}$$

Example 4.26 Show that

$$cot(sin^{-1}x) = \frac{\sqrt{1-x^2}}{x}, -1 \le x \le 1 \; and \; x \ne 0$$

Solution: Let $sin^{-1}x = \theta$

Then, $\sin\theta = x$

$$\frac{opp}{hyp} = \frac{x}{1}$$

By using Pythagoras Theorem,

$$hyp^2 = opp^2 + adj^2$$

$$1^2 = x^2 + adj^2$$
$$1 = x^2 + adj^2$$
$$adj^2 = 1 - x^2$$
$$adj = \sqrt{1 - x^2}$$
$$\cos\theta = \frac{\text{adj}}{\text{hyp}}$$
$$= \frac{\sqrt{1-x^2}}{1}$$
$$\cot\theta = \frac{\cos\theta}{\sin\theta}$$
$$= \frac{\sqrt{1-x^2}}{x}$$
$$\therefore cot(sin^{-1}x) = cot(\theta)$$
$$= \frac{\sqrt{1-x^2}}{x}, \text{ hence proved.}$$

Example 4.27 Solve

$$tan^{-1}2x + tan^{-1}3x = \frac{\pi}{4}, if\ 6x^2 < 1$$

Solution: We know that

$$tan^{-1}A + tan^{-1}B = tan^{-1}\left(\frac{A+B}{1-AB}\right)$$

Given $tan^{-1}2x + tan^{-1}3x = \frac{\pi}{4}$

$$\therefore tan^{-1}\left(\frac{2x+3x}{1-(2x)(3x)}\right) = \frac{\pi}{4}$$
$$\left(\frac{5x}{1-6x^2}\right) = \tan\frac{\pi}{4}$$
$$\frac{5x}{1-6x^2} = 1$$
$$5x = 1 - 6x^2$$
$$6x^2 + 5x - 1 = 0$$
$$6x^2 + 6x - x - 1 = 0$$
$$6x(x+1) - 1(x+1) = 0$$
$$(x+1)(6x-1) = 0$$

Gives $x = -1$, or $6x = 1 \Rightarrow x = \frac{1}{6}$

Since $x = -1$ does not satisfy $6x^2 < 1$

$x = \frac{1}{6}$, is the only solution.

Example 4.28 Solve

$$tan^{-1}\left(\frac{x-1}{x-2}\right) + tan^{-1}\left(\frac{x+1}{x+2}\right) = \frac{\pi}{4}$$

Solution: We know that

$$tan^{-1}A + tan^{-1}B = tan^{-1}\left(\frac{A+B}{1-AB}\right)$$

$$\therefore tan^{-1}\left(\frac{x-1}{x-2}\right) + tan^{-1}\left(\frac{x+1}{x+2}\right)$$
$$= tan^{-1}\left[\frac{\left(\frac{x-1}{x-2}\right)+\left(\frac{x+1}{x+2}\right)}{1-\left(\frac{x-1}{x-2}\right)\left(\frac{x+1}{x+2}\right)}\right]$$
$$= tan^{-1}\left[\frac{\frac{(x-1)(x+2)+(x+1)(x-2)}{(x-2)(x+2)}}{\frac{(x-2)(x+2)-(x-1)(x+1)}{(x-2)(x+2)}}\right]$$
$$= tan^{-1}\left[\frac{(x-1)(x+2)+(x+1)(x-2)}{(x-2)(x+2)-(x-1)(x+1)}\right]$$
$$= tan^{-1}\left[\frac{x^2+2x-x-2+x^2-2x+x-2}{(x^2-4)-(x^2-1)}\right]$$
$$= tan^{-1}\left[\frac{x^2+x-2+x^2-x-2}{x^2-4-x^2+1}\right]$$
$$= tan^{-1}\left[\frac{2x^2-4}{-3}\right]$$

Given $tan^{-1}\left(\frac{x-1}{x-2}\right) + tan^{-1}\left(\frac{x+1}{x+2}\right) = \frac{\pi}{4}$

$$\therefore tan^{-1}\left[\frac{2x^2-4}{-3}\right] = \frac{\pi}{4}$$
$$\left[\frac{2x^2-4}{-3}\right] = tan\frac{\pi}{4}$$
$$\left[\frac{2x^2-4}{-3}\right] = 1$$
$$2x^2 - 4 = -3$$
$$2x^2 = -3 + 4$$
$$2x^2 = 1$$
$$x^2 = \frac{1}{2}$$
$$x = \pm\frac{1}{\sqrt{2}}$$

Example 4.29 Solve

$$cos\left[sin^{-1}\left(\frac{x}{\sqrt{1+x^2}}\right)\right] = sin\left[cot^{-1}\left(\frac{3}{4}\right)\right]$$

Solution: Let $sin^{-1}\left(\frac{x}{\sqrt{1+x^2}}\right) = \theta$

Then, $\sin\theta = \left(\frac{x}{\sqrt{1+x^2}}\right)$

So, $\frac{\text{opp}}{\text{hyp}} = \frac{x}{\sqrt{1+x^2}}$

By using Pythagoras Theorem,

$$hyp^2 = opp^2 + adj^2$$
$$\sqrt{1+x^2}^2 = x^2 + adj^2$$
$$1 + x^2 = x^2 + adj^2$$

