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2. Complex Number

$z = a+ib$ be a complex number

$Re(z) = a$, $Im(z) = b$

Note :

* $\sqrt{-1} = i$
* $i^2 = -1$
* $i^3 = -i$
* $i^4 = 1$
* $(i)^{-1} = -i$
* $(i)^{-2} = -1$
* $(i)^{-3} = i$
* $(i)^{-4} = 1$

Exercise 2.1

Simplify the following:

① $i^{1947} + i^{1950}$

Solution:

$i^{1947} + i^{1950} = i^3 + i^2 = -i - 1 = -1 - i$

② $i^{1948} - i^{-1869}$

Solution:

$$i^{1948} - i^{-1869} = 1 - \frac{1}{i^{1869}} = 1 - \frac{1}{i}$$

$$= 1 - \frac{\cancel{i^4}^{\,3}}{\cancel{i}}$$

$$= 1 - i^3$$

$$= 1 + i$$

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③ $\sum_{n=1}^{12} i^n$

Solution:

$$\sum_{n=1}^{12} i^n = (i^1+i^2+i^3+i^4)+(i^5+i^6+i^7+i^8)+(i^9+i^{10}+i^{11}+i^{12})$$

$$=(i^1+i^2+i^3+i^4)+(i^1+i^2+i^3+i^4)+(i^1+i^2+i^3+i^4)$$

$$=(\cancel{i}-\cancel{1}-\cancel{i}+\cancel{1})+(\cancel{i}-\cancel{1}-\cancel{i}+\cancel{1})+(\cancel{i}-\cancel{1}-\cancel{i}+\cancel{1})$$

$$=0+0+0$$

$$=0$$

④ $i^{59}+\frac{1}{i^{59}}$

Solution:

$$i^{59}+\frac{1}{i^{59}} = i^{56}i^3+\frac{1}{i^{56}i^3} = i^3+\frac{1}{i^3} = i^3+\frac{i^4}{i^3}$$

$$= i^3+i = -1+i$$

⑤ $i\,i^2 i^3 \ldots\ldots i^{2000}$

Solution:

$$i\,i^2 i^3\ldots\ldots i^{2000} = i^{1+2+3+\cdots+2000} \quad \left[1+2+3+\cdots+n=\frac{n(n+1)}{2}\right]$$

$$= i^{\frac{2000\times 2001}{2}}$$

$$= i^{1000\times 2001}$$

$$= i^{4\times 250\times 2001}$$

$$= i^{4k} \text{ where } k=250\times 2001$$

$$=1$$

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(6) $\sum_{n=1}^{10} i^{n+50}$

Solution:

$$\sum_{n=1}^{10} i^{n+50} = i^{51}+i^{52}+i^{53}+\dots+i^{60}$$

$$= i^{50}(i^1+i^2+i^3+\dots+i^{10})$$

$$= i^2(i^1+i^2+i^3+i^4+\dots+\underbrace{i^9}_{i}+\underbrace{i^{10}}_{i^2})$$

$$= -1(0+0+i+i^2)$$

$$= -1(-1+i)$$

$$= 1-i$$

Rectangular form:

Rectangular form of a complex number is

$x+iy$ (or $x+yi$), wher x, y are real numbers.

Definition:

Two complex numbers $z_1 = x_1+iy$, $z_2 = x_2+iy_2$ are said to be equal iff

$$Re(z_1) = Re(z_2) \text{ and } Im(z_1) = Im(z_2)$$

ie $x_1 = x_2$ and $y_1 = y_2$.

Exercise 2.2

(1) Evaluate the following if $z = 5-2i$ and $w = -1+3i$

(i) $z+w$

Solution:

$$z+w = (5-2i)+(-1+3i)$$

$$= 5-2i-1+3i$$

$$= 4+i$$

(ii) $z-iw$

Solution:

$$z-iw = 5-2i-i(-1+3i)$$

$$= 5-2i+i-3\underbrace{i^2}_{-1}$$

$$= 5-i+3$$

$$= 8-i$$

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(iii) $2z+3w$

Solution:

$$2z+3w = 2(5-2i)+3(-1+3i)$$
$$= 10-4i-3+9i$$
$$= 7+5i$$

(iv) zw

Solution:

$$zw = (5-2i)(-1+3i)$$
$$= -5+15i+2i-6\underbrace{i^2}_{-1}$$
$$= -5+17i+6$$
$$= 1+17i$$

(v) $z^2+2zw+w^2$

Solution:

$$z^2+2zw+w^2 = (5-2i)^2+2(5-2i)(-1+3i)+(-1+3i)^2$$
$$= 25-20i+4\underbrace{i^2}_{-1}+2(-5+15i+2i-6\underbrace{i^2}_{-1})+(1-6i+9\underbrace{i^2}_{-1})$$
$$= 25-20i-4+2(-5+17i+6)+(1-6i-9)$$
$$= 25-20i-4-10+34i+12+1-6i-9$$
$$= 15+8i$$

(vi) $(z+w)^2$

Solution:

$$(z+w)^2 = (5-2i-1+3i)^2$$
$$= (4+i)^2$$
$$= 16+8i+i^2$$
$$= 16+8i-1$$
$$= 15+8i$$

(2) Given the complex number $z=2+3i$ represent the complex numbers in Argand diagram:

(i) z, iz and $z+iz$

Solution:

$$z = 2+3i$$
$$iz = i(2+3i)$$
$$= 2i+3\underbrace{i^2}_{-1}$$
$$= -3+2i$$
$$z+iz = -1+5i$$

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(ii)

$(z_1 z_2) z_3 = [(1-3i)(-4i)](5)$

$= (-4i + 12i^2) 5$

$= (-4i - 12) 5$

$= -20i - 60$ —— ①

$z_1 (z_2 z_3) = (1-3i)[(-4i)5]$

$= (1-3i)(-20i)$

$= -20i + 60i^2$

$= -20i - 60$ —— ②

From ① and ②

$(z_1 z_2) z_3 = z_1 (z_2 z_3)$.

② If $z_1 = 3$, $z_2 = -7i$, $z_3 = 5+4i$ show that

(i) $z_1(z_2+z_3) = z_1 z_2 + z_1 z_3$ (ii) $(z_1+z_2) z_3 = z_1 z_3 + z_2 z_3$

Solution:

$z_1(z_2+z_3) = 3(-7i+5+4i)$

$= 3(-3i+5)$

$= -9i + 15$

$= 15 - 9i$ —— ①

$z_1 z_2 + z_1 z_3 = 3(-7i) + 3(5+4i)$

$= -21i + 15 + 12i$

$= 15 - 9i$ —— ②

From ① and ②

$z_1(z_2+z_3) = z_1 z_2 + z_1 z_3$.

