Real number system

Theorem :

Prove that $\sqrt{3}$ is a not a rational number

Assume that $\sqrt{3}$ is a rational number

This is can be written in the form of $\frac{m}{n} = \sqrt{3}$

(since rational number is of the form p/q)

(m, n are +ve integer with no common factor greater than 1)

$\frac{m}{n} = \sqrt{3}$ **(Squaring on both sides)**

$\frac{m^2}{n^2} = 3$ $\Rightarrow$ **$m^2 = 3n^2$**

$\Rightarrow 2m^2 = 6n^2 = 3(2n^2) = 2(3n^2)$

$\Rightarrow$ **m^2 is an even number (Since multiples of 2)**

$\Rightarrow$ **m is even**

$\Rightarrow$ **Let m = 2k (because m is even)**

Substitute m= 2k in (i)

$(2k)^2 = 3n^2$

$\Rightarrow$ **$4k^2 = 3n^2$**

$\Rightarrow$ **n is also even number (multiples of 4)**

$\Rightarrow$**both m ,n are even numbers having common factor 2 >1**

Which contradicts our fact

Therefore our assumption is wrong

(this called Contradiction method of proving the result)

Hence $\sqrt{3}$ is an irrational number

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1. Classify each element of $\{\sqrt{7}, \frac{-1}{4}, 0, 3.14, 4, \frac{22}{7}\}$ as a member of $\mathbb{N}, \mathbb{Q}, \mathbb{R} - \mathbb{Q}$ or $\mathbb{Z}$.

$\sqrt{7}$ **is an irrational number** **($\sqrt{7} \in$ R-Q)**

$-\frac{1}{4}$ **is a rational number which is of the form p/q** **($-\frac{1}{4} \in$ Q)**

0 is an integer **($0 \in$ Z and also $0 \in$ Q)**

3.14 is a terminating decimal number **($3.14 \in Q$)**

4 is a natural number / Integer / rational number **($4 \in N$, $4 \in Z$, $4 \in Q$)**

$\frac{22}{7}$ **is of the form p/q** **($\frac{22}{7} \in Q$**

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2. Prove that $\sqrt{2}$ is not a rational number

Proof:

Assume that $\sqrt{2}$ is a rational number

(this called Contradiction method of proving the result)

This can be written in the form of $\frac{m}{n} = \sqrt{2}$ (since rational number is of the form p/q)

(m, n are +ve integer with no common factor greater than 1)

$\frac{m}{n} = \sqrt{2}$ **(Squaring on both sides)**

$\frac{m^2}{n^2} = 2 \Rightarrow$ **$m^2 = 2n^2$ ---------------(i)**

$\Rightarrow$ m^2 is an even number (Since multiples of 2)

$\Rightarrow$ m is even

$\Rightarrow$ Let m = 2k (because m is even)

Substitute m= 2k in (i)

$(2k)^2 = 2n^2$

$\Rightarrow$ $4k^2 = 2n^2$

$\Rightarrow$ $2k^2 = n^2$

$\Rightarrow$ n is also even number

$\Rightarrow$ m ,n are even numbers having common factor 2 >1

Which contradicts our fact

Therefore our assumption is wrong

Hence $\sqrt{2}$ is an irrational number

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3. Are there two distinct irrational numbers such that their difference is a rational number? Justify.

Consider two distinct irrational numbers ($a+\sqrt{b}$) and ($c+\sqrt{b}$) a, b,c $\in N$

difference of the above irrational = (a+$\sqrt{b}$) - (c+$\sqrt{b}$) =(a-c)= rational number

4. Find two irrational numbers such that their sum is a rational number. Can you find two irrational numbers whose product is a rational number.

Consider two distinct irrational numbers (a+$\sqrt{b}$) and (a-$\sqrt{b}$) a, b $\in$N

Sum of the above irrational = (a+$\sqrt{b}$) + (a-$\sqrt{b}$) = 2a = rational number

5. Find a positive number smaller than $\frac{1}{2^{1000}}$. Justify.

We know that 1000 < 1001

$$2^{1000} < 2^{1001}$$

$$\frac{1}{2^{1000}} > \frac{1}{2^{1001}}$$ **(By taking reciprocal the inequality gets reversed)**

$\frac{1}{2^{1001}}$ **is a positive number smaller than** $\frac{1}{2^{1000}}$

Absolute value

Example 2.1

Solve $|2x-17| = 3$ **(Since** $|x| = \begin{cases} x & x \ge 0 \\ -x & x < 0 \end{cases}$