$adj^2 = 1 + x^2 - x^2$

$= 1$

$adj = 1$

$\cos\theta = \frac{\text{adj}}{\text{hyp}}$

$= \frac{1}{\sqrt{1+x^2}}$

$\text{LHS} = cos\left[sin^{-1}\left(\frac{x}{\sqrt{1+x^2}}\right)\right]$

$= cos\,\theta$

$= \frac{1}{\sqrt{1+x^2}}$

Similarly, let $cot^{-1}\left(\frac{3}{4}\right) = \theta$

$\cot\theta = \left(\frac{3}{4}\right)$

So, $\frac{\text{adj}}{\text{opp}} = \frac{3}{4}$

By using Pythagoras Theorem,

$hyp^2 = opp^2 + adj^2$

$hyp^2 = 4^2 + 3^2$

$= 16 + 9$

$= 25$

$hyp = 5$

$\sin\theta = \frac{\text{opp}}{\text{hyp}}$

$= \frac{4}{5}$

$\text{RHS} = sin\left[cot^{-1}\left(\frac{3}{4}\right)\right]$

$= sin\,\theta$

$= \frac{4}{5}$

Given

$cos\left[sin^{-1}\left(\frac{x}{\sqrt{1+x^2}}\right)\right] = sin\left[cot^{-1}\left(\frac{3}{4}\right)\right]$

$\therefore \frac{1}{\sqrt{1+x^2}} = \frac{4}{5}$

Squaring, $\frac{1}{1+x^2} = \frac{16}{25}$

$16(1+x^2) = 25$

$16 + 16x^2 = 25$

$16x^2 = 25 - 16$

$16x^2 = 9$

$x^2 = \frac{9}{16}$

$x = \pm\frac{3}{4}$

EXERCISE 4.5

1. Find the value, if it exists. If not, give the reason for non-existence.

$(i)\ sin^{-1}(cos\pi)$

Solution: $sin^{-1}(cos\pi) = sin^{-1}(-1)$

$= -\frac{\pi}{2}$

Since we know that $sin\left(-\frac{\pi}{2}\right) = -\sin\frac{\pi}{2} = -1$

$(ii)\ tan^{-1}\left[sin\left(-\frac{5\pi}{2}\right)\right]$

Solution: $sin\left(-\frac{5\pi}{2}\right) = -sin\left(\frac{5\pi}{2}\right)$

$= -sin\left(2\pi + \frac{\pi}{2}\right)$

$= -sin\left(\frac{\pi}{2}\right)$

$= sin\left(-\frac{\pi}{2}\right) = -1$

$tan^{-1}\left[sin\left(-\frac{5\pi}{2}\right)\right] = tan^{-1}[-1]$

We know $tan^{-1}(-1) = -tan^{-1}(1)$

$= -\frac{\pi}{4}$

So, $tan^{-1}\left[sin\left(-\frac{5\pi}{2}\right)\right] = tan^{-1}\left[\left(-\frac{\pi}{4}\right)\right]$

$= -\frac{\pi}{4}$

$(iii)\ sin^{-1}(sin\,5)$

Solution: $sin^{-1}[sin\theta] = \theta$, as $\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

So, 5 is not in the interval, but

$5 - 2\pi \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

$\sin 5 = sin(2\pi + (5 - 2\pi))$

$= sin\,(5 - 2\pi)$

$\therefore sin^{-1}[sin5] = sin^{-1}[sin(5 - 2\pi)]$

$= 5 - 2\pi$

2. Find the value of the expression in terms of x, with the help of a reference triangle.

$(i)\ sin\,[cos^{-1}(1 - x)]$

Solution: Let $cos^{-1}(1-x)=\theta$

So, $\cos\theta=(1-x)$

$\frac{\text{adj}}{\text{hyp}}=\frac{1-x}{1}$

By using Pythagoras Theorem,

$$hyp^2=opp^2+adj^2$$
$$1^2=opp^2+(1-x)^2$$
$$1=opp^2+1+x^2-2x$$
$$opp^2=1-x^2+2x-1$$
$$opp^2=2x-x^2$$
$$opp=\sqrt{2x-x^2}$$
$$\sin\theta=\frac{\text{opp}}{\text{hyp}}$$
$$=\frac{\sqrt{2x-x^2}}{1}$$
$$=\sqrt{2x-x^2}$$

$\therefore sin\,[cos^{-1}(1-x)]=sin\,[\theta]$

$$=\sqrt{2x-x^2}$$

$(ii)\ cos\,[tan^{-1}(3x-1)]$

Solution: Let $tan^{-1}(3x-1)=\theta$

So, $\tan\theta=(3x-1)$

$\frac{\text{opp}}{\text{adj}}=\frac{(3x-1)}{1}$

By using Pythagoras Theorem,

$$hyp^2=opp^2+adj^2$$
$$=(3x-1)^2+1^2$$
$$=9x^2+1-6x+1$$
$$=9x^2-6x+2$$
$$hyp=\sqrt{(9x^2-6x+2)}$$

and $\cos\theta=\frac{\text{adj}}{\text{hyp}}$

$$=\frac{1}{\sqrt{(9x^2-6x+2)}}$$

$\therefore cos\,[tan^{-1}(3x-1)]=cos\,[\theta]$

$$=\frac{1}{\sqrt{(9x^2-6x+2)}}$$

$(iii)\ tan\left[sin^{-1}\left(x+\frac{1}{2}\right)\right]$

Solution: Let $sin^{-1}\left(x+\frac{1}{2}\right)=\theta$

Then, $\sin\theta=\left(x+\frac{1}{2}\right)$

$$=\frac{2x+1}{2}$$

So, $\frac{\text{opp}}{\text{hyp}}=\frac{2x+1}{2}$

By using Pythagoras Theorem,

$$hyp^2=opp^2+adj^2$$
$$2^2=(2x+1)^2+adj^2$$
$$4=4x^2+1+4x+adj^2$$
$$adj^2=4-4x^2-1-4x$$
$$=3-4x-4x^2$$
$$adj=\sqrt{(3-4x-4x^2)}$$
$$\tan\theta=\frac{\text{opp}}{\text{adj}}=\frac{2x+1}{\sqrt{(3-4x-4x^2)}}$$