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(ii) $(z_1+z_2)z_3 = (3-7i)(5+4i)$

$= 15+12i-35i-28i^2$ $\;\; i^2 \to -1$

$= 15-23i+28$

$= 43-23i$ —— ①

$z_1z_3+z_2z_3 = 3(5+4i)+(-7i)(5+4i)$

$= 15+12i-35i-28i^2$

$= 15-23i+28$

$= 43-23i$ —— ②

From ① and ②

$(z_1+z_2)z_3 = z_1z_3+z_2z_3$

③ If $z_1=2+5i$, $z_2=-3-4i$, $z_3=1+i$ find the additive and multiplicative inverse of z_1 z_2 and z_3.

Solution:

$z_1=2+5i$, $z_2=-3-4i$, $z_3=1+i$

Additive inverse

$-z_1 = -(2+5i)$

$= -2-5i$

$-z_2 = -(-3-4i)$

$= 3+4i$

$-z_3 = -(1+i)$

$= -1-i$

Multiplicative inverse $\left[\begin{array}{l} z = x+iy \\ z^{-1} = \frac{x}{x^2+y^2} + i\frac{-y}{x^2+y^2} \end{array}\right]$

$z_1^{-1} = \frac{2}{2^2+5^2} + i\frac{(-5)}{2^2+5^2} = \frac{2}{4+25} - \frac{5i}{4+25} = \frac{2}{29} - \frac{5i}{29}$

$= \frac{1}{29}(2-5i)$

$z_2^{-1} = \frac{-3}{3^2+4^2} - i\frac{(-4)}{3^2+4^2} = \frac{-3}{9+16} + \frac{4i}{9+16} = \frac{-3}{25} + \frac{4i}{25}$

$= \frac{1}{25}(-3+4i)$

$z_3^{-1} = \frac{1}{1^2+1^2} + i\frac{(-1)}{1^2+1^2} = \frac{1}{2} - \frac{i}{2}$

$= \frac{1}{2}(1-i)$

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* The commutative property under addition

$$z_1+z_2=z_2+z_1$$

* Inverse property under multiplication.

$z=x+iy$ then

$$z^{-1}=\frac{x}{x^2+y^2}+i\,\frac{-y}{x^2+y^2}$$

* complex numbers obey the laws of indices.

(i) $z^m z^n = z^{m+n}$

(ii) $\frac{z^m}{z^n}=z^{m-n},\ z\neq 0$

(iii) $(z^m)^n=z^{mn}$

(iv) $(z_1 z_2)^m = z_1^m z_2^m$

Exercise 2·3

① If $z_1=1-3i$, $z_2=-4i$ and $z_3=5$, show that

(i) $(z_1+z_2)+z_3=z_1+(z_2+z_3)$ (ii) $(z_1z_2)z_3=z_1(z_2z_3)$

solution:

(i) $(z_1+z_2)+z_3 = (1-3i+(-4i))+5$

$= 1-3i-4i+5$

$= 6-7i$ — ①

$z_1+(z_2+z_3) = 1-3i+(-4i+5)$

$= 1-3i-4i+5$

$= 6-7i$ — ②

① = ②

$(z_1+z_2)+z_3 = z_1+(z_2+z_3)$.

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(ii) z, $-iz$ and $z-iz$

Solution:

$$z = 2+3i$$

$$-iz = -i(2+3i)$$

$$= -2i + 3i^2$$

$$= -3-2i$$

$$z - iz = (2+3i) - (-3-2i)$$

$$= 2+3i+3+2i$$

$$= 5+5i$$

---

3. Find the values of the real numbers x and y if the complex numbers $(3-i)x-(2-i)y+2i+5$ and $2x+(-1+2i)y+3+2i$ are equal.

Solution:

$$(3-i)x-(2-i)y+2i+5 = 2x+(-1+2i)y+3+2i$$

$$3x - ix - 2y + iy + 2i + 5 = 2x - y + 2iy + 3 + 2i$$

$$(3x-2y+5) - i(x-y-2) = (2x-y+3) + i(2y+2)$$

Equating Real, Imaginary part,

$$3x-2y+5 = 2x-y+3$$

$$x-y+2 = 0$$

$$x - y = -2 \quad \text{—— (1)}$$

$$-(x-y-2) = 2y+2$$

$$-x+y+2 = 2y+2$$

$$-x-y = 0 \quad \text{—— (2)}$$

(1) + (2) $\Rightarrow$

$$-2y = -2$$

$$\boxed{y=1}$$

$y = 1$ sub in (1)

$$x - 1 = -2$$

$$x = -2+1$$

$$\boxed{x=-1}$$

$\therefore x = -1,\ y = 1.$

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Conjugate of a complex number:

The conjugate of the complex number $x+iy$ is defined as the complex number $x-iy$

Properties:

* $\overline{z_1+z_2} = \overline{z_1}+\overline{z_2}$
* $\overline{z_1-z_2} = \overline{z_1}-\overline{z_2}$
* $\overline{z_1 z_2} = \overline{z_1}\,\overline{z_2}$
* $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\overline{z_1}}{\overline{z_2}}$
* $Re(z) = \frac{z+\overline{z}}{2}$
* $Im(z) = \frac{z-\overline{z}}{2i}$
* $\overline{(z^n)} = (\overline{z})^n$
* z is real iff $z=\overline{z}$
* z is purely imaginary iff $z=-\overline{z}$
* $\overline{\overline{z}} = z$

Exercise 2.4

① Write the following in the rectangular form:

(i) $\overline{(5+9i)+(2-4i)}$

Solution:

$$\overline{(5+9i)+(2-4i)} = \overline{(5+9i)}+\overline{(2-4i)}$$
$$= 5-9i+2+4i$$
$$= 7\cancel{+5}i$$
$$= 7-5i$$

(ii) $\frac{10-5i}{6+2i}$

Solution:

$$\frac{10-5i}{6+2i} = \frac{10-5i}{6+2i}\times\frac{6-2i}{6-2i}$$
$$= \frac{60-20i-30i+10i^2}{6^2-2^2i^2} \quad [\because i^2=-1]$$
$$= \frac{60-50i-10}{36+4}$$
$$= \frac{50-50i}{40} = \frac{5\cancel{0}(1-i)}{4\cancel{0}} = \frac{5}{4}(1-i)$$