2x-17 = ±3

2x-17 =3 (or) 2x-17 = -3

2x= 20 2x= 14

x = 10 x= 7

Solution set = { 10 , 7 }

Example 2.2

Solve $3|x-2|+7 = 19$

$$3|x-2| = 19-7$$

3 $|x-2|$ **= 12**

÷ 3 $|x-2| = 4$

(x-2) = ± 4

Therefore x-2 =4 (or) x-2 = -4

x= 6 (or) x = -2 $\therefore$ **Solution set = { 6 , -2}**

Example 2.3

Solve $|2x-3| = |x-5|$

Note : $|u| = |v| \Leftrightarrow u = v \quad u = -v$ **(use this)**

2x-3 = x-5 (or) 2x-3 = -(x-5)

2x-x = -5 +3 (Or) 2x-3 = -x +5

x = -2 (or) 2x+x = 5+3 =8

3 x = 8

x= $\frac{8}{3}$

Solution Set = { -2 , $\frac{8}{3}$ **}**

Example 2.4

Solve $|x-9| < 2$ **(Note:** $|x| < r \Leftrightarrow -r < x < r$**) (use this note)**

-2 < x-9 < 2

Add 9 on both sides

-2 + 9 <x< 2+9

7 < x < 11

x$\in$ **(7,11)**

Example 2.5

Solve: $\left|\frac{2}{x-4}\right| > 1, \quad x \neq 4$

Since $\left|\frac{2}{x-4}\right| > 1$

2 > $|x-4|$

Ie., $|x-4|$**<2**

-2 < (x-4) < 2

Adding 4 throughout the inequality

4-2 < (x-4) < 4 +2

2 < x < 6 and x ≠4

So the solution set is (2,4) U (4,6)

Exercise 2.2

Solve : (i) $|3-x|<7.$

3-x < ±7

-7 < 3-x < 7

Add -3 0n both sides

-3-7 > -3+3-x > 7-3

-10 > -x > 4

Multiply with (- sign) the inequality will be reversed

10 > x < -4

i.,e -4 < x < 10

hence the value of x ∈ (-4,10)

Solve : (ii) $|4x-5|\geq -2.$

$|4x-5|+2\geq 0$ (Note : If $|x|>r$ and if r <0 for every x ∈ R)

Even x=0 , x= +ve x = - ve the inequality satisfies

The in equality satisfies for all values of x

Hence x∈ R (or) x ∈ (-∞ , ∞)

Solve for x: (iii) $\left|3-\frac{3}{4}x\right|\leq\frac{1}{4}.$

Note $|x|\leq r \Leftrightarrow -r\leq x\leq r$

$$-\frac{1}{4}\leq\left|3-\frac{3}{4}x\right|\leq\frac{1}{4}$$

Add -3 on both sides we have

$$-\frac{1}{4}-3\leq\left|3-\frac{3}{4}x-3\right|\leq\frac{1}{4}-3$$ (taking LCM)

$$-\frac{13}{4}\leq-\frac{3x}{4}\leq-\frac{11}{4}$$

Multiply with (- sign) the inequality will be reversed

$$\frac{13}{4} \geq \frac{3x}{4} \geq \frac{11}{4}$$

Multiply by $\frac{4}{3}$ **we have** $\left(\frac{13}{4}\right)\frac{4}{3} \geq \frac{4}{3}\left(\frac{3x}{4}\right) \geq \frac{11}{4}\frac{4}{3} \Rightarrow \frac{13}{3} \geq x \geq \frac{11}{3}$

$$\frac{11}{3} \leq x \leq \frac{13}{3}$$

The solution set is $x \in \left[\frac{11}{3}, \frac{13}{3}\right]$

Solve for x: (iv) $|x| - 10 < -3$.

Add 10 on both sides

$$|x| - 10 + 10 < -3 + 10$$

$|x| < 7$ **(Note:** $|x| < r \Leftrightarrow -r < x < r$ **) (use this note)**

$$-7 < x < 7$$

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2. Solve $\frac{1}{|2x-1|} < 6$ and express the solution using the interval notation.

By taking reciprocal the inequality will be reversed

$$|2x-1| > \frac{1}{6}$$

2x- 1 > $\pm\frac{1}{6}$

Add 1 on both sides

2x -1 +1 > 1$\pm\frac{1}{6}$

2x > $\frac{5}{6}$ **(OR) x <** $\frac{7}{6}$ **(Divide by 2 we get)**

X > $\frac{5}{12}$ **(OR) x <** $\frac{7}{12}$

$-\infty$ $\frac{5}{12}$ $\frac{7}{12}$ ∞

There fore x∈ $\left(-\infty, \frac{5}{12}\right) U \left(\frac{7}{12}, \infty\right)$

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3. Solve $-3|x| + 5 \leq -2$ and graph the solution set in a number line.

