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UNIT - 5

Two Dimensional Analytical Geometry-II

Chapter5
Two Dimensional Analytical Geometry-II

Example 5.1 Find the general equation of a circle with centre $(-3,-4)$ and radius 3 units.

Solution: Equation of the circle with centre (h,k) and radius r is $(x-h)^2+(y-k)^2=r^2$

Given: Centre $(h,k)=(-3,-4)$, radius $r=3$

$\therefore$ Equation of the circle

$$[x-(-3)]^2+[y-(-4)]^2=3^2$$
$$(x+3)^2+(y+4)^2=9$$
$$x^2+6x+9+y^2+8y+16=9$$
$$x^2+y^2+6x+8y+25=9$$
$$x^2+y^2+6x+8y+25-9=0$$
$$x^2+y^2+6x+8y+16=0$$

....

Example 5.2 Find the equation of the circle described on the chord $3x+y+5=0$ of the circle $x^2+y^2=16$ as diameter.

Solution:

The chord $3x+y+5=0$ is the diameter.

So, its centre is $\left(-\frac{3\lambda}{2},-\frac{\lambda}{2}\right)$

Since centre lies on the diameter

$$3\left(-\frac{3\lambda}{2}\right)+\left(-\frac{\lambda}{2}\right)+5=0$$

Multiplying by 2

$$-9\lambda-\lambda+10=0$$
$$-10\lambda+10=0$$
$$-10\lambda=-10$$
$$\lambda=1$$

Equation of the circle passing through the points of intersection of the chord and circle is

$$x^2+y^2-16+\lambda(3x+y+5)=0$$

Substituting $\lambda=1$

$$x^2+y^2-16+1(3x+y+5)=0$$
$$x^2+y^2-16+3x+y+5=0$$
$$x^2+y^2+3x+y-11=0$$

Example 5.3 Determine whether $x+y-1=0$ is the equation of a diameter of the circle $x^2+y^2-6x+4y+c=0$ for all possible values of c.

Solution:

General Equation of the circle with $(-g,-f)$ as center is $x^2+y^2+2gx+2fy+c=0$

Comparing, $x^2+y^2-6x+4y+c=0$

$$2g=-6\Rightarrow g=-3$$
$$2f=4\Rightarrow f=2$$

Hence centre $(-g,-f)=(3,-2)$

Given $x+y-1=0$ is the diameter

Substituting $x=3, y=-2$

$$3-2-1=0$$
$$3-3=0$$

So, $x+y-1=0$ is the diameter equation for all possible values of c.

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Example 5.4 Find the general equation of the circle whose diameter is the line segment joining the points $(-4,-2)$ and $(1,1)$.

Solution: Equation of the circle with (x_1,y_1) and (x_2,y_2) as end points of the diameter is

$$(x-x_1)(x-x_2)+(y-y_1)(y-y_2)=0$$

Given $(x_1,y_1)=(-4,-2)$ and $(x_2,y_2)=(1,1)$

So, $[x-(-4)](x-1)+[y-(-2)](y-1)=0$

$$(x+4)(x-1)+(y+2)(y-1)=0$$
$$x^2-x+4x-4+y^2-y+2y-2=0$$
$$x^2+3x-4+y^2+y-2=0$$
$$x^2+y^2+3x+y-6=0$$

....

Example 5.5 Examine the position of the point $(2,3)$ with respect to the circle $x^2+y^2-6x-8y+12=0$.

Solution: Given $(x_1,y_1)=(2,3)$

Substituting the values in

$x^2+y^2-6x-8y+12$, we get

$2^2 + 3^2 - 6(2) - 8(3) + 12$

$= 4 + 9 - 12 - 24 + 12$

$= 25 - 36$

$= -11 < 0$

Hence the point $(2, 3)$ lies inside the circle.

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Example 5.6 The line $3x + 4y - 12 = 0$ meets the coordinate axes at A and B. Find the equation of the circle drawn on AB as diameter

Solution: Let the line $3x + 4y - 12 = 0$ cuts the x axis at A and y axis at B.

On x axis y = 0, so $3x + 4(0) - 12 = 0$

$3x - 12 = 0$

$3x = 12$

$x = 4$

Hence A $(4, 0)$

On y axis x = 0, so $3(0) + 4y - 12 = 0$

$4y - 12 = 0$

$4y = 12$

$y = 3$

Hence B $(0, 3)$

Given AB is the diameter.

Equation of the circle with (x_1, y_1) and (x_2, y_2) as end points of the diameter is

$$(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0$$

Given $(x_1, y_1) = (4, 0)$ and $(x_2, y_2) = (0, 3)$

So, $[x - (4)](x - 0) + [y - (0)](y - 3) = 0$

$(x - 4)(x) + (y)(y - 3) = 0$

$x^2 - 4x + y^2 - 3y = 0$

$x^2 + y^2 - 4x - 3y = 0$

....

Example 5.7 A line $3x + 4y + 10 = 0$ cuts a chord of length 6 units on a circle with centre of the circle $(2, 1)$. Find the equations of the circle in general form.

Solution: Let AB be the chord.

Given length of the chord AB = 6

If M is the midpoint, then AM = BM = 3

We know that perpendicular distance from any point (x_1, y_1) to the line $ax + by + c = 0$ is

$$= \frac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}}$$

The distance from $(2, 1)$ to the line $3x + 4y + 10 = 0$ is

$$\text{CM} = \frac{|3(2) + 4(1) + 10|}{\sqrt{3^2 + 4^2}}$$

$$= \frac{|6 + 4 + 10|}{\sqrt{9 + 16}}$$

$$= \frac{|20|}{\sqrt{25}}$$

$$= \frac{20}{5}$$

$$= 4$$

By Pythagoras Theorem,

$$hyp^2 = opp^2 + adj^2$$

$$AC^2 = CM^2 + AM^2$$

$= 4^2 + 3^2$

$= 16 + 9$

$= 25$

Radius AC = 5

Equation of the circle with centre (h, k) and radius r is $(x - h)^2 + (y - k)^2 = r^2$

Given: Centre $(h, k) = (2, 1)$, radius $r = 5$

∴ Equation of the circle

$[x - (2)]^2 + [y - (1)]^2 = 5^2$

$(x - 2)^2 + (y - 1)^2 = 25$

$x^2 - 4x + 4 + y^2 - 2y + 1 = 25$

$x^2 + y^2 - 4x - 2y + 5 = 25$

$x^2 + y^2 - 4x - 2y + 5 - 25 = 0$

$x^2 + y^2 - 4x - 2y - 20 = 0$

....

Example 5.8 A circle of radius 3 units touches both the axes. Find the equations of all possible circles formed in the general form.

Solution: As the circle touches both the axes, the distance of the centre from both the axes is 3 units, centre can be $(\pm3,\pm3)$ and hence there are four circles with radius 3,

Equation of the circle with centre (h,k) and radius r is $(x-h)^2+(y-k)^2=r^2$

and the required equations of the four circles with Centre $(h,k)=(\pm3,\ \pm3)$, radius $r=3$

∴ Equation of the circles

$$[x-(\pm3)]^2+[y-(\pm3)]^2=3^2$$
$$(x\pm3)^2+(y\pm3)^2=9$$
$$x^2\pm6x+9+y^2\pm6y+9=9$$
$$x^2+y^2\pm6x\pm6y+18=9$$
$$x^2+y^2\pm6x\pm6y+18-9=0$$
$$x^2+y^2\pm6x\pm6y+9=0$$

<u>Example 5.9</u> Find the centre and radius of the circle $3x^2+(a+1)y^2+6x-9y+a+4=0$.

Solution: For a second degree equation to represent a circle is co-eff of x^2= co-eff of y^2

Hence 3 = a +1

a +1= 3

a = 3 – 1

a = 2

$\therefore\ 3x^2+(a+1)y^2+6x-9y+a+4=0$

becomes

$$3x^2+(2+1)y^2+6x-9y+2+4=0$$
$$3x^2+3y^2+6x-9y+6=0$$

Dividing by 3

$$x^2+y^2+2x-3y+2=0$$

General Equation of the circle with $(-g,-f)$ as center is $x^2+y^2+2gx+2fy+c=0$

Comparing, $x^2+y^2+2x-3y+2=0$

$$2g=2\Rightarrow g=1$$
$$2f=-3\Rightarrow f=-\frac{3}{2}$$

Hence centre is $\left(1,-\frac{3}{2}\right)$

Radius r $=\sqrt{g^2+f^2-c}$

$$=\sqrt{(1)^2+\left(-\frac{3}{2}\right)^2-2}$$
$$=\sqrt{1+\frac{9}{4}-2}$$
$$=\sqrt{\frac{9}{4}-1}$$
$$=\sqrt{\frac{9-4}{4}}$$

Radius r $=\sqrt{\frac{5}{4}}$

<u>Example 5.10</u> Find the equation of the circle passing through the points (1,1) , (2,−1), and (3, 2).

Solution: General Equation of the circle is

$x^2+y^2+2gx+2fy+c=0$

It passes through $(1,1)$ hence

$$1^2+1^2+2g(1)+2f(1)+c=0$$
$$1+1+2g+2f+c=0$$
$$2+2g+2f+c=0$$
$$2g+2f+c=-2\ldots(1)$$

It passes through $(2,-1)$ hence

$$2^2+(-1)^2+2g(2)+2f(-1)+c=0$$
$$4+1+4g-2f+c=0$$
$$5+4g-2f+c=0$$
$$4g-2f+c=-5\ldots(2)$$

It passes through $(3,2)$ hence

$$3^2+2^2+2g(3)+2f(2)+c=0$$
$$9+4+6g+4f+c=0$$
$$13+6g+4f+c=0$$
$$6g+4f+c=-13\ldots(3)$$

(2) − (1) gives $\quad 2g-4f=-3\ldots\ldots(4)$

(3) − (2) gives $\quad 2g+6f=-8\ldots\ldots(5)$

(5) − (4) gives $\quad 10f=-5$

$$f=-\frac{5}{10}$$
$$\therefore f=-\frac{1}{2}$$

Substituting $f=-\frac{1}{2}$ in $2g-4f=-3$

$$2g-4\left(-\frac{1}{2}\right)=-3$$
$$2g+2=-3$$
$$2g=-3-2$$
$$2g=-5$$
$$\therefore g=-\frac{5}{2}$$

Substituting $f=-\frac{1}{2}$ and $g=-\frac{5}{2}$

in $6g+4f+c=-13$

$$6\left(-\frac{5}{2}\right)+4\left(-\frac{1}{2}\right)+c=-13$$
$$3(-5)+2(-1)+c=-13$$
$$-15-2+c=-13$$
$$-17+c=-13$$
$$c=-13+17$$
$$c=4$$

∴ Equation of the circle is

$$x^2+y^2+2gx+2fy+c=0$$
$$x^2+y^2+2\left(-\frac{5}{2}\right)x+2\left(-\frac{1}{2}\right)y+4=0$$
$$x^2+y^2-5x-y+4=0$$

....

<u>Example 5.11</u> Find the equations of the tangent and normal to the circle $x^2+y^2=25$ at $P(-3,\ 4)$.

Solution: Equation the circle with centre at origin and radius a is $x^2+y^2=a^2$

(i) Equation of the tangent at (x_1,y_1) is

$$xx_1+yy_1=a^2$$

(ii) Equation of the normal at (x_1,y_1) is

$$xy_1-yx_1=0$$

(i) Equation of the tangent at $(-3,4)$ is

$$x(-3)+y(4)=25$$
$$-3x+4y-25=0$$
$$3x-4y+25=0$$

(ii) Equation of the normal at $(-3,4)$ is

$$xy_1-yx_1=0$$
$$x(4)-y(-3)=0$$
$$4x+3y=0$$

<u>Example 5.12</u> If $y=4x+\ c$ is a tangent to the circle $x^2+y^2=9$, find c.

Solution: $y=mx+\ c$ is the tangent to the circle $x^2+y^2=a^2$is $c^2=\ a^2(1+m^2)$

Comparing $m=4$ and $a^2=9$

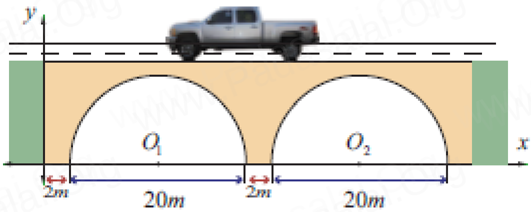
Hence $c^2=\ 9(1+4^2)$

$$c^2=9(1+4^2)$$
$$=9(1+16)$$
$$c^2=9(17)$$
$$c=\pm 3\sqrt{(17)}$$

....

<u>Example 5.13</u> A road bridge over an irrigation canal have two semi circular vents each with a span of 20m and the supporting pillars of width 2m. Use diagram to write the equations that model the arches.

Solution:



Let O_1,O_2 be the centers of the two given semi circles.

First vent center is $(12,0)$ and radius r = 10

Equation of the circle with centre $(h,k)$and radius r is $(x-h)^2+(y-k)^2=r^2$

$$\text{Hence } (x-12)^2+(y-0)^2=10^2$$
$$x^2-24x+144+y^2=100$$
$$x^2+y^2-24x+144-100=0$$
$$x^2+y^2-24x+44=0$$

Second vent center is $(34,0)$ and radius r = 10

Equation of the circle with centre $(h,k)$and radius r is $(x-h)^2+(y-k)^2=r^2$

$$\text{Hence } (x-34)^2+(y-0)^2=10^2$$
$$x^2-68x+1156+y^2=100$$
$$x^2+y^2-68x+1156-100=0$$
$$x^2+y^2-68x+1056=0$$

EXERCISE 5.1

1. Obtain the equation of the circles with radius 5 cm and touching x-axis at the origin in general form.

Solution: Given radius r =5 cm

The circle touches x axis, its centre is $(0, \pm 5)$

Equation of the circle with centre $(h, k)$and radius r is $(x-h)^2+(y-k)^2=r^2$

$$\text{Hence } (x-0)^2+(y\pm 5)^2=5^2$$
$$x^2+y^2\pm 10y+25=25$$
$$x^2+y^2\pm 10y+25-25=0$$
$$x^2+y^2\pm 10y=0$$

....

2. Find the equation of the circle with centre $(2,-1)$ and passing through the point $(3,6)$ in standard form.

Solution: Given center C $(2,-1)$ and passing through the point A $(3,\ 6)$

$$\text{Radius AC} = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$$
$$=\sqrt{(3-2)^2+(6+1)^2}$$
$$=\sqrt{(1)^2+(7)^2}$$
$$=\sqrt{1+49}$$
$$\text{r} = \sqrt{50}$$

Equation of the circle with centre $(h, k)$and radius r is $(x-h)^2+(y-k)^2=r^2$

We have center C $(2,-1)$ and $r^2=50$

$$\therefore (x-h)^2+(y-k)^2=r^2$$
$$(x-2)^2+(y+1)^2=50$$

....

3. Find the equation of circles that touch both the axes and pass through $(-4,-2)$ in general form.

Solution: The circles that touch both the axes. Hence its centre will beC $(r,\ r)$ and radius r. The circle passes through A $(-4,-2)$

$$\text{Radius AC} = \sqrt{(r+4)^2+(r+2)^2}$$
$$=\sqrt{r^2+8r+16+r^2+4r+4}$$
$$=\sqrt{2r^2+12r+20}$$
$$r^2=2r^2+12r+20$$
$$2r^2+12r+20-r^2=0$$
$$r^2+12r+20=0$$
$$(r+10)(r+2)=0$$
$$r=-10 \text{ or } r=-2$$

We know that Equation of the circle, centre $(h, k)$and radius r is $(x-h)^2+(y-k)^2=r^2$

(i) When $r=-10$, centre $(-10,-10)$

$$(x+10)^2+(y+10)^2=(-10)^2$$
$$x^2+20x+100+y^2+20y+100=100$$
$$x^2+y^2+20x+20y+200=100$$
$$x^2+y^2+20x+20y+200-100=0$$
$$x^2+y^2+20x+20y+100=0$$

(ii) When $r=-2$, centre $(-2,-2)$

$$(x+2)^2+(y+2)^2=(-2)^2$$
$$x^2+4x+4+y^2+4y+4=4$$
$$x^2+y^2+4x+4y+8=4$$
$$x^2+y^2+4x+4y+8-4=0$$
$$x^2+y^2+4x+4y+4=0$$

....

4. Find the equation of the circle with centre $(2,3)$ and passing through the intersection of the lines $3x-2y-1=0$ and $4x+y-27=0$.