(iii) $\overline{3i} + \frac{1}{2-i}$

Solution:

$$\overline{3i} + \frac{1}{2-i} = -3i + \frac{1}{2-i} \times \frac{2+i}{2+i}$$
$$= -3i + \frac{2+i}{4+1}$$
$$= -3i + \frac{2+i}{5}$$
$$= \frac{-15i + 2 + i}{5}$$
$$= \frac{2-14i}{5}$$
$$= \frac{2}{5} - i\,\frac{14}{5}$$

② If $z = x+iy$ find the following in rectangular form

(i) $Re\left(\frac{1}{z}\right)$

Solution:

$$Re\left(\frac{1}{z}\right) = Re\left(\frac{1}{x+iy}\right)$$
$$= Re\left(\frac{1}{x+iy} \times \frac{x-iy}{x-iy}\right)$$
$$= Re\left(\frac{x-iy}{x^2-i^2y^2}\right)$$
$$= Re\left(\frac{x-iy}{x^2+y^2}\right)$$
$$= \frac{x}{x^2+y^2}$$

(ii) $Re(i\bar{z})$

Solution:

$$Re(i\bar{z}) = Re\left[i\left(\overline{x+iy}\right)\right]$$
$$= Re\left[i(x-iy)\right]$$
$$= Re(ix - i^2y) \quad [\because i^2 = -1]$$
$$= Re(ix+y)$$
$$= y.$$

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(4) The complex numbers u, v and w are related by $\frac{1}{u}=\frac{1}{v}+\frac{1}{w}$ If $v=3-4i$, $w=4+3i$ find u in rectangular form.

Solution:

$$\frac{1}{u}=\frac{1}{v}+\frac{1}{w}$$

$$=\frac{1}{3-4i}+\frac{1}{4+3i}$$

$$=\frac{1}{3-4i}\times\frac{3+4i}{3+4i}+\frac{1}{4+3i}\times\frac{4-3i}{4-3i}$$

$$=\frac{3+4i}{9+16}+\frac{4-3i}{16+9}$$

$$=\frac{3+4i}{25}+\frac{4-3i}{25}$$

$$=\frac{1}{25}(3+4i+4-3i)$$

$$\frac{1}{u}=\frac{1}{25}(7+i)$$

$$u=25\times\frac{1}{7+i}$$

$$=25\cdot\frac{1}{7+i}\times\frac{7-i}{7-i}$$

$$=\frac{25(7-i)}{49+1}$$

$$=\frac{25}{50}(7-i)$$

$$=\frac{1}{2}(7-i)$$

(5) Prove the following properties:

(i) z is real iff $z=\bar{z}$

Proof:

Let $z=x+iy$, $\bar{z}=x-iy$

$\therefore z=\bar{z}$

$\Leftrightarrow x+iy=x-iy$

$\Leftrightarrow 2y=0$

$\Leftrightarrow y=0$

$\Leftrightarrow$ z is real

(ii) Let $z=x+iy$

$Re(z)=x$, $Im(z)=y$

$\bar{z}=x-iy$

$z+\bar{z}=x+iy+x-iy$

$z+\bar{z}=2x$

$\frac{z+\bar{z}}{2}=x$

$Re(z)=x=\frac{z+\bar{z}}{2}$

$\therefore Re(z)=\frac{z+\bar{z}}{2}$

$z-\bar{z}=x+iy-x+iy$

$=2iy$

$\frac{z-\bar{z}}{2i}=y$

$Im(z)=y=\frac{z-\bar{z}}{2i}$

$\therefore Im(z)=\frac{z-\bar{z}}{2i}$

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⑦ S.T (i) $(2+i\sqrt{3})^{10} - (2-i\sqrt{3})^{10}$ is purely Imaginary

(ii) $\left(\frac{19-7i}{9+i}\right)^{12} + \left(\frac{20-5i}{7-6i}\right)^{12}$ is real.

Solution:

(i) let $z = (2+i\sqrt{3})^{10} - (2-i\sqrt{3})^{10}$

$\bar{z} = \overline{(2+i\sqrt{3})^{10} - (2-i\sqrt{3})^{10}}$

$= \overline{(2+i\sqrt{3})^{10}} - \overline{(2-i\sqrt{3})^{10}}$

$= (\overline{2+i\sqrt{3}})^{10} - (\overline{2-i\sqrt{3}})^{10}$

$= (2-i\sqrt{3})^{10} - (2+i\sqrt{3})^{10}$

$= -[(2+i\sqrt{3})^{10} - (2-i\sqrt{3})^{10}]$

$\bar{z} = -z$

$\Rightarrow z$ is purely Imaginary

(ii)

$\frac{19-7i}{9+i} = \frac{19-7i}{9+i} \times \frac{9-i}{9-i}$

$= \frac{171-19i-63i+7i^2}{9^2-i^2}$

$= \frac{171-82i-7}{81+1}$

$= \frac{164-82i}{82}$

$= \frac{\cancel{82}(2-i)}{\cancel{82}}$

$= 2-i$

$\frac{20-5i}{7-6i} = \frac{20-5i}{7-6i} \times \frac{7+6i}{7+6i}$

$= \frac{140+120i-35i-30i^2}{7^2-6^2i^2}$

$= \frac{140+85i+30}{49+36}$

$= \frac{170+85i}{85}$

$\frac{20-5i}{7-6i} = \frac{\cancel{85}(2+i)}{\cancel{85}} = 2+i$

$z = \left(\frac{19-7i}{9+i}\right)^{12} + \left(\frac{20-5i}{7-6i}\right)^{12}$

$z = (2-i)^{12} + (2+i)^{12}$

$\bar{z} = \overline{(2-i)^{12} + (2+i)^{12}}$

$= (\overline{2-i})^{12} + (\overline{2+i})^{12}$

$= (2+i)^{12} + (2-i)^{12}$

$\bar{z} = z$

$\Rightarrow z$ is real.

⑥ Find the least value of +ve integer n for which $(\sqrt{3}+i)^n$ (i) real (ii) purely imaginary.