$\Rightarrow -3|x|+5-5 \le -2-5$

$\Rightarrow$ **-3**$|x|$ **≤ -7**

÷ by (-3) the inequality will be reversed

$\Rightarrow \; |x| \ge \frac{7}{3}$

$\Rightarrow$ **x** $\ge \frac{7}{3}$ **(or)** **x**$\le -\frac{7}{3}$

$-\infty$ $\qquad -\frac{7}{3}$ $\qquad \frac{7}{3}$ $\qquad \infty$

$\therefore$ **x** $\in \left(-\infty, -\frac{7}{3}\right] U \left[\frac{7}{3}, \infty\right)$

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4. Solve $2|x+1|-6 \le 7$ and graph the solution set in a number line.

Add 6 on both sides

$2|x+1|-6+6 \le 7+6$

$2|x+1| \le 13$

÷ by 2 we get

$|x+1| \le \frac{13}{2}$ **(Note:** $|x|<r \Leftrightarrow -r<x<r$ **) (use this note)**

$-\frac{13}{2} \le x+1 \le \frac{13}{2}$ **(Add -1 on both sides)**

$-\frac{13}{2}-1 \le x+1-1 \le \frac{13}{2}-1$ **(n by taking LCM)**

$-\frac{15}{2} \le x \le \frac{11}{2}$

$-\infty$ $\qquad -\frac{15}{2}$ $\qquad \frac{11}{2}$

$\therefore$ **x** $\in \left[-\frac{15}{2}, \frac{11}{2}\right]$

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5. Solve $\frac{1}{5}|10x-2| < 1$.

Multiply with 5 on both sides

$|10x-2|<5$ **(Note:** $|x|<r \Leftrightarrow -r<x<r$ **) (use this note)**

$\Rightarrow$ **-5 <** $|10x-2|<5$

Add 2 through out the inequality

$-5+2<10x-2+2<5+2$

-3 <10x < 7

÷ by 10

$-\frac{3}{10}<x<\frac{7}{10}$

$-\infty \qquad -\frac{3}{10} \qquad \frac{7}{10} \qquad \infty$

$\therefore$ **x** $\in \left(-\frac{3}{10},\frac{7}{10}\right)$

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6. Solve $|5x-12|<-2$.

$|5x-2|+2<0$

$|5x-2|$ **is always +ve and also 2 is +ve**

$|5x-2|+2$ **is always +ve value (cannot be –ve)**

Hence the inequality has no solution

Linear inequalities

Example 2.5

Our monthly electricity bill contains a basic charge, which does not change with number of units used, and a charge that depends only on how many units we use. Let us say Electricity Board charges Rs. 110 as basic charge and charges Rs. 4 for each unit we use. If a person wants to keep his electricity bill below Rs. 250 , then what should be his electricity usage?

X denotes number of units used (Note that x ≥ 0)

Electricity bill = electiricty Board chrges + 4 (number of units used)

= Rs 110+4x

If the person wants his bill to be below Rs 250

We have to solve the inequality 110+4x <250

4x < 250 -110

4 x < 140

÷ 4 **x <** $\frac{140}{4}$ **=35**

Therfore $x < 35$

The person should keep his usage below 35 units

(so that his bill below Rs 250)

Example 2.7

Solve $3x-5 \leq x+1$ for x

$3x-x \leq 1+5$

$2x \leq 6$

Hence we have $x \leq 3$

X value should be less than or equal to 3

The Solution set is $x \in (-\infty, 3]$

Example 2.8

Solve the system of linear inequalities $3x-9 \geq 0$ and $4x-10 \leq 6$

$3x \geq 9$ and $4x \leq 6+10$

$4x \leq 16$

$x \geq 3$ and $x \leq 4$

$x \geq 3 \Rightarrow x \in [3, \infty)$ and $x \leq 4 \Rightarrow x \in (-\infty, 4]$

the intersection of these intervals gives you $x \in [3, 4]$

Example 2.9

A girl A is reading a book having 446 pages and she has already finished reading 271 pages. She wants to finish reading this book within a week . What is the minimum number of pages she should read per day to complete reading the book with in a week?

Let the number of pages = x (per day)

Total number of pages 446

she can read 7x pages with in a week (7 days for a week)

(7 (number of pages)= 7x)

she already finished reading 271 pages

hence $7x+271 \geq 446$

hence $7x \geq 446-271$

7x ≥ 17 5

÷by 7 x ≥ 25

⇒ she should read at least 25 pages per day (minimum number of pages to be read)

Exercise 2.3

1. Represent the following inequalities in the interval notation:
   (i) $x \geq -1$ and $x < 4$ (ii) $x \leq 5$ and $x \geq -3$
   (iii) $x < -1$ or $x < 3$ (iv) $-2x > 0$ or $3x - 4 < 11$.