Solution: Circle passes through the intersection of the lines

$$3x-2y-1=0 \quad ......(1)$$
$$4x+y-27=0 \quad ......(2)$$
$$2\times(2) \Rightarrow 8x+2y-54=0$$
$$(1) \Rightarrow 3x-2y-1=0$$
$$11x-55=0$$
$$11x=55$$
$$x=5$$

Substituting $x = 5$ in $3x - 2y - 1 = 0$

$$3(5) - 2y - 1 = 0$$
$$15 - 2y - 1 = 0$$
$$14 - 2y = 0$$
$$-2y = -14$$
$$y = 7$$

Hence A $(5,\ 7)$ is the point on the circle, and center C $(2, 3)$

Radius AC $= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

$$= \sqrt{(5-2)^2 + (7-3)^2}$$
$$= \sqrt{(3)^2 + (4)^2}$$
$$= \sqrt{9 + 16}$$
$$= \sqrt{25}$$
$$r = 5$$

We know that Equation of the circle, centre (h, k) and radius r is $(x-h)^2 + (y-k)^2 = r^2$

here $r = 5$, centre $(2,\ 3)$

$$(x-2)^2 + (y-3)^2 = 5^2$$
$$x^2 - 4x + 4 + y^2 - 6y + 9 = 25$$
$$x^2 + y^2 - 4x - 6y + 13 = 25$$
$$x^2 + y^2 - 4x - 6y + 13 - 25 = 0$$
$$x^2 + y^2 - 4x - 6y - 12 = 0$$

....

5. Obtain the equation of the circle for which $(3, 4)$ and $(2, -7)$ are the ends of a diameter.

Solution: Equation of the circle with (x_1, y_1) and (x_2, y_2) as end points of the diameter is

$(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0$

Given $(x_1, y_1) = (3, 4)$ and $(x_2, y_2) = (2, -7)$

So, $(x-3)(x-2) + (y-4)[y-(-7)] = 0$

$$(x-3)(x-2) + (y-4)(y+7) = 0$$
$$x^2 - 2x - 3x + 6 + y^2 + 7y - 4y - 28 = 0$$
$$x^2 - 5x + 6 + y^2 + 3y - 28 = 0$$
$$x^2 + y^2 - 5x + 3y - 22 = 0$$

....

6. Find the equation of the circle through the points $(1,0)$, $(-1,0)$, and $(0, 1)$.

Solution: General Equation of the circle is

$x^2 + y^2 + 2gx + 2fy + c = 0$

It passes through $(1, 0)$ hence

$$1^2 + 0 + 2g(1) + 2f(0) + c = 0$$
$$1 + 2g + 0 + c = 0$$
$$1 + 2g + c = 0$$
$$2g + c = -1 \ldots(1)$$

It passes through $(-1, 0)$ hence

$$(-1)^2 + 0 + 2g(-1) + 2f(0) + c = 0$$
$$1 + 0 - 2g + 0 + c = 0$$
$$1 - 2g + c = 0$$
$$-2g + c = -1 \ldots(2)$$

It passes through $(0, 1)$ hence

$$0 + 1 + 2g(0) + 2f(1) + c = 0$$
$$1 + 2f + c = 0$$
$$2f + c = -1 \ldots(3)$$

$(1) + (2)$ gives $\quad 2c = -2$

$$c = -1$$

Substituting $c = -1$ in $-2g + c = -1$

$$-2g - 1 = -1$$
$$-2g = -1 + 1$$
$$-2g = 0$$

Gives $g = 0$

Substituting $c = -1$ in $\quad 2f + c = -1$

$$-2f - 1 = -1$$
$$-2f = -1 + 1$$
$$-2f = 0$$

Gives $f = 0$

So, $g = 0, f = 0$ and $c = -1$

∴ Equation of the circle is

$$x^2 + y^2 + 2gx + 2fy + c = 0$$
$$x^2 + y^2 + 2(0)x + 2(0)y - 1 = 0$$
$$x^2 + y^2 - 1 = 0$$
$$x^2 + y^2 = 1$$

....

7. A circle of area 9π square units has two of its diameters along the lines $x+y=5$ and $x-y=1$. Find the equation of the circle.

Solution: Two diameters are

$$x+y=5$$
$$x-y=1$$

Adding $2x=6$

$$x=3$$

Substituting $x=3$ in $x+y=5$

$$3+y=5$$
$$y=5-3$$
$$y=2$$

Hence centre C $(3,2)$

Given Area of the circle $\pi r^2 = 9\pi$

gives $r^2 = 9$

Equation of the circle with centre $(h,k)$and radius r is $(x-h)^2+(y-k)^2=r^2$

Hence $(x-3)^2+(y-2)^2=9$

$$x^2-6x+9+y^2-4y+4=9$$
$$x^2+y^2-6x-4y+13-9=0$$
$$x^2+y^2-6x-4y+4=0$$

....

8. If $y=2\sqrt{2}x+\ c$ is a tangent to the circle $x^2+y^2=16$, find the value of c.

Solution: $y=mx+\ c$ is the tangent to the circle $x^2+y^2=a^2$is $c^2=\ a^2(1+m^2)$

Comparing $m=2\sqrt{2}$ and $a^2=16$

Hence $c^2=\ 16\left[1+\left(2\sqrt{2}\right)^2\right]$

$$c^2=16(1+8)$$
$$=16(9)$$
$$c=\pm 4\ (3)$$
$$c=\pm 12$$

....

9. Find the equation of the tangent and normal to the circle $x^2+y^2-6x+6y-8=0$ at $(2,2)$.

Solution: Equation of the tangent at (x_1,y_1) to the circle $x^2+y^2+2gx+2fy+c=0$ is

$$xx_1+yy_1+2g\left(\frac{x+x_1}{2}\right)+2f\left(\frac{y+y_1}{2}\right)+c=0$$

Given the point is $(2,2)$

The equation of the tangent is

$$xx_1+yy_1-6\left(\frac{x+x_1}{2}\right)+6\left(\frac{y+y_1}{2}\right)-8=0$$
$$xx_1+yy_1-3(x+x_1)+3(y+y_1)-8=0$$
$$x(2)+y(2)-3(x+2)+3(y+2)-8=0$$
$$2x+2y-3x-6+3y+6-8=0$$
$$-x+5y-8=0$$
$$x-5y+8=0$$

Normal is perpendicular to the tangent.

So, Normal is of the form $5x+y+k=0$

Normal passes through the point $(2,2)$

$$5(2)+2+k=0$$
$$10+2+k=0$$
$$12+k=0$$
$$k=-12$$

So, equation of the normal is $5x+y-12=0$

....

10. Determine whether the points $(-2,1)$, $(0,0)$, and $(-4,-3)$ lie outside, on or inside the circle $x^2+y^2-5x+2y-5=0$.

Solution:

(i) Given $(x_1,y_1)=(-2,1)$

Substituting the values in

$x^2+y^2-5x+2y-5$, we get

$(-2)^2+(1)^2-5(-2)+2(1)-5$

$=4+1+10+2-5$

$=17-5$

$=12>0$

So, $(-2,1)$lies outside the circle.

(ii) Given $(x_1,y_1)=(0,0)$

Substituting the values in

$x^2+y^2-5x+2y-5$, we get

$0+0-5(0)+2(0)-5$

$=0-5$

$=-5<0$

So, $(0,\ 0)$lies inside the circle.

(iii) Given $(x_1,y_1)=(-4,-3)$

Substituting the values in

$x^2+y^2-5x+2y-5$, we get

$(-4)^2+(-3)^2-5(-4)+2(-3)-5$

$=16+9+20-6-5$

$=45-11$

$=34>0$

So, $(-4,-3)$ lies outside the circle.

....

11. Find centre and radius of the circles.

(i) $x^2+(y+2)^2=0$.

Solution: Equation of the circle,

centre (h,k) and radius r is

$(x-h)^2+(y-k)^2=r^2$

Comparing center $(0,-2)$

and radius $r=0$

(ii) $x^2+y^2+6x-4y+4=0$.

Solution: General Equation of the circle

with $(-g,-f)$ as center is

$x^2+y^2+2gx+2fy+c=0$

Comparing, $x^2+y^2+6x-4y+4=0$

$2g=6 \Rightarrow g=3$

$2f=-4 \Rightarrow f=-2$ and $c=4$

Hence centre is $(-3,2)$

Radius r $=\sqrt{g^2+f^2-c}$

$=\sqrt{(-3)^2+(2)^2-4}$

$=\sqrt{9+4-4}$

$=\sqrt{9}$

$r=3$

(iii) $x^2+y^2-x+2y-3=0$.

Solution: General Equation of the circle

with $(-g,-f)$ as center is

$x^2+y^2+2gx+2fy+c=0$

Comparing, $x^2+y^2-x+2y-3=0$

$2g=-1 \Rightarrow g=-\frac{1}{2}$

$2f=2 \Rightarrow f=1$ and $c=-3$

Hence centre is $\left(\frac{1}{2},-1\right)$

Radius r $=\sqrt{g^2+f^2-c}$

$=\sqrt{\left(\frac{1}{2}\right)^2+(-1)^2+3}$

$=\sqrt{\frac{1}{4}+1+3}$

$=\sqrt{\frac{1}{4}+4}$

$=\sqrt{\frac{1+16}{4}}$

$=\sqrt{\frac{17}{4}}$

$r=\frac{\sqrt{17}}{2}$

(iv) $2x^2+2y^2-6x+4y+2=0$.

Solution: Given

$2x^2+2y^2-6x+4y+2=0$.

Dividing by 2

$x^2+y^2-3x+2y+1=0$.

General Equation of the circle with $(-g,-f)$ as center is $x^2+y^2+2gx+2fy+c=0$

Comparing, $x^2+y^2-3x+2y+1=0$

$2g=-3 \Rightarrow g=-\frac{3}{2}$

$2f=2 \Rightarrow f=1$ and $c=1$

Hence centre is $\left(\frac{3}{2},-1\right)$

Radius r $=\sqrt{g^2+f^2-c}$

$=\sqrt{\left(\frac{3}{2}\right)^2+(-1)^2-1}$

$=\sqrt{\frac{9}{4}+1-1}$

$=\sqrt{\frac{9}{4}}$

$r=\frac{3}{2}$

....

12. If the equation

$3x^2+(3-p)xy+qy^2-2px=8pq$

represents a circle, find p and q. Also determine the centre and radius of the circle.

Solution: For a second degree equation to represent a circle is co-eff of x^2= co-eff of y^2

Hence $3 = q$

and co-eff of $xy = 0$

Hence $3 - p = 0$

$p = 3$

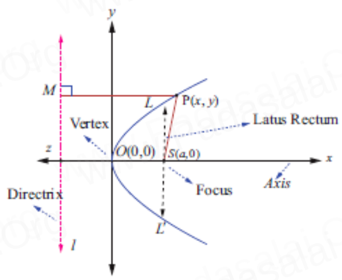
So $3x^2 + (3-p)xy + qy^2 - 2px = 8pq$

becomes $3x^2 + (3)y^2 - 2(3)x = 8(3)(3)$

Dividing by 3, $x^2 + y^2 - 2x - 24 = 0$

....

Example 5.14 Find the length of Latus rectum of the parabola $y^2 = 4ax$.



Solution: Equation of the parabola $y^2 = 4ax$

Latus rectum is $LL^{/}$, which passes through Focus $(a, 0)$

If we take LF= y_1, then L is (a, y_1)

Since L (a, y_1) on the parabola,

$${y_1}^2 = 4a(a)$$

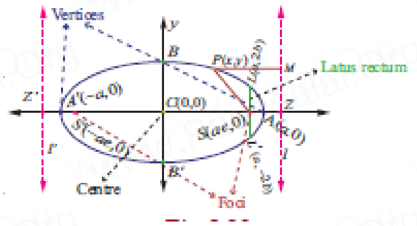
$${y_1}^2 = 4a^2 \text{ gives,}$$

$$y_1 = \pm 2a$$

The end points of the latus rectum are $(a, 2a)$ and $(a, -2a)$

Hence length of the latus rectum $LL^{/} = 4a$

Example 5.15 Find the length of Latus rectum of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.



Solution: Equation of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Latus rectum is $LL^{/}$, which passes through Focus $(ae, 0)$

If we take LS= y_1, then L is (ae, y_1)

Since L (ae, y_1) on the ellipse,

$$\frac{(ae)^2}{a^2} + \frac{{y_1}^2}{b^2} = 1$$

$$\frac{a^2e^2}{a^2} + \frac{{y_1}^2}{b^2} = 1$$

$$e^2 + \frac{{y_1}^2}{b^2} = 1$$

$$\frac{{y_1}^2}{b^2} = 1 - e^2$$

$${y_1}^2 = b^2(1 - e^2)$$

$$\text{Since } e^2 = \left(1 - \frac{b^2}{a^2}\right) \Rightarrow (1 - e^2) = \frac{b^2}{a^2}$$

$${y_1}^2 = b^2\left(\frac{b^2}{a^2}\right)$$

$$y_1 = \pm\frac{b^2}{a}$$

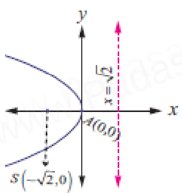
The end points of the latus rectum L and $L^{/}$ are $\left(ae, \frac{b^2}{a}\right)$ and $\left(ae, -\frac{b^2}{a}\right)$

Hence length of the latus rectum $LL^{/} = \frac{2b^2}{a}$

....

Example 5.14 Find the equation of the parabola with focus $(-\sqrt{2}, 0)$,and directrix $x = 2$.

Solution:



By the given data Parabola is open left, axis x axis, with vertex A $(0, 0)$ with $a = \sqrt{2}$
The equation of the Parabola open left, x axis with vertex A $(h, k)$is

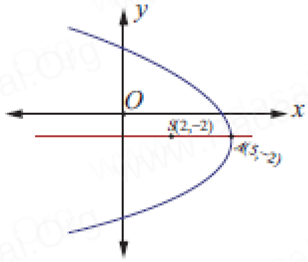
$$(y - k)^2 = -4a(x - h)$$

Hence $(y-0)^2 = -4\sqrt{2}(x-0)$

$$y^2 = -4\sqrt{2}x$$

Example 5.15 Find the equation of the parabola whose vertex is $(5,-2)$ and focus $(2,-2)$.

Solution:



Given vertex A $(5,-2)$ and focus S $(2,-2)$.

Distance between vertex and focus

$$AS = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$$
$$= \sqrt{(2-5)^2+(-2+2)^2}$$
$$= \sqrt{(-3)^2+(0)^2}$$
$$= \sqrt{9}$$
$$AS = a = 3$$

Parabola is open left, axis x axis, with vertex A $(5,-2)$ with $a=3$

The equation of the Parabola open left, x axis with vertex A $(h,k)$is

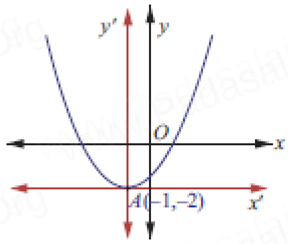
$$(y-k)^2 = -4a(x-h)$$

Hence $(y+2)^2 = -4(3)(x-5)$

$$y^2+4y+4 = -12(x-5)$$
$$y^2+4y+4 = -12x+60$$
$$y^2+4y+12x+4-60 = 0$$
$$y^2+4y+12x-56 = 0$$

Example 5.16 Find the equation of the parabola with vertex $(-1,-2)$, axis parallel to y-axis and passing through $(3,6)$.

Solution:



The parabola with vertex A $(-1,-2)$, axis parallel to y-axis, it is open upward.

The equation of the Parabola open up, y axis with vertex A $(h,k)$is

$$(x-h)^2 = 4a(y-k)$$

Hence $(x+1)^2 = 4a\,(y+2)$

Given the parabola passes through $(3,6)$

So, $(3+1)^2 = 4a\,(6+2)$

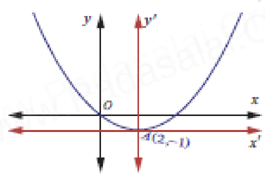
$$(4)^2 = 4a\,(8)$$
$$16 = 32a$$
$$32a = 16$$
$$a = \frac{16}{32}$$
$$a = \frac{1}{2}$$

Equation of the Parabola is

$$(x+1)^2 = 4\left(\frac{1}{2}\right)\,(y+2)$$
$$(x+1)^2 = 2\,(y+2)$$
$$x^2+2x+1 = 2y+4$$
$$x^2+2x-2y+1-4 = 0$$
$$x^2+2x-2y-3 = 0$$

Example 5.17 Find the vertex, focus, directrix, and length of the latus rectum of the parabola $x^2-4x-5y-1=0$.

Solution:



Equation of parabola $x^2-4x-5y-1=0$.

$$x^2-4x = 5y+1$$

Adding 4

$$x^2 - 4x + 4 = 5y + 1 + 4$$
$$x^2 - 4x + 4 = 5y + 5$$
$$(x-2)^2 = 5(y+1)$$

Comparing with $(x-h)^2 = 4a(y-k)$

the Parabola open up, y axis with

vertex A $(h,k) \Rightarrow A(2,-1)$

$4a = 5 \Rightarrow a = \frac{5}{4}$

Focus S $(0,a)$

$$x - 2 = 0 \Rightarrow x = 2$$
$$y + 1 = a$$
$$y = a - 1$$
$$= \frac{5}{4} - 1$$
$$y = \frac{5-4}{4} = \frac{1}{4}$$

Focus S $(0,a) \Rightarrow \left(2,\frac{1}{4}\right)$

Equation of directrix is $y + k + a = 0$

$$y - 1 + \frac{5}{4} = 0$$

Multiplying by 4, $4y - 4 + 5 = 0$

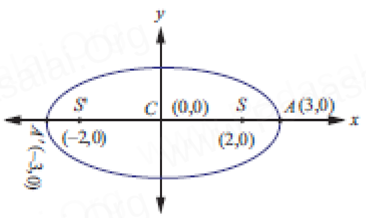
$$4y + 1 = 0$$

Length of Latus rectum 4a = 5

....