Solution:

$(\sqrt{3}+i)^n$

$(\sqrt{3}+i)^2 = (\sqrt{3})^2 + i2\sqrt{3} + i^2$

$= 3 + i2\sqrt{3} - 1$

$= 2 + i2\sqrt{3}$

$= 2(1+i\sqrt{3})$

$(\sqrt{3}+i)^3 = (\sqrt{3}+i)^2(\sqrt{3}+i)$

$= 2(1+i\sqrt{3})(\sqrt{3}+i)$

$= 2(\sqrt{3}+3i+i+i^2\sqrt{3})$

$= 2(\cancel{\sqrt{3}}+4i-\cancel{\sqrt{3}})$

$= 2(4i)$

$= 8i$ Imaginary

$n=3$, $(\sqrt{3}+i)^n$ is purely Imaginary.

$(\sqrt{3}+i)^6 = (\sqrt{3}+i)^3(\sqrt{3}+i)^3$

$= (8i)(8i)$

$= 64i^2 = -64$ real

$n=6$, $(\sqrt{3}+i)^n$ is real

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Modulus of a complex number

If $z = x+iy$ then $\sqrt{x^2+y^2}$ is called modulus of z. It is denoted by $|z|$

Note:

$z = x+iy$, $\bar{z} = x-iy$

$z\bar{z} = x^2+y^2 = |z|^2$

$|z|^2 = z\bar{z}$

properties:

* $|z| = |\bar{z}|$
* $|z_1+z_2| \le |z_1|+|z_2|$
* $|z_1 z_2| = |z_1||z_2|$
* $|z_1-z_2| \ge ||z_1|-|z_2||$
* $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$, $z_2 \ne 0$
* $|z^n| = |z|^n$
* $Re(z) \le |z|$
* $Im(z) \le |z|$
* $z = x+iy$

  $|z| = \sqrt{x^2+y^2}$.

Formula for finding square root of a complex number

$$\sqrt{a+ib} = \pm\left(\sqrt{\frac{|z|+a}{2}} + i\frac{b}{|b|}\sqrt{\frac{|z|-a}{2}}\right)$$

where $z = a+ib$, $b \ne 0$

Exercise 2.5

① Find the modulus of the following complex numbers.

(i) $\left|\frac{2i}{3+4i}\right|$

Solution:

$$\left|\frac{2i}{3+4i}\right| = \frac{|2i|}{|3+4i|}$$

$$= \frac{\sqrt{4}}{\sqrt{9+16}}$$

$$= \frac{\sqrt{4}}{\sqrt{25}} = \frac{2}{5}$$

(ii) $\frac{2-i}{1+i} + \frac{1-2i}{1-i}$

Solution:

$$\frac{2-i}{1+i} + \frac{1-2i}{1-i} = \frac{(2-i)(1-i)+(1+i)(1-2i)}{(1+i)(1-i)}$$

$$= \frac{2-2i-i+i^2+1-2i+i-2i^2}{1^2-i^2}$$

$$= \frac{2-4i+2}{1+1}$$

$$= \frac{4-4i}{2}$$

$$= \frac{\cancel{2}(2-2i)}{\cancel{2}}$$

$$= 2-2i$$

$$\left|\frac{2-i}{1+i} + \frac{1-2i}{1-i}\right| = |2-2i| = \sqrt{2^2+2^2}$$

$$= \sqrt{4+4}$$

$$= \sqrt{8}$$

$$= \sqrt{4\times 2}$$

$$= 2\sqrt{2}$$

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(iii) $(1-i)^{10}$

Solution:

$$|(1-i)^{10}| = |1-i|^{10}$$
$$= \left(\sqrt{1^2+1^2}\right)^{10}$$
$$= \left(\sqrt{1+1}\right)^{10}$$
$$= (\sqrt{2})^{10}$$
$$= 2^5$$
$$= 32$$

(iv) $2i(3-4i)(4-3i)$

Solution:

$$|2i(3-4i)(4-3i)|$$
$$= |2i|\,|3-4i|\,|4-3i|$$
$$= \sqrt{2^2}\cdot\sqrt{3^2+4^2}\cdot\sqrt{4^2+3^2}$$
$$= \sqrt{4}\cdot\sqrt{9+16}\cdot\sqrt{16+9}$$
$$= \sqrt{4}\cdot\sqrt{25}\cdot\sqrt{25}$$
$$= 2\times5\times5$$
$$= 50$$

(4) If $|z|=3$, S.T $7 \le |z+6-8i| \le 13$

Solution:

$$|z+6-8i| \le |z| + |6-8i|$$
$$\le |z| + \sqrt{36+64}$$
$$\le 3 + \sqrt{100}$$
$$\le 3+10$$
$$\le 13 \quad \text{—— (1)}$$

$$|z+6-8i| \ge |6-8i| - |z|$$
$$\ge \sqrt{36+64} - |z|$$
$$\ge \sqrt{100} - 3$$
$$\ge 10-3$$
$$\ge 7 \quad \text{—— (2)}$$

From (1) and (2)

$$7 \le |z+6-8i| \le 13$$

(5) If $|z|=1$ S.T $2 \le |z^2-3| \le 4$

Solution:

$$|z|=1$$
$$|z^2-3| \le |z|^2 + |3|$$
$$\le 1^2+3$$
$$\le 1+3$$
$$\le 4 \quad \text{—— (1)}$$

$$|z^2-3| = |-3+z^2|$$
$$\ge |-3| - |z^2|$$
$$\ge 3 - |z|^2$$
$$\ge 3-1$$
$$\ge 2 \quad \text{—— (2)}$$

From (1) and (2)

$$2 \le |z^2-3| \le 4.$$

(7) If z_1, z_2 and z_3 are three complex numbers S.t $|z_1|=1$, $|z_2|=2$, $|z_3|=3$ and $|z_1+z_2+z_3|=1$ S.T $|9z_1z_2 + 4z_1z_3 + z_2z_3| = 6$.

