2.COMPLEX NUMBERS

POWERS OF IMAGINARY UNIT

- $i^2 = -1, i^3 = -i, i^4 = 1$
- Division algorithm : $n = 4(q) + r$
- $i^n = (i)^{4q+r} = (i)^{4q}(i)^r = (i^4)^q(i)^r \Rightarrow if\ r = 0, i^n = 1\ ; if\ r = 1\ , i^n = i;$ $if\ r = 2, i^n = -1; if\ r = 3, i^n = -i$
- $\sqrt{ab} = \sqrt{a}\sqrt{b}$ is valid only if at least one of a, b is non-negative.
- For $y \in R, y^2 > 0$

EXERCISE 2.1

Simplify the following:

1.$i^{1947} + i^{1950}$

$1947 \div 4 = 4(486) + 3\ ; 1950 = 4(487) + 2$

$i^{1947} + i^{1950} = (i^4)^{486}(i)^3 + (i^4)^{487}(i)^2 = -i - 1 = -1 - i$

2.$i^{1948} - i^{-1869}$

$1948 \div 4 = 4(487) + 0\ ; 1869 \div 4 = 4(467) + 1$

$i^{1948} - i^{-1869} = i^{1948} - \frac{1}{i^{1869}} = (i^4)^{487}(i)^0 - \frac{1}{(i^4)^{467}i^1} = 1 - \frac{1}{i} = 1 - \left(\frac{1}{i} \times \frac{-i}{-i}\right)$

$$= 1 - \left(\frac{-i}{-i^2}\right) = 1 + i \quad (\because -i^2 = 1)$$

3.$\sum_{n=1}^{12} i^n$

$\sum_{n=1}^{12} i^n = i^1 + i^2 + i^3 + i^4 + i^5 + i^6 + i^7 + +i^8 + i^9 + i^{10} + i^{11} + i^{12}$

$= i^1 + i^2 + i^3 + i^4 + i^4 i + (i^2)^3 + (i^2)^3 i + (i^2)^4 + (i^3)^3 + (i^2)^5 + (i^2)^5 i + (i^2)^6$

$= i - 1 - i + 1 + i - 1 - i + 1 + i - 1 - i + 1$

$= 0$

4.$i^{59} + \frac{1}{i^{59}}$

$59 \div 4 = 4(14) + 3$

$i^{59} + \frac{1}{i^{59}} = (i^4)^{14}i^3 + \frac{1}{(i^4)^{14}i^3}$

$= -i + \frac{1}{-i} = i + \frac{1}{-i}\frac{i}{i}$

$= -i + \frac{i}{1} = i - i$

$= 0$

5.$i.\, i^2.\, i^3.\, i^4 \ldots\ldots i^{2000}$

$= i^{(1+2+3 \ldots\ldots +2000)}$

$= i^{\frac{2000\times 2001}{2}}$

$= i^{1000\times 2001}$

$= i^{1000}.\, i^{2001}$

$= (i^4)^{250}(i^4)^{500}i^1 \qquad (\because 2001 \div 4 = 4(500) + 1)$

$= 1.1.\, i$

$= i$

6.$\sum_{n=1}^{10} i^{n+50}$

$\sum_{n=1}^{10} i^{n+50} = i^{51} + i^{52} + i^{53} + \cdots + i^{60}$

$= i^{50}(i^1 + i^2 + i^3 + i^4 + i^5 + i^6 + i^7 + i^8 + i^9 + i^{10})$

$= (i^4)^{12} i^2 (i - 1 - i + 1 + i - 1 - i + 1 + i - 1)$

$= -1(i - 1)$

$= 1 - i$

EXAMPLE 2.1 Simplify the following:

(i) i^7 (ii) i^{1729} (iii) $i^{-1924} + i^{2018}$ (iv) $\sum_{n=1}^{102} i^n$ (v) $i.\, i^2.\, i^3.\, i^4 \ldots\ldots i^{40}$

COMPLEX NUMBERS

- **Rectangular Form Of a Complex Number:** A complex number is of the form $z = x + iy, x, y \in R$. x is called the real part and y is called the imaginary part of the complex number.
- Two complex numbers $z_1 = a + ib, z_2 = c + id$ are said to be equal if and only if $Re(z_1) = Re(z_2)$ and $Im(z_1) = Im(z_2)$. ie. $a = c\ and\ b = d$
- **Scalar multiplication of complex numbers:** $If\ z = x + iy\ and\ k \in R\ then\ kz = k(x + iy) = kx + kyi$
- **Addition of complex numbers:** If $z_1 = a + ib, z_2 = c + id, then\ z_1 + z_2 = (a + c) + i(b + d)$
- **Subtraction of complex numbers**: If $z_1 = a + ib, z_2 = c + id, then\ z_1 - z_2 = (a - c) + i(b - d)$
- **Multiplication of complex numbers:** If $z_1 = a + ib, z_2 = c + id, then$ $z_1 z_2 = (ac - bd) + i(ad + bc)$
- Multiplication of a complex z by i successively gives a 90^0 counter clockwise rotation successively about the origin.

EXERCISE 2.2

1. Evaluate the following if $z = 5 - 2i$ and $w = -1 + 3i$

$(i) z + w\ \ (ii) z - iw\ \ (iii) 2z + 3w\ \ (iv) zw\ \ (v) z^2 + 2zw + w^2\ \ (vi)(z + w)^2$

$(i) z+w = 5-2i-1+3i = 4+i$

$(ii) z-iw = 5-2i-i(-1+3i) = 5-2i+i-3i^2 = 5-2i+i+3 = 8-i$

$\because i^2 = -1$

$(iii) 2z+3w = 2(5-2i) + 3(-1+3i) = 10-4i-3+9i = 7+5i$

$(iv)\ z_1z_2 = (ac-bd)+i(ad+bc); a=5, b=-2, c=-1, d=3$

$zw = (5-2i)(-1+3i) = [(5)(-1)-(-2)(3)] + i[(5)(3)+(-2)(-1)]$

$= [-5+6] + i[15+2] = 1+17i$

$(v) z^2 = (5-2i)^2 = 25-20i+4i^2 = 25-20i-4 = 21-20i$

$2zw = 2(5-2i)(-1+3i) = 2[1+17i]$ from (iv)

$2zw = 2+34i$

$w^2 = (-1+3i)^2 = 1-6i+9i^2 = 1-6i-9 = -8-6i$

$z^2+2zw+w^2 = 21-20i+2+34i-8-6i = 15+8i$

$(vi)(z+w)^2 = (4+i)^2$ from(i)

$= 16+8i+i^2 = 16+8i-1 = 15+8i$

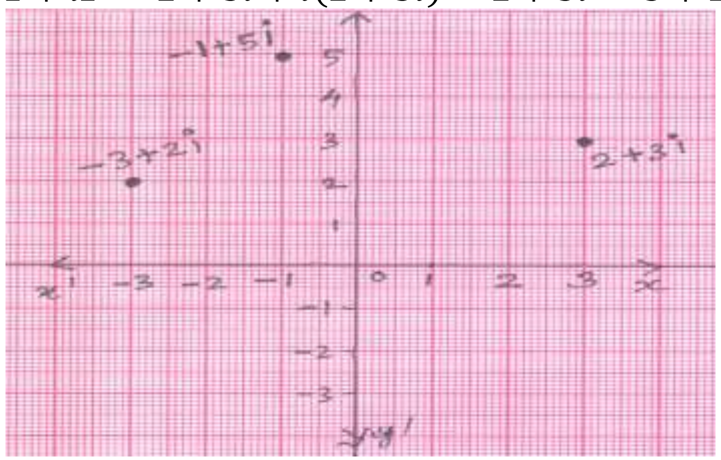
2. Given the complex number $z = 2+3i$, represent the complex numbers in Argand diagram.

$(i) z, iz \text{ and } z+iz \quad (ii) z, -iz \text{ and } z-iz$

$(i) z = 2+3i = (2,3)$

$iz = i(2+3i) = 2i+3i^2 = -3+2i = (-3,2)$

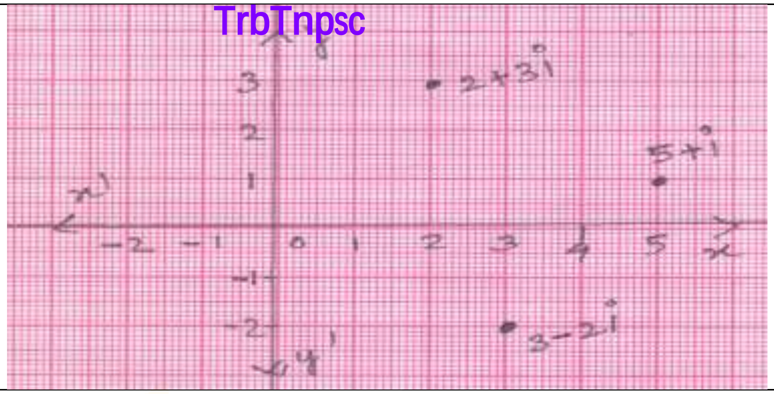
$z+iz = 2+3i+i(2+3i) = 2+3i-3+2i = -1+5i = (-1,5)$



$(i) z = 2+3i = (2,3)$

$-iz = -i(2+3i) = -2i-3i^2 = 3-2i = (3,-2)$

$z-iz = 2+3i-i(2+3i) = 2+3i+3-2i = 5+i = (5,1)$



3. Find the values of the real numbers x and y, if the complex numbers $(3-i)x-(2-i)y+2i+5$ and $2x+(-1+2i)y+3+2i$ are equal.

$z_1 = (3-i)x-(2-i)y+2i+5$

$= 3x-xi-2y+yi+2i+5$

$= (3x-2y+5)+i(-x+y+2)$

$z_2 = 2x+(-1+2i)y+3+2i$

$= 2x-y+2yi+3+2i$

$= (2x-y+3)+i(2y+2)$

Given : $z_1 = z_2 \Longrightarrow (3x-2y+5)+i(-x+y+2) = (2x-y+3)+i(2y+2)$

Equating the real and imaginary parts on both sides,

$3x-2y+5 = 2x-y+3$	$-x+y+2 = 2y+2$
$3x-2y+5-2x+y-3 = 0$	$-x+y+2-2y-2 = 0$
$x-y = -2 \to (1)$	$-x-y = 0 \to (2)$

$(1) + (2) \Longrightarrow -2y = -2 \Longrightarrow \boxed{y=1}$

Sub. $y = -1$ in $(1) \Longrightarrow x-1 = -2 \Longrightarrow \boxed{x=-1}$

EXAMPLE 2.2 Find the values of the real numbers x and y, if the complex numbers $(2+i)x+(1-i)y+2i-3$ and $x+(-1+2i)y+1+i$ are equal. $[Ans: x = 2 \text{ and } y = 1]$

PROPERTIES OF COMPLEX NUMBERS

Properties of complex numbers under addition:

I. Closure property: $\forall z_1, z_2 \in \mathbb{C}, (z_1+z_2) \in \mathbb{C}$

II. Commutative property: $\forall z_1, z_2 \in \mathbb{C}, z_1+z_2 = z_2+z_1$

III. Associative property: $\forall z_1, z_2, z_3 \in \mathbb{C}, (z_1+z_2)+z_3 = z_1+(z_2+z_3)$

IV. Additive identity: $\forall z \in \mathbb{C}, z+0 = 0+z = z \Longrightarrow 0 = 0+0i$ is an additive identity.

V. Additive inverse: $\forall z \in \mathbb{C}, -z \in \mathbb{C} \ni z + (-z) = (-z) + z = 0 \in \mathbb{C}, so -z$ is an additive identity.

Properties of complex numbers under multiplication:

I. Closure property: $\forall z_1, z_2 \in \mathbb{C}, (z_1.z_2) \in \mathbb{C}$

II. Commutative property: $\forall z_1, z_2 \in \mathbb{C}, z_1.z_2 = z_2.z_1$

III. Associative property: $\forall z_1, z_2, z_3 \in \mathbb{C}, (z_1.z_2).z_3 = z_1.(z_2.z_3)$

IV. Multiplicative identity: $\forall z \in \mathbb{C}, z.1 = 1.z = z \Longrightarrow 1 = 1 + 0i$ is a multiplicative identity.

V. Multiplicative inverse: $\forall z \in \mathbb{C} \ \exists \frac{1}{z} z \in \mathbb{C} \ni z.\frac{1}{z} = \frac{1}{z}.z = 1 + i0 \in \mathbb{C}, so \ \frac{1}{z}$ is multiplicative identity.