(i)

-∞ **∞**

x $\in [-1, 4)$

(ii)

∞

-∞ **X** $\in [-3, 5]$

(iii)

-∞ **∞**

x$\in (-\infty, 3)$

(iv) -2x >0 (or) 3x-4 <11

÷ by (-2) x <0 (or) 3x < 15

x < 5 (divide by 3)

-∞ 5 ∞

x ∈ (-∞ , 5)

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2. Solve $23x < 100$ when (i) x is a natural number, (ii) x is an integer.

X < $\frac{100}{23}$

X < 4.34

When x is natural number the values of x are 1,2,3,4

When x is integer the value of x = { ……………..-3,-2,-1,0,1,2,3,4}

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3. Solve $-2x \geq 9$ when (i) x is a real number, (ii) x is an integer, (iii) x is a natural number.

$\div$ (-2) $\quad x \leq -\frac{9}{2}$

$X \leq -4.5$

(i) X is real number

$X \in (-\infty, -4.5]$

(ii) When x is an integer

X= { ………….. -7,-6,-5}

(iii) X is natural number

(iv) There is no natural number less than -4.5

X has no values (x has no solution)

$X = \phi = \{ \ \}$

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4. Solve: (i) $\frac{3(x-2)}{5} \leq \frac{5(2-x)}{3}$.

By cross multiplication we get

$9(x-2) \leq 25(2-x)$

$9x - 18 \leq 50 - 25x$

$9x + 25x \leq 50 + 18$

$9x + 25x \leq 68$

$34x \leq 68$

$X \leq 2$

-∞ 2 ∞

$X \in (-\infty, 2]$

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4. Solve: (ii) $\frac{5-x}{3} < \frac{x}{2} - 4$.

$\frac{5-x}{3} < \frac{x-8}{2}$

By cross multiplication we get

$10 - 2x < 3x - 24$

$10 + 24 < 3x + 2x$

$34 < 5x$

$\frac{34}{5} < x$

$6.8 < x$

$-\infty$ ⟵⟶ ∞

$$X \in \left(\frac{34}{5}, \infty\right)$$

5. To secure A grade one must obtain an average of 90 marks or more in 5 subjects each of maximum 100 marks. If one scored 84, 87, 95, 91 in first four subjects, what is the minimum mark one scored in the fifth subject to get A grade in the course?

To get A grade $\Rightarrow$ Average ≥ 90

$$\frac{84+87+95+91+x}{5} \geq 90$$

357 +x $\geq$ 450 (cross multiply)

X $\geq$ 450 -357

X $\geq$ 93

Minimum marks one should scored 93 to get A grade in the course.

6. A manufacturer has 600 litres of a 12 percent solution of acid. How many litres of a 30 percent acid solution must be added to it so that the acid content in the resulting mixture will be more than 15 percent but less than 18 percent?

(i) Let x denotes the number of solution of acid
More than 15%

Given that 12% of 600 +30%of x > 15% (600+x)

$$\left(\frac{12}{100} \times 600\right) + \frac{30x}{100} > (600+x)\frac{15}{100}$$

7200+30x > 9000+15x (multiply by 100 through out)

30x -15x > 9000-7200

15 x > 1800

X > 120

(ii) Less than 18%

30% of x + 12%(600) < (600+x) 18%

30x +7200 < 10800+18x

30x – 18x < 10800 -7200

12 x < 3600

X < 300

7. Find all pairs of consecutive odd natural numbers both of which are larger than 10 and their sum is less than 40.

Let x be the small odd number

Another number = x +2

Given x>10 and also x+2 >10

Their Sum = x+(x+2) <40

2x < 40-2=38

X < 19

$\Rightarrow$ 10 <x < 19

Moreover x is odd

Therefore the possible values of x are 11, 13,15, 17

All possible pair of consecutive odd natural numbers

(11,13) (13,15) ,(15,17), (17,19)

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8. A model rocket is launched from the ground. The height h reached by the rocket after t seconds from lift off is given by $h(t) = -5t^2 + 100t, 0 \le t \le 20$. At what time the rocket is 495 feet above the ground?