Solution:

$|z_1|=1$, $|z_2|=2$, $|z_3|=3$ $\quad [z\bar{z}=|z|^2$

$z_1\bar{z}_1 = 1$	$z_2\bar{z}_2 = 4$	$z_3\bar{z}_3 = 9$
$z_1 = \frac{1}{\bar{z}_1}$	$z_2 = \frac{4}{\bar{z}_2}$	$z_3 = \frac{9}{\bar{z}_3}$

$$z_1+z_2+z_3 = \frac{1}{\bar{z}_1} + \frac{4}{\bar{z}_2} + \frac{9}{\bar{z}_3}$$
$$= \frac{\bar{z}_2\bar{z}_3 + 4\bar{z}_1\bar{z}_3 + 9\bar{z}_1\bar{z}_2}{\bar{z}_1\bar{z}_2\bar{z}_3}$$

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$$|z_1+z_2+z_3| = \frac{|\overline{z_2z_3+4z_1z_3+9z_1z_2}|}{|\overline{z_1z_2z_3}|} \qquad [\because |\bar{z}| = |z|]$$

$$1 = \frac{|z_2z_3+4z_1z_3+9z_1z_2|}{|z_1|\,|z_2|\,|z_3|}$$

$$1 = \frac{|9z_1z_2+4z_1z_3+z_2z_3|}{1\times 2\times 3}$$

$$6 = |9z_1z_2+4z_1z_3+z_2z_3|$$

$$\therefore |9z_1z_2+4z_1z_3+z_2z_3| = 6.$$

(9) S.T the equ $z^3+2\bar{z}=0$ has five solutions.

Solution:

$z^3 = -2\bar{z}$ —— (1)

$|z|^3 = |-2||\bar{z}|$

$|z|^3 = 2|\bar{z}|$

$|z|^3 = 2|z|$

$|z|^3 - 2|z| = 0$

$|z|\,(|z|^2-2) = 0$

$|z| = 0,\ |z|^2 = 2$

$z = 0,\ |z| = \pm\sqrt{2}$

$|z| = \sqrt{2},\ |z| = -\sqrt{2}$

$z\bar{z} = \sqrt{2},\ z\bar{z} = -\sqrt{2}$

$\bar{z} = \frac{\sqrt{2}}{z}$

(1) $\Rightarrow z^3 = -2\times\frac{\sqrt{2}}{z}$

$z^4 = -2\sqrt{2}$

It has 4 non zero solutions
Hence including zero solution.
There are five solutions.

(3) Which one of the pts $10-8i$, $11+6i$ is closest to $1+i$

Solution:

let $z = 1+i$

$z_1 = 10-8i$

$|z-z_1| = |1+i-(10-8i)|$

$= |1+i-10+8i|$

$= |-9+9i| = 9|-1+i|$

$= 9\sqrt{1+1} = 9\sqrt{2}$

$z_2 = 11+6i$

$|z-z_2| = |1+i-(11+6i)|$

$= |1+i-11-6i|$

$= |-10-5i|$

$= 5|-2-i| = 5\sqrt{4+1}$

$= 5\sqrt{2}$

$\therefore 11+6i$ is closest pt to $1+i$

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(10) Find the square roots of (i) $4+3i$ (ii) $-6+8i$ (iii) $-5-12i$

Solution:

(i) $4+3i$

$a=4,\ b=3$

$|4+3i| = \sqrt{16+9} = \sqrt{25} = 5$

$\sqrt{4+3i} = \pm\left(\sqrt{\frac{|z|+a}{2}} + i\frac{b}{|b|}\sqrt{\frac{|z|-a}{2}}\right)$

$= \pm\left(\sqrt{\frac{5+4}{2}} + i\frac{\cancel{3}}{\cancel{3}}\sqrt{\frac{5-4}{2}}\right)$

$= \pm\left(\sqrt{\frac{9}{2}} + i\sqrt{\frac{1}{2}}\right)$

$= \pm\left(\frac{3}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right)$

(iii) $-5-12i$

Solution:

$a=-5,\ b=-12$

$|-5-12i| = \sqrt{25+144} = \sqrt{169} = 13$

$\sqrt{-5-12i} = \pm\left(\sqrt{\frac{|z|+a}{2}} + i\frac{b}{|b|}\sqrt{\frac{|z|-a}{2}}\right)$

$= \pm\left(\sqrt{\frac{13-5}{2}} + i\left(-\frac{12}{12}\right)\sqrt{\frac{13+5}{2}}\right)$

$= \pm\left(\sqrt{\frac{8}{2}} + i(-1)\sqrt{\frac{18}{2}}\right)$

$= \pm\left(\sqrt{4} - i\sqrt{9}\right)$

$= \pm(2-3i).$

(ii) $-6+8i$

Solution:

$a=-6,\ b=8$

$|-6+8i| = \sqrt{36+64} = \sqrt{100} = 10$

$\sqrt{-6+8i} = \pm\left(\sqrt{\frac{|z|+a}{2}} + i\frac{b}{|b|}\sqrt{\frac{|z|-a}{2}}\right)$

$= \pm\left(\sqrt{\frac{10-6}{2}} + i\frac{\cancel{8}}{\cancel{8}}\sqrt{\frac{10+6}{2}}\right)$

$= \pm\left(\sqrt{\frac{4}{2}} + i\sqrt{\frac{16}{2}}\right)$

$= \pm\left(\sqrt{2} + i\sqrt{8}\right)$

$= \pm\left(\sqrt{2} + i\sqrt{4\times 2}\right)$

$= \pm\left(\sqrt{2} + i2\sqrt{2}\right).$

(8) Area of the triangle formed by the vertices z, iz and $z+iz$ is 50 sq. units. Find the value of $|z|$.

Solution:

let $z = x+iy$

$iz = i(x+iy) = ix+i^2y = ix-y$

$= -y+ix$

$z+iz = (x-y)+i(x+y)$

$|z+iz| = \sqrt{(x-y)^2+(x+y)^2} = \sqrt{2(x^2+y^2)}$

$|z-iz| = \sqrt{(x+y)^2+(y-x)^2} = \sqrt{2(x^2+y^2)}$

$|z+iz-iz| = |z| = \sqrt{x^2+y^2}$

$|z+iz-z| = |iz| = \sqrt{x^2+y^2}$

$z,\ iz,\ z+iz$ form a isosceles triangle.

Area $= 50$

$\frac{1}{2}\, b\times h = 50$

$\frac{1}{2}\sqrt{x^2+y^2}\sqrt{x^2+y^2} = 50$

$x^2+y^2 = 100$

$|z|^2 = 100$

$|z| = 10.$

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EXERCISE: 2.6

① If $z = x+iy$ is a complex number s.t $\left|\frac{z-4i}{z+4i}\right| = 1$ S.T the locus of z is real axis.