$\therefore tan\left[sin^{-1}\left(x+\frac{1}{2}\right)\right]=tan\,[\theta]$

$$=\frac{2x+1}{\sqrt{(3-4x-4x^2)}}$$

3. Find the value of

$(i)\ sin^{-1}\left(cos\left(sin^{-1}\left(\frac{\sqrt{3}}{2}\right)\right)\right)$

Solution: $sin^{-1}\left(\frac{\sqrt{3}}{2}\right)=\frac{\pi}{3}$

$$\therefore cos\left(sin^{-1}\left(\frac{\sqrt{3}}{2}\right)\right)=cos\left(\frac{\pi}{3}\right)$$
$$=\frac{1}{2}$$

Now, $sin^{-1}\left(cos\left(sin^{-1}\left(\frac{\sqrt{3}}{2}\right)\right)\right)=sin^{-1}\left(\frac{1}{2}\right)$

Since $sin\left(\frac{\pi}{6}\right)=\frac{1}{2}$

$$sin^{-1}\left(\frac{1}{2}\right)=\frac{\pi}{6}$$

$$\therefore sin^{-1}\left(cos\left(sin^{-1}\left(\frac{\sqrt{3}}{2}\right)\right)\right)=\frac{\pi}{6}$$

$(ii)\ cot\left(sin^{-1}\frac{3}{5}+sin^{-1}\frac{4}{5}\right)$

Solution: Let $sin^{-1}\left(\frac{3}{5}\right)=A$

Then, $\sin A=\frac{3}{5}$

$$\frac{\text{opp}}{\text{hyp}} = \frac{3}{5}$$

By using Pythagoras Theorem,

$$hyp^2 = opp^2 + adj^2$$
$$5^2 = 3^2 + adj^2$$
$$25 = 9 + adj^2$$
$$adj^2 = 25 - 9$$
$$= 16$$
$$adj = \sqrt{16} = 4$$
$$\therefore \tan A = \frac{\text{opp}}{\text{adj}} = \frac{3}{4}$$

if $sin^{-1}\left(\frac{4}{5}\right) = B$

Then, $\sin B = \frac{4}{5}$

$$\frac{\text{opp}}{\text{hyp}} = \frac{4}{5}$$

By using Pythagoras Theorem,

$$hyp^2 = opp^2 + adj^2$$
$$5^2 = 4^2 + adj^2$$
$$25 = 16 + adj^2$$
$$adj^2 = 25 - 16$$
$$= 9$$
$$adj = \sqrt{9} = 3$$
$$\therefore \tan B = \frac{\text{opp}}{\text{adj}} = \frac{4}{3}$$

$$\therefore cot\left(sin^{-1}\frac{3}{5} + sin^{-1}\frac{4}{5}\right) = cot\,(A+B)$$
$$= \frac{1}{\tan(A+B)}$$
$$\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$$
$$= \frac{\frac{3}{4}+\frac{4}{3}}{1 - \left(\frac{3}{4}\right)\left(\frac{4}{3}\right)}$$
$$= \frac{\frac{9+16}{12}}{1 - \frac{12}{12}}$$
$$= \frac{\frac{9+16}{12}}{1 - 1}$$
$$= \frac{\frac{25}{12}}{0}$$
$$\therefore \frac{1}{\tan(A+B)} = 0$$
$$\therefore cot\left(sin^{-1}\frac{3}{5} + sin^{-1}\frac{4}{5}\right) = cot\,(A+B)$$
$$= \frac{1}{\tan(A+B)}$$
$$= 0$$

(iii) $tan\left(sin^{-1}\frac{3}{5} + cot^{-1}\frac{3}{2}\right)$

Solution: Let $sin^{-1}\left(\frac{3}{5}\right) = A$

Then, $\sin A = \frac{3}{5}$

$$\frac{\text{opp}}{\text{hyp}} = \frac{3}{5}$$

By using Pythagoras Theorem,

$$hyp^2 = opp^2 + adj^2$$
$$5^2 = 3^2 + adj^2$$
$$25 = 9 + adj^2$$
$$adj^2 = 25 - 9$$
$$= 16$$
$$adj = \sqrt{16} = 4$$
$$\therefore \tan A = \frac{\text{opp}}{\text{adj}} = \frac{3}{4}$$

if $cot^{-1}\left(\frac{3}{2}\right) = B$

Then, $\cot B = \frac{3}{2}$

$$\therefore \tan B = \frac{2}{3}$$

$$\therefore tan\left(sin^{-1}\frac{3}{5} + cot^{-1}\frac{3}{2}\right) = tan\,(A+B)$$
$$= \frac{\tan A + \tan B}{1 - \tan A \tan B}$$
$$= \frac{\frac{3}{4}+\frac{2}{3}}{1 - \left(\frac{3}{4}\right)\left(\frac{2}{3}\right)}$$
$$= \frac{\frac{9+8}{12}}{1 - \frac{6}{12}}$$
$$= \frac{\frac{17}{12}}{\frac{12-6}{12}}$$
$$= \frac{17}{12} \times \frac{12}{6}$$
$$= \frac{17}{6}$$