Example 5.18 Find the equation of the ellipse with foci $(\pm 2,0)$, vertices $(\pm 3,0)$.

Solution:



Given foci S $(\pm 2,0)$, vertices A $(\pm 3,0)$

Distance between foci SS/=2C= 4 gives C=2

Distance between vertex AA/=2a= 6 gives a=3

Hence $b^2 = a^2 - c^2$

$$= 3^2 - 2^2$$
$$= 9 - 4$$
$$= 5$$

Major axis x axis, with center (0,0) *and* $a > b$

The equation of the ellipse is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$$\frac{x^2}{9} + \frac{y^2}{5} = 1$$

....

Example 5.19 Find the equation of the ellipse whose eccentricity is $\frac{1}{2}$, one of the foci is (2,3) and a directrix is $x = 7$. Also find the length of the major and minor axes of the ellipse.

Solution: Given eccentricity e=$\frac{1}{2}$ and

focus S=(2,3)

Equation of the directrix is $x = 7$

Let P (x,y) be any point.

Perpendicular distance PM $= (x-7)$

We know $\frac{SP}{PM} = e$

Hence $SP^2 = e^2 PM^2$

$$(x-2)^2 + (y-3)^2 = \frac{1}{4}(x-7)^2$$
$$x^2 - 4x + 4 + y^2 - 6y + 9 = \frac{1}{4}(x^2 - 14x + 49)$$
$$4(x^2 - 4x + y^2 - 6y + 13) = (x^2 - 14x + 49)$$
$$4x^2 - 16x + 4y^2 - 24y + 52 = x^2 - 14x + 49$$
$$4x^2 - x^2 - 16x + 14x + 4y^2 - 24y + 52 - 49 = 0$$
$$3x^2 - 2x + 4y^2 - 24y + 3 = 0$$
$$3\left(x^2 - \frac{2}{3}x\right) + 4(y^2 - 6y) + 3 = 0$$
$$3\left(x^2 - \frac{2}{3}x + \frac{1}{9} - \frac{1}{9}\right) + 4(y^2 - 6y + 9 - 9) + 3 = 0$$
$$3\left(x^2 - \frac{2}{3}x + \frac{1}{9}\right) - \frac{3}{9} + 4(y^2 - 6y + 9) - 36 + 3 = 0$$
$$3\left(x - \frac{1}{3}\right)^2 - \frac{1}{3} + 4(y-3)^2 - 33 = 0$$
$$3\left(x - \frac{1}{3}\right)^2 + 4(y-3)^2 - 33 - \frac{1}{3} = 0$$
$$3\left(x - \frac{1}{3}\right)^2 + 4(y-3)^2 - \left(\frac{99+1}{3}\right) = 0$$
$$3\left(x - \frac{1}{3}\right)^2 + 4(y-3)^2 - \frac{100}{3} = 0$$
$$3\left(x - \frac{1}{3}\right)^2 + 4(y-3)^2 = \frac{100}{3}$$

Let $x - \frac{1}{3} = X$ and $y - 3 = Y$

$$3X^2 + 4Y^2 = \frac{100}{3}$$
$$\frac{3X^2}{\frac{100}{3}} + \frac{4Y^2}{\frac{100}{3}} = 1$$

$$\frac{X^2}{\frac{100}{9}} + \frac{Y^2}{\frac{100}{12}} = 1$$

Comparing with $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, a > b$

$$a^2 = \frac{100}{9} \Rightarrow a = \frac{10}{3}$$

$$b^2 = \frac{100}{12} = \frac{25}{3} \Rightarrow b = \frac{5}{\sqrt{3}}$$

Length of the major axis $2a = \frac{20}{3}$

Length of the minor axis $2b = \frac{10}{\sqrt{3}}$

....

<u>Example 5.20</u> Find the foci, vertices and length of major and minor axis of the conic $4x^2 + 36y^2 + 40x - 288y + 532 = 0.$

Solution: Given

$$4x^2 + 36y^2 + 40x - 288y + 532 = 0.$$

$$4x^2 + 40x + 36y^2 - 288y + 532 = 0.$$

$$4(x^2 + 10x) + 36(y^2 - 8y) + 532 = 0$$

$$4(x^2 + 10x + 25 - 25)$$
$$+36(y^2 - 8y + 16 - 16) + 532 = 0$$

$$4(x^2 + 10x + 25) - 100$$
$$+36(y^2 - 8y + 16) - 576 + 532 = 0$$

$$4(x^2 + 10x + 25) + 36(y^2 - 8y + 16) = 144$$

$$4(x + 5)^2 + 36(y - 4)^2 = 144$$

Substituting $x + 5 = X$, and $y - 4 = Y$

$$4X^2 + 36Y^2 = 144$$

Dividing by 144, $\frac{4X^2}{144} + \frac{36Y^2}{144} = 1$

$$\frac{X^2}{36} + \frac{Y^2}{4} = 1$$

Major axis X axis,

Comparing with general form

$$a^2 = 36 \Rightarrow a = 6$$

$$b^2 = 4 \Rightarrow b = 2$$

$$(ae)^2 = a^2 - b^2$$
$$= 36 - 4$$
$$= 32$$
$$= 16 \times 2$$
$$ae = 4\sqrt{2}$$

X , Y axis		x , y axis
Center C(0, 0)		Center (– 5, 4)
$X = 0$ $x + 5 = 0$ $x = -5$	$Y = 0$ $y - 4 = 0$ $y = 4$	
Vertex $A(\pm a, 0) = (\pm 6, 0)$		Vertex $A(1, 4)$
$X = 6$ $x + 5 = 6$ $x = 6 - 5$ $x = 1$	$Y = 0$ $y - 4 = 0$ $y = 4$	
$X = -6$ $x + 5 = -6$ $x = -6 - 5$ $x = -11$	$Y = 0$ $y - 4 = 0$ $y = 4$	Vertex $A^1(-11,4)$
Foci $S(\pm ae, 0) = (\pm 4\sqrt{2}, 0)$		Focus $S(4\sqrt{2} - 5,4)$
$X = 4\sqrt{2}$ $x + 5 = 4\sqrt{2}$ $x = 4\sqrt{2} - 5$	$Y = 0$ $y - 4 = 0$ $y = 4$	
$X = -4\sqrt{2}$ $x + 5 = -4\sqrt{2}$ $x = -4\sqrt{2} - 5$	$Y = 0$ $y - 4 = 0$ $y = 4$	Focus $S^1(-4\sqrt{2} - 5,4)$

Length of the major axis $2a = 12$

Length of the minor axis $2b = 4$

....

<u>Example 5.21</u> For the ellipse $4x^2 + y^2 + 24x - 2y + 21 = 0$, find the centre, vertices, and the foci. Also prove that the length of latus rectum is 2 .

Solution: $4x^2 + y^2 + 24x - 2y + 21 = 0$

$$4x^2 + 24x + y^2 - 2y + 21 = 0$$

$$4(x^2 + 6x) + 1(y^2 - 2y) + 21 = 0$$

$$4(x^2 + 6x + 9 - 9) + 1(y^2 - 2y + 1 - 1) + 21 = 0$$

$$4(x^2 + 6x + 9) - 36 + 1(y^2 - 2y + 1) - 1 + 21 = 0$$

$$4(x + 3)^2 + 1(y - 1)^2 - 16 = 0$$

$$4(x + 3)^2 + 1(y - 1)^2 = 16$$

Substituting $x + 3 = X$, and $y - 1 = Y$

$$4X^2 + 1Y^2 = 16$$

Dividing by 16, $\frac{4X^2}{16} + \frac{Y^2}{16} = 1$

$$\frac{X^2}{4} + \frac{Y^2}{16} = 1$$

Major axis Y axis,

Comparing with general form

$$a^2 = 16 \Rightarrow a = 4$$
$$b^2 = 4 \Rightarrow b = 2$$
$$(ae)^2 = a^2 - b^2$$
$$= 16 - 4$$
$$= 12$$
$$= 4 \times 3$$
$$ae = 2\sqrt{3}$$

X , Y axis		x , y *axis*
Center C(0, 0)		Center (– 3, 1)
$X = 0$ $x + 3 = 0$ $x = -3$	$Y = 0$ $y - 1 = 0$ $y = 1$	
Vertex A$(0, \pm a) = (0, \pm 4)$		Vertex $A(-3, 5)$
$X = 0$ $x + 3 = 0$ $x = -3$	$Y = 4$ $y - 1 = 4$ $y = 4 + 1$ $y = 5$	
$X = 0$ $x + 3 = 0$ $x = -3$	$Y = -4$ $y - 1 = -4$ $y = -4 + 1$ $y = -3$	Vertex $A^1(-3, -3)$
Foci S$(0, \pm ae) = (0, \pm 2\sqrt{3})$		Focus S$(-3, 2\sqrt{3} + 1)$
$X = 0$ $x + 3 = 0$ $x = -3$	$Y = 2\sqrt{3}$ $y - 1 = 2\sqrt{3}$ $y = 2\sqrt{3} + 1$	
$X = 0$ $x + 3 = 0$ $x = -3$	$Y = -2\sqrt{3}$ $y - 1 = -2\sqrt{3}$ $y = -2\sqrt{3} + 1$	Focus $S^1(-3, -2\sqrt{3} + 1)$
The length of the Latus rectum $= \frac{2b^2}{a} = \frac{2\times 2}{2} = 2$		

....

Example 5.22 Find the equation of the hyperbola with vertices $(0, \pm 4)$ and foci$(0, \pm 6)$.

Solution: From the given data the midpoint of the vertices center is at (0,0)

Vertices $A\ (0, \pm 4) \Rightarrow\ a = \pm 4$

Foci $S\ (0, \pm 6) \Rightarrow ae = \pm 6$

Hence Transverse axis is y axis.

$$\text{Since } b^2 = (ae)^2 - a^2$$
$$= (6)^2 - 4^2$$
$$= 36 - 16$$
$$b^2 = 20$$

Equation of the Hyperbola with Transverse axis y axis is

$$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1 \quad \text{gives}$$
$$\frac{y^2}{16} - \frac{x^2}{20} = 1$$

....

Example 5.23 Find the vertices, foci for the hyperbola $9x^2 - 16y^2 = 144$.

Solution: $9x^2 - 16y^2 = 144$

Dividing by 144,

$$\frac{9x^2}{144} - \frac{16Y^2}{144} = 1$$
$$\frac{x^2}{16} - \frac{Y^2}{9} = 1$$

Hence Transverse axis is x axis.

$$a^2 = 16, and\ b^2 = 9$$
$$\text{Since } b^2 = (ae)^2 - a^2$$
$$b^2 + a^2 = (ae)^2$$
$$9 + 16 = (ae)^2$$
$$(ae)^2 = 25$$
$$ae = 5$$

Vertex A $(\pm a, 0) = (\pm 4, 0)$

Foci S $(\pm ae, 0) = (\pm 5, 0)$

....

Example 5.24 Find the centre, foci, and eccentricity of the hyperbola

$$11x^2 - 25y^2 - 44x + 50y - 256 = 0$$

Solution: $11x^2 - 25y^2 - 44x + 50y - 256 = 0$

$$11x^2 - 44x - 25y^2 + 50y - 256 = 0$$
$$11(x^2 - 4x) - 25(y^2 - 2y) - 256 = 0$$
$$11(x^2 - 4x) - 25(y^2 - 2y) - 256 = 0$$
$$11(x^2 - 4x + 4 - 4) - 25(y^2 - 2y + 1 - 1) - 256 = 0$$
$$11(x^2 - 4x + 4) - 44 - 25(y^2 - 2y + 1) + 25 - 256 = 0$$
$$11(x^2 - 4x + 4) - 25(y^2 - 2y + 1) = 275$$
$$11(x - 2)^2 - 25(y - 1)^2 = 275$$

Substituting $x - 2 = X$, and $y - 1 = Y$

$$11X^2 - 25Y^2 = 275$$

Dividing by 275, $\frac{11X^2}{275} - \frac{25Y^2}{275} = 1$

$$\frac{X^2}{25} - \frac{Y^2}{11} = 1$$

Hence Transverse axis is X axis.

$a^2 = 25, and\ b^2 = 11$

Since $b^2 = (ae)^2 - a^2$

$b^2 + a^2 = (ae)^2$

$11 + 25 = (ae)^2$

$(ae)^2 = 36$

$ae = 6$

Eccentricity $e = \frac{6}{a} = \frac{6}{5}$

X , Y axis		x , y axis
Center C(0, 0)		Center (2, 1)
$X = 0$ $x - 2 = 0$ $x = 2$	Y = 0 y – 1 = 0 y = 1	
Foci $S(\pm ae, 0) = (\pm 6,0)$		Focus $S(8,1)$
$X = 6$ $x - 2 = 6$ $x = 6 + 2$ $x = 8$	Y = 0 y – 1 = 0 y = 1	
$X = -6$ $x - 2 = -6$ $x = -6 + 2$ $x = -4$	Y = 0 y – 1 = 0 y = 1	Focus $S^1(-4,1)$

....

<u>Example 5.25</u> The orbit of Halley's Comet (Fig. 5.51) is an ellipse 36.18 astronomical units long and by 9.12 astronomical units wide. Find its eccentricity.

Solution: Given 2a= 36.18 and 2b= 9.12

Hence a= 18.09 and b= 4.56

$(ae)^2 = a^2 - b^2$

$= 18.09^2 - 4.56^2$

$= (18.09 + 4.56) \times (18.09 - 4.56)$

$= (22.65) \times (13.53)$

$= 306.4545$

$ae = \sqrt{306.4545}$

$= 17.51$

$e = \frac{17.51}{18.09} = 0.97$

....

EXERCISE 5.2

1. Find the equation of the parabola in each of the cases given below:

(i) focus $(4, 0)$ and directrix $x = -4$.

Solution: focus $(4, 0)$

directrix $x = -4$

Hence the Parabola open right with a = 4

So, the equation is of the form $y^2 = 4ax$

$\therefore\ y^2 = 4(4)x$

$y^2 = 16x$

(ii) passes through $(2, -3)$ and symmetric about y-axis.

Solution: symmetric about y-axis.

So, the equation is of the form $x^2 = 4ay$

It passes through $(2, -3)$

Hence $(2)^2 = 4a(-3)$

$4 = -12a$

$\frac{4}{-12} = a$

$a = -\frac{1}{3}$

$\therefore\ x^2 = 4\left(-\frac{1}{3}\right)y$

$3x^2 = -4y$

(iii) vertex $(1, -2)$ and focus $(4, -2)$.

Solution:

Given vertex A $(1, -2)$ and focus S $(4, -2)$.

Distance between vertex and focus

$AS = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

$= \sqrt{(4 - 1)^2 + (-2 + 2)^2}$

$= \sqrt{(3)^2 + (0)^2}$

$= \sqrt{9}$

$AS = a = 3$

Parabola is open right, axis x axis, with vertex A $(1,-2)$ with $a=3$

The equation of the Parabola open right, x axis with vertex A (h,k) is

$$(y-k)^2 = 4a(x-h)$$

Hence $$(y+2)^2 = 4(3)(x-1)$$

$$y^2+4y+4 = 12(x-1)$$

$$y^2+4y+4 = 12x-12$$

$$y^2+4y-12x+4-12 = 0$$

$$y^2+4y-12x-8 = 0$$

(iv) end points of latus rectum $(4,-8)$ and $(4,8)$.

Solution: Given $(4,-8)$ and $(4,8)$ are end points of the Latus rectum. So its midpoint Focus is $(4,0)$

Hence the Parabola open right with a = 4

So, the equation is of the form $y^2 = 4ax$

$$\therefore\ y^2 = 4(4)x$$

$$y^2 = 16x$$

....

2. Find the equation of the ellipse in each of the cases given below:

(i) foci $(\pm 3,0), e=\frac{1}{2}$.

Solution: foci $S(\pm ae,0) = (\pm 3,0)$

$$\therefore ae = 3$$

$$\text{Given } e = \frac{1}{2}$$

$$a\left(\frac{1}{2}\right) = 3$$

$$a = 6$$

We know for ellipse

$$b^2 = a^2-(ae)^2$$

$$= 6^2-3^2$$

$$= 36-9$$

$$b^2 = 27$$

By the given data ellipse has Center at origin with major axis x axis is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1, a>b$$

$$\frac{x^2}{36}+\frac{y^2}{27} = 1$$

is the required equation of the ellipse.