Solution:

let $z = x+iy$

$\left|\frac{z-4i}{z+4i}\right| = 1$

$\frac{|z-4i|}{|z+4i|} = 1$

$|z-4i| = |z+4i|$

$|x+iy-4i| = |x+iy+4i|$

$|x+i(y-4)| = |x+i(y+4)|$

$\sqrt{x^2+(y-4)^2} = \sqrt{x^2+(y+4)^2}$

squaring on both sides,

$x^2+(y-4)^2 = x^2+(y+4)^2$

$\cancel{x^2}+\cancel{y^2}-8y+\cancel{16} = \cancel{x^2}+\cancel{y^2}+8y+\cancel{16}$

$16y = 0$

$y = 0$

The locus of z is real axis.

② If $z = x+iy$ is a complex number s.t $Im\left(\frac{2z+1}{iz+1}\right) = 0$ S.T the locus of z is $2x^2+2y^2+x-2y = 0$

Solution:

let $z = x+iy$

$Im\left(\frac{2z+1}{iz+1}\right) = 0$

$Im\left(\frac{2(x+iy)+1}{i(x+iy)+1}\right) = 0$

$Im\left[\frac{2x+i2y+1}{ix+i^2y+1}\right] = 0$

$Im\left[\frac{(2x+1)+i2y}{(1-y)+ix}\right] = 0$

$Im\left[\frac{(2x+1)+i2y}{(1-y)+ix} \times \frac{(1-y)-ix}{(1-y)-ix}\right] = 0$

$Im\left[\frac{(2x+1)(1-y)-ix(2x+1)+i2y(1-y)+2xy}{(1-y)^2+x^2}\right] = 0$

$\frac{-x(2x+1)+2y(1-y)}{(1-y)^2+x^2} = 0$

$-2x^2-x+2y-2y^2 = 0$

x'ly by (-)

$2x^2+2y^2+x-2y = 0$

③ obtain the cartesian form the locus of $z = x+iy$ in each of the following cases:

(i) $[Re(iz)]^2 = 3$

Solution:

let $z = x+iy$

$[Re(i(x+iy))]^2 = 3$

$[Re(ix+i^2y)]^2 = 3$

$[Re(ix-y)]^2 = 3$

$(-y)^2 = 3$

$y^2 = 3.$

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(ii) $Im[(1-i)z+1]=0$

Solution:

let $z=x+iy$

$Im[(1-i)z+1]=0$

$Im[(1-i)(x+iy)+1]=0$

$Im[x+iy-ix-i^2y+1]=0$

$Im[x-ix+iy+y+1]=0$

$Im[(x+y+1)+i(y-x)]=0$

$y-x=0$

x'ly by (-)

$x-y=0$

(iii) $|z+i|=|z-1|$

Solution:

let $z=x+iy$

$|z+i|=|z-1|$

$|x+iy+i|=|x+iy-1|$

$|x+i(y+1)|=|(x-1)+iy|$

$\sqrt{x^2+(y+1)^2}=\sqrt{(x-1)^2+y^2}$

squaring on both sides.

$x^2+(y+1)^2=(x-1)^2+y^2$

$\cancel{x^2}+\cancel{y^2}+2y+1=\cancel{x^2}-2x+1+\cancel{y^2}$

$2y+\cancel{1}=-2x+\cancel{1}$

$2x+2y=0$

$\div$ by 2

$x+y=0$

(iv) $\bar{z}=z^{-1}$

Solution:

let $z=x+iy$

$\bar{z}=z^{-1}$

$\bar{z}=\frac{1}{z}$

$\overline{(x+iy)}=\frac{1}{x+iy}$

$(x-iy)(x+iy)=1$

$x^2+y^2=1$

(4) S.T the following equations represent a circle and find its centre and radius.

(i) $|z-2-i|=3$

Solution:

$|z-2-i|=3$

$|z-(2+i)|=3$

It is of the form $|z-z_0|=r$

centre and radius are $(2,1)$ and 3.

(or)

centre and radius are $(2+i)$, 3

(ii) $|2z+2-4i|=2$

Solution:

$|2z+2-4i|=2$

$2|z+1-2i|=2$

$|z-(-1+2i)|=\frac{2}{2}$

$|z-(-1+2i)|=1$

It is of the form $|z-z_0|=r$

centre $=(-1,2)$ or $(-1+2i)$

radius $=1$

(iii) $|3z-6+12i|=8$

Solution:

$|3z-6+12i|=8$

$3|z-2+4i|=8$

$|z-(2-4i)|=\frac{8}{3}$

It is of the form $|z-z_0|=r$

centre $=(2-4i)$ or $(2,-4)$

radius $=\frac{8}{3}$.

⑤ obtain the cartesian eqn for the locus of $z=x+iy$ in each of the following cases:

(i) $|z-4|=16$

Solution:

$|z-4|=16$

$z=x+iy$

$|x+iy-4|=16$

$|(x-4)+iy|=16$

$\sqrt{(x-4)^2+y^2}=16$

$(x-4)^2+y^2=16^2$

$x^2-8x+16+y^2=256$

$x^2+y^2-8x+16-256=0$

$x^2+y^2-8x-240=0$.

(ii) $|z-4|^2-|z-1|^2=16$

Solution:

$|z-4|^2-|z-1|^2=16$

Let $z=x+iy$

$|x+iy-4|^2-|x+iy-1|^2=16$

$|(x-4)+iy|^2-|(x-1)+iy|^2=16$

$\left(\sqrt{(x-4)^2+y^2}\right)^2-\left(\sqrt{(x-1)^2+y^2}\right)^2=16$

$(x-4)^2+y^2-((x-1)^2+y^2)=16$

$x^2-8x+16+y^2-(x^2-2x+1+y^2)=16$

$\cancel{x^2}-8x+\cancel{16}+\cancel{y^2}-\cancel{x^2}+2x-1-\cancel{y^2}=\cancel{16}$

$-6x-1=0$

x'ly by (−)

$6x+1=0$

Polar form of a complex numbers

$z=x+iy=r(\cos\theta+i\sin\theta)=r\,cis\,\theta$

$r=|z|=\sqrt{x^2+y^2}$

$x=r\cos\theta,\ y=r\sin\theta$.