- If $z = x + iy$ then its multiplicative inverse is $z^{-1} = \left(\frac{x}{x^2+y^2}\right) + i\left(-\frac{y}{x^2+y^2}\right)$
- Distributive property: $\forall z_1, z_2, z_3 \in \mathbb{C}, (z_1 + z_2).z_3 = z_1z_3 + z_2z_3$ (or)
  - $\forall z_1, z_2, z_3 \in \mathbb{C}, z_1.(z_2 + z_3) = z_1z_2 + z_1z_3$
- Complex numbers obey the laws of indices:

$(i) z^m z^n = z^{m+n}$ $(ii) \frac{z^m}{z^n} = z^{m-n}$

$(iii) (z^m)^n = z^m z^n$ $(iv) (z_1z_2)^m = z_1{}^m z_2{}^m$

EXERCISE 2.3

1.If $z_1 = 1 - 3i, z_2 = -4i$ and $z_3 = 5$, show that

$(i)(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$ $(ii)(z_1z_2)z_3 = z_1(z_2z_2)$

$(i)(z_1 + z_2) + z_3 = (1 - 3i - 4i) + 5 = 1 - 7i + 5 = 6 - 7i \to (1)$

$z_1 + (z_2 + z_3) = 1 - 3i + (-4i + 5) = 1 - 3i - 4i + 5 = 6 - 7i \to (2)$

From (1) & (2), $(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$

$(ii)(z_1z_2)z_3 = [(1 - 3i)(-4i)]5 = (-4i + 12i^2)5$

$= (-4i - 12)5 = -60 - 20i \to (1)$

$z_1(z_2z_2) = (1 - 3i)[(-4i)(5)] = (1 - 3i)[-20i]$

$= -20i + 60i^2 = -60 - 20i \to (2)$

From (1) & (2), $(z_1z_2)z_3 = z_1(z_2z_2)$

2.If $z_1 = 3, z_2 = -7i$ and $z_3 = 5 + 4i$, show that

$(i) z_1.(z_2 + z_3) = z_1z_2 + z_1z_3$ $(ii)(z_1 + z_2).z_3 = z_1z_3 + z_2z_3$

$(i) z_1.(z_2 + z_3) = 3(-7i + 5 + 4i) = 3(5 - 3i) = 15 - 9i \to (1)$

$z_1z_2 + z_1z_3 = (3)(-7i) + (3)(5 + 4i) = -21i + 15 + 12i = 15 - 9i \to (2)$

From (1) & (2), $z_1.(z_2 + z_3) = z_1z_2 + z_1z_3$

$(ii)(z_1 + z_2).z_3 = (3 - 7i)(5 + 4i)$

$= 15 + 12i - 35i - 28i^2 = 15 + 12i - 35i + 28 = 43 - 23i \to (1)$

$z_1z_3 + z_2z_3 = (3)(5 + 4i) + (-7i)(5 + 4i)$

$= 15 + 12i - 35i - 28i^2 = 15 + 12i - 35i + 28 = 43 - 23i \to (2)$

From (1) & (2), $(z_1 + z_2).z_3 = z_1z_3 + z_2z_3$

3.If $z_1 = 2 + 5i, z_2 = -3 - 4i$ and $z_3 = 1 + i$, find the additive and multiplicative inverse of z_1, z_2 and z_3

Additive inverse of z is $-z$.

So, Additive inverse of $z_1 = 2 + 5i$ is $-z_1 = -2 - 5i$,

Additive inverse of $z_2 = -3 - 4i$ is- $z_2 = 3 + 4i$

Additive inverse of $z_3 = 1 + i$ is $-z_3 = -1 - i$

Multiplicative inverse of $\mathrm{z = x + iy}$ is $\mathrm{z^{-1}} = \left(\frac{x}{x^2+y^2}\right) + i\left(-\frac{y}{x^2+y^2}\right)$

Multiplicative inverse of $z_1 = 2 + 5i$ is $z_1{}^{-1} = \left(\frac{2}{2^2+5^2}\right) + i\left(-\frac{5}{2^2+5^2}\right) = \frac{2}{29} - \frac{5}{29}i$

Multiplicative inverse of $z_2 = -3 - 4i$ is $z_2{}^{-1} = \left(\frac{-3}{(-3)^2+(-4)^2}\right) + i\left(-\frac{-4}{(-3)^2+(-4)^2}\right)$

$= -\frac{3}{25} + \frac{4}{25}i$

Multiplicative inverse of $z_3 = 1 + i$ is $z^{-1} = \left(\frac{1}{1^2+1^2}\right) + i\left(-\frac{1}{1^2+1^2}\right) = \frac{1}{2} - \frac{1}{2}i$

CONJUGATE OF A COMPLEX NUMBER

- The conjugate of the complex number $z = x + iy$ is $\bar{z} = x - iy$.
- $z\bar{z} = x^2 + y^2$
- The conjugate is useful in division of complex numbers.

PROPERTIES OF COMPLEX CONJUGATES:

✓ $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$	✓ $\mathrm{Im}(z) = \frac{z-\bar{z}}{2i}$
✓ $\overline{z_1 - z_2} = \overline{z_1} - \overline{z_2}$	✓ $\overline{(z^n)} = (\bar{z})^n$
✓ $\overline{z_1.z_2} = \overline{z_1}.\overline{z_2}$	✓ z is real if and only if $z = \bar{z}$
✓ $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\overline{z_1}}{\overline{z_2}}, z_2 \neq 0$	✓ z is purely imaginary if and only if $z = -\bar{z}$
✓ $\mathrm{Re}(z) = \frac{z+\bar{z}}{2}$	✓ $\bar{\bar{z}} = z$

EXERCISE 2.4

1.Write the following in the rectangular form:

$(i)\overline{(5+9\imath)+(2-4\imath)}$ $(ii)\frac{10-5i}{6+2i}$ $(iii)\overline{3\imath}+\frac{1}{2-i}$

$(i)\overline{(5+9\imath)+(2-4\imath)}=\overline{5+9\imath}+\overline{2-4\imath}=5-9i+2+4i=7-5i$

$(ii)\frac{10-5i}{6+2i}=\frac{10-5i}{6+2i}\times\frac{6-2i}{6-2i}=\frac{60-20i-30i+10i^2}{6^2+2^2}$

$=\frac{60-50i-10}{40}=\frac{50-50i}{40}=\frac{5}{4}-\frac{5}{4}i$

$(iii)\overline{3\imath}+\frac{1}{2-i}=-3i+\frac{1}{2-i}\times\frac{2+i}{2+i}=-3i+\frac{2+i}{2^2+1^2}=-3i+\frac{2}{5}+\frac{1}{5}i$

$=\frac{2}{5}+\frac{-15i+i}{5}=\frac{2}{5}-\frac{14}{5}i$

2.If $z=x+iy$, find the following in rectangular form:

$(i)Re\left(\frac{1}{z}\right)$ $(ii)Re(i\bar{z})$ $(iii)Im(3z+4\bar{z}-4i)$

$(i)\frac{1}{z}=\frac{1}{x+iy}\times\frac{x-iy}{x-iy}=\frac{x-iy}{x^2+y^2}=\frac{x}{x^2+y^2}-\frac{y}{x^2+y^2}i$

$\therefore Re\left(\frac{1}{z}\right)=Re\left(\frac{x}{x^2+y^2}-\frac{y}{x^2+y^2}i\right)=\frac{x}{x^2+y^2}$

$(ii)i\bar{z}=i(\overline{x+\imath y})=i(x-iy)=ix-i^2y=y+ix$

$\therefore Re(i\bar{z})=Re(-y+ix)=y$

$(iii)3z+4\bar{z}-4i=3(x+iy)+4\overline{x+\imath y}-4i=3x+3yi+4(x-iy)-4i$

$=3x+3yi+4x-4yi-4i=7x-yi-4i=7x+i(-y-4)$

$\therefore Im(3z+4\bar{z}-4i)=Im\big(7x+i(-y-4)\big)=-y-4$

3. If $z_1=2-i$ and $z_2=-4+3i$, find the inverse of z_1z_2 and $\frac{z_1}{z_2}$

$z_1z_2=(2-i)(-4+3i)=-8+6i+4i-3i^2=-8+10i+3=-5+10i$

Inverse of $z_1z_2=\left(\frac{x}{x^2+y^2}\right)+i\left(-\frac{y}{x^2+y^2}\right)=\left(\frac{-5}{(-5)^2+10^2}\right)+i\left(-\frac{10}{(-5)^2+10^2}\right)$

$=\frac{-5}{125}-\frac{10}{125}i=-\frac{1}{25}-\frac{2}{25}i$

$\frac{z_1}{z_2}=\frac{2-i}{-4+3i}=\frac{2-i}{-4+3i}\times\frac{-4-3i}{-4-3i}=\frac{-8-6i+4i+3i^2}{(-4)^2+3^2}=\frac{-8-2i-3}{25}=-\frac{11}{25}-\frac{2}{25}i$

Inverse of $\frac{z_1}{z_2}=\left(\frac{x}{x^2+y^2}\right)+i\left(-\frac{y}{x^2+y^2}\right)$

$=\left(\frac{-11/25}{(-11/25)^2+(-2/25)^2}\right)+i\left(-\frac{-2/25}{(-11/25)^2+(-2/25)^2}\right)$

$=-\frac{11}{25\times\frac{125}{625}}+\frac{2}{25\times\frac{125}{625}}i=-\frac{11}{5}+\frac{2}{5}i$

4. The complex numbers u,v and w are related by $\frac{1}{u}=\frac{1}{v}+\frac{1}{w}$. If $v=3-4i$ and

$w=4+3i$, find u in the rectangular form.

$\frac{1}{u}=\frac{1}{v}+\frac{1}{w}\Longrightarrow\frac{1}{u}=\frac{1}{3-4i}+\frac{1}{4+3i}$

$=\left(\frac{1}{3-4i}\times\frac{3+4i}{3+4i}\right)+\left(\frac{1}{4+3i}\times\frac{4-3i}{4-3i}\right)$

$=\frac{3+4i}{3^2+4^2}+\frac{4-3i}{4^2+3^2}$

$=\frac{1}{25}(3+4i+4-3i)$

$\Longrightarrow\frac{1}{u}=\frac{7+i}{25}$

$u=\frac{25}{7+i}=\frac{25}{7+i}\times\frac{7-i}{7-i}=\frac{25(7-i)}{7^2+1^2}=\frac{25(7-i)}{50}=\frac{7-i}{2}=\frac{7}{2}-\frac{1}{2}i$

5.Prove the following properties:

(i)z is real if and only if $z=\bar{z}$ $(ii)Re(z)=\frac{z+\bar{z}}{2}$ $(iii)Im(z)=\frac{z-\bar{z}}{2i}$

(i)z is real $\Longrightarrow z=x+0i$

$\bar{z}=\overline{x+0\imath}=x-0i=x\Longrightarrow$ z is real if and only if $z=\bar{z}$

(ii) Let $z=x+iy$; $Re(z)=x\to(1)$

$\Longrightarrow\bar{z}=x-iy$

$z+\bar{z}=x+iy+x-iy=2x\Longrightarrow\frac{z+\bar{z}}{2}=x\to(2)$

From (1) & (2), $Re(z)=\frac{z+\bar{z}}{2}$

(iii) Let $z=x+iy$; $Im(z)=y\to(1)$

$\Longrightarrow\bar{z}=x-iy$

$z-\bar{z}=x+iy-x+iy=2yi\Longrightarrow\frac{z-\bar{z}}{2i}=y\to(2)$

From (1) & (2), $Im(z)=\frac{z-\bar{z}}{2i}$

6.Find the least value of the positive integer n for which $\left(\sqrt{3}+i\right)^n$

(i)real (ii)purely imaginary

$\left(\sqrt{3}+i\right)^3=3\sqrt{3}+9i-3\sqrt{3}-i=8i$

$\left(\sqrt{3}+i\right)^6=\left(\left(\sqrt{3}+i\right)^3\right)^2=(8i)^2=-64$

(i)When $n=6,\left(\sqrt{3}+i\right)^n$ is real

(ii)When $n=3,\left(\sqrt{3}+i\right)^n$ is purely imaginary

7.Show that $(i)\left(2+i\sqrt{3}\right)^{10}-\left(2-i\sqrt{3}\right)^{10}$ is purely imaginary.

$(ii)\left(\frac{19-7i}{9+i}\right)^{12}+\left(\frac{20-5i}{7-6i}\right)^{12}$ is real.

$(i)\ z=\left(2+i\sqrt{3}\right)^{10}-\left(2-i\sqrt{3}\right)^{10}$

$\bar{z}=\overline{\left(2+i\sqrt{3}\right)^{10}-\left(2-i\sqrt{3}\right)^{10}}=\overline{\left(2+i\sqrt{3}\right)^{10}}-\overline{\left(2-i\sqrt{3}\right)^{10}}$

$=\left(\overline{2+i\sqrt{3}}\right)^{10}-\left(\overline{2-i\sqrt{3}}\right)^{10}=\left(2-i\sqrt{3}\right)^{10}-\left(2+i\sqrt{3}\right)^{10}$

$=-\left\{\left(2+i\sqrt{3}\right)^{10}-\left(2-i\sqrt{3}\right)^{10}\right\}=-z$

$\therefore\left(2+i\sqrt{3}\right)^{10}-\left(2-i\sqrt{3}\right)^{10}$ is purely imaginary.

$(ii)\ \frac{19-7i}{9+i}=\frac{19-7i}{9+i}\times\frac{9-i}{9-i}=\frac{171-19i-63i+7i^2}{9^2+1^2}=\frac{164-82i}{82}=\frac{82(2-i)}{82}=2-i$

$\frac{20-5i}{7-6i}=\frac{20-5i}{7-6i}\times\frac{7+6i}{7+6i}=\frac{140+120i-35i-30i^2}{7^2+6^2}=\frac{170+85i}{85}=2+i$

$z=\left(\frac{19-7i}{9+i}\right)^{12}+\left(\frac{20-5i}{7-6i}\right)^{12}=(2-i)^{12}-(2+i)^{12}$

$\bar{z}=\overline{(2-i)^{12}-(2+i)^{12}}=\overline{(2-i)^{12}}-\overline{(2+i)^{12}}=(2+i)^{12}-(2-i)^{12}=z$

$\therefore\left(\frac{19-7i}{9+i}\right)^{12}+\left(\frac{20-5i}{7-6i}\right)^{12}$ is real.

EXAMPLE 2.8 Show that $(i)\left(2+i\sqrt{3}\right)^{10}+\left(2-i\sqrt{3}\right)^{10}$ is real and $(ii)\left(\frac{19+9ii}{5-3i}\right)^{15}-\left(\frac{8+i}{1+2i}\right)^{15}$ is purely imaginary.

EXAMPLE 2.3 Write $\frac{3+4i}{5-12i}$ in the x + iy form , hence find its real and imaginary parts.