4. Prove that

(i) $tan^{-1}\frac{2}{11} + tan^{-1}\frac{7}{24} = tan^{-1}\frac{1}{2}$

Solution: We know that

$$tan^{-1}A + tan^{-1}B = tan^{-1}\left(\frac{A+B}{1-AB}\right)$$

$\therefore tan^{-1}\frac{2}{11} + tan^{-1}\frac{7}{24}$

$$= tan^{-1}\left[\frac{\left(\frac{2}{11}\right)+\left(\frac{7}{24}\right)}{1-\left(\frac{2}{11}\right)\left(\frac{7}{24}\right)}\right]$$

$$= tan^{-1}\left[\frac{\frac{48+77}{11\times 24}}{1-\frac{14}{11\times 24}}\right]$$

$$= tan^{-1}\left[\frac{\frac{125}{264}}{\frac{264-14}{264}}\right]$$

$$= tan^{-1}\left[\frac{\frac{125}{264}}{\frac{250}{264}}\right]$$

$$= tan^{-1}\left[\frac{125}{264}\times\frac{264}{250}\right]$$

$$= tan^{-1}\left[\frac{1}{2}\right]$$

(ii) $sin^{-1}\frac{3}{5} - cos^{-1}\frac{12}{13} = sin^{-1}\frac{16}{65}$

Solution: Let $sin^{-1}\left(\frac{3}{5}\right) = A$ and

$$cos^{-1}\frac{12}{13} = B$$

$$\therefore sin^{-1}\frac{3}{5} - cos^{-1}\frac{12}{13} = A + B$$

To prove $A - B = sin^{-1}\frac{16}{65}$

Given $sin^{-1}\left(\frac{3}{5}\right) = A$

So, $\sin A = \frac{3}{5}$

$$\frac{\text{opp}}{\text{hyp}} = \frac{3}{5}$$

By using Pythagoras Theorem,

$$hyp^2 = opp^2 + adj^2$$
$$5^2 = 3^2 + adj^2$$
$$25 = 9 + adj^2$$
$$adj^2 = 25 - 9$$
$$= 16$$
$$adj = \sqrt{16} = 4$$
$$\cos A = \frac{\text{adj}}{\text{hyp}}$$
$$= \frac{4}{5}$$

and $cos^{-1}\frac{12}{13} = B$

$$\cos B = \frac{12}{13} = \frac{adj}{hyp}$$

By using Pythagoras Theorem,

$$hyp^2 = opp^2 + adj^2$$
$$13^2 = opp^2 + 12^2$$
$$opp^2 = 13^2 - 12^2$$
$$= 169 - 144$$
$$= 25$$
$$opp = 5$$
$$\sin B = \frac{\text{opp}}{\text{hyp}}$$
$$= \frac{5}{13}$$

We know $\sin(A-B) = \sin A\cos B - \cos A\sin B$

$$= \frac{3}{5}\left(\frac{12}{13}\right) - \frac{4}{5}\left(\frac{5}{13}\right)$$
$$= \frac{36}{65} - \frac{20}{65}$$
$$= \frac{36-20}{65}$$
$$\sin(A-B) = \frac{16}{65}$$

$\therefore A - B = sin^{-1}\left(\frac{16}{65}\right)$ Proved.

5. Prove that

$$tan^{-1}x + tan^{-1}y + tan^{-1}z = tan^{-1}\left[\frac{x+y+z-xyz}{1-xy-yz-zx}\right]$$

Solution: We know that

$$tan^{-1}x + tan^{-1}y = tan^{-1}\left(\frac{x+y}{1-xy}\right)$$

$= tan^{-1}(A)$, Where $A = \frac{x+y}{1-xy}$

$\therefore tan^{-1}x + tan^{-1}y + tan^{-1}z$

$$= tan^{-1}A + tan^{-1}z$$

$$= tan^{-1}\left(\frac{A+Z}{1-AZ}\right)$$

$$= tan^{-1}\left[\frac{\frac{x+y}{1-xy}+Z}{1-\left(\frac{x+y}{1-xy}\right)Z}\right]$$

$$= tan^{-1}\left[\frac{\frac{x+y+z(1-xy)}{1-xy}}{1-\left(\frac{xz+yz}{1-xy}\right)}\right]$$

$$= tan^{-1}\left[\frac{\frac{x+y+z-xyz}{1-xy}}{\frac{(1-xy)-(xz+yz)}{1-xy}}\right]$$

$$= tan^{-1}\left[\frac{\frac{x+y+z-xyz}{1-xy}}{\frac{1-xy-xz-yz}{1-xy}}\right]$$

$$= tan^{-1}\left[\frac{x+y+z-xyz}{1-xy}\times\frac{1-xy}{1-xy-xz-yz}\right]$$

$$= tan^{-1}\left[\frac{x+y+z-xyz}{1-xy-xz-yz}\right] \text{ Hence proved.}$$

6. If $tan^{-1}x + tan^{-1}y + tan^{-1}z = \pi$,

show that $x+y+z = xyz$

Solution: Already proved that

$$tan^{-1}x + tan^{-1}y + tan^{-1}z = tan^{-1}\left[\frac{x+y+z-xyz}{1-xy-xz-yz}\right]$$

$$\pi = tan^{-1}\left[\frac{x+y+z-xyz}{1-xy-xz-yz}\right]$$

$$\tan\pi = \left[\frac{x+y+z-xyz}{1-xy-xz-yz}\right]$$

$$0 = \left[\frac{x+y+z-xyz}{1-xy-xz-yz}\right]$$

$$\therefore x+y+z-xyz = 0$$

$$x+y+z = xyz$$

7. Prove that

$$tan^{-1}x + tan^{-1}\frac{2x}{1-x^2} = tan^{-1}\frac{3x-x^3}{1-3x^2},$$

$|x| < \frac{1}{\sqrt{3}}$

Solution: We know that

$$tan^{-1}A + tan^{-1}B = tan^{-1}\left(\frac{A+B}{1-AB}\right)$$

$$\therefore tan^{-1}x + tan^{-1}\frac{2x}{1-x^2} = tan^{-1}\left(\frac{x+\frac{2x}{1-x^2}}{1-(x)\left(\frac{2x}{1-x^2}\right)}\right)$$