(ii) foci $(0,\pm 4)$ and end points of major axis are $(0,\pm 5)$

Solution: foci $S(0,\pm ae) = (0,\pm 4)$

$$\therefore ae = 4$$

Vertices $A(0,\pm a) = (0,\pm 5)$

$$\therefore a = 5$$

We know for ellipse

$$b^2 = a^2-(ae)^2$$

$$= 5^2-4^2$$

$$= 25-16$$

$$b^2 = 9$$

By the given data ellipse has Center at origin with major axis y axis is

$$\frac{x^2}{b^2}+\frac{y^2}{a^2} = 1, a>b$$

$$\frac{x^2}{9}+\frac{y^2}{25} = 1$$

is the required equation of the ellipse.

(iii) length of latus rectum 8, eccentricity $=\frac{3}{5}$ and major axis on x-axis.

Solution: Given LLR = 8

$$\frac{2b^2}{a} = 8$$

$$2b^2 = 8a$$

$$b^2 = 4a$$

We know for ellipse

$$b^2 = a^2-(ae)^2$$

$$b^2 = a^2(1-e^2)$$

$$4a = a^2\left(1-\frac{9}{25}\right)$$

$$4 = a\left(\frac{25-9}{25}\right)$$

$$4 = a\left(\frac{16}{25}\right)$$

$$a = 4\left(\frac{25}{16}\right)$$

$$a = \frac{25}{4}$$

$$\therefore a^2 = \frac{625}{16}$$

Since $b^2 = 4a$ gives

$$= 4\left(\frac{25}{4}\right)$$

$$b^2 = 25$$

By the given data ellipse has Center at origin with major axis x axis is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, a > b$$

$$\frac{x^2}{\frac{625}{16}} + \frac{y^2}{25} = 1$$

$$\frac{16x^2}{625} + \frac{y^2}{25} = 1$$

is the required equation of the ellipse.

(iv) length of latus rectum 4 , distance between foci $4\sqrt{2}$ and major axis as y- axis.

Solution: Given LLR = 4

$$\frac{2b^2}{a} = 4$$

$$2b^2 = 4a$$

$$b^2 = 2a$$

Distance between Foci = $4\sqrt{2}$

$$2ae = 4\sqrt{2}$$

$$\therefore ae = 2\sqrt{2}$$

We know for ellipse

$$b^2 = a^2 - (ae)^2$$

$$2a = a^2 - \left(2\sqrt{2}\right)^2$$

$$2a = a^2 - 8$$

$$a^2 - 2a - 8 = 0$$

$$(a-4)(a+2) = 0$$

$a - 4 = 0, gives\ a = 4$ and

$a + 2 = 0, gives\ a = -2$

Since $a \neq -2, We\ get\ \ a = 4$

$$\therefore b^2 = a^2 - (ae)^2$$

$$= 16 - 8$$

$$\therefore b^2 = 8$$

By the given data ellipse has Center at origin with major axis y axis is

$$\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1, a > b$$

$$\frac{x^2}{8} + \frac{y^2}{16} = 1$$

is the required equation of the ellipse.

....

3. Find the equation of the hyperbola in each of the cases given below:

(i) foci $(\pm 2, 0)$, eccentricity $= \frac{3}{2}$.

Solution: foci $S(\pm ae, 0) = (\pm 2, 0)$

$$\therefore ae = 2$$

Given $e = \frac{3}{2}$

$$a\left(\frac{3}{2}\right) = 2$$

$$a = \frac{4}{3} \qquad \therefore a^2 = \frac{16}{9}$$

We know for hyperbola

$$b^2 = (ae)^2 - a^2$$

$$= 2^2 - \left(\frac{4}{3}\right)^2$$

$$= 4 - \frac{16}{9}$$

$$b^2 = \frac{36-16}{9}$$

$$b^2 = \frac{20}{9}$$

By the given data Hyperbola has Center at origin with Transverse axis x axis.

$$\text{Hence } \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

$$\frac{x^2}{\frac{16}{9}} - \frac{y^2}{\frac{20}{9}} = 1$$

$$\frac{9x^2}{16} - \frac{9y^2}{20} = 1$$

is the required equation of the hyperbola.

(ii) Centre $(2, 1)$, one of the foci $(8, 1)$ and corresponding directrix $x = 4$.

Solution: Given Centre $(h,k) = (2,1)$

foci $S(8,1)$

Distance between centre and one of the foci

$$ae = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$$

$$(ae)^2 = (x_2-x_1)^2+(y_2-y_1)^2$$

$$= (8-2)^2+(1-1)^2$$

$$= 6^2+0$$

$$\therefore a^2e^2 = 36$$

Distance between centre and dirctix

$$\frac{a}{e} = 2$$

$$a = 2e$$

$$\therefore a^2 = 4e^2$$

Hence $(4e^2)e^2 = 36$

$$4e^4 = 36$$

$$e^4 = 9$$

$$e^2 = 3$$

Substituting $e^2 = 3$ in $a^2 = 4e^2$ we get

$$a^2 = 4(3)$$

$$a^2 = 12$$

We know for hyperbola

$$b^2 = (ae)^2 - a^2$$

$$= 36 - 12$$

$$b^2 = 24$$

By the given data Hyperbola with

Center at (h,k) with Transverse axis x axis is

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$$

Given Centre $(h,k) = (2,1)$

$$\frac{(x-2)^2}{12} - \frac{(y-1)^2}{24} = 1$$

is the required equation of the hyperbola.

(iii) passing through $(5,-2)$ and length of the transverse axis along x axis and of length 8 units.

Solution:

Length of the transverse axis 2a = 8

Hence a = 4

By the given data Hyperbola with Transverse axis x axis. Hence $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

$$\therefore \frac{x^2}{16} - \frac{y^2}{b^2} = 1$$

It passing through $(5,-2)$

$$\frac{(5)^2}{16} - \frac{(-2)^2}{b^2} = 1$$

$$\frac{25}{16} - \frac{4}{b^2} = 1$$

$$\frac{25}{16} = 1 + \frac{4}{b^2}$$

$$\frac{25}{16} - 1 = \frac{4}{b^2}$$

$$\frac{4}{b^2} = \frac{25-16}{16}$$

$$\frac{4}{b^2} = \frac{9}{16}$$

$$64 = 9b^2$$

$$b^2 = \frac{64}{9}$$

$$\therefore \frac{x^2}{16} - \frac{y^2}{\frac{64}{9}} = 1$$

$$\frac{x^2}{16} - \frac{9y^2}{64} = 1$$

is the required equation of the hyperbola.

....

4. Find the vertex, focus, equation of directrix and length of the latus rectum of the following:

(i) $y^2 = 16x$.

Solution: Comparing with $y^2 = 4ax$

4a = 16, hence a = 4

The parabola open right.

Vertex $(0,0)$

Focus S $(a,0) = (4,0)$

Latus rectum $x = a, that\ is\ x = 4$

So, Directrix $x = -a, that\ is\ x = -4$

Length of the Latus rectum = 4a = 16

(ii) $x^2 = 24y$.

Solution: Comparing with $x^2 = 4ay$

4a = 24, hence a = 6

The parabola open up.

Vertex $(0, 0)$

Focus S $(0, a) = (0, 6)$

Latus rectum $y = a, that\ is\ y = 6$

So, Directrix $y = -a, that\ is\ y = -6$

Length of the Latus rectum = 4a = 24

(iii) $y^2 = -8x$

Solution: Comparing with $y^2 = -4ax$

4a = 8, hence a = 2

The parabola open left.

Vertex $(0, 0)$

Focus S $(-a, 0) = (-2, 0)$

Latus rectum $x = -a, that\ is\ x = -2$

So, Directrix $x = a, that\ is\ x = 2$

Length of the Latus rectum = 4a = 8

(iv) $x^2 - 2x + 8y + 17 = 0$.

Solution: $x^2 - 2x + 8y + 17 = 0$

$$x^2 - 2x = -8y - 17$$

Adding 1 on both sides

$$x^2 - 2x + 1 = -8y - 17 + 1$$

$$x^2 - 2x + 1 = -8y - 16$$

$$(x - 1)^2 = -8(y + 2)$$

Substituting $(x - 1) = X$ and $(y + 2) = Y$

$$X^2 = -8Y$$

Comparing with $x^2 = -4ay$

4a = 8, hence a = 2

The parabola open down.

Vertex $(0, 0)$

$X = 0, x - 1 = 0\ gives\ x = 1$

$Y = 0, y + 2 = 0\ gives\ y = -2$

Required Vertex $(1, -2)$

Focus S $(0, -a) = (0, -2)$

$X = 0, x - 1 = 0\ gives\ x = 1$

$Y = -2, y + 2 = -2\ gives\ y = -4$

Required Focus S $(1, -4)$

Latus rectum $Y = -a$

$$y + 2 = -2$$

$$y = -4$$

Directrix $Y = a$

$$y + 2 = 2$$

$$y = 0$$

Length of the Latus rectum = 4a = 8

(v) $y^2 - 4y - 8x + 12 = 0$.

Solution: $y^2 - 4y - 8x + 12 = 0$

$$y^2 - 4y = 8x - 12$$

Adding 4 on both sides

$$y^2 - 4y + 4 = 8x - 12 + 4$$

$$y^2 - 4y + 4 = 8x - 8$$

$$(y - 2)^2 = 8(x - 1)$$

Substituting $(x - 1) = X$ and $(y - 2) = Y$

$$Y^2 = 8X$$

4a = 8, hence a = 2

The parabola open right. Vertex $(0, 0)$

$X = 0, x - 1 = 0\ gives\ x = 1$

$Y = 0, y - 2 = 0\ gives\ y = 2$

Required Vertex $(1, 2)$

Focus S $(a, 0) = (2, 0)$

$X = 2\ x - 1 = 2\ gives\ x = 3$

$Y = 0, y - 2 = 0\ gives\ y = 2$

Required Focus S $(3, 2)$

Latus rectum $X = a$

$$x - 1 = 2$$

$$x = 3$$

Directrix $X = -a$

$$x - 1 = -2$$

$$x = -1$$

Length of the Latus rectum = 4a = 8

5. Identify the type of conic and find centre, foci, vertices, and directrices of each of the following:

(i) $\frac{x^2}{25} + \frac{y^2}{9} = 1$

Solution: $\frac{x^2}{25}+\frac{y^2}{9}=1$ is in the form

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

Here $a^2=25$ and $b^2=9$

We know $e^2=\frac{a^2-b^2}{a^2}$

$$=\frac{25-9}{25}$$

$$=\frac{16}{25}$$

$e=\frac{4}{5}$ and a $=5$

Hence $ae=5\times\frac{4}{5}$

$ae=4$ and

$$\frac{a}{e}=\frac{5}{\frac{4}{5}}=5\times\frac{5}{4}$$

$$\frac{a}{e}=\frac{25}{4}$$

Major axis x axis.

Center $=(0,0)$

Foci $=(\pm ae,0)=(\pm 4,0)$

Vertices $=(\pm a,0)=(\pm 5,0)$

Directrix $x=\pm\frac{a}{e}=\pm\frac{25}{4}$

(ii) $\frac{x^2}{3}+\frac{y^2}{10}=1$

Solution: $\frac{x^2}{3}+\frac{y^2}{10}=1$ is in the form

$$\frac{x^2}{b^2}+\frac{y^2}{a^2}=1$$

Here $a^2=10$ and $b^2=3$

We know $e^2=\frac{a^2-b^2}{a^2}$

$$=\frac{10-3}{10}$$

$$=\frac{7}{10}$$

$e=\frac{\sqrt{7}}{\sqrt{10}}$ and a $=\sqrt{10}$

Hence $ae=\sqrt{10}\times\frac{\sqrt{7}}{\sqrt{10}}$

$ae=\sqrt{7}$ and

$$\frac{a}{e}=\frac{\sqrt{10}}{\frac{\sqrt{7}}{\sqrt{10}}}=\sqrt{10}\times\frac{\sqrt{10}}{\sqrt{7}}$$

$$\frac{a}{e}=\frac{10}{\sqrt{7}}$$

Major axis y axis.

Center $=(0,0)$

Foci $=(0,\pm ae)=(0,\pm\sqrt{7})$

Vertices $=(0,\pm a)=(0,\pm\sqrt{10})$

Directrix $y=\pm\frac{a}{e}=\pm\frac{10}{\sqrt{7}}$

(iii) $\frac{x^2}{25}-\frac{y^2}{144}=1$

Solution: $\frac{x^2}{25}-\frac{y^2}{144}=1$ is in the form

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$$

Here $a^2=25$ and $b^2=144$

We know $e^2=\frac{a^2+b^2}{a^2}$

$$=\frac{25+144}{25}$$

$$=\frac{169}{25}$$

$e=\frac{13}{5}$ and a $=5$

Hence $ae=5\times\frac{13}{5}$

$ae=13$ and

$$\frac{a}{e}=\frac{5}{\frac{13}{5}}=5\times\frac{5}{13}$$

$$\frac{a}{e}=\frac{25}{13}$$

Transverse axis x axis.

Center $=(0,0)$

Foci $=(\pm ae,0)=(\pm 13,0)$

Vertices $=(\pm a,0)=(\pm 5,0)$

Directrix $x=\pm\frac{a}{e}=\pm\frac{25}{13}$

(iv) $\frac{y^2}{16}-\frac{x^2}{9}=1$

Solution: $\frac{y^2}{16}-\frac{x^2}{9}=1$ is in the form

$$\frac{y^2}{a^2}-\frac{x^2}{b^2}=1$$

Here $a^2=16$ and $b^2=9$

We know $e^2=\frac{a^2+b^2}{a^2}$

$$= \frac{16+9}{16}$$

$$= \frac{25}{16}$$

$$e = \frac{5}{4} \text{ and } a = 4$$

Hence $ae = 4 \times \frac{5}{4}$

$ae = 5$ and

$$\frac{a}{e} = \frac{4}{\frac{5}{4}} = 4 \times \frac{4}{5}$$

$$\frac{a}{e} = \frac{16}{5}$$

Transverse axis y axis.

Center $= (0,0)$

Foci $= (0, \pm ae) = (0, \pm 5)$

Vertices $= (0, \pm a) = (0, \pm 4)$

Directrix $y = \pm \frac{a}{e} = \pm \frac{16}{5}$

....

6. Prove that the length of the latus rectum of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is $\frac{2b^2}{a}$.

Solution: The Latus rectum of $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ passes through Focus $(\pm ae, 0)$

So, let L (ae, y) lies on $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

$$\therefore \frac{a^2e^2}{a^2} - \frac{y^2}{b^2} = 1$$

$$e^2 - \frac{y^2}{b^2} = 1$$

$$e^2 - 1 = \frac{y^2}{b^2}$$

$$y^2 = b^2(e^2 - 1)$$

Since $e^2 = 1 + \frac{b^2}{a^2}$,

$$y^2 = b^2\left(1 + \frac{b^2}{a^2} - 1\right)$$

$$y^2 = b^2\left(\frac{b^2}{a^2}\right)$$

Hence $y = \pm \frac{b^2}{a}$

That is the end point of the latus rectum

L $\left(ae, \frac{b^2}{a}\right)$ and $L^{/}\left(ae, \frac{b^2}{a}\right)$

So the length of the latus rectum is $\frac{2b^2}{a}$

....

7. Show that the absolute value of difference of the focal distances of any point P on the hyperbola is the length of its transverse axis.

Solution: We know $\frac{SP}{PM} = e$

$$SP = e(PM)$$

$$= e\left(x - \frac{a}{e}\right)$$

$$= ex - a$$

Similarly $\frac{S^{/}P}{PM^{/}} = e$

$$S^{/}P = e(PM^{/})$$

$$= e\left(x + \frac{a}{e}\right)$$

$$= ex + a$$

Difference of the focal distances $= S^{/}P - SP$

$$= (ex + a) - (ex - a)$$

$$= ex + a - ex + a$$

$= 2a$ which is nothing but the length of its transverse axis.

....

8. Identify the type of conic and find centre, foci, vertices, and directrices of each of the following :

(i) $\frac{(x-3)^2}{225} + \frac{(y-4)^2}{289} = 1$

Solution: The conic is ellipse.