$\frac{3+4i}{5-12i}=\frac{3+4i}{5-12i}\times\frac{5+12i}{5+12i}=\frac{15+36i+20i-48}{5^2+12^2}=\frac{-33+56i}{169}=-\frac{33}{169}+\frac{56}{169}i$

Real part $=-\frac{33}{169}$ and imaginary part$=\frac{56}{169}$

EXAMPLE 2.4 Simplify $\left(\frac{1+i}{1-i}\right)^3-\left(\frac{1-i}{1+i}\right)^3$

$\frac{1+i}{1-i}=\frac{1+i}{1-i}\times\frac{1+i}{1+i}=\frac{1+i+i+i^2}{1^2+1^2}=\frac{1+2i-1}{2}=\frac{2i}{2}=i$

$\frac{1-i}{1+i}=\left(\frac{1+i}{1-i}\right)^{-1}=-i$

$\left(\frac{1+i}{1-i}\right)^3-\left(\frac{1-i}{1+i}\right)^3=(i)^3-(-i)^3=-i-i=-2i$

EXAMPLE 2.5 If $\frac{z+3}{z-5i}=\frac{1+4i}{2}$, find the complex number z.

$\frac{z+3}{z-5i}=\frac{1+4i}{2}$

$\Longrightarrow 2(z+3)=(1+4i)(z-5i)$

$\Longrightarrow 2z+6=z-5i+4zi-20i^2$

$\Longrightarrow 2z-z-4zi=-5i+20-6$

$\Longrightarrow z-4zi=14-5i$

$\Longrightarrow z(1-4i)=14-5i$

$\Longrightarrow z=\frac{14-5i}{1-4i}$

$=\frac{14-5i}{1-4i}\times\frac{1+4i}{1+4i}=\frac{14+56i-5i-20i^2}{1^2+4^2}=\frac{14+51i+20}{17}=\frac{34+51i}{17}=\frac{17(2+3i)}{17}=2+3i$

EXAMPLE 2.6 If $z_1=3-2i$ and $z_2=6+4i$, find $\frac{z_1}{z_2}$.

$\frac{z_1}{z_2}=\frac{3-2i}{6+4i}=\frac{3-2i}{6+4i}\times\frac{6-4i}{6-4i}=\frac{18-12i-12i-8}{6^2+4^2}=\frac{10-24i}{52}=\frac{2(5-12i)}{52}=\frac{5-12i}{26}$

EXAMPLE 2.7 Find z^{-1}, if $z=(2+3i)(1-i)$.

$z=(2+3i)(1-i)=2-2i+3i-3i^2=2+i+3=5+i$

$z^{-1}=\left(\frac{x}{x^2+y^2}\right)+i\left(-\frac{y}{x^2+y^2}\right)=\left(\frac{5}{5^2+1^2}\right)+i\left(-\frac{1}{5^2+1^2}\right)=\frac{5}{26}-\frac{1}{26}i$

MODULUS OF A COMPLEX NUMBER

- If $z=x+iy$, then the modulus of z is $|z|=\sqrt{x^2+y^2}$
- $z\bar{z}=|z|^2$
- **PROPERTIES OF MODULUS OF A COMPLEX NUMBER:**
  - ✓ $|Z|=|\bar{Z}|$
  - ✓ $|z_1+z_2|\le|z_1|+|z_2|$ (Triangle inequality)
  - ✓ $|z_1-z_2|\ge|z_1|-|z_2|$
  - ✓ $|z_1z_2|=|z_1||z_2|$
  - ✓ $\left|\frac{z_1}{z_2}\right|=\frac{|z_1|}{|z_2|}, z_2\ne 0$
  - ✓ $|z^n|=|z|^n$, n is an integer.
  - ✓ $Re(z)\le|z|$
  - ✓ $Im(z)\le|z|$
  - ✓ $\left||z_1|-|z_2|\right|\le|z_1+z_2|\le|z_1|+|z_2|$
- The distance between the two points z_1 and z_2 in the complex plane is $|z_1-z_2|$ (or) $\sqrt{(x_1-x_2)^2+(y_1-y_2)^2}$
- Formula for finding the square root of the complex number $z=a+ib$ is

$\sqrt{a+ib}=\pm\left(\sqrt{\frac{|z|+a}{2}}+i\frac{b}{|b|}\sqrt{\frac{|z|-a}{2}}\right),$

- If b is negative $\frac{b}{|b|}=-1$, x and y have different signs.
- If b is positive $\frac{b}{|b|}=1$, x and y have same signs.

Exercise 2.5

1.Find the modulus of the following complex numbers:

$(i)\frac{2i}{3+4i}\quad (ii)\frac{2-i}{1+i}+\frac{1-2i}{1-i}\quad (iii)(1-i)^{10}\quad (iv)2i(3-4i)(4-3i)$

$$|z|=\sqrt{x^2+y^2}$$

$(i)\left|\frac{2i}{3+4i}\right|=\frac{|2i|}{|3+4i|}=\frac{\sqrt{0^2+2^2}}{\sqrt{3^2+4^2}}=\frac{\sqrt{4}}{\sqrt{25}}=\frac{2}{5}$

$(ii)\left|\frac{2-i}{1+i}+\frac{1-2i}{1-i}\right|=\left|\frac{(1-i)(2-i)+(1+i)(1-2i)}{(1+i)(1-i)}\right|=\left|\frac{2-i-2i+i^2+1-2i+i-2i^2}{1^2+1^2}\right|$

$=\left|\frac{2-i-2i-1+1-2i+2}{2}\right|=\left|\frac{4-4i}{2}\right|=|2-2i|=\sqrt{2^2+2^2}=\sqrt{8}=2\sqrt{2}$

$(iii)|(1-i)^{10}|=|1-i|^{10}=\left(\sqrt{1^2+1^2}\right)^{10}=\sqrt{2}^{10}=2^5=32$

$(iv)|2i(3-4i)(4-3i)|=|2i||3-4i||4-3i|=\sqrt{0^2+2^2}\sqrt{3^2+4^2}\sqrt{4^2+3^2}$
$=\sqrt{4}\sqrt{25}\sqrt{25}=2.5.5=50$

EXAMPLE 2.9 If $z_1=3+4i$, $z_2=5-12i$ and $z_3=6+8i$, find $|z_1|,|z_2|,|z_3|,|z_1+z_2|,|z_2-z_3|$ and $|z_1+z_3|$.

EXAMPLE 2.10 Find the following : $(i)\left|\frac{2+i}{-1+2i}\right|\quad (ii)\left|\overline{(1+i)}(2+3i)(4i-3)\right|$
$(iii)\left|\frac{i(2+i)^3}{(1+i)^2}\right|$

2.For ant two complex numbers z_1 and z_2, such that $|z_1|=|z_2|=1$ and $z_1z_2\neq -1$ then show that $\frac{z_1+z_2}{1+z_1z_2}$ is a real number.

Given $|z_1|=|z_2|=1$

$|z_1|=1\Rightarrow |z_1|^2=1\Rightarrow z_1\bar{z_1}=1\Rightarrow \bar{z_1}=\frac{1}{z_1}$

$lll^{ly}\ \bar{z_2}=\frac{1}{z_2}$

Let $w=\frac{z_1+z_2}{1+z_1z_2}\Rightarrow \bar{w}=\overline{\left(\frac{z_1+z_2}{1+z_1z_2}\right)}=\frac{\overline{z_1}+\overline{z_2}}{1+\overline{z_1z_2}}=\frac{\overline{z_1}+\overline{z_2}}{1+\overline{z_1}\,\overline{z_2}}=\frac{\frac{1}{z_1}+\frac{1}{z_2}}{1+\frac{1}{z_1}\frac{1}{z_2}}=\frac{\frac{z_1+z_2}{z_1z_2}}{\frac{1+z_1z_2}{z_1z_2}}=\frac{z_1+z_2}{1+z_1z_2}=w$

Since $\bar{w}=w, w$ is purely real.

3.Which one of the points $10-8i, 11+6i$ is closest to $1+i$.

Distance between the two points z_1 and $z_2=\sqrt{(x_1-x_2)^2+(y_1-y_2)^2}$

$z=1+i, z_1=10-8i$

Distance between the two points z and $z_1=\sqrt{(1-10)^2+(1+8)^2}$
$=\sqrt{81+81}=\sqrt{162}=\sqrt{2\times 81}=9\sqrt{2}$

$z=1+i, z_2=11+6i$

Distance between the two points z and $z_2=\sqrt{(1-11)^2+(1-6)^2}$
$=\sqrt{100+125}=\sqrt{125}=\sqrt{5\times 25}=5\sqrt{5}$

$\because 5\sqrt{5}<9\sqrt{2}, 11+6i$ is the closest point to $1+i$

EXAMPLE 2.11 Which one of the points $i,-2+i$ and 3 is farthest from the origin?

4.If $|z|=3$, show that $7\leq|z+6-8i|\leq 13$.

Let $z_1=z$ and $z_2=6-8i$

W.K.T, $||z_1|-|z_2||\leq|z_1+z_2|\leq|z_1|+|z_2|$

$\Rightarrow\left|3-\sqrt{6^2+8^2}\right|\leq|z+6-8i|\leq 3+\sqrt{6^2+8^2}$

$\Rightarrow\left|3-\sqrt{100}\right|\leq|z+6-8i|\leq 3+\sqrt{100}$

$\Rightarrow|3-10|\leq|z+6-8i|\leq 3+10$

$\Rightarrow|-7|\leq|z+6-8i|\leq 13$

$\Rightarrow 7\leq|z+6-8i|\leq 13$

EXAMPLE 2.13 If $|z|=2$, show that $3\leq|z+3+4i|\leq 7$.

5. If $|z|=1$, show that $2\leq|z^2-3|\leq 4$.

Let $z_1=z^2$ and $z_2=-3$

W.K.T, $||z_1|-|z_2||\leq|z_1+z_2|\leq|z_1|+|z_2|$

$\Rightarrow||z|^2-|-3||\leq|z^2-3|\leq|z|^2+|-3|$

$\Rightarrow|1^2-3|\leq|z^2-3|\leq 1^2+3$

$\Rightarrow|-2|\leq|z^2-3|\leq 4$

$\Rightarrow 2\leq|z^2-3|\leq 4$

6.If $\left|z-\frac{2}{z}\right|=2$, show that the greatest and least value of $|z|$ are $\sqrt{3}+1$ and $\sqrt{3}-1$ respectively.

$\left|z-\frac{2}{z}\right|=2 \Rightarrow 2=\left|z-\frac{2}{z}\right| \geq \left||z|-\left|\frac{2}{z}\right|\right|$

$\Rightarrow \left||z|-\left|\frac{2}{z}\right|\right| \leq 2 \Rightarrow -2 \leq |z|-\left|\frac{2}{z}\right| \leq 2$ $\boxed{\because |x-a| \leq r \Rightarrow -r \leq x-a \leq r}$

$\Rightarrow -2 \leq |z|-\frac{2}{|z|} \leq 2 \Rightarrow -2 \leq \frac{|z|^2-2}{|z|} \leq 2 \Rightarrow -2|z| \leq |z|^2-2 \leq 2|z| \rightarrow (1)$

From (1) $\Rightarrow -2|z| \leq |z|^2-2$

$\Rightarrow 2|z|-2|z| \leq |z|^2-2+2|z|$

$\Rightarrow 0 \leq |z|^2+2|z|-2$

$\Rightarrow |z|^2+2|z|-2 \geq 0$

$\Rightarrow \left[|z|-\left(\sqrt{3}+1\right)\right]\left[|z|-\left(\sqrt{3}-1\right)\right] \geq 0$

$\Rightarrow |z| \geq \left(\sqrt{3}+1\right)\ ;\ |z| \leq \left(-\sqrt{3}-1\right)$

$\Rightarrow \boxed{|z| \geq \left(\sqrt{3}-1\right) \rightarrow (2)}$

From (1) $\Rightarrow |z|^2-2 \leq 2|z|$

$\Rightarrow |z|^2-2|z|-2 \leq 0$

$\Rightarrow \left[|z|-\left(1+\sqrt{3}\right)\right]\left[|z|-\left(1-\sqrt{3}\right)\right] \leq 0$

$\Rightarrow \left(1-\sqrt{3}\right) \leq |z| \leq \left(1+\sqrt{3}\right)$

$\Rightarrow \boxed{|z| \leq \left(1+\sqrt{3}\right) \rightarrow (3)}$

From (2) & (3), $\sqrt{3}-1 \leq |z| \leq \sqrt{3}+1$

Hence the greatest and least value of $|z|$ are $\sqrt{3}+1$ and $\sqrt{3}-1$ respectively.

7.If z_1, z_2 *and* z_3 are three complex numbers such that $|z_1|=1, |z_2|=2, |z_3|=3$ and $|z_1+z_2+z_3|=1$, show that $|9z_1z_2+4z_1z_3+z_2z_3|=6$.