Given h(t) = -5t²+100t $0 \le t \le 20$

Height must be lies between 0 and 495

$0 \le h(t) \le 495$

$0 \le -5t^2+100t \le 495$

$0 \le -5t^2+100t - 495 \le 0$

ie., $-5t^2+100t - 495 = 0$

Divide by (-5) $t^2-20t+99=0$

(t-11) (t-9) = 0

t = 11 t = 9

When t = 11 sec (or) 9 sec the rocket is above the ground

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9. A plumber can be paid according to the following schemes: In the first scheme he will be paid rupees 500 plus rupees 70 per hour, and in the second scheme he will paid rupees 120 per hour. If he works x hours, then for what value of x does the first scheme give better wages?

The plumber works x hours

Wages from the first scheme = 500+70x

Wages from the second scheme=120x

First scheme give better wages mean $\Rightarrow 500+70x > 120x$

$\Rightarrow 500 > 120x-70x$

$\Rightarrow 500 > 50x$

$\Rightarrow 10 > x$

Therefore the value of x is less than 10

X = 1,2,3,4,5,6,7,8,9

---

10. A and B are working on similar jobs but their monthly salaries differ by one more than Rs. 6000. If B earn rupees 27000 per month then what are the possibilities of A's salary per month?

Let the salary of A = x

There are two possibilities (i) Salary of A < salary of B

(ii) Salary of A >salary of B

(i)

Therefore the salary A will be = 6000 +x (differ more than Rs.6000)

Salary of B is Rs.27,000

Salary of A < salary of B

$6000+x < 27000$

$X < 27000-6000$

$X < 21000$

The salary of A is less than Rs 21,000

(OR)

(ii)

The salary of A will be = x-6000 (differ less than Rs.6000)

Salary of A >salary of B

Salary of A > 27000

$x-6,000 > 27000$

$x > 33000$

The salary of B is greater than Rs 33,000

Quadratic function

Example 2.10

If a and b are the roots of the equation x^2-px+q=0 find the value of $\frac{1}{a}+\frac{1}{b}$

Sum of the roots = a+b = -(-p) = p

Product of the roots = ab = q

$$\therefore \quad \frac{1}{a}+\frac{1}{b}=\frac{a+b}{ab}=\frac{p}{q}$$

Example 2.11

Find the complete set of values of a for which the quadratic x^2-ax+a+2=0 has equal roots.

For equal roots the condition is D = b^2 -4ac =0

$$(-a)^2-4(1)(a+2)=0$$

$$a^2-4a-8=0$$

this cannot be factorized

$$\text{use } x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

$$=\frac{4\pm\sqrt{48}}{2}=\frac{4\pm\sqrt{16(3)}}{2}=\frac{2\left(2\pm2\sqrt{3}\right)}{2}=\left(2\pm2\sqrt{3}\right)$$

The value of a = { $2+2\sqrt{3}$, $2-2\sqrt{3}$ }

Example 2.12

Find the Number of solution of $x^2 + |x-1|=1$

Case (i) for x ≥ 1 , $|x-1|=x-1$

$$x^2+x-1=1$$

$$x^2+x-2=0 \quad \text{(On factorizing)}$$

$$(x+2)(x-1)=0$$

$$x=-2 \quad \text{(or)} \quad x=1$$

As x≥ 1 ⇒ the solution is x =1

Case (ii) for x < 1 $|x-1|=-(x-1)=-x+1$

$$x^2-x+1=1$$

$$x^2-x=0$$

x (x-1)=0

x= 0 (or) x =1

The solution set is { 0,1}

Hence the equation has two solution

Exercise 2.4

1. Construct a quadratic equation with roots 7 and -3.

Sum = 7-3=4

Product = (7) (-3) = -21

The quadratic equation is x^2-(Sum)x + Product = 0

x^2-4x-21= 0

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2. A quadratic polynomial has one of its zeros $1+\sqrt{5}$ and it satisfies $p(1) = 2$. Find the quadratic polynomial.

$1+\sqrt{5}$ **is one root**

The other root is $1-\sqrt{5}$

Since P(x) = a [(x- $(1+\sqrt{5})$ (x -($1-\sqrt{5}$)]

= a (x-1-$\sqrt{5}$) (x-1+$\sqrt{5}$)

= a [$(x-1)^2$ – 5]

P(x)= a (x^2-2x-4)

Since P(1) =2 given

(1) =a (1^2-2(1) -4 = 2
a (1-2-4) =2

5a= 2

a= $-\frac{2}{5}$

therefore the Polynomial = P(x) = $-\frac{2}{5}$ **(x^2-2x-4)**

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3. If α and β are the roots of the quadratic equation $x^2 + \sqrt{2}x + 3 = 0$, form a quadratic polynomial with zeroes $\frac{1}{\alpha}, \frac{1}{\beta}$.