$$= tan^{-1}\left(\frac{\frac{x(1-x^2)+2x}{1-x^2}}{1-\left(\frac{2x^2}{1-x^2}\right)}\right)$$

$$= tan^{-1}\left(\frac{\frac{x-x^3+2x}{1-x^2}}{\frac{1-x^2-2x^2}{1-x^2}}\right)$$

$$= tan^{-1}\left(\frac{\frac{3x-x^3}{1-x^2}}{\frac{1-3x^2}{1-x^2}}\right)$$

$$= tan^{-1}\left(\frac{3x-x^3}{1-x^2}\times\frac{1-x^2}{1-3x^2}\right)$$

$$= tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right)$$

8. Simplify: $tan^{-1}\frac{x}{y} - tan^{-1}\frac{x-y}{x+y}$

Solution: We know that

$$tan^{-1}A - tan^{-1}B = tan^{-1}\left(\frac{A-B}{1+AB}\right)$$

$$\therefore tan^{-1}\frac{x}{y} - tan^{-1}\frac{x-y}{x+y} = tan^{-1}\left[\frac{\left(\frac{x}{y}\right)-\left(\frac{x-y}{x+y}\right)}{1+\left(\frac{x}{y}\right)\left(\frac{x-y}{x+y}\right)}\right]$$

$$= tan^{-1}\left[\frac{\frac{x(x+y)-y(x-y)}{x+y}}{\frac{y(x+y)+x(x-y)}{x+y}}\right]$$

$$= tan^{-1}\left[\frac{\frac{x^2+xy-xy+y^2}{x+y}}{\frac{xy+y^2+x^2-xy}{x+y}}\right]$$

$$= tan^{-1}\left[\frac{\frac{x^2+y^2}{x+y}}{\frac{x^2+y^2}{x+y}}\right]$$

$$= tan^{-1}\left[\frac{x^2+y^2}{x+y}\times\frac{x+y}{x^2+y^2}\right]$$

$$= tan^{-1}[1]$$

$$= \frac{\pi}{4}\ \left[because \tan\frac{\pi}{4} = 1\right]$$

$$\therefore tan^{-1}\frac{x}{y} - tan^{-1}\frac{x-y}{x+y} = \frac{\pi}{4}$$

9. Solve:

$(i)\ sin^{-1}\frac{5}{x} + sin^{-1}\frac{12}{x} = \frac{\pi}{2}$

Solution:

Given $sin^{-1}\frac{5}{x} + sin^{-1}\frac{12}{x} = \frac{\pi}{2}$

$$\therefore sin^{-1}\frac{5}{x} = \frac{\pi}{2} - sin^{-1}\frac{12}{x}$$

$$sin^{-1}\frac{5}{x} = cos^{-1}\frac{12}{x}$$

Let $sin^{-1}\frac{5}{x} = \theta$

Then $\sin\theta = \frac{5}{x}$

Similarly if $cos^{-1}\frac{12}{x} = \theta$

Then $\cos\theta = \frac{12}{x}$

We know that $sin^2\theta + cos^2\theta = 1$

$$\therefore \left(\frac{5}{x}\right)^2 + \left(\frac{12}{x}\right)^2 = 1$$

$$\frac{25}{x^2} + \frac{144}{x^2} = 1$$

$$\frac{25+144}{x^2} = 1$$

$$\frac{169}{x^2} = 1$$

$$x^2 = 169$$

$$x = 13$$

(ii) $2tan^{-1}x = cos^{-1}\frac{1-a^2}{1+a^2} - cos^{-1}\frac{1-b^2}{1+b^2},$

Solution: Let $a = \tan\theta_1$

Then $tan^{-1}a = \theta_1$

$$Now\ \frac{1-a^2}{1+a^2} = \frac{1-(\tan\theta_1)^2}{1+(\tan\theta_1)^2}$$

$$= \cos 2\theta_1$$

Since, $\cos 2\theta = \frac{1-(\tan\theta)^2}{1+(\tan\theta)^2}$

Similarly if $b = \tan\theta_2$

Then $tan^{-1}b = \theta_2$

$$Now\ \frac{1-b^2}{1+b^2} = \frac{1-(\tan\theta_2)^2}{1+(\tan\theta_2)^2}$$

$$= \cos 2\theta_2$$

$$2tan^{-1}x = cos^{-1}\frac{1-a^2}{1+a^2} - cos^{-1}\frac{1-b^2}{1+b^2}$$

$$= cos^{-1}(\cos 2\theta_1) - cos^{-1}(\cos 2\theta_2)$$

$= 2\theta_1 - 2\theta_2$, dividing by 2

$$tan^{-1}x = \theta_1 - \theta_2$$

$$= tan^{-1}a - tan^{-1}b$$

$$= tan^{-1}\left(\frac{a-b}{1+ab}\right)$$

$$x = \frac{a-b}{1+ab}$$

(iii) $2tan^{-1}(\cos x) = tan^{-1}(2\ cosec\ x)$

Solution: We know that

$$tan^{-1}A + tan^{-1}B = tan^{-1}\left(\frac{A+B}{1-AB}\right)$$

$$2tan^{-1}(\cos x) = tan^{-1}(\cos x) + tan^{-1}(\cos x)$$

$$= tan^{-1}\left(\frac{\cos x + \cos x}{1-(\cos x)(\cos x)}\right)$$

$$= tan^{-1}\left(\frac{2\cos x}{1-cos^2 x}\right)$$

$$= tan^{-1}\left(\frac{2\cos x}{sin^2 x}\right)$$

Given $2tan^{-1}(\cos x) = tan^{-1}(2\ cosec\ x)$

$$\therefore tan^{-1}\left(\frac{2\cos x}{sin^2 x}\right) = tan^{-1}(2\ cosec\ x)$$