Substituting $x - 3 = X$, and $y - 4 = Y$

$$\frac{X^2}{225} + \frac{Y^2}{289} = 1$$

Major axis Y axis,

Comparing with general form

$$a^2 = 289 \Rightarrow a = 17$$

$$b^2 = 225 \Rightarrow b = 15$$

$$(ae)^2 = a^2 - b^2$$

$$= 289 - 225$$

$$= 64$$

$$ae = 8 \text{ and } e = \frac{ae}{a} = \frac{8}{17}$$

X , Y axis	x, y axis
Center C(0, 0)	Center

$X = 0$ $x - 3 = 0$ $x = 3$	$Y = 0$ $y - 4 = 0$ $y = 4$	$(3, 4)$
Vertex $A(0, \pm a) = (0, \pm 17)$		Vertex $A(3,21)$
$X = 0$ $x - 3 = 0$ $x = 0 + 3$ $x = 3$	$Y = 17$ $y - 4 = 17$ $y = 17+4$ $y = 21$	
$X = 0$ $x - 3 = 0$ $x = 0 + 3$ $x = 3$	$Y = -17$ $y - 4 = -17$ $y = -17+4$ $y = -13$	Vertex $A^1(3, -13)$
Foci $S(0, \pm ae) = (0, \pm 8)$		Focus $S(3,12)$
$X = 0$ $x - 3 = 0$ $x = 0 + 3$ $x = 3$	$Y = 8$ $y - 4 = 8$ $y = 8+4$ $y = 12$	
$X = 0$ $x - 3 = 0$ $x = 0 + 3$ $x = 3$	$Y = -8$ $y - 4 = -8$ $y = -8+4$ $y = -4$	Focus $S^1(3, -4)$

Equation of diectrices $Y = \pm \frac{a}{e}$

$$y - 4 = \pm \frac{17}{\frac{8}{17}} = \pm \frac{289}{8}$$

Equation of directrix1 $y = \frac{289}{8} + 4$

$$y = \frac{289+32}{8}$$

$$y = \frac{321}{8} \text{ and}$$

Equation of directrix2 $y = -\frac{289}{8} + 4$

$$y = \frac{-289+32}{8}$$

$$y = \frac{-257}{8}$$

(ii) $\frac{(x+1)^2}{100} + \frac{(y-2)^2}{64} = 1$

Solution: The conic is ellipse.

Substituting $x + 1 = X$, and $y - 2 = Y$

$$\frac{X^2}{100} + \frac{Y^2}{69} = 1$$

Major axis X axis,

Comparing with general form

$$a^2 = 100 \Rightarrow a = 10$$

$$b^2 = 64 \Rightarrow b = 8$$

$$(ae)^2 = a^2 - b^2$$

$$= 100 - 64$$

$$= 36$$

$$ae = 6 \text{ and } e = \frac{ae}{a} = \frac{6}{10} = \frac{3}{5}$$

X , Y axis		x , y *axis*
Center C(0, 0)		Center $(-1, 2)$
$X = 0$ $x + 1 = 0$ $x = -1$	$Y = 0$ $y - 2 = 0$ $y = 2$	
Vertex $A(\pm a, 0) = (\pm 10, 0)$		Vertex $A(9, 2)$
$X = 10$ $x + 1 = 10$ $x = 10 - 1$ $x = 9$	$Y = 0$ $y - 2 = 0$ $y = 2$	
$X = -10$ $x + 1 = -10$ $x = -10 - 1$ $x = -11$	$Y = 0$ $y - 2 = 0$ $y = 2$	Vertex $A^1(-11, 2)$
Foci $S(\pm ae, 0) = (\pm 6, 0)$		Focus $S(5, 2)$
$X = 6$ $x +$ $1 = 6$ $x = 6 - 1$ $x = 5$	$Y = 0$ $y - 2 = 0$ $y = 2$	
$X = -6$ $x + 1 = -6$ $x = -6 - 1$ $x = -7$	$Y = 0$ $y - 2 = 0$ $y = 2$	Focus $S^1(-4, 2)$

Equation of diectrices $X = \pm \frac{a}{e}$

$$x + 1 = \pm \frac{10}{\frac{3}{5}} = \pm \frac{50}{3}$$

Equation of directrix1 $x = \frac{50}{3} - 1$

$$x = \frac{50 - 3}{3}$$

$$x = \frac{47}{3} \text{ and}$$

Equation of directrix2 $x = -\frac{50}{3} - 1$

$$x = \frac{-50 - 3}{3}$$

$$y = \frac{-53}{3}$$

(iii) $\frac{(x+3)^2}{225} - \frac{(y-4)^2}{64} = 1$

Solution: The conic is hyperbola.

Substituting $x + 3 = X$, and $y - 4 = Y$

$$\frac{X^2}{225} - \frac{Y^2}{64} = 1$$

Major axis X axis,

Comparing with general form

$a^2 = 225 \Rightarrow a = 15$

$b^2 = 64 \Rightarrow b = 8$

$$(ae)^2 = a^2 + b^2$$
$$= 225 + 64$$
$$= 289$$
$$ae = 17$$

$e = \frac{ae}{a} = \frac{17}{15}$ and

$$\frac{a}{e} = \frac{15}{\frac{17}{15}} = 15 \times \frac{15}{17} = \frac{225}{17}$$

Transverse axis X axis.

X , Y axis		$x\,,y\;axis$
Center C(0, 0)		
$X = 0$ $x + 3 = 0$ $x = -3$	$Y = 0$ $y - 4 = 0$ $y = 4$	Center $(-3, 4)$
Vertex $A(\pm a, 0) = (\pm 15, 0)$		
$X = 15$ $x + 3 = 15$ $x = 15 - 3$ $x = 12$	$Y = 0$ $y - 4 = 0$ $y = 4$	Vertex $A(12, 4)$
$X = -15$ $x + 3 = -15$ $x = -15 - 3$ $x = -18$	$Y = 0$ $y - 4 = 0$ $y = 4$	Vertex $A^1(-18, 4)$
Foci $S(\pm ae, 0) = (\pm 17, 0)$		
$X = 17$ $x + 3 = 17$ $x = 17 - 3$ $x = 14$	$Y = 0$ $y - 4 = 0$ $y = 4$	Focus $S(14, 4)$
$X = -17$ $x + 3 = -17$ $x = -17 - 3$ $x = -20$	$Y = 0$ $y - 4 = 0$ $y = 4$	Focus $S^1(-20, 4)$
Equation of diectrices $X = \pm \frac{a}{e}$ $x + 3 = \pm \frac{225}{17}$ Equation of directrix1 $x = \frac{225}{17} - 3$ $x = \frac{225 - 51}{17}$ $x = \frac{174}{51}$ and Equation of directrix2 $x = -\frac{225}{17} - 3$ $x = \frac{-225 - 51}{17}$ $y = \frac{-276}{17}$		

(iv) $\frac{(y-2)^2}{25} - \frac{(x+1)^2}{16} = 1$

Solution: The conic is hyperbola.

Substituting $y - 2 = Y$ and $x + 1 = X$

$$\frac{Y^2}{25} - \frac{X^2}{16} = 1$$

Major axis Y axis,

Comparing with general form

$a^2 = 25 \Rightarrow a = 5$

$b^2 = 16 \Rightarrow b = 4$

$$(ae)^2 = a^2 + b^2$$
$$= 25 + 16$$
$$= 41$$
$$ae = \sqrt{41}$$

$e = \frac{ae}{a} = \frac{\sqrt{41}}{5}$ and

$$\frac{a}{e} = \frac{5}{\frac{\sqrt{41}}{5}} = \frac{25}{\sqrt{41}}$$

Transverse axis Y axis.

X , Y axis		$x\,,y\;axis$
Center C(0, 0)		
$X = 0$ $x + 1 = 0$ $x = -1$	$Y = 0$ $y - 2 = 0$ $y = 2$	Center $(-1,2)$
Vertex $A(0, \pm 5) = (0, \pm 5)$		
$X = 0$ $x + 1 = 0$ $x = -1$	$Y = 5$ $y - 2 = 5$ $y = 5+2$ $y = 7$	Vertex $A(-1,7)$
$X = 0$ $x + 1 = 0$ $x = -1$	$Y = -5$ $y - 2 = -5$ $y = -5+2$ $y = -3$	Vertex $A^1(-1, -13)$
Foci $S(0, \pm ae) = (0, \pm\sqrt{41})$		
$X = 0$ $x + 1 = 0$ $x = -1$	$Y = \sqrt{41}$ $y - 2 = \sqrt{41}$ $y = \sqrt{41}+2$	Focus $S(-1, \sqrt{41} + 2)$
$X = 0$ $x + 1 = 0$ $x = -1$	$Y = -\sqrt{41}$ $y - 2 = -\sqrt{41}$ $y = -\sqrt{41}+2$	Focus $S^1(-1, \sqrt{41} - 2)$
Equation of diectrices $Y = \pm \frac{a}{e}$ $y - 2 = \pm \frac{25}{\sqrt{41}}$ $y = \pm \frac{25}{\sqrt{41}} + 2$		

(v) $18x^2 + 12y^2 - 144x + 48y + 120 = 0$

Solution:

$$18x^2 + 12y^2 - 144x + 48y + 120 = 0$$
$$18x^2 - 144x + 12y^2 + 48y + 120 = 0$$
$$18(x^2 - 8x) + 12(y^2 + 4y) + 120 = 0$$
$$18(x^2 - 8x + 16 - 16) + 12(y^2 + 4y + 4 - 4) + 120 = 0$$
$$18(x^2 - 8x + 16) - 288 + 12(y^2 + 4y + 4) - 48 + 120 = 0$$
$$18(x^2 - 8x + 16) + 12(y^2 + 4y + 4) - 336 + 120 = 0$$
$$18(x^2 - 8x + 16) + 12(y^2 + 4y + 4) - 216 = 0$$
$$18(x^2 - 8x + 16) + 12(y^2 + 4y + 4) = 216$$
$$18(x-4)^2 + 12(y+2)^2 = 216$$

Substituting $(x-4) = X$ and $(y+2) = Y$

$$18X^2 + 12Y^2 = 216$$

Dividing by 216

$$\frac{18X^2}{216} + \frac{12Y^2}{216} = 1$$
$$\frac{X^2}{12} + \frac{Y^2}{18} = 1$$

Major axis Y axis,

Comparing with general form

$$a^2 = 18 \Rightarrow a = 3\sqrt{2}$$
$$b^2 = 12 \Rightarrow b = 2\sqrt{3}$$
$$(ae)^2 = a^2 - b^2$$
$$= 18 - 12$$
$$= 6$$
$$ae = \sqrt{6}$$
$$e = \frac{ae}{a} = \frac{\sqrt{6}}{3\sqrt{2}} = \frac{\sqrt{3}\sqrt{2}}{\sqrt{3}\sqrt{3}\sqrt{2}} = \frac{1}{\sqrt{3}}$$
$$\frac{a}{e} = \frac{3\sqrt{2}}{\frac{1}{\sqrt{3}}} = 3\sqrt{2} \times \frac{\sqrt{3}}{1} = 3\sqrt{6}$$

X , Y axis		x , y axis
Center C(0, 0)		Center (4, −2)
$X = 0$ $x - 4 = 0$ $x = 4$	$Y = 0$ $y + 2 = 0$ $y = -2$	
Vertex $A(0, \pm a) = (0, \pm 3\sqrt{2})$		Vertex $A(4, 3\sqrt{2} - 2)$
$X = 0$ $x - 4 = 0$ $x = 4$	$Y = 3\sqrt{2}$ $y + 2 = 3\sqrt{2}$ $y = 3\sqrt{2} - 2$	
$X = 0$ $x - 4 = 0$ $x = 4$	$Y = -3\sqrt{2}$ $y + 2 = -3\sqrt{2}$ $y = -3\sqrt{2} - 2$	Vertex $A^1(4, -3\sqrt{2} - 2)$
Foci $S(0, \pm ae) = (0, \pm\sqrt{6})$		Focus $S(4, \sqrt{6} - 2)$
$X = 0$ $x - 4 = 0$ $x = 4$	$Y = \sqrt{6}$ $y + 2 = \sqrt{6}$ $y = \sqrt{6} - 2$	
$X = 0$ $x - 4 = 0$ $x = 4$	$Y = -\sqrt{6}$ $y + 2 = -\sqrt{6}$ $y = -\sqrt{6} - 2$	Focus $S^1(4, -\sqrt{6} - 2)$
Equation of diectrices $Y = \pm\frac{a}{e}$ $y + 2 = \pm 3\sqrt{6}$ Equation of directrix1 $y = 3\sqrt{6} - 2$ and Equation of directrix2 $y = -3\sqrt{6} - 2$		

(vi) $9x^2 - y^2 - 36x - 6y + 18 = 0$

Solution: $9x^2 - y^2 - 36x - 6y + 18 = 0$

$$9x^2 - 36x - y^2 - 6y + 18 = 0$$
$$(9x^2 - 36x) - (y^2 + 6y) + 18 = 0$$
$$9(x^2 - 4x) - 1(y^2 + 6y) + 18 = 0$$
$$9(x^2 - 4x + 4 - 4) - 1(y^2 + 6y + 9 - 9) + 18 = 0$$
$$9(x^2 - 4x + 4) - 36 - 1(y^2 + 6y + 9) + 9 + 18 = 0$$
$$9(x^2 - 4x + 4) - 1(y^2 + 6y + 9) - 36 + 27 = 0$$
$$9(x^2 - 4x + 4) - 1(y^2 + 6y + 9) - 9 = 0$$
$$9(x^2 - 4x + 4) - 1(y^2 + 6y + 9) = 9$$
$$9(x-2)^2 - (y+3)^2 = 9$$

Substituting $(x-2) = X$ and $(y+3) = Y$

$$9X^2 - Y^2 = 9$$

Dividing by 9

$$\frac{9X^2}{9} - \frac{Y^2}{9} = 1$$
$$\frac{X^2}{1} - \frac{Y^2}{9} = 1$$

Major axis X axis,

Comparing with general form

$$a^2 = 1 \Rightarrow a = 1$$
$$b^2 = 9 \Rightarrow b = 3$$
$$(ae)^2 = a^2 + b^2$$
$$= 1 + 9$$
$$= 10$$
$$ae = \sqrt{10}$$
$$e = \frac{ae}{a} = \frac{\sqrt{10}}{1} = \sqrt{10} \text{ and}$$
$$\frac{a}{e} = \frac{1}{\sqrt{10}}$$

Transverse axis X axis.

X , Y axis		$x\,,y$ *axis*
Center C(0, 0)		Center (2,−3)
$X = 0$ $x-2=0$ $x=2$	$Y=0$ $y+3=0$ $y=-3$	
Vertex $A(\pm a,0)=(\pm 1,0)$		Vertex $A(3,-3)$
$X = 1$ $x-2=1$ $x=3$	$Y=0$ $y+3=0$ $y=-3$	
$X=-1$ $x-2=-1$ $x=-1+2$ $x=1$	$Y=0$ $y+3=0$ $y=-3$	Vertex $A^1(1,-3)$
Foci $S(\pm ae,0)=(\pm\sqrt{10},0)$		Focus $S(\sqrt{10}+2,-3)$
$X=\sqrt{10}$ $x-2=\sqrt{10}$ $x=\sqrt{10}+2$	$Y=0$ $y+3=0$ $y=-3$	
$X=-\sqrt{10}$ $x-2=-\sqrt{10}$ $x=-\sqrt{10}+2$	$Y=0$ $y+3=0$ $y=-3$	Focus $S^1(-\sqrt{10}+2,-3)$
Equation of diectrices $X=\pm\frac{a}{e}$ $x-2=\pm\frac{1}{\sqrt{10}}$ Equation of directrix1 $x=\frac{1}{\sqrt{10}}+2$ Equation of directrix2 $x=-\frac{1}{\sqrt{10}}+2$		

Example 5.26 Identify the type of the conic for the following equations:

(1) $16y^2=-4x^2+64$

Solution: $16y^2=-4x^2+64$

$$16y^2+4x^2-64=0$$

Comparing with

$$Ax^2+Bxy+Cy^2+Dx+Ey+F=0$$

We get A = 4 and C= 16

Here $A\neq C$, also A and C are same signs.

So the conic is an ellipse

(2) $x^2+y^2=-4x-y+4$

Solution: $x^2+y^2=-4x-y+4$

$$x^2+y^2+4x+y-4=0$$

Comparing with

$$Ax^2+Bxy+Cy^2+Dx+Ey+F=0$$

We get A = C= 1 also B=0

So the conic is a circle.

(3) $x^2-2y=x+3$

Solution: $x^2-2y-x-3=0$

Comparing with

$$Ax^2+Bxy+Cy^2+Dx+Ey+F=0$$

We get B = C= 0

So the conic is a parabola.

(4) $4x^2-9y^2-16x+18y-29=0$

Solution: $4x^2-9y^2-16x+18y-29=0$

Comparing with

$$Ax^2+Bxy+Cy^2+Dx+Ey+F=0$$

We get A = 4 and C= −9

Here $A\neq C$, also A and C are opposite signs.

So the conic is a hyperbola.

EXERCISE 5.3

Identify the type of conic section for each of the equations.

1. $2x^2-4y^2=7$

Solution: $2x^2-4y^2=7$

Comparing with

$$Ax^2+Bxy+Cy^2+Dx+Ey+F=0$$

We get A = 2 and C= −4

Here $A\neq C$, also A and C are opposite signs.

So the conic is a hyperbola.

2. $3x^2+3y^2-4x+3y+10=0.$

Solution: $3x^2+3y^2-4x+3y+10=0.$

Comparing with

$$Ax^2+Bxy+Cy^2+Dx+Ey+F=0$$

We get A = C= 3 also B=0

So the conic is a circle.