Principal Argument of a complex number:

I Quadrant : $\theta=\alpha$ $(+,+)$

IInd Quadrant : $\theta=\pi-\alpha$ $(-,+)$

IIIrd Quadrant : $\theta=\alpha-\pi$ $(-,-)$

IVth Quadrant : $\theta=-\alpha$. $(+,-)$

Properties:

* $\arg(z_1z_2)=\arg z_1+\arg z_2$
* $\arg\left(\frac{z_1}{z_2}\right)=\arg z_1-\arg z_2$
* $\arg(z^n)=n\arg z$.
* The alternative form of $\cos\theta+i\sin\theta$ is $\cos(2k\pi+\theta)+i\sin(2k\pi+\theta)$

Euler's formula:

$e^{i\theta}=\cos\theta+i\sin\theta$

polar form

$z=re^{i\theta}$.

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Exercise 2.7

1. Write in polar form of the following complex numbers.

(i) $2+i2\sqrt{3}$

Solution:

Let $2+i2\sqrt{3} = r(\cos\theta + i\sin\theta)$

$$r=\sqrt{2^2+(2\sqrt{3})^2}=\sqrt{4+12}=\sqrt{16}=4$$

$$\alpha = \tan^{-1}\left|\frac{y}{x}\right| = \tan^{-1}\left|\frac{2\sqrt{3}}{2}\right| = \tan^{-1}\sqrt{3}$$

$$\alpha = \frac{\pi}{3}$$

$\therefore 2+i2\sqrt{3}$ lies on the 1^{st} quadrant

$$\theta = \alpha = \frac{\pi}{3}$$

$$\therefore 2+i2\sqrt{3} = 4\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)$$

$$=4\left[\cos\left(2k\pi+\frac{\pi}{3}\right)+i\sin\left(2k\pi+\frac{\pi}{3}\right)\right]$$

$k\in\mathbb{Z}$.

(ii) $3-i\sqrt{3}$

Solution:

$3-i\sqrt{3} = r(\cos\theta+i\sin\theta)$

$$r=\sqrt{(3)^2+(\sqrt{3})^2}=\sqrt{9+3}=\sqrt{12}=2\sqrt{3}$$

$$\alpha=\tan^{-1}\left|\frac{-\sqrt{3}}{3}\right|=\tan^{-1}\frac{1}{\sqrt{3}}=\frac{\pi}{6}$$

$\therefore 3-i\sqrt{3}$ lies on the 4^{th} quadrant

$$\theta=-\alpha=-\frac{\pi}{6}$$

$$3-i\sqrt{3}=2\sqrt{3}\left[\cos\left(-\frac{\pi}{6}\right)+i\sin\left(-\frac{\pi}{6}\right)\right]$$

$$=2\sqrt{3}\left[\cos\left(2k\pi-\frac{\pi}{6}\right)+i\sin\left(2k\pi-\frac{\pi}{6}\right)\right]$$

$k\in\mathbb{Z}$.

(iii) $-2-i2$

Solution:

$-2-i2 = r(\cos\theta+i\sin\theta)$

$$r=\sqrt{(-2)^2+(-2)^2}=\sqrt{4+4}=\sqrt{8}=2\sqrt{2}$$

$$\alpha=\tan^{-1}\left|\frac{-2}{-2}\right|=\tan^{-1}1=\frac{\pi}{4}$$

$\therefore -2-i2$ lies on the 3^{rd} quadrant

$$\theta=\frac{\pi}{4}-\pi=-\frac{3\pi}{4}$$

$$-2-i2=2\sqrt{2}\left[\cos\left(-\frac{3\pi}{4}\right)+i\sin\left(-\frac{3\pi}{4}\right)\right]$$

$$=2\sqrt{2}\left[\cos\left(2k\pi-\frac{3\pi}{4}\right)+i\sin\left(2k\pi-\frac{3\pi}{4}\right)\right],\ k\in\mathbb{Z}$$

(iv) $\dfrac{i-1}{\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}}$

Solution:

$-1+i=r(\cos\theta+i\sin\theta)$

$$r=\sqrt{1+1}=\sqrt{2}$$

$$\alpha=\tan^{-1}\left|\frac{1}{-1}\right|=\tan^{-1}1=\frac{\pi}{4}$$

$\therefore -1+i$ lies on the 2^{nd} quadrant

$$\theta=\pi-\alpha=\pi-\frac{\pi}{4}=\frac{3\pi}{4}$$

$$-1+i=\sqrt{2}\left(\cos\frac{3\pi}{4}+i\sin\frac{3\pi}{4}\right)$$

$$\frac{i-1}{\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}}=\frac{\sqrt{2}\left(\cos\frac{3\pi}{4}+i\sin\frac{3\pi}{4}\right)}{\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}}$$

$$=\frac{\sqrt{2}\,e^{i\frac{3\pi}{4}}}{e^{i\frac{\pi}{3}}}$$

$$=\sqrt{2}\,e^{i\frac{3\pi}{4}}\cdot e^{-i\frac{\pi}{3}}$$

$$=\sqrt{2}\,e^{i\left(\frac{3\pi}{4}-\frac{\pi}{3}\right)}$$

$$=\sqrt{2}\,e^{i\left(\frac{5\pi}{12}\right)}$$

$$=\sqrt{2}\left[\cos\frac{5\pi}{12}+i\sin\frac{5\pi}{12}\right]$$

$$=\sqrt{2}\left[\cos\left(2k\pi+\frac{5\pi}{12}\right)+i\sin\left(2k\pi+\frac{5\pi}{12}\right)\right]$$

$k\in\mathbb{Z}$.

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(2) Find the rectangular form of the complex numbers.

(i) $\left(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right)\left(\cos\frac{\pi}{12}+i\sin\frac{\pi}{12}\right)$

Solution:

$\left(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right)\left(\cos\frac{\pi}{12}+i\sin\frac{\pi}{12}\right)$

$= e^{i\pi/6}\cdot e^{i\pi/12}$

$= e^{i\left(\frac{\pi}{6}+\frac{\pi}{12}\right)}$

$= e^{i\left(\frac{18\pi}{72}\right)}$

$= e^{i\pi/4}$

$= \cos\frac{\pi}{4}+i\sin\frac{\pi}{4}$

$= \frac{1}{\sqrt{2}}+i\frac{1}{\sqrt{2}}$

$= \frac{1}{\sqrt{2}}(1+i)$.