$$\frac{2\cos x}{sin^2 x} = 2\ cosec\ x$$

$$\frac{\cos x}{(\sin x)(\sin x)} = \frac{1}{(\sin x)}$$

$$\frac{\cos x}{\sin x} = 1$$

$$\cos x = \sin x$$

Gives, $x = \frac{\pi}{4}$

(iv) $cot^{-1}x - cot^{-1}(x+2) = \frac{\pi}{12}$

Solution: $cot^{-1}x - cot^{-1}(x+2)$

$$= tan^{-1}\frac{1}{x} - tan^{-1}\frac{1}{(x+2)}$$

We know that

$$tan^{-1}A - tan^{-1}B = tan^{-1}\left(\frac{A-B}{1+AB}\right)$$

$$\therefore tan^{-1}\frac{1}{x} - tan^{-1}\frac{1}{(x+2)}$$

$$= tan^{-1}\left[\frac{\left(\frac{1}{x}\right) - \left(\frac{1}{x+2}\right)}{1+\left(\frac{1}{x}\right)\left(\frac{1}{x+2}\right)}\right]$$

$$= tan^{-1}\left[\frac{\frac{1(x+2)-x}{x(x+2)}}{\frac{1+x(x+2)}{x(x+2)}}\right]$$

$$= tan^{-1}\left[\frac{\frac{x+2-x}{x(x+2)}}{\frac{1+x^2+2x}{x(x+2)}}\right]$$

$$= tan^{-1}\left[\frac{2}{x(x+2)} \times \frac{x(x+2)}{x^2+2x+1}\right]$$

$$= tan^{-1}\left[\frac{2}{x^2+2x+1}\right]$$

$$\frac{\pi}{12} = \frac{180}{12} = 15 = 45 - 30$$

$$\therefore cot^{-1}x - cot^{-1}(x+2) = \frac{\pi}{12}$$

$$tan^{-1}\left[\frac{2}{x^2+2x+1}\right] = (45-30)$$

$$\left[\frac{2}{x^2+2x+1}\right] = tan(45-30)$$

$$= \frac{\tan 45 - \tan 30}{1 + \tan 45 \tan 30}$$

$$= \frac{1 - \frac{1}{\sqrt{3}}}{1 + 1\left(\frac{1}{\sqrt{3}}\right)}$$

$$= \frac{\frac{\sqrt{3}-1}{\sqrt{3}}}{\frac{\sqrt{3}+1}{\sqrt{3}}}$$

$$= \frac{\sqrt{3}-1}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}+1}$$

$$= \frac{\sqrt{3}-1}{\sqrt{3}+1}$$

$$= \frac{\sqrt{3}-1}{\sqrt{3}+1} \times \frac{\sqrt{3}+1}{\sqrt{3}+1}$$

$$= \frac{3+\sqrt{3}-\sqrt{3}-1}{(\sqrt{3}+1)^2}$$

$$= \frac{2}{(\sqrt{3}+1)^2}$$

$$\frac{2}{(x+1)^2} = \frac{2}{(\sqrt{3}+1)^2}$$

$$\therefore (x+1)^2 = (\sqrt{3}+1)^2$$

$$x + 1 = \sqrt{3} + 1$$

$$x = \sqrt{3} + 1 - 1$$

$$x = \sqrt{3}$$

10. Find the number of solution of the equation

$$tan^{-1}(x-1) + tan^{-1}x + tan^{-1}(x+1) = tan^{-1}(3x)$$

Solution: Given

$$tan^{-1}(x-1) + tan^{-1}x + tan^{-1}(x+1) = tan^{-1}(3x)$$

$$\therefore tan^{-1}(x-1) + tan^{-1}(x+1) = tan^{-1}(3x) - tan^{-1}x$$

We know that

$$tan^{-1}A + tan^{-1}B = tan^{-1}\left(\frac{A+B}{1-AB}\right)$$

$$\text{LHS} = tan^{-1}(x-1) + tan^{-1}(x+1)$$

$$= tan^{-1}\left(\frac{(x-1)+(x+1)}{1-(x-1)(x+1)}\right)$$

$$= tan^{-1}\left(\frac{x-1+x+1}{1-(x^2-1)}\right)$$

$$= tan^{-1}\left(\frac{2x}{1-x^2+1}\right)$$

$$= tan^{-1}\left(\frac{2x}{2-x^2}\right)$$

We know that

$$tan^{-1}A - tan^{-1}B = tan^{-1}\left(\frac{A-B}{1+AB}\right)$$

$$\text{RHS} = tan^{-1}(3x) - tan^{-1}x$$

$$= tan^{-1}\left(\frac{(3x)-(x)}{1+(3x)(x)}\right)$$

$$= tan^{-1}\left(\frac{2x}{1+3x^2}\right)$$

LHS = RHS

$$\therefore tan^{-1}\left(\frac{2x}{2-x^2}\right) = tan^{-1}\left(\frac{2x}{1+3x^2}\right)$$

$$\frac{2x}{2-x^2} = \frac{2x}{1+3x^2}$$

$$2x(1+3x^2) = 2x(2-x^2)$$

$$2x + 6x^3 = 4x - 2x^3$$

$$6x^3 + 2x^3 = 4x - 2x$$

$$8x^3 = 2x \quad \text{dividing by } 2x$$

$$4x^2 = 1$$

$$x^2 = \frac{1}{4}$$

$$x = \pm\frac{1}{2}$$

EXERCISE 4.6

Choose the correct or the most suitable answer from the given four alternatives.