3. $3x^2+2y^2=14$

Solution: $3x^2+2y^2-14=0$

Comparing with

$Ax^2+Bxy+Cy^2+Dx+Ey+F=0$

We get A = 3 and C= 2

Here $A \neq C$, also A and C are same signs.

So the conic is an ellipse

....

4. $x^2+y^2+x-y=0$

Solution: $x^2+y^2+x-y=0.$

Comparing with

$Ax^2+Bxy+Cy^2+Dx+Ey+F=0$

We get A = C= 1 also B=0

So the conic is a circle.

....

5. $11x^2-25y^2-44x+50y-256=0$

Solution: $11x^2-25y^2-44x+50y-256=0$

Comparing with

$Ax^2+Bxy+Cy^2+Dx+Ey+F=0$

We get A = 11 and C= −25

Here $A \neq C$, also A and C are opposite signs.

So the conic is a hyperbola.

....

6. $y^2+4x+3y+4=0$

Solution: $y^2+4x+3y+4=0$

Comparing with

$Ax^2+Bxy+Cy^2+Dx+Ey+F=0$

We get A = C= 0

So the conic is a parabola.

....

FORMULAE:

1. $y=mx+c$ is the tangent to the parabola if $c=\frac{a}{m}$

2. $y=mx+c$ is the tangent to the ellipse if $c^2=a^2m^2+b^2$ and the point of contact is $\left(-\frac{a^2m}{c},\frac{b^2}{c}\right)$

3. $y=mx+c$ is the tangent to the hyperbola if $c^2=a^2m^2-b^2$ and the point of contact is $\left(-\frac{a^2m}{c},-\frac{b^2}{c}\right)$

4. The parametric form of the parabola $y^2=4ax$ is $(at^2,2at)$

5. The parametric form of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is $(a\cos\theta, b\sin\theta)$

6. The parametric form of the hyperboa $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is $(a\sec\theta, b\tan\theta)$

7. The equation of the tangent at (x_1,y_1) to the parabola $y^2=4ax$ is $yy_1=4a\left(\frac{x+x_1}{2}\right)$

8. Equation of the tangent with slope m at (x_1,y_1) is $y-y_1=m(x-x_1)$

9. Equation of the normal with slope m at (x_1,y_1) is $y-y_1=-\frac{1}{m}(x-x_1)$

....

Example 5.27

Find the equations of tangent and normal to the parabola $x^2+6x+4y+5=0$ at $(1,-3)$.

Solution: The equation of the tangent at (x_1,y_1) to the parabola $y^2=4ax$ is $yy_1=4a\left(\frac{x+x_1}{2}\right)$

Given $x^2+6x+4y+5=0$

Tangent equation is

$$xx_1+6\left(\frac{x+x_1}{2}\right)+4\left(\frac{y+y_1}{2}\right)+5=0$$

$$xx_1+3(x+x_1)+2(y+y_1)+5=0$$

Substituting $x_1=1, y_1=-3$

$$x(1)+3(x+1)+2(y-3)+5=0$$

$$x+3x+3+2y-6+5=0$$

$$4x+2y-6+8=0$$

$$4x+2y+2=0$$

Dividing by 2,

The equation of tangent is $2x+y+1=0$

Normal is the form $x-2y+k=0$

It passes through $(1,-3)$

Hence $1-2(-3)+k=0$

$1+6+k=0$

$$7 + k = 0$$
$$k = -7$$

The equation of normal is $x - 2y - 7 = 0$

....

Example 5.28
Find the equations of tangent and normal to the ellipse $x^2 + 4y^2 = 32$ when $\theta = \frac{\pi}{4}$.

Solution: Given $x^2 + 4y^2 = 32$

Dividing by 32, $\frac{x^2}{32} + \frac{y^2}{8} = 1$

$a^2 = 32$ gives

$a = \sqrt{32} = \sqrt{16 \times 2} = 4\sqrt{2}$

$b^2 = 8$ gives

$b = \sqrt{8} = \sqrt{4 \times 2} = 2\sqrt{2}$

The parametric form of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is $(a\cos\theta, b\sin\theta)$

when $\theta = \frac{\pi}{4}$,

$$x = a\cos\theta = 4\sqrt{2}\cos\left(\frac{\pi}{4}\right) = 4\sqrt{2} \times \frac{1}{\sqrt{2}} = 4$$

$$y = b\sin\theta = 2\sqrt{2}\cos\left(\frac{\pi}{4}\right) = 2\sqrt{2} \times \frac{1}{\sqrt{2}} = 2$$

It passes through $(4, 2)$

The equation of the tangent at (x_1, y_1) to the given ellipse is $xx_1 + 4yy_1 = 32$

Substituting $x_1 = 4, y_1 = 2$

$$x(4) + 4y(2) = 32$$
$$4x + 8y - 32 = 0$$

$\div$ by 4, The equn. of tangent is $x + 2y - 8 = 0$

Normal is of the form $2x - y + k = 0$

It passes through $(4, 2)$

Hence $2(4) - 2 + k = 0$

$$8 - 2 + k = 0$$
$$6 + k = 0$$
$$k = -6$$

The equation of normal is $2x - y - 6 = 0$

....

EXERCISE 5.4

1. Find the equations of the two tangents that can be drawn from $(5, 2)$ to the ellipse $2x^2 + 7y^2 = 14$.

Solution: Given $2x^2 + 7y^2 = 14$

Dividing by 14, $\frac{2x^2}{14} + \frac{7y^2}{14} = 1$

$$\frac{x^2}{7} + \frac{y^2}{2} = 1$$

$a^2 = 7$ and $b^2 = 2$

$y = mx + c$ is the tangent to the ellipse if $c^2 = a^2m^2 + b^2$

So, $c^2 = 7m^2 + 2$

$$c = \pm\sqrt{7m^2 + 2}$$

Tangent equation is $y = mx \pm \sqrt{7m^2 + 2}$

It passes through $(5, 2)$

$$2 = m(5) \pm \sqrt{7m^2 + 2}$$
$$2 - 5m = \pm\sqrt{7m^2 + 2}$$

Squaring, $(2 - 5m)^2 = 7m^2 + 2$

$$4 + 25m^2 - 20m = 7m^2 + 2$$
$$25m^2 - 7m^2 - 20m - 2 + 4 = 0$$
$$18m^2 - 20m + 2 = 0$$

$\div$ by 2, $9m^2 - 10m + 1 = 0$

$$9m^2 - 9m - m + 1 = 0$$
$$9m(m - 1) - 1(m - 1) = 0$$
$$(9m - 1)(m - 1) = 0$$

$$9m - 1 = 0 \text{ gives } 9m = 1 \text{ that is } m = \frac{1}{9}$$

$$m - 1 = 0 \text{ gives } m = 1$$

(i) When $m = \frac{1}{9}$ at $(5, 2)$

Equation of the tangent is $y - y_1 = m(x - x_1)$

$$y - 2 = \frac{1}{9}(x - 5)$$
$$9(y - 2) = 1(x - 5)$$
$$9y - 18 = x - 5$$
$$x - 5 - 9y + 18 = 0$$
$$x - 9y + 13 = 0$$

(ii) When $m = 1$ at $(5, 2)$

Equation of the tangent is $y - y_1 = m(x - x_1)$

$$y - 2 = 1(x - 5)$$

$$y-2=x-5$$
$$x-5-y+2=0$$
$$x-y-3=0$$

....

2. Find the equations of tangents to the hyperbola $\frac{x^2}{16}-\frac{y^2}{64}=1$ which are parallel to $10x-3y+9=0$.

Solution:

Tangent is parallel to $10x-3y+9=0$

$$10x+9=3y$$
$$3y=10x+9$$
$$y=\frac{10}{3}x+\frac{9}{3}$$

Hence slope $m=\frac{10}{3}$

Given Hyperbola is $\frac{x^2}{16}-\frac{y^2}{64}=1$

$a^2=16$ and $b^2=64$

$y=mx+c$ is the tangent to the ellipse if $c^2=a^2m^2-b^2$

So, $c^2=16m^2-64$

$$=16\left(\frac{10}{3}\right)^2-64$$
$$=16\left(\frac{100}{9}\right)-64$$
$$=\frac{1600-576}{9}$$
$$=\frac{1024}{9}$$
$$c=\pm\frac{32}{3}$$

Equation of Tangent is $y=\frac{10}{3}x\pm\frac{32}{3}$

Multiply by 3, $3y=10x\pm32$

Hence $10x-3y+32=0$ and

$10x-3y-32=0$

are the equation of 2 tangents.

....

3. Show that the line $x-y+4=0$ is a tangent to the ellipse $x^2+3y^2=12$. Also find the coordinates of the point of contact.

Solution: Given $x-y+4=0$

$$x+4=y$$
$$y=x+4$$

Comparing with $y=mx+c$

$m=1, and\ c=4$

Given $x^2+3y^2=12$

Dividing by 12, $\frac{x^2}{12}+\frac{y^2}{4}=1$

$a^2=12$ gives

$b^2=4$ gives

$y=mx+c$ is the tangent to the ellipse if $c^2=a^2m^2+b^2$

$$4^2=12(1)^2+4$$
$$16=12(1)+4$$
$$16=12+4$$
$$16=16$$

So, $x-y+4=0$ is the tangent.

The point of contact is $\left(-\frac{a^2m}{c},\frac{b^2}{c}\right)$

$$=\left(-\frac{12(1)}{4},\frac{4}{4}\right)$$
$$=(-3,1)$$

....

4. Find the equation of the tangent to the parabola $y^2=16x$ perpendicular to $2x+2y+3=0$.

Solution: Given $2x+2y+3=0$

$$2y=-2x-3$$

Dividing by 2 $y=-x+\frac{3}{2}$

Comparing with $y=mx+c$

$m=-1, and\ c=\frac{3}{2}$

$\therefore$ Perpendicular slope $=1$

Given $y^2=16x$

Comparing with $y^2=4ax$

$$4a=16$$
$$a=4$$

$y=mx+c$ is the tangent to the parabola if $c=\frac{a}{m}$

$$\therefore c=\frac{4}{1}=4$$

So tangent equation is $y=(1)x+4$

$$y = x + 4$$
$$x - y + 4 = 0$$

....

5. Find the equation of the tangent at $t = 2$ to the parabola $y^2 = 8x$. (Hint: use parametric form)

Solution: Given $y^2 = 8x$

Comparing with $y^2 = 4ax$

$$4a = 8$$
$$a = 2$$

The parametric form of the parabola $y^2 = 4ax$ is $(at^2, 2at)$

At t =2, $x = at^2 = 2(2)^2 = 2 \times 4 = 8$

$$y = 2at = 2 \times 2 \times 2 = 8$$
$$(x_1, y_1) = (8, 8)$$

The equation of the tangent at (x_1, y_1) to the parabola $y^2 = 4ax$ is $yy_1 = 4a\left(\frac{x + x_1}{2}\right)$

$$y(8) = 4(2)\left(\frac{x + 8}{2}\right)$$
$$8y = 4(x + 8)$$
$$4x - 8y + 32 = 0$$

Dividing by 4,

Tangent equation is $x - 2y + 8 = 0$

....

6. Find the equations of the tangent and normal to hyperbola $12x^2 - 9y^2 = 108$ at $\theta = \frac{\pi}{3}$. (Hint: use parametric form)

Solution: Solution: Given $12x^2 - 9y^2 = 108$

Dividing by 108, $\frac{12x^2}{108} - \frac{9y^2}{108} = 1$

$$\frac{x^2}{9} - \frac{y^2}{12} = 1$$
$$a^2 = 9 \text{ gives}$$
$$a = \sqrt{9} = 3$$
$$b^2 = 12 \text{ gives}$$
$$b = \sqrt{12} = \sqrt{4 \times 3} = 2\sqrt{3}$$

The parametric form of the ellipse $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is $(a\sec\theta, b\tan\theta)$

when $\theta = \frac{\pi}{3}$,

$$x = a\sec\theta = 3\sec\left(\frac{\pi}{3}\right) = 3 \times \frac{2}{1} = 6$$
$$y = b\tan\theta = 2\sqrt{3}\tan\left(\frac{\pi}{3}\right) = 2\sqrt{3} \times \frac{\sqrt{3}}{1} = 6$$

It passes through $(6, 6)$

The equation of the tangent at (x_1, y_1) to the given ellipse is $12xx_1 - 9yy_1 = 108$

Substituting $x_1 = 6, y_1 = 6$

$$12x(6) - 9y(6) = 108$$
$$72x - 54y - 108 = 0$$

$\div$ by 18,

The equation of tangent is $4x - 3y - 6 = 0$

Normal is of the form $3x + 4y + k = 0$

It passes through $(6, 6)$

Hence $3(6) + 4(6) + k = 0$

$$18 + 24 + k = 0$$
$$42 + k = 0$$
$$k = -42$$

The equation of normal is $3x + 4y - 42 = 0$

....

7. Prove that the point of intersection of the tangents at 't_1' and 't_2' on the parabola $y^2 = 4ax$ is $[at_1t_2, a(t_1 + t_2)]$.

Solution: Equation of the tangent to parabola $y^2 = 4ax$ at t is, $yt = x + at^2$

Hence at t_1 is $yt_1 = x + at_1{}^2$ .....(1)

at t_2 is $yt_2 = x + at_2{}^2$ .....(2)

(2) – (1) gives

$$yt_2 - yt_1 = x + at_2{}^2 - x - at_1{}^2$$
$$y(t_2 - t_1) = a(t_2{}^2 - t_1{}^2)$$
$$y(t_2 - t_1) = a(t_2 + t_1)(t_2 - t_1)$$
$$y = \frac{a(t_2 + t_1)(t_2 - t_1)}{(t_2 - t_1)}$$
$$y = a(t_2 + t_1)$$

Substituting $y = a(t_2 + t_1)$ in $yt_1 = x + at_1{}^2$

$$a(t_2 + t_1)t_1 = x + at_1{}^2$$
$$at_1t_2 + at_1{}^2 = x + at_1{}^2$$

$$at_1t_2 + at_1^2 - at_1^2 = x$$

$$x = at_1t_2$$

Hence the point of intersection

is $[at_1t_2, a(t_1 + t_2)]$.

....

8. If the normal at the point 't_1' on the parabola $y^2 = 4ax$ meets the parabola again at the point 't_2', then prove that $t_2 = -\left(t_1 + \frac{2}{t_1}\right)$

Solution: Equation of the normal to parabola $y^2 = 4ax$ at t is,

$$y + xt = at^3 + 2at$$

Normal at the point 't_1' is

$$y + xt_1 = at_1^3 + 2at_1$$

This normal meets $y^2 = 4ax$ at 't_2'

Hence $x = at_2^2$ *and* $y = 2at_2$

Substituting the values in

$$y + xt_1 = at_1^3 + 2at_1$$

$$2at_2 + (at_2^2)t_1 = at_1^3 + 2at_1$$

$$2at_2 + at_2^2t_1 = at_1^3 + 2at_1$$

$$(2at_2 - 2at_1) = (at_1^3 - at_2^2t_1)$$

$$2a(t_2 - t_1) = at_1(t_1^2 - t_2^2)$$

$$2a(t_2 - t_1) = -at_1(t_2^2 - t_1^2)$$

$$2a(t_2 - t_1) = -at_1(t_2 - t_1)(t_2 + t_1)$$

$$2 = -t_1(t_2 + t_1)$$

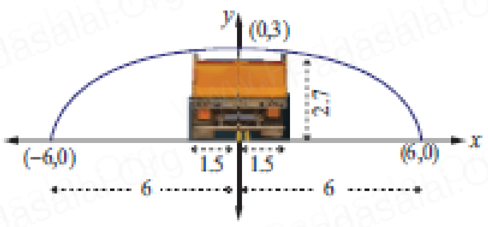
$$-\frac{2}{t_1} = t_2 + t_1$$

$$-\frac{2}{t_1} - t_1 = t_2$$

$$t_2 = -\left(\frac{2}{t_1} + t_1\right)$$

....

Example 5.30 A semielliptical archway over a one-way road has a height of 3m and a width of 12m. The truck has a width of 3m and a height of 2.7m. Will the truck clear the opening of the archway?



Solution:

The bridge is in the form of semi ellipse with width $2a = 12$ mt. Hence a = 6 and height at the centre $b = 3$ mt.

The equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ becomes

$$\frac{x^2}{6^2} + \frac{y^2}{3^2} = 1$$

$$\frac{x^2}{36} + \frac{y^2}{9} = 1 \quad ....(1)$$

The truck has a width of 3m and a height of 2.7m. passing through the arch.