$|z_1|=1 \Rightarrow |z_1|^2=1 \Rightarrow z_1\bar{z_1}=1 \Rightarrow z_1=\frac{1}{\bar{z_1}}$ $\boxed{\because z\bar{z}=|z|^2}$

$|z_2|=2 \Rightarrow |z_2|^2=4 \Rightarrow z_2\bar{z_2}=4 \Rightarrow z_2=\frac{4}{\bar{z_2}}$

$|z_3|=3 \Rightarrow |z_3|^2=9 \Rightarrow z_3\bar{z_3}=9 \Rightarrow z_3=\frac{9}{\bar{z_3}}$

$z_1+z_2+z_3=\frac{1}{\bar{z_1}}+\frac{4}{\bar{z_2}}+\frac{9}{\bar{z_3}}$

$\Rightarrow z_1+z_2+z_3=\frac{\overline{z_2z_3}+4\overline{z_1z_3}+9\overline{z_1z_2}}{\overline{z_1z_2z_3}}$

$\Rightarrow |z_1+z_2+z_3|=\left|\frac{\overline{z_2z_3}+4\overline{z_1z_3}+9\overline{z_1z_2}}{\overline{z_1z_2z_3}}\right|$

$\Rightarrow |z_1+z_2+z_3|=\frac{|\overline{\overline{z_2z_3}+4\overline{z_1}\bar{z_3}+9\overline{z_1z_2}}|}{|\overline{\overline{z_1z_2z_3}}|}$

$\Rightarrow |z_1+z_2+z_3|=\frac{|9z_1z_2+4z_1z_3+z_2z_3|}{|z_1z_2z_3|}$ $\boxed{\because \overline{\overline{z}}=z}$

$\Rightarrow |z_1+z_2+z_3|=\frac{|9z_1z_2+4z_1z_3+z_2z_3|}{|z_1||z_2||z_3|}$ $\boxed{\because |z_1z_2|=|z_1||z_2|}$

$\Rightarrow 1=\frac{|9z_1z_2+4z_1z_3+z_2z_3|}{1.2.3}$

$\Rightarrow |9z_1z_2+4z_1z_3+z_2z_3|=6$

EXAMPLE 2.12 If z_1, z_2 *and* z_3 are three complex numbers such that $|z_1|=|z_2|=|z_3|=|z_1+z_2+z_3|=1$, find the value of $\left|\frac{1}{z_1}+\frac{1}{z_2}+\frac{1}{z_3}\right|$.

EXAMPLE 2.15 If z_1, z_2 *and* z_3 are three complex numbers such that $|z_1|=|z_2|=|z_3|=r>0$ and $z_1+z_2+z_3 \neq 0$. Prove that $\left|\frac{z_1z_2+z_2z_3+z_3z_1}{z_1+z_2+z_3}\right|=r$

$|z_1|=r \Rightarrow |z_1|^2=r^2 \Rightarrow z_1\bar{z_1}=r^2 \Rightarrow z_1=\frac{r^2}{\bar{z_1}}$ $\boxed{\because z\bar{z}=|z|^2}$

$|z_2|=r \Rightarrow |z_2|^2=r^2 \Rightarrow z_2\bar{z_2}=r^2 \Rightarrow z_2=\frac{r^2}{\bar{z_2}}$

$|z_3|=r \Rightarrow |z_3|^2=r^2 \Rightarrow z_3\bar{z_3}=r^2 \Rightarrow z_3=\frac{r^2}{\bar{z_3}}$

$z_1+z_2+z_3=\frac{r^2}{\bar{z_1}}+\frac{r^2}{\bar{z_2}}+\frac{r^2}{\bar{z_3}}=r^2\left(\frac{\overline{z_2z_3}+\overline{z_1z_3}+\overline{z_1z_2}}{\overline{z_1z_2z_3}}\right)$

$\Rightarrow |z_1+z_2+z_3|=|r^2|\left|\frac{\overline{z_2z_3}+\overline{z_1z_3}+\overline{z_1z_2}}{\overline{z_1z_2z_3}}\right|=r^2\frac{|\overline{\overline{z_2z_3}+\overline{z_1z_3}+\overline{z_1z_2}}|}{|\overline{\overline{z_1z_2z_3}}|}=r^2\frac{|z_1z_2+z_2z_3+z_3z_1|}{|z_1||z_2||z_3|}$

$\Rightarrow |z_1+z_2+z_3|=r^2\frac{|z_1z_2+z_2z_3+z_3z_1|}{r.r.r} \Rightarrow |z_1+z_2+z_3|=\frac{|z_1z_2+z_2z_3+z_3z_1|}{r}$

$\Rightarrow \left|\frac{z_1z_2+z_2z_3+z_3z_1}{z_1+z_2+z_3}\right|=r$

8.If the area of the triangle formed by the vertices z, iz *and* $z+iz$ is 50 square units, find the value of $|z|$.

Let $z=a+ib=(a,b) \Rightarrow |z|=\sqrt{a^2+b^2}$

Vertices : $z=(a,b)$

$iz=i(a+ib)=ia+i^2b=-b+ia=(-b,a)$

$z+iz=a+ib-b+ia=(a-b)+i(a+b)=(a-b,a+b)$

Area of the triangle = 50 sq. units

$\Longrightarrow \frac{1}{2}\begin{vmatrix} a & -b & a-b & a \\ b & a & a+b & b \end{vmatrix} = 50$

$\Longrightarrow (a^2 - ab - b^2 + ab - b^2) - (-b^2 + a^2 - ab + a^2 + ab) = 50 \times 2$

$\Longrightarrow a^2 - 2b^2 + b^2 - 2a^2 = 100$

$\Longrightarrow -a^2 - b^2 = 100$

$\Longrightarrow -(a^2 + b^2) = \pm 100$

$\Longrightarrow a^2 + b^2 = \mp 100$

$\Longrightarrow \sqrt{a^2 + b^2} = \mp 10$

$\Longrightarrow |z| = 10$ ($\because a^2 + b^2$ is not negative)

NOTE: This can also done by using distance formula, find the length of base and height and use $\boldsymbol{Area\ of\ the\ \Delta = \frac{1}{2}bh}$

EXAMPLE 2.14 Show that the points $1, -\frac{1}{2} + i\frac{\sqrt{3}}{2}$ and $-\frac{1}{2} - i\frac{\sqrt{3}}{2}$ are the vertices of an equilateral triangle.

Let $z_1 = 1, z_2 = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$ and $z_3 = -\frac{1}{2} - i\frac{\sqrt{3}}{2}$

The length of the sides of the triangle are

$|z_1 - z_2| = \left|1 - \left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)\right| = \left|\frac{3}{2} - i\frac{\sqrt{3}}{2}\right| = \sqrt{\left(\frac{3}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} = \sqrt{\frac{9}{4} + \frac{3}{4}} = \sqrt{\frac{12}{4}} = \sqrt{3}$

$|z_2 - z_3| = \left|\left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) - \left(-\frac{1}{2} - i\frac{\sqrt{3}}{2}\right)\right| = \left|-\frac{1}{2} + i\frac{\sqrt{3}}{2} + \frac{1}{2} + i\frac{\sqrt{3}}{2}\right| = \sqrt{\left(\frac{2\sqrt{3}}{2}\right)^2} = \sqrt{3}$

$|z_3 - z_1| = \left|\left(-\frac{1}{2} - i\frac{\sqrt{3}}{2}\right) - 1\right| = \left|-\frac{3}{2} - i\frac{\sqrt{3}}{2}\right| = \sqrt{\left(\frac{3}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} = \sqrt{\frac{9}{4} + \frac{3}{4}} = \sqrt{3}$

Since all the three sides are equal, the given points form an equilateral triangle.

EXAMPLE 2.18 Given the complex number $= 3 + 2i$, represent the complex numbers z, iz and $z + iz$ in one Argand diagram. Show that these complex numbers form the vertices of an isosceles right triangle.

Given : $z = 3 + 2i = A(3,2)$

$iz = i(3 + 2i) = -2 + 3i = B(-2,3)$

$z + iz = 3 + 2i - 2 + 3i = 1 + 5i = C(1,5)$

The length of the sides of the triangle are

$AB = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} = \sqrt{(3+2)^2 + (2-3)^2} = \sqrt{25 + 1} = \sqrt{26}$

$BC = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} = \sqrt{(-2-1)^2 + (3-5)^2} = \sqrt{9 + 4} = \sqrt{13}$

$CA = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} = \sqrt{(1-3)^2 + (5-2)^2} = \sqrt{4 + 9} = \sqrt{13}$

$AB^2 = 26, BC^2 = 13, CA^2 = 13$

Since $(i) BC = CA = \sqrt{13}$ and $(ii) BC^2 + CA^2 = 13 + 13 = 26 = AB^2$, So the given vertices form an isosceles right triangle.

9.Show that the equation $z^3 + 2\bar{z} = 0$ has five solutions.

$z^3 + 2\bar{z} = 0$

$\Longrightarrow z^3 = -2\bar{z}$

$\Longrightarrow |z^3| = |-2||\bar{z}|$

$\Longrightarrow |z^3| = 2|z|$

$\Longrightarrow |z|^3 - 2|z| = 0$

$\Longrightarrow |z|(|z|^2 - 2) = 0 \Longrightarrow \begin{cases} |z| = 0 \Longrightarrow z = 0 \ is\ one\ of\ the\ solution \\ |z|^2 - 2 = 0 \Longrightarrow z\bar{z} - 2 = 0 \Longrightarrow \boxed{\bar{z} = \frac{2}{z}} \end{cases}$

Put $\bar{z} = \frac{2}{z}$ in $z^3 + 2\bar{z} = 0 \Longrightarrow z^3 + 2\left(\frac{2}{z}\right) = 0 \Longrightarrow z^4 + 4 = 0$, which provides four solution.

Hence $z^3 + 2\bar{z} = 0$ has five solutions.

EXAMPLE 2.16 Show that the equation $z^2 = \bar{z}$ has four solutions.

10.Find the square root of $(i) 4 + 3i\ \ (ii) -6 + 8i\ \ (iii) -5 - 12i$.

$$\boxed{\sqrt{a + ib} = \pm\left(\sqrt{\frac{|z| + a}{2}} + i\frac{b}{|b|}\sqrt{\frac{|z| - a}{2}}\right)}$$

$(i) 4 + 3i$

$a = 4, b = 3, |z| = \sqrt{4^2 + 3^2} = \sqrt{25} = 5$

$\sqrt{4 + 3i} = \pm\left(\sqrt{\frac{5+4}{2}} + i\frac{3}{3}\sqrt{\frac{5-4}{2}}\right) = \pm\left(\frac{3}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right)$

$(ii) -6 + 8i$

$a = -6, b = 8, |z| = \sqrt{6^2 + 8^2} = \sqrt{100} = 10$

$\sqrt{-6 + 8i} = \pm\left(\sqrt{\frac{10-6}{2}} + i\frac{8}{8}\sqrt{\frac{10+6}{2}}\right) = \pm(\sqrt{2} + i\sqrt{8}) = \pm(\sqrt{2} + i2\sqrt{2})$

$(iii) -5 - 12i$ 4

$a=-5, b=-12, |z|=\sqrt{5^2+12^2}=\sqrt{169}=13$

$\sqrt{-5-12i}=\pm\left(\sqrt{\frac{13-5}{2}}+i\frac{-12}{12}\sqrt{\frac{13+5}{2}}\right)=\pm(\sqrt{4}-i\sqrt{9})=\pm(2-3i)$

EXAMPLE 2.17 Find the square root of $6+8i$

GEOMETRY AND LOCUS OF COMPLEX NUMBERS

- A circle is defined as the locus of a point which moves in a plane such that its distance from a fixed point in that plane is always a constant. The fixed point is the centre and the constant distant is the radius of the circle.
- Equation of complex form of a circle is $|z-z_0|=r$.
- $|z-z_0|<r$ represents the points interior of the circle.
- $|z-z_0|>r$ represents the points exterior of the circle.
- $x^2+y^2=r^2$ represents a circle with centre at the origin and the radius r units.
- $r=\sqrt{x^2+y^2}$
- If $z=\frac{a+ib}{c+id}$ then $\text{Re}(Z)=\frac{ac+bd}{c^2+d^2}$
- If $z=\frac{a+ib}{c+id}$ then $\text{Im}(Z)=\frac{bc-ad}{c^2+d^2}$

EXERCISE 2.6

1.If $z=x+iy$ is a complex number such that $\left|\frac{z-4i}{z+4i}\right|=1$. Show that the locus of z is real axis.

Given: $z=x+iy$

$\left|\frac{z-4i}{z+4i}\right|=1\Rightarrow|z-4i|=|z+4i|$

$\Rightarrow|x+iy-4i|=|x+iy+4i|$

$\Rightarrow|x+i(y-4)|=|x+i(y+4)|$

$\Rightarrow\sqrt{x^2+(y-4)^2}=\sqrt{x^2+(y-4)^2}$

Squaring on both sides,

$\Rightarrow x^2+(y-4)^2=x^2+(y-4)^2$

$\Rightarrow x^2+y^2-8y+16=x^2+y^2+8y+16$

$\Rightarrow 8y=0$

$\Rightarrow y=0$

∴ The locus of z is real axis.

2. If $z=x+iy$ is a complex number such that $Im\left(\frac{2z+1}{iz+1}\right)=0$, show that the locus of z is $2x^2+2y^2+x-2y=0$.