1. *The value of* $sin^{-1}(cosx)$, $0 \le x \le \pi$ *is*

(1) $\pi - x$ (2) $x - \frac{\pi}{2}$

(3) $\boldsymbol{\frac{\pi}{2} - x}$ (4) $x - \pi$

2. If $sin^{-1}x + sin^{-1}y = \frac{2\pi}{3}$

then $cos^{-1}x + cos^{-1}y$ is equal to

(1) $\frac{2\pi}{3}$ **(2)** $\boldsymbol{\frac{\pi}{3}}$ (3) $\frac{\pi}{6}$ (4) π

3. $sin^{-1}\frac{3}{5} - cos^{-1}\frac{12}{13} + sec^{-1}\frac{5}{3} - cosec^{-1}\frac{13}{12}$

is equal to

(1) π (2) π **(3) 0** (4) $tan^{-1}\frac{12}{65}$

4. If $sin^{-1}x = 2sin^{-1}\alpha$ has a solution, then

(1) $\boldsymbol{|\alpha| \le \frac{1}{\sqrt{2}}}$ (2) $|\alpha| \ge \frac{1}{\sqrt{2}}$

(3) $|\alpha| < \frac{1}{\sqrt{2}}$ (4) $|\alpha| > \frac{1}{\sqrt{2}}$

5. $sin^{-1}(cosx) = \frac{\pi}{2} - x$ is valid for

(1) $-\pi \le x \le 0$ **(2)** $\boldsymbol{0 \le x \le \pi}$

(3) $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$ (4) $-\frac{\pi}{4} \le x \le \frac{3\pi}{4}$

6. If $sin^{-1}x + sin^{-1}y + sin^{-1}z = \frac{3\pi}{2}$, the value

$x^{2017} + y^{2018} + z^{2019} - \frac{9}{x^{101} + y^{101} + z^{101}}$ is

(1) 0 (2) 1 (3) 2 (4) 3

7. If $cot^{-1}x = \frac{2\pi}{5}$ for some $x \in R$,

the value of $tan^{-1}x$ is

(1) $-\frac{\pi}{10}$ (2) $\frac{\pi}{5}$ **(3) $\frac{\pi}{10}$** (4) $-\frac{\pi}{5}$

8. The domain of the function defined by

$f(x) = sin^{-1}\sqrt{x-1}$ is

(1) $[1, 2]$ (2) $[-1,1]$

(3) $[0,1]$ (4) $[-1,0]$

9 If $x = \frac{1}{5}$, the value of

$cos(cos^{-1}x + 2sin^{-1}x)$ is

(1) $-\sqrt{\frac{24}{25}}$ (2) $\sqrt{\frac{24}{25}}$ (3) $\frac{1}{5}$ **(4) $-\frac{1}{5}$**

10. $tan^{-1}\left(\frac{1}{4}\right) + tan^{-1}\left(\frac{2}{9}\right)$ is equal to

(1) $\frac{1}{2}cos^{-1}\left(\frac{3}{5}\right)$ (2) $\frac{1}{2}sin^{-1}\left(\frac{3}{5}\right)$

(3) $\frac{1}{2}tan^{-1}\left(\frac{3}{5}\right)$ **(4) $tan^{-1}\left(\frac{1}{2}\right)$**

11. If the function

$f(x) = sin^{-1}\sqrt{x^2-3}$, then x belongs to

(1) $[-1,1]$ (2) $[\sqrt{2}, 2]$

(3) $[-2, -\sqrt{2}] \cup [\sqrt{2}, 2]$

(4) $[-2, -\sqrt{2}] \cap [\sqrt{2}, 2]$

12. If $cot^{-1}2$ and $cot^{-1}3$ are two angles of a triangle, then the third angle is

(1) $\frac{\pi}{4}$ **(2) $\frac{3\pi}{4}$** (3) $\frac{\pi}{6}$ (4) $\frac{\pi}{3}$

13. $sin^{-1}\left(tan\frac{\pi}{4}\right) - sin^{-1}\left(\sqrt{\frac{3}{x}}\right) = \frac{\pi}{6}$

Then x is a root of the equation

(1) $x^2 - x - 6 = 0$ **(2) $x^2 - x - 12 = 0$**

(3) $x^2 + x - 12 = 0$ (4) $x^2 + x - 6 = 0$

14. $sin^{-1}(2cos^2x - 1) + cos^{-1}(1 - 2sin^2x) =$

(1) $\frac{\pi}{2}$ (2) $\frac{\pi}{3}$ (3) $\frac{\pi}{4}$ (4) $\frac{\pi}{6}$

15. If $cot^{-1}(\sqrt{sin\alpha}) + tan^{-1}(\sqrt{sin\alpha}) = u$,

then $cos\ 2u$ is equal to

(1) $tan^2\alpha$ (2) 0 **(3) -1** (4) $tan\ 2\alpha$

16. If $|x| \le 1$, then

$2tan^{-1}x - sin^{-1}\frac{2x}{1+x^2}$ is equal to

(1) $tan^{-1}x$ (2) $sin^{-1}x$ **(3) 0** (4) π

17. The equation

$tan^{-1}x - cot^{-1}x = tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$ has

(1) no solution **(2) unique solution**

(3) two solutions

(4) infinite number of solutions

18. If $sin^{-1}x + cot^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{2}$, then x is equal to

(1) $\frac{1}{2}$ **(2) $\frac{1}{\sqrt{5}}$** (3) $\frac{2}{\sqrt{5}}$ (4) $\frac{\sqrt{3}}{2}$