Let when $x = 1.5$, y_1 be the height of the arch

Hence $(1.5, y_1)$ is the point on (1)

So, $$\frac{\left(\frac{3}{2}\right)^2}{36} + \frac{y_1^2}{9} = 1$$

$$\frac{\frac{9}{4}}{36} + \frac{y_1^2}{9} = 1$$

$$\frac{9}{4} \times \frac{1}{36} + \frac{y_1^2}{9} = 1$$

$$\frac{9}{144} + \frac{y_1^2}{9} = 1$$

$$\frac{y_1^2}{9} = 1 - \frac{9}{144}$$

$$= \frac{144-9}{144}$$

$$y_1^2 = 9 \times \frac{135}{144}$$

$$= \frac{135}{16}$$

$$y_1 = \sqrt{\frac{135}{16}}$$

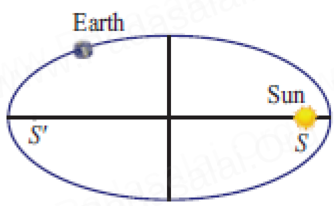
$$y_1 = \frac{11.62}{4}$$

$$y_1 = 2.90$$

The height of the arch is 2.90 mt at 1.5 mt from the centre. The height of the truck is 2.7 mt. only hence it passes through the bridge.

....

Example 5.31 The maximum and minimum distances of the Earth from the Sun respectively are 152×10^6 km and 94.5×10^6km.



The

Sun is at one focus of the elliptical orbit. Find the distance from the Sun to the other focus.

Solution:

The sun is at one of the focus at S.

Let the other focus be $S^{/}$

We have to find the distance between them

$$S^{/}S = 2ae$$

Given

The maximum distance of the Earth from the Sun $\quad a + ae = 152 \times 10^{6}$ km and

The minimum distance of the Earth from the Sun $\quad a - ae = 94.5 \times 10^{6}$ km and

Hence $(a + ae) - (a - ae) = (152 - 94.5)10^{6}$

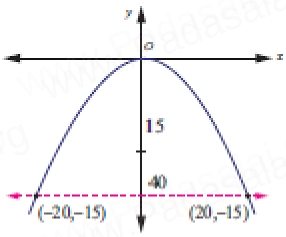
$$a + ae - a + ae = (57.5)10^{6}$$

$$\therefore 2ae = (57.5)10^{6}$$

The distance from the Sun to the other focus $= (57.5)10^{6}$km.

Example 5.32 A concrete bridge is designed as a parabolic arch. The road over bridge is $40m$ long and the maximum height of the arch is $15m$. Write the equation of the parabolic arch.

Solution:



The parabola is open down with vertex at O

The equation is $\quad x^{2} = -4ay$

$(20, -15)$ is a point lies on it.

So, $(20)^{2} = -4a(-15)$

$$400 = 60a$$

$$a = \frac{400}{60} = \frac{20}{3}$$

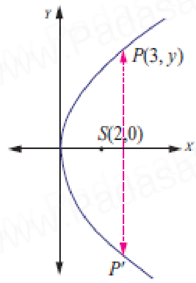
$$\therefore x^{2} = -4\left(\frac{20}{3}\right)y$$

$$3x^{2} = -80y$$

$$3x^{2} + 80y = 0$$

Example 5.33 The parabolic communication antenna has a focus at $2m$ distance from the vertex of the antenna. Find the width of the antenna $3m$ from the vertex.

Solution:



Let the parabola open right with vertex at O.

The equation is $\quad y^{2} = 4ax$

Given focus at a = 2 and the depth VC= 3

Let CP $= y_{1}$

Hence P $(3, y_{1})$ lies on $y^{2} = 4ax$

So, $\quad {y_{1}}^{2} = 4(2)(3)$

$$y_{1} = 2(\sqrt{2})(\sqrt{3})$$

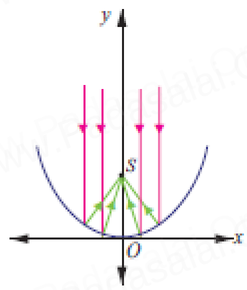
$$CP = y_{1} = 2\sqrt{6}$$

width of the antenna $PP^{/} = 2 \times 2\sqrt{6}$

$\therefore$ width of the antenna $= 4\sqrt{6}$ mt.

Example 5.34 The equation $y = \frac{1}{32}x^{2}$ models cross sections of parabolic mirrors that are used for solar energy. There is a heating tube located at the focus of each parabola; how high is this tube located above the vertex of the parabola?

Solution:



The equation of the mirror $y = \frac{1}{32}x^2$

$$\therefore 32y = x^2$$

Comparing with $x^2 = 4ay$

$$4a = 32$$

$$a = 8$$

heating tube located at the focus $a = 8$ from O.

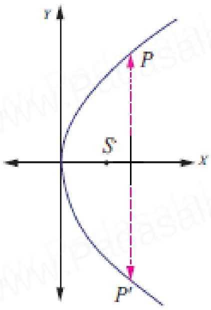
....

Example 5.35 A search light has a parabolic reflector (has a cross section that forms a 'bowl'). The parabolic bowl is 40 cm wide from rim to rim and 30 cm deep. The bulb is located at the focus .

(1) What is the equation of the parabola used for reflector?

(2) How far from the vertex is the bulb to be placed so that the maximum distance covered?

Solution:



Let the parabola open right with vertex at O.

The equation is $y^2 = 4ax$

Given width $PP^{/} = 40$ and the depth VC= 30

Hence P (30, 20) lies on $y^2 = 4ax$

So, $20^2 = 4(a)(30)$

$$400 = 120a$$

$$a = \frac{400}{120} = \frac{10}{3}$$

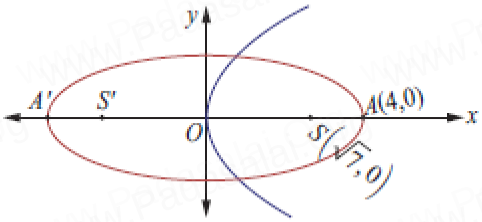
(i) The equation of the parabola used for reflector $y^2 = 4\left(\frac{10}{3}\right)x$

(ii) Bulb is placed at the distance of $a = \frac{10}{3}$ cm

....

Example 5.36 An equation of the elliptical part of an optical lens system is $\frac{x^2}{16} + \frac{y^2}{9} = 1$. The parabolic part of the system has a focus in common with the right focus of the ellipse .The vertex of the parabola is at the origin and the parabola opens to the right. Determine the equation of the parabola.

Solution:



To find the equation of the parabola drawn at the focus of the given ellipse as the focus.

$$\frac{x^2}{16} + \frac{y^2}{9} = 1$$

Here $a^2 = 16$ and $b^2 = 9$

$$(ae)^2 = a^2 - b^2$$

$$= 16 - 9$$

$$= 7$$

$$ae = \sqrt{7}$$

Then focus of the ellipse $S = (ae, 0) = (\sqrt{7}, 0)$

Focus of the Parabola is $(\sqrt{7}, 0)$ that is $a = \sqrt{7}$

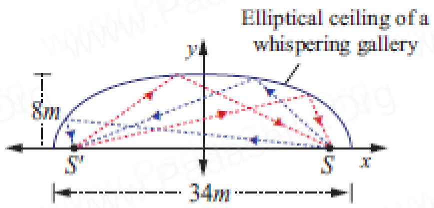
Equation of the parabola open right is

$$y^2 = 4(\sqrt{7})x$$

....

Example 5.37 A room 34 m long is constructed to be a whispering gallery. The room has an elliptical ceiling, as shown in Fig. 5.64. If the maximum height of the ceiling is 8 m, determine where the foci are located.

Solution:



From the diagram 2a=34 so a = 17

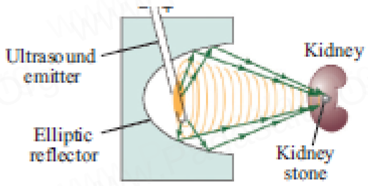
Height of the ceiling b = 8

$$(ae)^2 = a^2 - b^2$$
$$= 289 - 64$$
$$= 225$$
$$ae = \sqrt{225} = 15$$

Hence the foci are located on either side 15 mt from the centre along its major axis.

Example 5.38 If the equation of the ellipse is $\frac{(x-11)^2}{484} + \frac{y^2}{64} = 1$ (x and y are measured in centimeters) where to the nearest centimeter, should the patient's kidney stone be placed so that the reflected sound hits the kidney stone?

Solution:



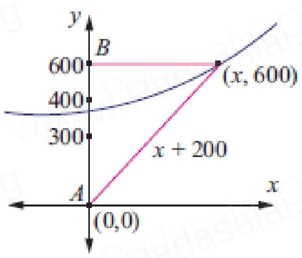
Given $\frac{(x-11)^2}{484} + \frac{y^2}{64} = 1$

Here $a^2 = 484$ and $b^2 = 64$

$$(ae)^2 = a^2 - b^2$$
$$= 484 - 64$$
$$= 420$$
$$ae = \sqrt{420} = 20.5$$

the patient's kidney stone be placed 20.5 cm away from the centre of the ellipse.

Example 5.39 Two coast guard stations are located 600 km apart at points A (0,0) and B (0, 600) . A distress signal from a ship at P is received at slightly different times by two stations. It is determined that the ship is 200 km farther from station A than it is from station B. Determine the equation of hyperbola that passes through the location of the ship.



Solution:

Given A (0,0) and B (0, 600) are 2 coast guard stations. They are the foci of the hyperbola.

So its midpoint C (0, 300) is the centre of the hyperbola with transverse axis y axis.

Its equation is $\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$

Substituting C (h, k)= C (0, 300)

$$\frac{(y-300)^2}{a^2} - \frac{x^2}{b^2} = 1 \quad \ldots . (1)$$

The ship is 200 km farther from station A than it is from station B.

So, D (0, 400) is the point lies on (1)

$$\frac{(400-300)^2}{a^2} - \frac{0}{b^2} = 1$$
$$\frac{(100)^2}{a^2} = 1$$
$$a^2 = 10000$$

The distance between two foci $2ae = 600$

$$\therefore ae = 300$$

For hyperbola $(ae)^2 = a^2 + b^2$

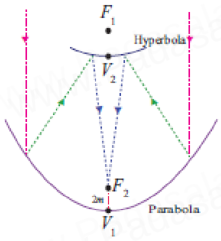
$$(300)^2 = 10000 + b^2$$
$$b^2 = 90000 - 10000$$
$$b^2 = 80000$$

The equation of hyperbola that passes through the location of the ship is $\frac{(y-300)^2}{10000} - \frac{x^2}{80000} = 1$

Example 5.40 Certain telescopes contain both parabolic mirror and a hyperbolic mirror. In the telescope shown in figure the parabola and hyperbola share focus F_1 which is 14m above the vertex of the parabola. The hyperbola's second focus F_2 is 2m above the parabola's vertex. The vertex of the hyperbolic mirror is

$1m$ below F_1. Position a coordinate system with the origin at the centre of the hyperbola and with the foci on the y-axis. Then find the equation of the hyperbola.

Solution:



Given

V_1, F_1 are the vertex and focus of the parabola.

F_1, F_2 are the foci of the hyperbola.

By the data $F_1F_2 = 2ae = 12$

$\therefore ae = 6$

$F_2V_1 = 2a = 10$

$\therefore a = 5$

For hyperbola $(ae)^2 = a^2 + b^2$

$(6)^2 = 25 + b^2$

$b^2 = 36 - 25$

$b^2 = 11$

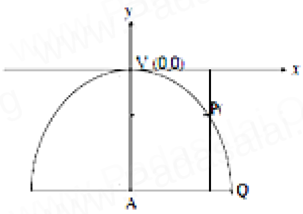
Hence equation of the hyperbola with transverse axis y axis , center at origin is

$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$ gives $\frac{y^2}{25} - \frac{x^2}{11} = 1$

EXERCISE 5.5

1. A bridge has a parabolic arch that is $10m$ high in the centre and $30m$ wide at the bottom. Find the height of the arch $6m$ from the centre, on either sides.

Solution:



The equation of the parabolic arch

$$x^2 = -4ay$$

P (15, −10) lies on it

$$15^2 = -4a(-10)$$

$$225 = 40a$$

$$\frac{225}{10} = 4a$$

$\therefore x^2 = -4ay$ becomes

$$x^2 = -\frac{225}{10}y \quad \text{...... (1)}$$

The point B $(6, -y_1)$ lies on (1)

$$6^2 = -\frac{225}{10}(-y_1)$$

$$36 = \frac{225}{10}y_1$$

$$\frac{360}{225} = y_1$$

$$\frac{8}{5} = y_1$$

Required height of the bridge $= 10 - y_1$

$$= 10 - \frac{8}{5}$$

$$= \frac{50-8}{5}$$

$$= \frac{42}{5}$$

$$= 8.4\text{mt}$$

2. A tunnel through a mountain for a four lane highway is to have a elliptical opening. The total width of the highway (not the opening) is to be $16m$, and the height at the edge of the road must be sufficient for a truck $4m$ high to clear if the highest point of the opening is to be $5m$ approximately . How wide must the opening be?

Solution:

The tunnel opening is in the form of an ellipse.

Given $AA' = 2a = 16$. So, $OA = a = 8$

Height at the centre $OB = b = 5$

Hence the equation of the ellipse with major axis y axis is $\frac{y^2}{a^2} + \frac{x^2}{b^2} = 1$

$$\therefore \frac{y^2}{64} + \frac{x^2}{25} = 1$$

Let $AC = 4$

Hence $C'(4, y)$ lies on the ellipse

$$\frac{y^2}{64} + \frac{x^2}{25} = 1$$

$$\therefore \frac{y^2}{64} + \frac{4^2}{25} = 1$$

$$\frac{y^2}{64} + \frac{16}{25} = 1$$

$$\frac{y^2}{64} = 1 - \frac{16}{25}$$

$$= \frac{25-16}{25}$$

$$\frac{y^2}{64} = \frac{9}{25}$$

$$y^2 = \frac{64 \times 9}{25}$$

$$y = \frac{8 \times 3}{5} = \frac{24}{5}$$

$$y = 4.8$$

The required wide for the opening $2y = 9.6$ mt.

3. At a water fountain, water attains a maximum height of $4m$ at horizontal distance of $0.5\ m$ from its origin. If the path of water is a parabola, find the height of water at a horizontal distance of $0.75\ m$ from the point of origin.

Solution: Equation of the parabola with vertex $V(h, k)$ open down is

$$(x - h)^2 = -4a(y - k)$$

Given vertex $V(0.5, 4)$

So, $(x - 0.5)^2 = -4a(y - 4)$

Origin $O(0, 0)$ is a point on it.

$$\therefore (0 - 0.5)^2 = -4a(0 - 4)$$

$$(-0.5)^2 = -4a(-4)$$

$$0.25 = 16a$$

$$\frac{0.25}{16} = a$$

So, $(x - 0.5)^2 = -4\left(\frac{0.25}{16}\right)(y - 4)$

$$(x - 0.5)^2 = -\left(\frac{0.25}{4}\right)(y - 4)$$

When horizontal distance of $0.75\ m$ from the point of origin, the OD is 7.5

Let $DE = y_1$

Then $E\ (0.75, y_1)$ lies on the parabola

$$(0.75 - 0.5)^2 = -\left(\frac{0.25}{4}\right)(y_1 - 4)$$

$$(0.25)^2 = -\left(\frac{0.25}{4}\right)(y_1 - 4)$$

$$\frac{(0.25)(0.25)\times 4}{(0.25)} = -(y_1 - 4)$$

$$(0.25) \times 4 = -(y_1 - 4)$$

$$1 = -y_1 + 4$$

$$y_1 = -1 + 4$$

$$y_1 = 3$$

The height of water at a horizontal distance of $0.75\ m$ *from* the point of origin is 3 mt.

4. An engineer designs a satellite dish with a parabolic cross section. The dish is $5m$ wide at the opening, and the focus is placed $1.2\ m$ from the vertex

(a) Position a coordinate system with the origin at the vertex and the x-axis on the parabola's axis of symmetry and find an equation of the parabola.

(b) Find the depth of the satellite dish at the vertex.

Solution: Dish wide $AB = 5$

OC is the depth. Let OC = x_1

Focus a = 1.2

Let the parabola open right.

So, $y^2 = 4ax$

$= 4(1.2)x$

$y^2 = 4.8x \quad ....(1)$

AC = BC= 2.5

A $(x_1, 2.5)$ lies on the equation.

$(2.5)^2 = 4.8x_1$

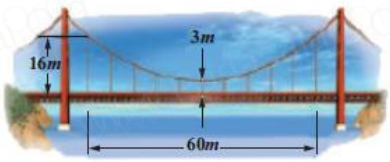
$\frac{(2.5)\times(2.5)}{(4.8)} = x_1$

Depth $x_1 = 1.3\, mt \quad ....(2)$

....

5. Parabolic cable of a 60 m portion of the road bed of a suspension bridge are positioned as shown below. Vertical Cables are to be spaced every 6 m along this portion of the roadbed. Calculate the lengths of first two of these vertical cables from the vertex.

Solution:



From the given diagram, parabola is open up.

Let O be the origin and V be the vertex.