Given: $z=x+iy$

$\frac{2z+1}{iz+1}=\frac{2(x+iy)+1}{i(x+iy)+1}=\frac{(2x+1)+i2y}{(1-y)+ix}$

$\text{Im}(Z)=\frac{bc-ad}{c^2+d^2}$

Here $a=2x+1, b=2y, c=1-y, d=x$

$Im\left(\frac{2z+1}{iz+1}\right)=0\Rightarrow\frac{2y(1-y)-x(2x+1)}{(1-y)^2+x^2}=0\Rightarrow 2y-2y^2-2x^2-x=0$

$\Rightarrow 2x^2+2y^2+x-2y=0$

3.Obtain the Cartesian form of the locus of $z=x+iy$ in each of the following cases:

$(i)[Re(iz)]^2=3$ $(ii)Im[(1-i)z+1]=0$ $(iii)|z+i|=|z-1|$ $(iv)\bar{z}=z^{-1}$

Given: $z=x+iy$

$(i) iz=i(x+iy)=y-ix$

$[Re(iz)]^2=3$

$\Rightarrow[Re(y-ix)]^2=3$

$\Rightarrow y^2=3$

$(ii)(1-i)z+1=(1-i)(x+iy)+1=x+iy-ix+y+1$

$=(x+y+1)+i(y-x)$

$Im[(1-i)z+1]=0$

$\Rightarrow Im[(x+y+1)+i(y-x)]=0$

$\Rightarrow y-x=0\ (or) x-y=0$

(iii) $|z+i|=|z-1|$

$\Rightarrow|x+iy+i|=|x+iy-1|$

$\Rightarrow|x+i(y+1)|=|(x-1)+iy|$

$\Rightarrow\sqrt{(x)^2+(y+1)^2}=\sqrt{(x-1)^2+(y)^2}$

Squaring on both sides,

$\Rightarrow(x)^2+(y+1)^2=(x-1)^2+(y)^2$

$\Rightarrow x^2+y^2+2y+1=x^2-2x+1+y^2$

$\Rightarrow 2x+2y=0$

$\Rightarrow x+y=0$

$(iv)\bar{z}=z^{-1}\Rightarrow\bar{z}=\frac{1}{z}\Rightarrow z\bar{z}=1\Rightarrow(x+iy)(x-iy)=1\Rightarrow x^2+y^2=1$

4.Show that the following equations represent a circle, and find its centre and radius: $(i)|z-2-i|=3$ $(ii)|2z+2-4i|=2$ $(iii)|3z-6+12i|=8$

$$\boxed{Equation\ of\ the\ circle\ is\ |z-z_0|=r}$$

$(i)|z-2-i|=3 \implies |z-(2+i)|=3$

Centre $=2+i=(2,1)$; Radius $=3\ units$

$(ii)|2z+2-4i|=2 \implies 2\,|z+1-2i|=2 \implies |z-(-1+2i)|=1$

Centre $=-1+2i=(-1,2)$; Radius $=1\ unit$

$(iii)|3z-6+12i|=8 \implies 3|z-2+4i|=8 \implies |z-(2-4i)|=\frac{8}{3}$

Centre $=2-4i=(2,-4)$; Radius $=\frac{8}{3}\ units$

EXAMPLE 2.19 Show that $|3z-5+i|=4$ represent a circle, and find its centre and radius.

5. Obtain the Cartesian form of the locus of $z=x+iy$ in each of the following cases: $(i)|z-4|=16$ $(ii)|z-4|^2-|z-1|^2=16$

Given: $z=x+iy$

$(i)|z-4|=16 \implies |x+iy-4|=16$

$\implies |(x-4)+iy|=16$

$\implies \sqrt{(x-4)^2+y^2}=16$

Squaring on both sides,

$\implies (x-4)^2+y^2=16^2$

$\implies x^2-8x+16+y^2=256$

$\implies x^2+y^2-8x-240=0$ is the locus of z.

$(ii)|z-4|^2-|z-1|^2=16 \implies |x+iy-4|^2-|x+iy-1|^2=16$

$\implies |(x-4)+iy|^2-|(x-1)+iy|^2=16$

$\implies \left(\sqrt{(x-4)^2+y^2}\right)^2-\sqrt{(x-1)^2+y^2}^{\,2}=16$

$\implies (x-4)^2+y^2-[(x-1)^2+y^2]=16$

$\implies x^2-8x+16+y^2-x^2+2x-1-y^2=16$

$\implies -6x-1=0$

$\implies 6x+1=0$ is the locus of z.

EXAMPLE 2.21 Obtain the Cartesian form of the locus of z in each of the following cases: $(i)|z|=|z-i|$ $(ii)|2z-3-i|=3$

EXAMPLE 2.20 Show that $|z+2-i|<2$ represents interior points of a circle. Find its centre and radius.

Consider $|z+2-i|=2 \implies |z-(-2+i)|=2.$

Centre $=-2+i=(-2,1)$; Radius $=2\ units$

$\therefore |z+2-i|<2$ represents all points inside the circle.

POLAR AND EULER FORM OF A COMPLEX NUMBER

- Polar form of $z=x+iy$ is $z=r(\cos\theta+i\sin\theta)$ (or) $z=r cis\theta$
- $Modulus\ r=\sqrt{x^2+y^2}$
- $\alpha=tan^{-1}\left(\frac{y}{x}\right)$
- $\arg z=\theta$; $\begin{cases}\theta=\alpha\ ; if\,(+,+)\ \theta\ lies\ in\ the\ Ist\ quadrant\\ \theta=\pi-\alpha\ ; if\,(-,+)\ \theta\ lies\ in\ the\ 2nd\ quadrant\\ \theta=-\pi+\alpha\ ; if\,(-,-)\ \theta\ lies\ in\ the\ 3rd\ quadrant\\ \theta=-\alpha\ ; if\,(+,-)\ \theta\ lies\ in\ the\ 4th\ quadrant\end{cases}$
- In general $\arg z=Arg\ z+2n\pi\ ,n\in z.$
- Properties of arguments:
  - ✓ $arg(z_1z_2)=arg(z_1)+arg(z_2)$
  - ✓ $arg\left(\frac{z_1}{z_2}\right)=arg(z_1)-arg(z_2)$
  - ✓ $arg(z^n)=n\ argz$
- Some of the principal argument and argument:

z	1	i	-1	$-i$
$Arg\ z$	0	$\frac{\pi}{2}$	π	$-\frac{\pi}{2}$
$argz$	$2n\pi$	$2n\pi+\frac{\pi}{2}$	$2n\pi+\pi$	$2n\pi-\frac{\pi}{2}$

- Euler's form of the complex number: $z=re^{i\theta}$; $e^{i\theta}=\cos\theta+i\sin\theta$
- Properties of polar form:

- $z=r(\cos\theta+i\sin\theta) \implies z^{-1}=\frac{1}{r}(\cos\theta-i\sin\theta)$
- If $z_1=r_1(\cos\theta_1+i\sin\theta_1), z_2=r_2(\cos\theta_2+i\sin\theta_2)$ $then$ $z_1z_2=r_1r_2[\cos(\theta_1+\theta_2)+i\sin(\theta_1+\theta_2)]$
- If $z_1=r_1(\cos\theta_1+i\sin\theta_1), z_2=r_2(\cos\theta_2+i\sin\theta_2)$ $then$ $\frac{z_1}{z_2}=\frac{r_1}{r_2}[\cos(\theta_1-\theta_2)+i\sin(\theta_1-\theta_2)]$

❖ Some of the useful polar forms :

$1 = cis(0)\ ;\ -1 = cis(\pi)\ ; i = cis\left(\frac{\pi}{2}\right);\ -i = cis\left(-\frac{\pi}{2}\right)$

EXAMPLE 2.22 Find the modulus and principal argument of the following complex numbers: $(i)\sqrt{3}+i\ \ (ii)-\sqrt{3}+i\ \ (iii)-\sqrt{3}-i\ \ (iv)\sqrt{3}-i$

$(i)\sqrt{3}+i$

$Modulus\ r = \sqrt{x^2+y^2} = \sqrt{\sqrt{3}^2+1^2} = \sqrt{4} = 2$

$\alpha = tan^{-1}\left|\frac{y}{x}\right| = tan^{-1}\left|\frac{1}{\sqrt{3}}\right| = \frac{\pi}{6}$

Since the complex number $\sqrt{3}+i$ lies in the first quadrant, principal argument $\theta = \alpha \Longrightarrow \theta = \frac{\pi}{6}$

$(ii)-\sqrt{3}+i$

$Modulus\ r = \sqrt{x^2+y^2} = \sqrt{\left(-\sqrt{3}\right)^2+1^2} = \sqrt{4} = 2$

$\alpha = tan^{-1}\left(\frac{y}{x}\right) = tan^{-1}\left|\frac{1}{-\sqrt{3}}\right| = \frac{\pi}{6}$

Since the complex number $-\sqrt{3}+i$ lies in the second quadrant, principal argument $\theta = \pi-\alpha \Longrightarrow \theta = \pi-\frac{\pi}{6} = \frac{5\pi}{6}$

$(iii)-\sqrt{3}-i$

$Modulus\ r = \sqrt{x^2+y^2} = \sqrt{\left(-\sqrt{3}\right)^2+(-1)^2} = \sqrt{4} = 2$

$\alpha = tan^{-1}\left(\frac{y}{x}\right) = tan^{-1}\left|\frac{-1}{-\sqrt{3}}\right| = \frac{\pi}{6}$

Since the complex number $-\sqrt{3}-i$ lies in the third quadrant, principal argument $\theta = -\pi+\alpha \Longrightarrow \theta = -\pi+\frac{\pi}{6} = -\frac{5\pi}{6}$

$(iv)\sqrt{3}-i$

$Modulus\ r = \sqrt{x^2+y^2} = \sqrt{\sqrt{3}^2+(-1)^2} = \sqrt{4} = 2$

$\alpha = tan^{-1}\left(\frac{y}{x}\right) = tan^{-1}\left|\frac{-1}{\sqrt{3}}\right| = \frac{\pi}{6}$

Since the complex number $\sqrt{3}-i$ lies in the fourth quadrant, principal argument $\theta = -\alpha \Longrightarrow \theta = -\frac{\pi}{6}$

EXAMPLE 2.24 Find the principal argument $Arg\ z$, when $z = \frac{-2}{1+i\sqrt{3}}$

$argz = arg\left(\frac{-2}{1+i\sqrt{3}}\right) = arg(-2) - arg\left(1+i\sqrt{3}\right) \to (1)$

$arg(-2) = tan^{-1}\left|\frac{y}{x}\right| = tan^{-1}\left|\frac{0}{-2}\right| = 0$

Since the complex number -2 lies in the second quadrant, principal argument $\theta = \pi-\alpha \Longrightarrow \theta = \pi - 0 = \pi$

$arg\left(1+i\sqrt{3}\right) = tan^{-1}\left|\frac{y}{x}\right| = tan^{-1}\left|\frac{\sqrt{3}}{1}\right| = \frac{\pi}{3}$

Since the complex number $1+i\sqrt{3}$ lies in the first quadrant, principal argument $\theta = \alpha \Longrightarrow \theta = \frac{\pi}{3}$

$(1) \Longrightarrow arg\left(\frac{-2}{1+i\sqrt{3}}\right) = \pi - \frac{\pi}{3} = \frac{2\pi}{3}$

EXERCISE 2.7

1.Write in polar form of the following complex numbers:

$(i)2+i2\sqrt{3}\quad (ii)3-i\sqrt{3}\quad (iii)-2-i2\quad (iv)\frac{i-1}{\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}}$

$(i)2+i2\sqrt{3} = rcis(\theta) \to (1)$

$r = \sqrt{x^2+y^2} = \sqrt{(2)^2+\left(2\sqrt{3}\right)^2} = \sqrt{4+12} = \sqrt{16} = 4$

$\alpha = tan^{-1}\left|\frac{y}{x}\right| = tan^{-1}\left|\frac{2\sqrt{3}}{2}\right| = \frac{\pi}{3}$

Principal argument lies in the first quadrant $\theta = \alpha = \frac{\pi}{3}$

$(1) \Longrightarrow 2+i2\sqrt{3} = 4cis\left(\frac{\pi}{3}\right)\ = 4cis\left(2k\pi+\frac{\pi}{3}\right), k \in z$

$(ii)3-i\sqrt{3} = rcis(\theta) \to (1)$

$r = \sqrt{x^2+y^2} = \sqrt{(3)^2+\left(-\sqrt{3}\right)^2} = \sqrt{9+3} = \sqrt{4\times 3} = 2\sqrt{3}$

$\alpha = tan^{-1}\left|\frac{y}{x}\right| = tan^{-1}\left|\frac{-\sqrt{3}}{3}\right| = tan^{-1}\left|\frac{1}{\sqrt{3}}\right| = \frac{\pi}{6}$

Principal argument lies in the 4th quadrant $\theta = -\alpha = -\frac{\pi}{6}$

$(1) \Longrightarrow 3-i\sqrt{3} = 2\sqrt{3}cis\left(-\frac{\pi}{6}\right)\ = 2\sqrt{3}cis\left(2k\pi-\frac{\pi}{6}\right), k \in z$

$(iii)-2-i2 = rcis(\theta) \to (1)$

$r = \sqrt{x^2+y^2} = \sqrt{(-2)^2+(-2)^2} = \sqrt{4+4} = \sqrt{2\times 4} = 2\sqrt{2}$

$\alpha = tan^{-1}\left|\frac{y}{x}\right| = tan^{-1}\left|-\frac{-2}{2}\right| = \frac{\pi}{4}$

Principal argument lies in the 3rd quadrant $\theta = -\pi + \alpha = -\pi + \frac{\pi}{4} = -\frac{3\pi}{4}$

$(1) \Longrightarrow -2 - i2 = 2\sqrt{2}cis\left(-\frac{3\pi}{4}\right) \;\; = 2\sqrt{2}cis\left(2k\pi - \frac{3\pi}{4}\right), k \in z$

$(iv)\ \frac{i-1}{\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}}$

$i - 1 = -1 + i = rcis(\theta) \rightarrow (1)$

$r = \sqrt{x^2+y^2} = \sqrt{(-1)^2+(1)^2} = \sqrt{2}$

$\alpha = tan^{-1}\left|\frac{y}{x}\right| = tan^{-1}\left|\frac{1}{-1}\right| = \frac{\pi}{4}$

Principal argument lies in the 2[nd] quadrant $\theta = \pi - \alpha \Longrightarrow \theta = \pi - \frac{\pi}{4} = \frac{3\pi}{4}$

$(1) \Longrightarrow i - 1 = \sqrt{2}cis\left(\frac{3\pi}{4}\right)$

$\frac{i-1}{\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}} = \frac{\sqrt{2}cis\left(\frac{3\pi}{4}\right)}{cis\left(\frac{\pi}{3}\right)} = \sqrt{2}cis\left(\frac{3\pi}{4} - \frac{\pi}{3}\right) = \sqrt{2}cis\left(\frac{5\pi}{12}\right) \;\; = \sqrt{2}cis\left(2k\pi + \frac{5\pi}{12}\right), k \in z$

EXAMPLE 2.23 Represent the complex number $(i) -1 - i \;\; (ii) 1 + i\sqrt{3}$ in polar form.