Here a =1 b= $\sqrt{2}$ **c= 3**

Sum of the roots = α+β = $-\frac{b}{a}$ **=** $-\sqrt{2}$

Product of the roots = αβ = $\frac{c}{a}$ **= 3**

Now the given roots are $\frac{1}{\alpha}$ **and** $\frac{1}{\beta}$

Sum of the new roots = $\frac{1}{\alpha}+\frac{1}{\beta} = \frac{\alpha+\beta}{\alpha\beta} = -\frac{\sqrt{2}}{3}$

Product of the new roots = $\frac{1}{\alpha}\frac{1}{\beta} = \frac{1}{\alpha\beta} = \frac{1}{3}$

The new quadratic equation = P(x) = $x^2 - \left(-\frac{\sqrt{2}}{3}\right)x + \frac{1}{3} = 0$

$$3x^2+\sqrt{2}x+1=0$$

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4. If one root of $k(x-1)^2 = 5x-7$ is double the other root, show that $k = 2$ or -25.

The quadratic equation is $k(x^2-2x+1)=5x-7$

$$Kx^2-2kx+k-5x+7=0$$

$$Kx^2-(2k+5)x+(k+7)=0$$

Here $a = k \quad b = -(2k+5) \quad c = k+7$

Since one root is double the other

Let one root be = α and other roots be= 2α (double the other)

Sum of the roots = $\alpha + 2\alpha = \frac{2k+5}{k}$

$$3\alpha = \frac{2k+5}{k}$$

$$\alpha = \frac{2k+5}{3k} \quad \text{---------------(i)}$$

Product of the roots = $\alpha(2\alpha) = 2\alpha^2 = \frac{k+7}{k}$ **---------------(ii)**

Substitute (i) in (ii)

$$2\left(\frac{2k+5}{3k}\right)^2 = \frac{k+7}{k}$$

$$2\left(\frac{4k^2+25+20k}{9k^2}\right)=\frac{k+7}{k}$$

$$2\left(\frac{4k^2+25+20k}{9k}\right)= K+7 \quad \textbf{(cancelling k on either side)}$$

$8k^2+50+40k = 9k(k+7)$

$8k^2+50+40k = 9k^2+63k$

$9k^2 - 8k^2 - 40k + 63k + 50 = 0$

$K^2 + 23k - 50 = 0$

$(k+25)\ (k-2) = 0$

$K = -25$ (or) $k = 2$

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5. If the difference of the roots of the equation $2x^2-(a+1)x+a-1=0$ is equal to their product, then prove that $a=2$.

Here a= 2 , b= -(a+1), c = a-1

Since sum of the roots = $\alpha+\beta = -\frac{b}{a} = \frac{(a+1)}{2}$

Product of the roots = $\alpha\beta = \frac{c}{a} = \frac{a-1}{2}$

The difference of the roots = their product

$\alpha - \beta = \alpha\beta$

Squaring on both sides we get

$(\alpha-\beta)^2 = (\alpha\beta)^2$

$\alpha^2+\beta^2-2\alpha\beta = (\alpha\beta)^2$

$\alpha^2+\beta^2+2\alpha\beta-4\alpha\beta = (\alpha\beta)^2$

$(\alpha+\beta)^2 - 4\alpha\beta = (\alpha\beta)^2$

$$\left(\frac{a+1}{2}\right)^2-4\left(\frac{a-1}{2}\right)=\frac{(a-1)^2}{4}$$

$$\left(\frac{a^2+1+2a}{4}\right)-2(a-1)=\frac{(a^2+1-2a)}{4}$$

$$\left(\frac{a^2+1+2a}{4}\right)-\frac{(a^2+1-2a)}{4}=2(a-1) \quad \textbf{(on simplification)}$$

$a = 2a-2$

2a-a = 2

a=2

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6. Find the condition that one of the roots of $ax^2 + bx + c$ may be (i) negative of the other, (ii) thrice the other, (iii) reciprocal of the other.

Let α and β are the roots of $ax^2+bx+c=0$

Sum = $-\frac{b}{a}$ Product = $\frac{c}{a}$

(i) One root is negative of other

Let the roots be α and -α

Sum of the roots = α-α = $-\frac{b}{a}$

$0 = -\frac{b}{a} \Rightarrow -b = 0 \Rightarrow b=0$

This is the required condition

(ii) One root is thrice the other

Let one root be α

The other root is 3α (∴ thrice the other)

Sum of the roots = α + 3α = $-\frac{b}{a}$

$4\alpha = -\frac{b}{a} \Rightarrow \alpha = -\frac{b}{4a}$ ________(1)

But Product of the roots =(α) (3α) = $\frac{c}{a}$

$3\alpha^2 = \frac{c}{a}$ -----------------(2)