19. If $sin^{-1}\frac{x}{5} + cosec^{-1}\frac{5}{4} = \frac{\pi}{2}$,

then the value of x is

(1) 4 (2) 5 (3) 2 **(4) 3**

20. $sin(tan^{-1}x), |x| < 1$ is equal to

(1) $\frac{x}{\sqrt{1-x^2}}$ (2) $\frac{1}{\sqrt{1-x^2}}$

(3) $\frac{1}{\sqrt{1+x^2}}$ **(4) $\frac{x}{\sqrt{1+x^2}}$**

FORMULAE

Property – I

1. $sin^{-1}[sin(\theta)] = \theta$ if $\theta \in \left[-\frac{\pi}{2},\frac{\pi}{2}\right]$
2. $cos^{-1}[cos(\theta)] = \theta$ if $\theta \in [0,\pi]$
3. $tan^{-1}[tan(\theta)] = \theta$ if $\theta \in \left[-\frac{\pi}{2},\frac{\pi}{2}\right]$
4. $cosec^{-1}[cosec(\theta)] = \theta$ if $\theta \in \left[-\frac{\pi}{2},\frac{\pi}{2}\right] \setminus \{0\}$
5. $sec^{-1}[sec(\theta)] = \theta$ if $\theta \in [0,\pi] \setminus \left\{\frac{\pi}{2}\right\}$
6. $cot^{-1}[cot(\theta)] = \theta$ if $\theta \in [0,\pi]$

Property – II

1. $sin^{-1}[sin(x)] = x$ if $x \in [-1,1]$
2. $cos^{-1}[cos(x)] = x$ if $x \in [-1,1]$
3. $tan^{-1}[tan(x)] = x$ if $x \in R$
4. $cosec^{-1}[cosec(x)] = x$ if $x \in R \setminus (-1,1)$
5. $sec^{-1}[sec(x)] = x$ if $x \in R \setminus (-1,1)$
6. $cot^{-1}[cot(x)] = x$ if $x \in R$

Property – III [Reciprocal Inverse Identities]

1. $sin^{-1}\left(\frac{1}{x}\right) = cosec\ x$ if $x \in R \setminus (-1,1)$
2. $cos^{-1}\left(\frac{1}{x}\right) = sec\ x$ if $x \in R \setminus (-1,1)$
3. $tan^{-1}\left(\frac{1}{x}\right) = \begin{cases} cot^{-1}x & if\ x > 0 \\ -\pi + cot^{-1}x & if\ x < 0 \end{cases}$

Property – IV [Reflection Identities]

1. $sin^{-1}(-x) = -sin^{-1}x$ if $x \in [-1,1]$
2. $cos^{-1}(-x) = \pi - cos^{-1}x$ if $x \in [-1,1]$
3. $tan^{-1}(-x) = -tan^{-1}x$ if $x \in R$
4. $cosec^{-1}(-x)$
   $= -cosec^{-1}x$ if $|x| \geq 1\ or\ x \in R \setminus (-1,1)$
5. $sec^{-1}(-x)$
   $= -sec^{-1}x$ if $|x| \geq 1\ or\ x \in R \setminus (-1,1)$
6. $cot^{-1}(-x) = -cot^{-1}x$ if $x \in R$

Property –V [co function Inverse Identities]

1. $sin^{-1}x + cos^{-1}x = \frac{\pi}{2}, x \in [-1,1]$
2. $tan^{-1}x + cot^{-1}x = \frac{\pi}{2}, x \in R$
3. $cosec^{-1}x + sec^{-1}x = \frac{\pi}{2}, x \in R \setminus (-1,1)$

1. $If\ sin^{-1}(x) = \theta\ then\ \sin\theta = x$
2. $sin(-\theta) = -\sin\theta$
3. The range of the principal value of $sin^{-1}(x) = \left[-\frac{\pi}{2},\frac{\pi}{2}\right]$
4. The domain of $sin^{-1}(x) = [-1,1]$
5. $sin(\pi-\theta) = \sin\theta$
6. $sin(\pi+\theta) = -\sin\theta$
7. Period of the function $sin\ x\ is\ 2\pi$
8. Amplitude of $sin\ x\ is\ 1$
9. $\pi\ radians = 180^0$
10. $sin^{-1}[sin(\theta)] = \theta$
11. If $a \leq |x| \leq b, \Rightarrow x \in [-b,-a] \cup [a,b]$
12. $\sin(A+B) = \sin A\cos B + \cos A\sin B$
13. $\sin(A-B) = \sin A\cos B - \cos A\sin B$
14. $If\ cos^{-1}(x) = \theta\ then\ \cos\theta = x$
15. $cos(-\theta) = \cos\theta$
16. The range of the principal value of $cos^{-1}(x) = [0,\pi]$
17. The domain of $cos^{-1}(x) = [-1,1]$
18. $cos(\pi-\theta) = -\cos\theta$
19. $cos(\pi+\theta) = -\cos\theta$
20. $cos^{-1}[cos(\theta)] = \theta$

21. $\cos(A+B) = \cos A \cos B - \sin A \sin B$

22. $\cos(A-B) = \cos A \cos B + \sin A \sin B$

23. $cos^{-1}(x) + sin^{-1}(x) = \frac{\pi}{2}$

24. $\tan(-\theta) = -\tan\theta$

25. $\sin 2\theta = \frac{2\tan\theta}{1+tan^2\theta}$

26. $\cos 2\theta = \frac{1-tan^2\theta}{1+tan^2\theta}$

27. $tan^{-1}A + tan^{-1}B = tan^{-1}\left[\frac{A+B}{1-AB}\right]$

28. $tan^{-1}A - tan^{-1}B = tan^{-1}\left(\frac{A-B}{1+AB}\right)$

29. $\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$

30.

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
$\sin\theta$	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
$\tan\theta$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	∞

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