2.Find the rectangular form of the complex numbers:

$(i)\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)\left(\cos\frac{\pi}{12} + i\sin\frac{\pi}{12}\right) \qquad (ii)\ \frac{\cos\frac{\pi}{6} - i\sin\frac{\pi}{6}}{2\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)}$

$(i)\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)\left(\cos\frac{\pi}{12} + i\sin\frac{\pi}{12}\right)$

$= \text{cis}\left(\frac{\pi}{6} + \frac{\pi}{12}\right)$

$= \text{cis}\left(\frac{3\pi}{12}\right) = cis\left(\frac{\pi}{4}\right)$

$= \cos\frac{\pi}{4} + i\sin\frac{\pi}{4}$

$= \frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}$

$(ii)\ \frac{\cos\frac{\pi}{6} - i\sin\frac{\pi}{6}}{2\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)}$

$= \frac{1}{2}cis\left(-\frac{\pi}{6} - \frac{\pi}{3}\right) = \frac{1}{2}cis\left(-\frac{3\pi}{6}\right) = \frac{1}{2}cis\left(-\frac{\pi}{2}\right) = \frac{1}{2}\left[\cos\left(-\frac{\pi}{2}\right) + i\sin\left(-\frac{\pi}{2}\right)\right]$

$= \frac{1}{2}\left(\cos\frac{\pi}{2} - i\sin\frac{\pi}{2}\right)$

$= \frac{1}{2}(0 - i)$

$= -\frac{1}{2}i$

EXAMPLE 2.25 Find the product $\frac{3}{2}\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right).6\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right)$ in rectangular form.

EXAMPLE 2.26 Find the quotient $\frac{2\left(\cos\frac{9\pi}{4}+i\sin\frac{9\pi}{4}\right)}{4\left(\cos\left(\frac{-3\pi}{2}\right)+i\sin\left(\frac{-3\pi}{2}\right)\right)}$ in rectangular form.

3.If $(x_1+iy_1)(x_2+iy_2)\ldots..(x_n+iy_n) = a + ib$, show that (i) $(x_1^2+y_1^2)(x_2^2+y_2^2)\ldots..(x_n^2+y_n^2) = a^2+b^2$ (ii) $\sum_{r=1}^{n} tan^{-1}\left(\frac{b_r}{a_r}\right) = tan^{-1}\left(\frac{b}{a}\right) + 2k\pi, k\epsilon Z6$

$(i)(x_1+iy_1)(x_2+iy_2)\ldots..(x_n+iy_n) = a + ib$

Taking modulus on both sides,

$\Longrightarrow |(x_1+iy_1)(x_2+iy_2)\ldots..(x_n+iy_n)| = |a+ib|$

$\Longrightarrow |x_1+iy_1||x_2+iy_2|\ldots\ldots|x_n+iy_n| = |a+ib|$

$\Longrightarrow \sqrt{(x_1^2+y_1^2)}\sqrt{(x_2^2+y_2^2)}\ldots..\sqrt{(x_n^2+y_n^2)} = \sqrt{a^2+b^2}$

Squaring on both sides,

$\Longrightarrow (x_1^2+y_1^2)(x_2^2+y_2^2)\ldots..(x_n^2+y_n^2) = a^2+b^2$

$(i)(x_1+iy_1)(x_2+iy_2)\ldots..(x_n+iy_n) = a + ib$

Taking argument on both sides,

$\Longrightarrow arg[(x_1+iy_1)(x_2+iy_2)\ldots..(x_n+iy_n)] = arg(a+ib)$

$\Longrightarrow tan^{-1}\left(\frac{y_1}{x_1}\right) + tan^{-1}\left(\frac{y_2}{x_2}\right) + \cdots. + tan^{-1}\left(\frac{y_n}{x_n}\right) = tan^{-1}\left(\frac{b}{a}\right)$

$\Longrightarrow \sum_{r=1}^{n} tan^{-1}\left(\frac{b_r}{a_r}\right) = tan^{-1}\left(\frac{b}{a}\right) + 2k\pi, k\epsilon Z6$

4.If $\frac{1+z}{1-z} = \cos 2\theta + i\sin 2\theta$, show that $z = i\tan\theta$.

$\frac{1+z}{1-z} = \cos 2\theta + i\sin 2\theta$

Add 1 on both sides, $\frac{1+z}{1-z} + 1 = (1+\cos 2\theta) + i\sin 2\theta$

$\Longrightarrow \frac{1+z+1-z}{1-z} = 2cos^2\theta + i2\sin\theta\cos\theta$

$\Longrightarrow \frac{2}{1-z} = 2\cos\theta\,[\cos\theta + i\sin\theta]$

$\div\ by\ 2 \Longrightarrow \frac{1}{1-z} = \cos\theta\,[\cos\theta + i\sin\theta]$

$\Longrightarrow \frac{1}{\cos\theta[\cos\theta+i\sin\theta]} = 1 - z$

$$\Rightarrow z = 1 - \frac{1}{\cos\theta[\cos\theta + i\sin\theta]}$$

$$\Rightarrow z = 1 - \frac{1}{\cos\theta[\cos\theta + i\sin\theta]} \times \frac{\cos\theta - i\sin\theta}{\cos\theta - i\sin\theta}$$

$$\Rightarrow z = 1 - \frac{\cos\theta - i\sin\theta}{\cos\theta(\cos^2\theta + \sin^2\theta)}$$

$$\Rightarrow z = 1 - \left[\frac{\cos\theta}{\cos\theta} - i\frac{\sin\theta}{\cos\theta}\right]$$

$$\Rightarrow z = 1 - 1 + i\tan\theta$$

$$\Rightarrow z = i\tan\theta$$

5.If $cos\alpha + cos\beta + cos\gamma = sin\alpha + sin\beta + sin\gamma = 0$, then show that

(i) $cos3\alpha + cos3\beta + cos3\gamma = 3\cos(\alpha + \beta + \gamma)$

(ii)$sin3\alpha + sin3\beta + sin3\gamma = 3\sin(\alpha + \beta + \gamma)$

Let $a = cis\alpha, b = cis\beta, c = cis\gamma$

W.K.T, If $a + b + c = 0 \text{ then } a^3 + b^3 + c^3 = 3abc$

$a + b + c = (cos\alpha + cos\beta + cos\gamma) + i(sin\alpha + sin\beta + sin\gamma) = 0$

$a^3 + b^3 + c^3 = 3abc$

$\Rightarrow (cis\alpha)^3 + (cis\beta)^3 + (cis\gamma)^3 = 3cis\alpha.cis\beta.cis\gamma$

$\Rightarrow cis(3\alpha) + cis(3\beta) + cis(3\gamma) = 3cis(\alpha + \beta + \gamma)$

$\Rightarrow (cos3\alpha + cos3\beta + cos3\gamma) + i(sin3\alpha + sin3\beta + sin3\gamma) = 3(\cos(\alpha + \beta + \gamma)) + i3(\sin(\alpha + \beta + \gamma))$

Equating the real and imaginary parts on both sides,

$cos3\alpha + cos3\beta + cos3\gamma = 3\cos(\alpha + \beta + \gamma)$

$sin3\alpha + sin3\beta + sin3\gamma = 3\sin(\alpha + \beta + \gamma)$

6.If $z = x + iy$ and $arg\left(\frac{z-i}{z+2}\right) = \frac{\pi}{4}$, then show that $x^2 + y^2 + 3x - 3y + 2 = 0$.

Given: $Z = x + iy$

$arg\left(\frac{z-i}{z+2}\right) = \frac{\pi}{4}$

$$\Rightarrow arg(z - i) - arg(z + 2) = \frac{\pi}{4}$$

$$\Rightarrow arg(x + iy - i) - arg(x + iy + 2) = \frac{\pi}{4}$$

$$\Rightarrow arg\big(x + i(y - 1)\big) - arg(x + 2 + iy) = \frac{\pi}{4}$$

$$\Rightarrow tan^{-1}\frac{y-1}{x} - tan^{-1}\frac{y}{x+2} = \frac{\pi}{4}$$

$$\Rightarrow tan^{-1}\frac{\frac{y-1}{x} - \frac{y}{x+2}}{1 + \left(\frac{y-1}{x} \times \frac{y}{x+2}\right)} = \frac{\pi}{4}$$

$$\Rightarrow \frac{\frac{\big((y-1)\big)(x+2) - xy}{x(x+2)}}{1 + \frac{y^2 - y}{x(x+2)}} = \tan\frac{\pi}{4}$$

$$\Rightarrow \frac{\frac{xy + 2y - x - 2 - xy}{(x+2)}}{\frac{x^2 + 2x + y^2 - y}{(x+2)}} = 1$$

$$\Rightarrow \frac{-x + 2y - 2}{x^2 + y^2 + 2x - y} = 1$$

$$\Rightarrow x^2 + y^2 + 2x - y + x - 2y + 2 = 0$$

$$\Rightarrow x^2 + y^2 + 3x - 3y + 2 = 0$$

EXAMPLE 2.27 If $z = x + iy$ and $arg\left(\frac{z-1}{z+1}\right) = \frac{\pi}{2}$, then show that $x^2 + y^2 = 1$.

De MOIVRE'S THEOREM AND ITS APPLICATIONS

- De Moivre's theorem: Given any complex number $\cos\theta + i\sin\theta$ and any integer n, $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$
- Some results:
  - ✓ $(\cos\theta - i\sin\theta)^n = \cos n\theta - i\sin n\theta$
  - ✓ $(\cos\theta + i\sin\theta)^{-n} = \cos n\theta - i\sin n\theta$
  - ✓ $(\cos\theta - i\sin\theta)^{-n} = \cos n\theta + i\sin n\theta$
  - ✓ $\sin\theta + i\cos\theta = i(\cos\theta - i\sin\theta)$
- Formula for finding n^{th} root of a complex number: $z^{1/n} = r^{1/n}cis\left(\frac{\theta + 2k\pi}{n}\right), k = 0,1, \dots, (n-1)$
- $\omega^3 = 1 \Rightarrow 1 + \omega + \omega^2 = 0$
- The sum of all the n^{th} roots of unity is 0.$(ie., 1 + \omega + \omega^2 + \cdots. + \omega^{n-1} = 0)$
- The product of all the n^{th} roots of unity is $(-1)^{n-1}$. $(ie., 1 + \omega + \omega^2 + \cdots. + \omega^{n-1} = (-1)^{n-1})$
- All the n roots of n^{th} roots of unity are in geometrical progression.
- All the n roots of n^{th} roots of unity lie on the circumference of a circle whose

centre is at the origin and radius equal to 1 and these roots divide the circle into n equal parts and form a polygon of n sides.

EXERCISE 2.8

1.If $\omega \neq 1$ is a cube root of unity, then show that $\frac{a+b\omega+c\omega^2}{b+c\omega+a\omega^2}+\frac{a+b\omega+c\omega^2}{c+a\omega+b\omega^2}=-1$.

$$\frac{a+b\omega+c\omega^2}{b+c\omega+a\omega^2}+\frac{a+b\omega+c\omega^2}{c+a\omega+b\omega^2}$$
$$=\left(\frac{a+b\omega+c\omega^2}{b+c\omega+a\omega^2}\times\frac{\omega}{\omega}\right)+\left(\frac{a+b\omega+c\omega^2}{c+a\omega+b\omega^2}\times\frac{\omega^2}{\omega^2}\right)$$
$$=\frac{\omega(a+b\omega+c\omega^2)}{a\omega^3+c\omega^2+b\omega}+\frac{\omega^2(a+b\omega+c\omega^2)}{a\omega^3+b\omega^4+c\omega^2}$$

$\boxed{\omega^3=1\ ;\ \ 1+\omega+\omega^2=0}$

$$=\frac{\omega(a+b\omega+c\omega^2)}{a+b\omega+c\omega^2}+\frac{\omega^2(a+b\omega+c\omega^2)}{a+b\omega+c\omega^2}$$
$$=\omega+\omega^2$$
$$=-1$$

2.Show that $\left(\frac{\sqrt{3}}{2}+\frac{i}{2}\right)^5+\left(\frac{\sqrt{3}}{2}-\frac{i}{2}\right)^5=-\sqrt{3}$

$\frac{\sqrt{3}}{2}+\frac{i}{2}=rcis(\theta)\rightarrow(1)$

$r=\sqrt{x^2+y^2}=\sqrt{\left(\frac{\sqrt{3}}{2}\right)^2+\left(\frac{1}{2}\right)^2}=\sqrt{\frac{3}{4}+\frac{1}{4}}=\sqrt{1}=1$

$\alpha=tan^{-1}\left|\frac{y}{x}\right|=tan^{-1}\left|\frac{1/2}{\sqrt{3}/2}\right|=\frac{\pi}{6}$

Principal argument lies in the first quadrant $\theta=\alpha=\frac{\pi}{6}$

$(1)\Longrightarrow\frac{\sqrt{3}}{2}+\frac{i}{2}=cis\left(\frac{\pi}{6}\right)$

$\Longrightarrow\left(\frac{\sqrt{3}}{2}+\frac{i}{2}\right)^5=\left[cis\left(\frac{\pi}{6}\right)\right]^5=cis\left(\frac{5\pi}{6}\right)=\cos\left(\frac{5\pi}{6}\right)+i\sin\left(\frac{5\pi}{6}\right)$

Similarly, $\left(\frac{\sqrt{3}}{2}-\frac{i}{2}\right)^5=\cos\left(\frac{5\pi}{6}\right)-i\sin\left(\frac{5\pi}{6}\right)$

$\left(\frac{\sqrt{3}}{2}+\frac{i}{2}\right)^5+\left(\frac{\sqrt{3}}{2}-\frac{i}{2}\right)^5=\cos\left(\frac{5\pi}{6}\right)+i\sin\left(\frac{5\pi}{6}\right)+\cos\left(\frac{5\pi}{6}\right)-i\sin\left(\frac{5\pi}{6}\right)$