Given OV = 3 m, V(0, 3)

Equation of the parabola with vertex V(h, k) open down is $(x-h)^2 = 4a(y-k)$

Given vertex V(0, 3)

So, $(x-0)^2 = 4a(y-3)$

$x^2 = 4a(y-3)$

Width of the bridge AB = 60 m

Hence AO = OB = 30 m

Height of the support AC = 16 m,

$\therefore C(-30, 16)$ lies on $x^2 = 4a(y-3)$

$(-30)^2 = 4a(16-3)$

$900 = 4a(13)$

$4a = \frac{900}{13}$

$\therefore x^2 = \frac{900}{13}(y-3)$

Let DE and FG two vertical cables such that OD = 6 m and OF = 12 m

Let length DE = y_1.

$\therefore E(6, y_1)$ lies on $x^2 = \frac{900}{13}(y-3)$

$\therefore (6)^2 = \frac{900}{13}(y_1 - 3)$

$\frac{36\times13}{900} = y_1 - 3$

$y_1 - 3 = \frac{4\times13}{100}$

$= \frac{52}{100}$

$y_1 - 3 = 0.52$

$y_1 = 0.52 + 3$

$y_1 = 3.52$ m

Length of first vertical cable from the vertex DE = 3.52 m.

Let length FG = y_2.

$\therefore G(12, y_2)$ lies on $x^2 = \frac{900}{13}(y-3)$

$\therefore (12)^2 = \frac{900}{13}(y_2 - 3)$

$\frac{144\times13}{900} = y_2 - 3$

$y_2 - 3 = \frac{16\times13}{100}$

$= \frac{208}{100}$

$y_2 - 3 = 2.08$

$y_2 = 2.08 + 3$

$y_2 = 5.08$ m

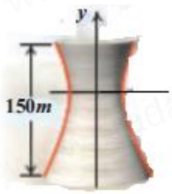
Length of second vertical cable from the vertex FG = 5.08 m.

....

6. Cross section of a Nuclear cooling tower is in

the shape of a hyperbola with equation $\frac{x^2}{30^2} - \frac{y^2}{44^2} = 1$. The tower is 150 m tall and the distance from the top of the tower to the centre of the hyperbola is half the distance from the base of the tower to the centre of the hyperbola. Find the diameter of the top and base of the tower.

Solution:



Given $\frac{x^2}{30^2} - \frac{y^2}{44^2} = 1$

Let x_1 be the radius at the top.

Then $(x_1, 50)$ lies the equation.

$$\therefore \frac{{x_1}^2}{30^2} - \frac{(50)^2}{44^2} = 1$$

$$\frac{{x_1}^2}{30^2} = 1 + \frac{(50)^2}{44^2}$$

$$= \frac{44^2+(50)^2}{44^2}$$

$${x_1}^2 = 30^2\left(\frac{44^2+50^2}{44^2}\right)$$

$$= \frac{30^2}{44^2}(44^2 + 50^2)$$

$$x_1 = \frac{30}{44}\sqrt{(44^2 + 50^2)}$$

$$= \frac{30}{44}\sqrt{1936 + 2500}$$

$$= \frac{30}{44}\sqrt{4436}$$

$$= \frac{30}{44}(66.60)$$

$$= \frac{1998}{44}$$

$$x_1 = 45.41$$

Hence the diameter at the top

$$2x_1 = 45.41 \times 2 = 90.82 \text{ m}$$

Let x_2 be the radius at the base.

Then $(x_2, 100)$ lies the equation.

$$\therefore \frac{{x_2}^2}{30^2} - \frac{(100)^2}{44^2} = 1$$

$$\frac{{x_2}^2}{30^2} = 1 + \frac{(100)^2}{44^2}$$

$$= \frac{44^2+(100)^2}{44^2}$$

$${x_2}^2 = 30^2\left(\frac{44^2+100^2}{44^2}\right)$$

$$= \frac{30^2}{44^2}(44^2 + 100^2)$$

$$x_2 = \frac{30}{44}\sqrt{(44^2 + 100^2)}$$

$$= \frac{30}{44}\sqrt{1936 + 10000}$$

$$= \frac{30}{44}\sqrt{11936}$$

$$= \frac{30}{44}(109.2)$$

$$= \frac{3276}{44}$$

$$x_2 = 74.49$$

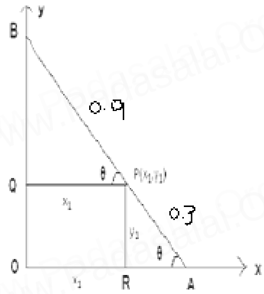
Hence the diameter at the base

$$2x_2 = 74.49 \times 2 = 148.98 \text{ m}$$

....

7. A rod of length 1.2 m moves with its ends always touching the coordinate axes. The locus of a point P on the rod, which is 0.3 m from the end in contact with x-axis is an ellipse. Find the eccentricity.

Solution:



Let $P(x_1, y_1)$ be the point on the iron rod.

AP = 0.3 mts. and PB = 0.9 mts.

Δ APR and Δ BPQ are similar.

From right angle Δ BPQ

$$\cos\theta = \frac{x_1}{0.9}$$

From right angle Δ APR

$$\sin\theta = \frac{y_1}{0.3}$$

We know that $cos^2\theta + sin^2\theta = 1$

Hence $\frac{{x_1}^2}{0.9^2} + \frac{{y_1}^2}{0.3^2} = 1$

$a^2 = (0.9)^2$ and $b^2 = (0.3)^2$

We know that $(ae)^2 = a^2 - b^2$

$$(a^2)e^2 = (0.9)^2 - (0.3)^2$$

$$(0.81)e^2 = 0.81 - 0.09$$

$$(0.81)e^2 = 0.72$$

$$(81)e^2 = 72$$

$$e^2 = \frac{72}{81}$$

$$= \frac{36\times 2}{81}$$

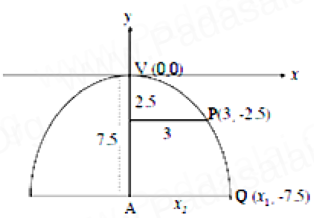
$$e = \frac{6\times\sqrt{2}}{9}$$

$$e = \frac{2\sqrt{2}}{3}$$

....

8. Assume that water issuing from the end of a horizontal pipe, 7.5 m above the ground, describes a parabolic path. The vertex of the parabolic path is at the end of the pipe. At a position 2.5 m below the line of the pipe, the flow of water has curved outward 3 m beyond the vertical line through the end of the pipe. How far beyond this vertical line will the water strike the ground?

Solution:



Equation of the parabolic path of the water

$$x^2 = -4ay$$

P (3, −2 .5) lies on the equation

Sub $x = 3, y = -2.5$

$$3^2 = -4a\,(-2.5)$$

$$9 = 10a$$

$$a = \frac{9}{10}$$

The equation is $x^2 = -4\left(\frac{9}{10}\right)y$

Let AQ $= x_1$

∴ Q$(x_1, -7.5)$ lies on the equation

$${x_1}^2 = -4\left(\frac{9}{10}\right)y$$

$$x_1^2 = -4\left(\frac{9}{10}\right)(-7.5)$$

$$x_1^2 = -4\left(\frac{9}{10}\right)\left(-\frac{75}{10}\right)$$

$$x_1^2 = 4\left(\frac{9}{10}\right)\left(\frac{25\times 3}{10}\right)$$

$$x_1 = 2\left(\frac{3}{10}\right)(5\sqrt{3})$$

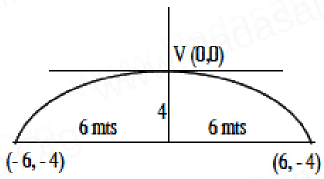
$$x_1 = 3\sqrt{3}\text{ mt.}$$

The water strikes the ground 3 $\sqrt{3}$ mt. away from A.

....

9. On lighting a rocket cracker it gets projected in a parabolic path and reaches a maximum height of 4 m when it is 6 m away from the point of projection. Finally it reaches the ground 12 m away from the starting point. Find the angle of projection.

Solution:



The equation of the parabolic path

$$x^2 = -4ay$$

It passes through $(6, -4)$

$$6^2 = -4a(-4)$$

$$36 = 16a$$

$$\frac{36}{16} = a$$

$$a = \frac{9}{4}$$

$\therefore\ x^2 = -4ay$ becomes

$$x^2 = -4\left(\frac{9}{4}\right)y$$

$$x^2 = -9y$$

To find the slope at $(-6,-4)$

$$2x = -9\frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{-2x}{9}$$

sub $x = -6$

$$\frac{dy}{dx} = \frac{-2(-6)}{9} = \frac{12}{9} = \frac{4}{3}$$

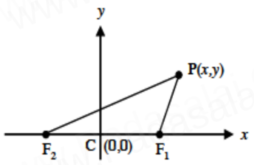
$$\tan\theta = \frac{4}{3}$$

$$\theta = tan^{-1}\left(\frac{4}{3}\right)$$

....

10. Points A and B are $10\,km$ apart and it is determined from the sound of an explosion heard at those points at different times that the location of the explosion is 6 km closer to A than B. Show that the location of the explosion is restricted to a particular curve and find an equation of it.

Solution:



A and B be the focus of the hyperbola.

P is any point on it.

By focal property $F_1P - F_2P = 2a = 6$

So, $a = 3$

and $F_1F_2 = 2ae = 10$

Hence, $ae = 5$

We know that $(ae)^2 = a^2 + b^2$

$$(5)^2 = 3^2 + b^2$$

$$25 = 9 + b^2$$

$$b^2 = 25 - 9$$

$$b^2 = 16$$

The required equation $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is

$$\frac{x^2}{9} - \frac{y^2}{16} = 1$$

....

EXERCISE 5.6

Choose the most appropriate answer.

1. The equation of the circle passing through $(1,5)$ and $(4,1)$ and touching y-axis is $x^2 + y^2 - 5x - 6y + 9 + \lambda(4x + 3y - 19) = 0$ where λ is equal to .....

(1) $0, -\frac{40}{9}$ (2) 0 (3) $\frac{40}{9}$ (4) $-\frac{40}{9}$

2. The eccentricity of the hyperbola whose latus rectum is 8 and conjugate axis is equal to half the distance between the foci is ......

(1) $\frac{4}{3}$ (2) $\frac{4}{\sqrt{3}}$ **(3) $\frac{2}{\sqrt{3}}$** (4) $\frac{3}{2}$

3. The circle $x^2 + y^2 = 4x + 8y + 5$ intersects the line $3x - 4y = m$ at two distinct points if

(1) $15 < m < 65$ (2) $35 < m < 85$

(3) $-85 < m < -35$ **(4) $-35 < m < 15$**

4. The length of the diameter of the circle which touches the x-axis at the point $(1, 0)$ and passes through the point $(2, 3)$.......

(1) $\frac{6}{5}$ (2) $\frac{5}{3}$ **(3) $\frac{10}{3}$** (4) $\frac{3}{5}$

5. The radius of the circle

$3x^2 + by^2 + 4bx - 6by + b^2 = 0$ is . . . . . .

(1) 1 (2) 3 **(3) $\sqrt{10}$** (4) $\sqrt{11}$

6. The centre of the circle inscribed in a square formed by the lines $x^2 - 8x - 12 = 0$ and $y^2 - 14y + 45 = 0$ is

(1) (4, 7) (2) (7, 4) (3) (9, 4) (4) (4, 9)

7. The equation of the normal to the circle $x^2 + y^2 - 2x - 2y + 1= 0$ which is parallel to the line $2x + 4y = 3$ is

(1) $x + 2y = 3$ (2) $x + 2y + 3 = 0$

(3) $2x + 4y + 3 = 0$ (4) $x - 2y + 3 = 0$

8. If P (x, y) be any point on $16x^2 + 25y^2 = 400$ with foci F_1 $(3, 0)$and F_2 $(-3, 0)$ then PF_1+ PF_2 is

(1) 8 (2) 6 **(3) 10** (4) 12

9. The radius of the circle passing through the point $(6, 2)$ two of whose diameter are $x + y = 6$ and $x + 2y = 4$ is

(1) 10 **(2) $2\sqrt{5}$** (3) 6 (4) 4

10. The area of quadrilateral formed with foci of the hyperbolas $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ and $\frac{x^2}{a^2} - \frac{y^2}{b^2} = -1$ is

(1) $4(a^2 + b^2)$ **(2) $2(a^2 + b^2)$**

(3) $a^2 + b^2$ (4) $\frac{1}{2}(a^2 + b^2)$

11. If the normals of the parabola $y^2 = 4x$ drawn at the end points of its latus rectum are tangents to the circle $(x - 3)^2 + (y + 2)^2 = r^2$, then the value of r^2 is

(1) 2 (2) 3 (3) 1 (4) 4

12. If $x + y = k$ is a normal to the parabola $y^2 = 12x$, then the value of k is

(1) 3 (2) -1 (3) 1 **(4) 9**

13. The ellipse $E_1 : \frac{x^2}{9} + \frac{y^2}{4} = 1$is inscribed in a rectangle R whose sides are parallel to the coordinate axes. Another ellipse E_2 passing through the point $(0, 4)$ 4circumscribes the rectangle R. The eccentricity of the ellipse is

(1) $\frac{\sqrt{2}}{2}$ (2) $\frac{\sqrt{3}}{2}$ **(3) $\frac{1}{2}$** (4) $\frac{3}{4}$

14. Tangents are drawn to the hyperbola $\frac{x^2}{9} - \frac{y^2}{4} = 1$parallel to the straight line $2x - y = 1$. One of the points of contact of tangents on the hyperbola is

(1) $\left(\frac{9}{2\sqrt{2}}, \frac{-1}{\sqrt{2}}\right)$ (2) $\left(\frac{-9}{2\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$

(3) $\left(\frac{9}{2\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ (4) $(3\sqrt{3}, -2\sqrt{2})$

15. The equation of the circle passing through the foci of the ellipse $\frac{x^2}{16} + \frac{y^2}{9} = 1$having centre at (0, 3) is

(1) $x^2 + y^2 - 6y - 7 = 0$

(2) $x^2 + y^2 - 6y + 7 = 0$

(3) $x^2 + y^2 - 6y - 5 = 0$

(4) $x^2 + y^2 - 6y + 5 = 0$

16. Let C be the circle with centre at $(1, 1)$ and radius $=1$. If T is the circle centered at $(0, y)$ passing through the origin and touching the circle C externally, then the radius of T is equal to

(1) $\frac{\sqrt{3}}{\sqrt{2}}$ (2) $\frac{\sqrt{3}}{2}$ (3) $\frac{1}{2}$ **(4) $\frac{1}{4}$**

17. Consider an ellipse whose centre is of the origin and its major axis is along x-axis. If its eccentrcity is $\frac{3}{5}$ and the distance between its foci is 6, then the area of the quadrilateral inscribed in the ellipse with diagonals as major and minor axis of the ellipse is

(1) 8 (2) 32 (3) 80 **(4) 40**

18. Area of the greatest rectangle inscribed in the ellipse $\frac{x^2}{16} + \frac{y^2}{9} = 1$ is

(1) 2ab (2) ab (3) $\sqrt{ab}$ (4) $\frac{a}{b}$

19. An ellipse has OB as semi minor axes, F and F/ its foci and the angle FBF/ is a right angle. Then the eccentricity of the ellipse is

(1) $\frac{1}{\sqrt{2}}$ (2) $\frac{1}{2}$ (3) $\frac{1}{4}$ (4) $\frac{1}{\sqrt{3}}$

20. The eccentricity of the ellipse $(x-3)^2 + (y-4)^2 = \frac{y^2}{9}$ is

(1) $\frac{\sqrt{3}}{2}$ **(2) $\frac{1}{3}$** (3) $\frac{1}{3\sqrt{2}}$ (4) $\frac{1}{\sqrt{3}}13$

21. If the two tangents drawn from a point P to the parabola $y^2 = 4x$ are at right angles then the locus of P is

(1) $2x + 1 = 0$ **(2) $x = -1$**

(3) $2x - 1 = 0$ (4) $x = 1$

22. The circle passing through $(1, -2)$ and touching the axis of x at $(3, 0)$ passing through the point

(1) $(-5, 2)$ (2) $(2, -5)$

(3) $(5, -2)$ (4) $(-2, 5)$

23. The locus of a point whose distance from $(-2, 0)$ is 23 times its distance from the line $x = \frac{-9}{2}$ is

(1) a parabola (2) a hyperbola

(3) an ellipse (4) a circle

24. The values of m for which the line $y = mx + 2\sqrt{5}$ touches the hyperbola $16x^2 - 9y^2 = 144$ are the roots of $x^2 - (a+b)x - 4 = 0$, then the value of $(a+b)$ is

(1) 2 (2) 4 **(3) 0** (4) -2

25. If the coordinates at one end of a diameter of the circle $x^2 + y^2 - 8x - 4y + c = 0$ $(11, 2)$, the coordinates of the other end are

(1) $(-5, 2)$ **(2) $(-3, 2)$**

(3) $(5, -2)$ (4) $(-2, 5)$