From (1) and (2) we should remove α

Substitute (1) in (2) we get

$$\frac{c}{a} = 3\left(-\frac{b}{4a}\right)^2$$

$\frac{c}{a} = 3\left(-\frac{b^2}{16a^2}\right)$ (By cancelling a in the denominator and cross multiply)

We get $3b^2 = 16ac$ This is the required condition

(iii) Roots are reciprocal of the other

Let one root be α

Other root be $\frac{1}{\alpha}$

The Product of the roots = $(\alpha)\left(\frac{1}{\alpha}\right) = \frac{c}{a}$

By cancelling α we get $1 = \frac{c}{a}$

Hence the required condition is c = a

---

7. If the equations $x^2 - ax + b = 0$ and $x^2 - ex + f = 0$ have one root in common and if the second equation has equal roots, then prove that $ae = 2(b + f)$.

Let α and β be the two roots

Sum of the roots = α+β = a --------(1) Product of the roots = αβ = b --(2)
(for the 1st equation)

Sum of the roots = α+β = e ---------(3) Product of the roots = αβ = f ------------(4)

Since the roots are equal β = α

(2) ⇒ α+α = e ⇒ 2α=e (4) ⇒ α^2 = f

LHS = ae = (α+β) 2α= 2α^2 +2αβ

= 2 (α^2+αβ)

= 2 (f+ b) = RHS

---

8. Discuss the nature of roots of (i) $-x^2 + 3x + 1 = 0$, (ii) $4x^2 - x - 2 = 0$, (iii) $9x^2 + 5x = 0$.

(i) $-x^2+3x+1=0$
a= -1 b = 3 c=1
Discriminant = b^2-4ac
= $(3)^2-4(-1)(1)$ = 9+4 =13 >= the roots are real and equal

(ii) $4x^2-x-2=0$
a= 4 b= -1 c =-2
Discriminant = b^2-4ac
= $(-1)^2$ -4 (4) (-2) =1+32 = 33 >0 the roots are real and unequal

(iii) $9x^2+5x=0$
a= 9 b= 5 c= 0
Discriminant = b^2-4ac
= $(5)^2$- (4) (9) (0) = 25 the roots are real ,unequal and rational

---

9. Without sketching the graphs, find whether the graphs of the following functions will intersect the x-axis and if so in how many points.
(i) $y = x^2 + x + 2$, (ii) $y = x^2 - 3x - 7$, (iii) $y = x^2 + 6x + 9$.

(i) $y= x^2+x+2$
a=1 b=1 c= 2
D = 1-8 =-7 <0
The graph will not intersect the x axis

(ii) $y = x^2-3x-7$

$a= 1 \quad b= -3 \quad c= -7$

$D = 9 -4(1)(-7) = 9+28=37 >0$

The graph will intersect the x axis at two points

(iii) $y= x^2+6x+9=0$

$a=1 \quad b=6 \quad c= 9$

$D = 36-36 =0$

The graph will meets the x axis at only one point

10. Write $f(x) = x^2 + 5x + 4$ in completed square form.

x^2+5x+4 **Add and subtract** $\left(\frac{5}{2}\right)^2$

$$= x^2+5x+\left(\frac{5}{2}\right)^2-\left(\frac{5}{2}\right)^2+4$$

$$\Rightarrow \left(x+\frac{5}{2}\right)^2+4-\frac{25}{4}$$

$$\Rightarrow \left(x+\frac{5}{2}\right)^2+\left(\frac{16-25}{4}\right)$$

$$\Rightarrow \left(x+\frac{5}{2}\right)^2+\left(\frac{-9}{4}\right)=\left(x+\frac{5}{2}\right)^2+\left(\frac{3}{2}\right)^2$$ **(completed square form)**