$=2\cos\left(\frac{5\pi}{6}\right)=2\cos\left(\frac{5\times180}{6}\right)=2\cos150=-2\cos30=-2\times\frac{\sqrt{3}}{2}=-\sqrt{3}$

EXAMPLE 2.31 Simplify $(i)(1+i)^{18}\quad(ii)\left(-\sqrt{3}+3i\right)^{31}$

3.Find the value of $\left(\frac{1+\sin\frac{\pi}{10}+i\cos\frac{\pi}{10}}{1+\sin\frac{\pi}{10}-i\cos\frac{\pi}{10}}\right)^{10}$

Let $z=\sin\frac{\pi}{10}+i\cos\frac{\pi}{10}\ \Longrightarrow\frac{1}{z}=\sin\frac{\pi}{10}-i\cos\frac{\pi}{10}$

$\left(\frac{1+\sin\frac{\pi}{10}+i\cos\frac{\pi}{10}}{1+\sin\frac{\pi}{10}-i\cos\frac{\pi}{10}}\right)^{10}=\left(\frac{1+z}{1+\frac{1}{z}}\right)^{10}=\left(\frac{1+z}{1+z}\times z\right)^{10}=z^{10}$

$z^{10}=\left(\sin\frac{\pi}{10}+i\cos\frac{\pi}{10}\right)^{10}=\left[i\left(\cos\frac{\pi}{10}-i\sin\frac{\pi}{10}\right)\right]^{10}$

$=i^{10}\left[\cos\left(10\times\frac{\pi}{10}\right)-i\sin\left(10\times\frac{\pi}{10}\right)\right]$

$=-1[\cos\pi-i\sin\pi]$

$=-1(-1-0)$

$=1$

EXAMPLE 2.29 Simplify $\left(\sin\frac{\pi}{6}+i\cos\frac{\pi}{6}\right)^{18}$

EXAMPLE 2.30 Simplify $\left(\frac{1+\cos2\theta+i\sin2\theta}{1+\cos2\theta-i\sin2\theta}\right)^{30}$

4.If $2\cos\alpha=x+\frac{1}{x}$ and $2\cos\beta=y+\frac{1}{y}$, show that

$(i)\frac{x}{y}+\frac{y}{x}=2\cos(\alpha-\beta)\quad(ii)xy-\frac{1}{xy}=2i\sin(\alpha+\beta)$

$(iii)\frac{x^m}{y^n}-\frac{y^n}{x^m}=2i\sin(m\alpha-n\beta)\quad(iv)x^my^n+\frac{1}{x^my^n}=2\cos(m\alpha+n\beta)$

$x+\frac{1}{x}=2\cos\alpha\Longrightarrow x=x=\cos\alpha+i\sin\alpha$

$y+\frac{1}{y}=2\cos\beta\Longrightarrow y=\cos\beta+i\sin\beta$

$(i)\ \frac{x}{y}=\frac{\cos\alpha+i\sin\alpha}{\cos\beta+i\sin\beta}=\cos(\alpha-\beta)+i\sin(\alpha-\beta)$

$\frac{y}{x}=\left(\frac{x}{y}\right)^{-1}=\cos(\alpha-\beta)-i\sin(\alpha-\beta)$

$\frac{x}{y}+\frac{y}{x}=2\cos(\alpha-\beta)$

$(ii)\ xy=(\cos\alpha+i\sin\alpha)(\cos\beta+i\sin\beta)=\cos(\alpha+\beta)+i\sin(\alpha+\beta)$

$\frac{1}{xy}=(xy)^{-1}=\cos(\alpha+\beta)-i\sin(\alpha+\beta)$

$xy-\frac{1}{xy}=2i\sin(\alpha+\beta)$

$(iii)\ x^m=(\cos\alpha+i\sin\alpha)^m=\cos m\alpha+i\sin m\alpha$

$$y^n=(\cos\beta+i\sin\beta)^n=\cos n\beta+i\sin n\beta$$
$$\frac{x^m}{y^n}=\frac{\cos m\alpha+i\sin m\alpha}{\cos n\beta+i\sin n\beta}=\cos(m\alpha-n\beta)+i\sin(m\alpha-n\beta)$$
$$\frac{y^n}{x^m}=\left(\frac{x^m}{y^n}\right)^{-1}=\cos(m\alpha-n\beta)-i\sin(m\alpha-n\beta)$$
$$\frac{x^m}{y^n}-\frac{y^n}{x^m}=2i\sin(m\alpha-n\beta)$$
$$(iv)x^m=(\cos\alpha+i\sin\alpha)^m=\cos m\alpha+i\sin m\alpha$$
$$y^n=(\cos\beta+i\sin\beta)^n=\cos n\beta+i\sin n\beta$$
$$x^my^n=(\cos m\alpha+i\sin m\alpha)(\cos n\beta+i\sin n\beta)$$
$$=\cos(m\alpha+n\beta)+i\sin(m\alpha+n\beta)$$
$$\frac{1}{x^my^n}=\cos(m\alpha+n\beta)-i\sin(m\alpha+n\beta)$$
$$x^my^n+\frac{1}{x^my^n}=2\cos(m\alpha+n\beta)$$

EXAMPLE 2.28 If $z=\cos\theta+i\sin\theta$, show that $z^n+\frac{1}{z^n}=2\cos n\theta$ and $z^n-\frac{1}{z^n}=2\sin n\theta$.

5.Solve the equation $z^3+27=0$

$z^3+27=0\Rightarrow z^3=-27\Rightarrow z=(-27)^{1/3}=(27\times -1)^{1/3}$

$z=3(-1)^{1/3}=3[cis(\pi)]^{1/3}=3[cis(2k\pi+\pi)]^{1/3}, k=0,1,2$

$=3cis(2k+1)\frac{\pi}{3}, k=0,1,2$

$=3cis\left(\frac{\pi}{3}\right), 3cis(\pi), 3cis\left(\frac{5\pi}{3}\right)$

6. If $\omega\neq 1$ is a cube root of unity, show that the roots of the equation $(z-1)^3+8=0$ are $-1, 1-2\omega, 1-2\omega^2$.

$(z-1)^3+8=0$

$\Rightarrow (z-1)^3=-8$

$\Rightarrow (z-1)^3=(-2)^3\Rightarrow \frac{(z-1)^3}{(-2)^3}=1$

$\Rightarrow \left(\frac{z-1}{-2}\right)^3=1$

$\Rightarrow \frac{z-1}{-2}=(1)^{1/3}$

$\Rightarrow z-1=-2(1)^{1/3}\Rightarrow z-1=-2[1\ (or)\ \omega\ \ (or)\ \omega^2]$

$$\Rightarrow\begin{cases} z-1=-2(1)\Rightarrow z-1=-2\Rightarrow z=-1\\ z-1=-2\omega\Rightarrow z=1-2\omega\\ z-1=-2\omega^2\Rightarrow z=1-2\omega^2\end{cases}$$

Thus the roots of the equation $(z-1)^3+8=0$ are $-1, 1-2\omega, 1-2\omega^2$.

EXAMPLE 2.34 Solve the equation $z^3+8i=0$, where $z\in\mathbb{C}$.

Note: $z^3+8i=0\Rightarrow z=(8i)^{1/3}\Rightarrow z=2(i)^{1/3}$

find polar form for i and use de Moivre'theorem

EXAMPLE 2.35 Find all cube roots of $\sqrt{3}+i$.

Note: $\left(\sqrt{3}+i\right)^{1/3}$; *find polar form for* $\sqrt{3}+i$ *and use de Moivre'theorem*

7.Find the value of $\sum_{k=1}^{8}\left(\cos\frac{2k\pi}{9}+i\sin\frac{2k\pi}{9}\right)$

Let $x=(1)^{1/9}\Rightarrow x=[cis(0)]^{1/9}=[cis(2k\pi+0)]^{1/9}, k=0,1,2,\dots,8$

$\Rightarrow x=cis\left(\frac{2k\pi}{9}\right), k=0,1,2,\dots,8$

Thus the roots are $cis(0), cis\left(\frac{2\pi}{9}\right), cis\left(\frac{4\pi}{9}\right), cis\left(\frac{6\pi}{9}\right), cis\left(\frac{8\pi}{9}\right), cis\left(\frac{10\pi}{9}\right)$, $cis\left(\frac{12\pi}{9}\right), cis\left(\frac{14\pi}{9}\right), cis\left(\frac{16\pi}{9}\right)$.

W.K.T, sum of all the roots Is equal to zero.

$$cis(0)+cis\left(\frac{2\pi}{9}\right)+cis\left(\frac{4\pi}{9}\right)+cis\left(\frac{6\pi}{9}\right)+cis\left(\frac{8\pi}{9}\right)+cis\left(\frac{10\pi}{9}\right)+cis\left(\frac{12\pi}{9}\right)+cis\left(\frac{14\pi}{9}\right)+cis\left(\frac{16\pi}{9}\right)=0$$

$$\Rightarrow 1+cis\left(\frac{2\pi}{9}\right)+cis\left(\frac{4\pi}{9}\right)+cis\left(\frac{6\pi}{9}\right)+cis\left(\frac{8\pi}{9}\right)+cis\left(\frac{10\pi}{9}\right)+cis\left(\frac{12\pi}{9}\right)+cis\left(\frac{14\pi}{9}\right)+cis\left(\frac{16\pi}{9}\right)=0$$

$$\Rightarrow cis\left(\frac{2\pi}{9}\right)+cis\left(\frac{4\pi}{9}\right)+cis\left(\frac{6\pi}{9}\right)+cis\left(\frac{8\pi}{9}\right)+cis\left(\frac{10\pi}{9}\right)+cis\left(\frac{12\pi}{9}\right)+cis\left(\frac{14\pi}{9}\right)+cis\left(\frac{16\pi}{9}\right)=-1$$

$$\Rightarrow \sum_{k=1}^{8}cis\left(\frac{2k\pi}{9}\right)=-1$$

$$\Rightarrow \sum_{k=1}^{8}\left(\cos\frac{2k\pi}{9}+i\sin\frac{2k\pi}{9}\right)=-1$$

8. If $\omega\neq 1$ is a cube root of unity, show that

$(i)(1-\omega+\omega^2)^6+(1+\omega-\omega^2)^6=128$

$(ii)(1+\omega)(1+\omega^2)(1+\omega^4)(1+\omega^8)\ldots.(1+\omega^{2^{11}})=1$

W.K.T, $\boxed{\omega^3=1\ \ ;\ \ 1+\omega+\omega^2=0}$

$(i)(1-\omega+\omega^2)^6+(1+\omega-\omega^2)^6$

$=(-\omega-\omega)^6+(-\omega^2-\omega^2)^6$

$=(-2\omega)^6+(-2\omega^2)^6=2^6\omega^6+2^6\omega^{12}=64(\omega^3)^2+64(\omega^3)^4=64+64=128$

$(ii)(1+\omega)(1+\omega^2)(1+\omega^4)(1+\omega^8)\ldots.(1+\omega^{2^{11}})$

$=(1+\omega)(1+\omega^2)(1+\omega^{2^2})(1+\omega^{2^3})\ldots.(1+\omega^{2^{11}})$

Clearly the above expression contains 12 terms,

$=(1+\omega)(1+\omega^2)(1+\omega^4)(1+\omega^8)\ldots..12\ terms$

$=[(1+\omega)(1+\omega^2)][(1+\omega)(1+\omega^2)]\ldots.6\ terms$

$=[(1+\omega)(1+\omega^2)]^6$

$=(1+\omega+\omega^2+\omega^3)^6$

$=(0+1)^6$

$=1$

9.If $z=2-2i$, find the rotation of z by θ radians in the counter clockwise direction about the origin when $(i)\theta=\frac{\pi}{3}$ $(ii)\theta=\frac{2\pi}{3}$ $(iii)\theta=\frac{3\pi}{2}$

$z=2-2i=rcis(\theta)\rightarrow(1)$

$r=\sqrt{x^2+y^2}=\sqrt{(2)^2+(-2)^2}=\sqrt{4+4}=\sqrt{4\times2}=2\sqrt{2}$

$\alpha=tan^{-1}\left|\frac{y}{x}\right|=tan^{-1}\left|\frac{-2}{2}\right|=\frac{\pi}{4}$

Principal argument lies in the 4th quadrant $\theta=-\alpha=-\frac{\pi}{4}$

$(1)\Longrightarrow z=2-2i=2\sqrt{2}cis\left(-\frac{\pi}{4}\right)$

$(i)when\ \theta=\frac{\pi}{3}\Longrightarrow z=2\sqrt{2}cis\left(-\frac{\pi}{4}\right)cis\left(\frac{\pi}{3}\right)$

$=2\sqrt{2}cis\left(-\frac{\pi}{4}+\frac{\pi}{3}\right)$

$=2\sqrt{2}cis\left(\frac{-3\pi+4\pi}{12}\right)$

$=2\sqrt{2}cis\left(\frac{\pi}{12}\right)$

$(ii)when\ \theta=\frac{2\pi}{3}\Longrightarrow z=2\sqrt{2}cis\left(-\frac{\pi}{4}\right)cis\left(\frac{2\pi}{3}\right)$

$=2\sqrt{2}cis\left(-\frac{\pi}{4}+\frac{2\pi}{3}\right)$

$=2\sqrt{2}cis\left(\frac{-3\pi+8\pi}{12}\right)$

$=2\sqrt{2}cis\left(\frac{5\pi}{12}\right)$

$(ii)when\ \theta=\frac{3\pi}{2}\Longrightarrow z=2\sqrt{2}cis\left(-\frac{\pi}{4}\right)cis\left(\frac{3\pi}{2}\right)$

$=2\sqrt{2}cis\left(-\frac{\pi}{4}+\frac{3\pi}{2}\right)$

$=2\sqrt{2}cis\left(\frac{-\pi+6\pi}{4}\right)$

$=2\sqrt{2}cis\left(\frac{5\pi}{4}\right)$

EXAMPLE 2.36 Suppose z_1,z_2 and z_3 are the vertices of an equilateral triangle inscribed in the circle $|z|=2$. If $z_1=1+i\sqrt{3}$, then find z_2 and z_3.

$|z|=2$ represents the circle with centre (0,0) and radius 2.

Let $A,B\ and\ C$ be the vertices of the given triangle. Since the vertices z_1,z_2 and z_3 form an equilateral triangle inscribed in the circle $|z|=2$, the sides of this triangle $AB,BC\ and\ CA$ subtend $\frac{2\pi}{3}$ radian at the origin.

We obtain z_2 and z_3 by the rotation of z_1 by $\frac{2\pi}{3}$ and $\frac{4\pi}{3}$ respectively.

Given: $z_1=1+i\sqrt{3}$

$z_2=z_1cis\left(\frac{2\pi}{3}\right)=(1+i\sqrt{3})\left[\cos\left(\frac{2\pi}{3}\right)+i\sin\left(\frac{2\pi}{3}\right)\right]$

$=(1+i\sqrt{3})[\cos(180-60)+i\sin(180-60)]$

$=(1+i\sqrt{3})(-\cos 60+i\sin 60)$

$=(1+i\sqrt{3})\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)$

$=-2$

$z_3=z_2cis\left(\frac{2\pi}{3}\right)=-2\left[\cos\left(\frac{2\pi}{3}\right)+i\sin\left(\frac{2\pi}{3}\right)\right]$

$=-2[\cos(180-60)+i\sin(180-60)]$

$=-2(-\cos 60+i\sin 60)$

$=-2\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)$

$=1-i\sqrt{3}$

10.Prove that the values of $\sqrt[4]{-1}$ are $\pm\frac{1}{\sqrt{2}}(1\pm i)$

Let $x=\sqrt[4]{-1}$

$\Rightarrow x=(-1)^{1/4} \Rightarrow x^4=-1 \Rightarrow x^4+1=0$
$\Rightarrow (x^4+2x^2+1)-(2)=0$
$\Rightarrow (x^2+1)^2-(\sqrt{2}x)^2=0$
$\Rightarrow (x^2+1+\sqrt{2}x)(x^2+1-\sqrt{2}x)=0$
$\Rightarrow (x^2+1+\sqrt{2}x)=0\ ;\ (x^2+1-\sqrt{2}x)=0$

$$\boxed{x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}}$$

$\Rightarrow x=\frac{-\sqrt{2}\pm\sqrt{(\sqrt{2}^2)-4(1)(1)}}{2(1)}\quad ;\quad \frac{-(-\sqrt{2})\pm\sqrt{(-\sqrt{2}^2)-4(1)(1)}}{2(1)}$
$\Rightarrow x=\frac{-\sqrt{2}\pm\sqrt{2-4}}{2}\quad ;\quad \frac{\sqrt{2}\pm\sqrt{2-4}}{2}$
$\Rightarrow x=\frac{-\sqrt{2}\pm\sqrt{-2}}{2}\quad ;\quad \frac{\sqrt{2}\pm\sqrt{-2}}{2}$
$\Rightarrow x=\frac{-\sqrt{2}\pm\sqrt{2i^2}}{2}\quad ;\quad \frac{\sqrt{2}\pm\sqrt{2i^2}}{2}$
$\Rightarrow x=\frac{-\sqrt{2}\pm\sqrt{2}i}{2}\quad ;\quad \frac{\sqrt{2}\pm\sqrt{\sqrt{2}i}}{2}$
$\Rightarrow x=\frac{-\sqrt{2}}{2}(1\pm i)\ ;\ \frac{\sqrt{2}}{2}(1\pm i)$
$\Rightarrow x=\pm\frac{1}{\sqrt{2}}(1\pm i)$

EXAMPLE 2.32 Find the cube roots of unity.

Let $z^3=1 \Rightarrow z=(1)^{1/3}=[cis(0)]^{1/3}$
$\Rightarrow z=[cis(2k\pi+0)]^{1/3}, k=0,1,2$
$\Rightarrow z=cis\left(\frac{2k\pi}{3}\right), k=0,1,2$
$\Rightarrow z=cis(0), cis\left(\frac{2\pi}{3}\right), cis\left(\frac{4\pi}{3}\right)$
$\Rightarrow z=cis(0), cis\left(\pi-\frac{\pi}{3}\right), cis\left(\pi+\frac{\pi}{3}\right)$
$\Rightarrow z=\cos 0+i\sin 0, \cos\left(\pi-\frac{\pi}{3}\right)+i\sin\left(\pi-\frac{\pi}{3}\right), \cos\left(\pi+\frac{\pi}{3}\right)+i\sin\left(\pi+\frac{\pi}{3}\right)$
$\Rightarrow z=1, -\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}, -\cos\frac{\pi}{3}-i\sin\frac{\pi}{3}$
$\Rightarrow z=1, -\frac{1}{2}+i\frac{\sqrt{3}}{2}, -\frac{1}{2}+i\frac{\sqrt{3}}{2} \Rightarrow z=1, \frac{-1+i\sqrt{3}}{2}, \frac{-1-i\sqrt{3}}{2}$

$\therefore$ The cube roots of unity $1, \omega, \omega^2$ are $1, \frac{-1+i\sqrt{3}}{2}, \frac{-1-i\sqrt{3}}{2}$

EXAMPLE 2.33 Find the fourth roots of unity.

Let $z^4=1 \Rightarrow z=(1)^{1/4}=[cis(0)]^{1/4}$
$\Rightarrow z=[cis(2k\pi+0)]^{1/4}, k=0,1,2,3$
$\Rightarrow z=cis\left(\frac{2k\pi}{4}\right), k=0,1,2,3$
$\Rightarrow z=cis(0), cis\left(\frac{2\pi}{4}\right), cis\left(\frac{4\pi}{4}\right), cis\left(\frac{6\pi}{4}\right)$
$\Rightarrow z=cis(0), cis\left(\frac{\pi}{2}\right), cis(\pi), cis\left(\frac{3\pi}{2}\right)$
$\Rightarrow z=\cos 0+i\sin 0, \cos\left(\frac{\pi}{2}\right)+i\sin\left(\frac{\pi}{2}\right), \cos(\pi)+i\sin(\pi), \cos\left(\frac{3\pi}{2}\right)+i\sin\left(\frac{3\pi}{2}\right)$
$\Rightarrow z=1, i, -1, -i$
$\therefore$ The fourth roots of unity $1, \omega, \omega^2, \omega^3$ are $1, i, -1, -i$.

1.Prove that the multiplicative inverse of a non- zero complex number $z=x+iy$ is $z^{-1}=\left(\frac{x}{x^2+y^2}\right)+i\left(-\frac{y}{x^2+y^2}\right)$

PROOF:
Let $z^{-1}=u+iv$ be the inverse of $z=x+iy$
W.K.T, $zz^{-1}=1 \Rightarrow (x+iy)(u+iv)=1$
$\Rightarrow (xu-yv)+i(xv+yu)=1+i0$
Equating the Real and Imaginary parts on both sides,
$xu-yv=1\rightarrow(1)\ ;\ xv+yu=0\rightarrow(2)$
$\Delta=\begin{vmatrix}x & -y\\ y & x\end{vmatrix}=x^2+y^2;\ \Delta_u=\begin{vmatrix}1 & -y\\ 0 & x\end{vmatrix}=x;\ \Delta_v=\begin{vmatrix}x & 1\\ y & 0\end{vmatrix}=-y$
$u=\frac{\Delta_u}{\Delta}=\frac{x}{x^2+y^2}\ ;\ v=\frac{\Delta_v}{\Delta}=\frac{-y}{x^2+y^2}$
$z^{-1}=u+iv \Rightarrow z^{-1}=\left(\frac{x}{x^2+y^2}\right)+i\left(-\frac{y}{x^2+y^2}\right)$

2.For any two complex numbers z_1 *and* z_2 prove that $\overline{z_1+z_2}=\overline{z_1}+\overline{z_2}$

PROOF:
Let $z_1=a+ib$ and $z_2=c+id$, $a,b,c,d\in R$
$\overline{z_1+z_2}=\overline{(a+ib)+(c+id)}$
$=\overline{(a+c)+i(b+d)}$
$=(a+c)-i(b+d)$
$=(a-ib)+(c-id)$

$= \bar{z}_1 + \bar{z}_2$

3.Prove that $\overline{z_1 . z_2} = \overline{z_1} . \overline{z_2}$ where $a, b, c, d \in R$.

PROOF:

Let $z_1 = a + ib$ and $z_2 = c + id$, $a, b, c, d \in R$

$\overline{z_1 . z_2} = \overline{(a + ib)(c + id)}$

$= \overline{(ac - bd) + i(ad + bc)}$

$= (ac - bd) - i(ad + bc) \rightarrow (1)$

$\overline{z_1} . \overline{z_2} = \overline{a + ib} . \overline{c + id}$

$= (a - ib)(c - id)$

$= (ac - bd) - i(ad + bc) \rightarrow (2)$

From (1) and (2), $\overline{z_1 . z_2} = \overline{z_1} . \overline{z_2}$

4.Prove that z is purely imaginary if and only if $z = -\bar{z}$

PROOF:

Let $z = x + iy \Longrightarrow \bar{z} = x - iy$

$z = -\bar{z} \Leftrightarrow x + iy = -(x - iy)$

$\Leftrightarrow x + iy = -x + iy$

$\Leftrightarrow 2x = 0$

$\Leftrightarrow x = 0$

$\Leftrightarrow z$ *is purely imaginary*

5.State and prove triangle inequality of two complex numbers (or)

For any two complex numbers z_1 and z_2 prove that $|z_1 + z_2| \leq |z_1| + |z_2|$

PROOF:

$|z_1 + z_2|^2 = (z_1 + z_2)\overline{(z_1 + z_2)}$ $\quad (\because z\bar{z} = |z|^2)$

$= (z_1 + z_2)(\bar{z}_1 + \bar{z}_2)$ $\quad (\because \overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2)$

$= z_1\bar{z}_1 + (z_1\bar{z}_2 + \bar{z}_1 z_2) + z_2\bar{z}_2$

$= z_1\bar{z}_1 + (z_1\bar{z}_2 + \overline{z_1\bar{z}_2}) + z_2\bar{z}_2$ $\quad (\because \bar{\bar{z}} = z)$

$= |z_1|^2 + 2Re(z_1\bar{z}_2) + |z_2|^2$ $\quad (\because z + \bar{z} = 2Re(z))$

$\leq |z_1|^2 + 2|z_1\bar{z}_2| + |z_2|^2$ $\quad (\because Re(z) \leq |z|)$

$= |z_1|^2 + 2|z_1||z_2| + |z_2|^2$ $\quad (\because \overline{z_1 . z_2} = \bar{z}_1 . \bar{z}_2 \; \& |z| = |\bar{z}|)$

$= (|z_1| + |z_2|)^2$

Taking square root on both sides,

$|z_1 + z_2|^2 \leq |z_1| + |z_2|$

6.For any two complex numbers z_1 and z_2 prove that $|z_1 z_2| = |z_1||z_2|$

PROOF:

$|z_1 z_2|^2 = (z_1 z_2)\overline{(z_1 z_2)}$ $\quad (\because z\bar{z} = |z|^2)$

$= (z_1)(z_2)(\bar{z}_1)(\bar{z}_2)$ $\quad (\because \overline{z_1 . z_2} = \bar{z}_1 . \bar{z}_2)$

$= (z_1\bar{z}_1)(z_2\bar{z}_2)$

$= |z_1|^2|z_2|^2$

Taking square root on both sides,

$|z_1 z_2| = |z_1||z_2|$

7.If $z = r(\cos\theta + i\sin\theta)$ then prove that $z^{-1} = \frac{1}{r}(\cos\theta - i\sin\theta)$

PROOF:

$z = r(\cos\theta + i\sin\theta)$

$z^{-1} = \frac{1}{z} = \frac{1}{r(\cos\theta + i\sin\theta)}$

$= \frac{1}{r}\frac{1}{\cos\theta + i\sin\theta} \times \frac{\cos\theta - i\sin\theta}{\cos\theta - i\sin\theta}$

$= \frac{1}{r} . \frac{\cos\theta - i\sin\theta}{cos^2\theta + sin^2\theta}$

$= \frac{1}{r}(\cos\theta - i\sin\theta)$

8. If $z_1 = r_1(\cos\theta_1 + i\sin\theta_1), z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$ *then* $z_1 z_2 = r_1 r_2[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)]$

PROOF:

$z_1 z_2 = [r_1(\cos\theta_1 + i\sin\theta_1)][r_2(\cos\theta_2 + i\sin\theta_2)]$

$= r_1 r_2[(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2)]$

$= r_1 r_2[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)]$

9. If $z_1 = r_1(\cos\theta_1 + i\sin\theta_1), z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$ *then* $\frac{z_1}{z_2} = \frac{r_1}{r_2}[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)]$

PROOF:

$\frac{z_1}{z_2} = \frac{r_1(\cos\theta_1 + i\sin\theta_1)}{r_2(\cos\theta_2 + i\sin\theta_2)}$

$= \frac{r_1}{r_2} . \frac{\cos\theta_1 + i\sin\theta_1}{\cos\theta_2 + i\sin\theta_2} \times \frac{\cos\theta_2 - i\sin\theta_2}{\cos\theta_2 - i\sin\theta_2}$

$= \frac{r_1}{r_2} . \frac{(\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2 - \cos\theta_1\sin\theta_2)}{\cos^2(\theta_2) + \sin^2(\theta_2)}$

$= \frac{r_1}{r_2}[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)]$