Quadratic inequalities

Example 13

Solve $3x^2+5x-2 \leq 0$

$3x^2+5x-2 =0$

$(3x-1)(x+2) =0$

$\{ 1/3, -2 \}$

$x \in [-2, 1/3]$

EXAMPLE 14

Solve $\sqrt{x+14} < x+2$

Squaring on both sides

$(x+14) < (x+2)^2$

$x+14 < x^2+4+4x$

$x^2+3x-10 >0$

$(x+5)(x-2) > 0$

The value of $x \in (-\infty,-5) \cup (2,\infty)$

EXAMPLE 15

Solve the equation $\sqrt{6-4x-x^2} = x+4$

Squaring on both sides

$6-4x-x^2 = (x+4)^2$

$6 - 4x - x^2 = x^2+16+8x$

$2x^2+12x+10 = 0$

Divide by 2 $x^2+6x+5=0$

$(x+5)(x+1) = 0$

$X = -5 \quad x = -1$

Here $(x+4) \geq 0$ only if $x = -1$ Hence $x = -1$

Exercise 2.5

Solve $2x^2+x-15 \leq 0$

$2x^2+6x-5x-15 \leq 0$

$2x(x+3) - 5(x+3) \leq 0$

$(2x-5)(x+3) \leq 0$

The value of x lies between $x = -3$ and $x = \frac{5}{2}$

$[-3, \frac{5}{2}]$

Solve : $-x^2 + 3x - 2 \geq 0$

When multiply with (-) the inequality get reversed

$x^2-3x+2 \leq 0$

$(x-2)(x-1) \leq 0$

The value of x lies between $x = 2$ and $x = 1$

$x \in [1, 2\}$

EXERCISE 2.6

1. Find the zeros of the polynomial function $f(x) = 4x^2 - 25$.

f(x) =0

$4x^2 - 25 = 0$ (OR) $4X^2 = 25$

$(2x-5)(2x-5) = 0$ $x^2 = \frac{25}{4}$

$x = \frac{5}{2}$ and $x = \frac{5}{2}$ $x = \pm\frac{5}{2}$

2. If $x = -2$ is one root of $x^3 - x^2 - 17x = 22$, then find the other roots of equation.

If x = -2 is one root

Use Synthetic division method

-2	1	-1	-17	-22
	0	-2	6	22
	1	-3	-11	0

$x^2 - 3x - 11 = 0$

$$X = \frac{3 \pm \sqrt{9 - 4(1)(-11)}}{2} = \frac{3 \pm \sqrt{53}}{2}$$

The solution $x = -2$ $x = \frac{3+\sqrt{53}}{2}$ $x = \frac{3-\sqrt{53}}{2}$

---

3. Find the real roots of $x^4 = 16$.

$x^4 - 16 = 0$

$(x^2+4)(x^2-4) = 0$

$x^2 = -4$ $x^2 = 4$

$x = \pm\sqrt{-4} = \pm 2i$ $x = \pm 2$

---

4. Solve $(2x+1)^2 - (3x+2)^2 = 0$.

Use $(a+b)(a-b) = (a^2 - b^2)$

$(2x+1+3x+2)\ (2x+1+-3x-2) = 0$

$(5x+3)(-x-1) = 0$

$X = -\frac{3}{5}$ $x = -1$

Example 16

Find a quadratic polynomial f(x) such that f(0) =1 , f(-2) =0 and f(1)=0

Let the quadratic polynomial f(x) = ax^2+bx+c

$f(0) = 0+0+c = 1 \quad \Rightarrow \quad c = 1$

$f(-2) = a(-2)^2+b(-2)+c = 0$

$\Rightarrow 4a - 2b + c = 0$ ---------------------(1)

Substitute c = 1 in (1) $\Rightarrow \quad 4a - 2b + 1 = 0$

$4a - 2b = -1$ ------------- (2)

$f(1) = 0$

$f(1) = a(1)^2 + b(1) + c = 0$

$a + b + 1 = 0$

$a + b = -1$ ______________ (3)

Solving (2) and (3)

We get $x = -\frac{1}{2} \qquad x = -\frac{1}{2}$

So we have the quadratic polynomial

$f(x) = -\frac{1}{2}x^2 - \frac{1}{2}x + 1$

Another method

$x = -2$ and $x = 1$

$x+2=0$ and $x-1=0$

$f(x) = a(x+2)(x-1)$

$f(0) = 1$

$-2a = 1 \quad a = ½$

$f(x) = ½ (x+2)(x-1) = -\frac{1}{2}x^2 - \frac{1}{2}x + 1$

Construct a cubic polynomial function having zeros at x= $\frac{2}{5}$ **, f(0) = -8**

If $1+\sqrt{3}$ **is a one root**

Then the other root is $1-\sqrt{3}$

sum of the roots = $1+\sqrt{3} + 1-\sqrt{3} = 2$

Product of the roots = $(1+\sqrt{3})(1-\sqrt{3}) = 1-3 = -2$

One factor is $x^2 -$ x(sum) + Product

$x^2 -2x-2$

The cubic polynomial will be

$$f(x) = a\left(x-\frac{2}{5}\right)(x^2-2x-2)$$

$$f(0) = a\left(-\frac{2}{5}\right)(-2) = -8 \Rightarrow a = 10$$

Hence the required polynomial is $f(x) = (-10)\left(x-\frac{2}{5}\right)(x^2-2x-2) = -10x^3+24x+12x-8$