(ii)

$\dfrac{\cos\frac{\pi}{6}-i\sin\frac{\pi}{6}}{2\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)}$

Solution:

$\dfrac{\cos\frac{\pi}{6}-i\sin\frac{\pi}{6}}{2\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)}$

$= \dfrac{e^{-i\frac{\pi}{6}}}{2e^{i\frac{\pi}{3}}}$

$= \frac{1}{2}e^{-i\pi/6}\cdot e^{-i\pi/3}$

$= \frac{1}{2}e^{-i\left(\frac{\pi}{6}+\frac{\pi}{3}\right)}$

$= \frac{1}{2}e^{-i\left(\frac{9\pi}{18}\right)}$

$= \frac{1}{2}e^{-i\left(\frac{\pi}{2}\right)}$

$= \frac{1}{2}\left(\cos\frac{\pi}{2}-i\sin\frac{\pi}{2}\right)$

$= \frac{1}{2}(0-i(1))$

$= -\frac{i}{2}$

(3) If $(x_1+iy_1)(x_2+iy_2)(x_3+iy_3)\cdots(x_n+iy_n)=a+ib$ S.T

(i) $(x_1^2+y_1^2)(x_2^2+y_2^2)(x_3^2+y_3^2)\cdots(x_n^2+y_n^2)=a^2+b^2$

(ii) $\sum_{r=1}^{n}\tan^{-1}\left(\frac{y_r}{x_r}\right)=\tan^{-1}\left(\frac{b}{a}\right)+2k\pi,\ k\in\mathbb{Z}$

Solution:

$(x_1+iy_1)(x_2+iy_2)(x_3+iy_3)\cdots(x_n+iy_n)=a+ib$

$|(x_1+iy_1)(x_2+iy_2)(x_3+iy_3)\cdots(x_n+iy_n)|=|a+ib|$

$|x_1+iy_1||x_2+iy_2||x_3+iy_3|\cdots|x_n+iy_n|=|a+ib|$

$\sqrt{x_1^2+y_1^2}\cdot\sqrt{x_2^2+y_2^2}\cdot\sqrt{x_3^2+y_3^2}\cdots\sqrt{x_n^2+y_n^2}=\sqrt{a^2+b^2}$

Squaring on both sides.

$(x_1^2+y_1^2)(x_2^2+y_2^2)(x_3^2+y_3^2)\cdots(x_n^2+y_n^2)=a^2+b^2$

(ii)

$\arg[(x_1+iy_1)(x_2+iy_2)(x_3+iy_3)\cdots(x_n+iy_n)]=\arg(a+ib)$

$\arg(x_1+iy_1)+\arg(x_2+iy_2)+\cdots+\arg(x_n+iy_n)=\arg(a+ib)$

$\tan^{-1}\left(\frac{y_1}{x_1}\right)+\tan^{-1}\left(\frac{y_2}{x_2}\right)+\cdots+\tan^{-1}\left(\frac{y_n}{x_n}\right)=\tan^{-1}\left(\frac{b}{a}\right)$

$\therefore \sum_{r=1}^{n}\tan^{-1}\left(\frac{y_r}{x_r}\right)=\tan^{-1}\left(\frac{b}{a}\right)+2k\pi,\ k\in\mathbb{Z}$.

(4) If $\frac{1+z}{1-z}=\cos2\theta+i\sin2\theta$ S.T $z=i\tan\theta$

Solution:

$\frac{1+z}{1-z}=\cos2\theta+i\sin2\theta$

$\frac{1+z}{1-z}+1=1+\cos2\theta+i\sin2\theta$

$\frac{1+\cancel{z}+1-\cancel{z}}{1-z}=2\cos^2\theta+i2\sin\theta\cos\theta$

$\frac{\cancel{2}}{1-z}=\cancel{2}\cos\theta(\cos\theta+i\sin\theta)$

$\frac{1}{\cos\theta(\cos\theta+i\sin\theta)}=1-z$

$z=1-\frac{1}{\cos\theta(\cos\theta+i\sin\theta)}$

$=1-\left[\frac{1}{\cos\theta}\times\frac{\cos\theta-i\sin\theta}{(\cos\theta+i\sin\theta)(\cos\theta-i\sin\theta)}\right]$

$=1-\left[\frac{1}{\cos\theta}\times\frac{\cos\theta-i\sin\theta}{\underbrace{\cos^2\theta+\sin^2\theta}_{1}}\right]$

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$$z = 1 - \left[\frac{1}{\cos\theta}(\cos\theta - i\sin\theta)\right]$$

$$= 1 - \left[\frac{\cos\theta}{\cos\theta} - i\frac{\sin\theta}{\cos\theta}\right]$$

$$= 1 - [1 - i\tan\theta]$$

$$= \cancel{1} - \cancel{1} + i\tan\theta$$

$$z = i\tan\theta$$

(6) If $z = x+iy$ and $\arg\left(\frac{z-i}{z+2}\right) = \frac{\pi}{4}$ then S.T $x^2+y^2+3x-3y+2=0$

solution:

let $z = x+iy$

$$\arg\left(\frac{z-i}{z+2}\right) = \frac{\pi}{4}$$

$$\arg(z-i) - \arg(z+2) = \frac{\pi}{4}$$

$$\arg(x+iy-i) - \arg(x+iy+2) = \frac{\pi}{4}$$

$$\arg[x+i(y-1)] - \arg[(x+2)+iy] = \frac{\pi}{4}$$

$$\tan^{-1}\frac{y-1}{x} - \tan^{-1}\frac{y}{x+2} = \frac{\pi}{4}$$

$$\tan^{-1}\left[\frac{\frac{y-1}{x} - \frac{y}{x+2}}{1+\left(\frac{y-1}{x}\right)\left(\frac{y}{x+2}\right)}\right] = \frac{\pi}{4}$$

$$\tan^{-1}\left[\frac{\frac{(y-1)(x+2)-yx}{\cancel{x(x+2)}}}{\frac{x(x+2)+y(y-1)}{\cancel{x(x+2)}}}\right] = \frac{\pi}{4}$$

$$\frac{(y-1)(x+2)-xy}{x(x+2)+y(y-1)} = \tan\frac{\pi}{4}$$

$$\frac{\cancel{xy}+2y-x-2-\cancel{xy}}{x^2+2x+y^2-y} = 1$$

$$2y-x-2 = x^2+2x+y^2-y$$

$$x^2+y^2+2x-y-2y+x+2=0$$

$$x^2+y^2+3x-3y+2=0$$

(5) If $\cos\alpha+\cos\beta+\cos\gamma = \sin\alpha+\sin\beta+\sin\gamma = 0$ then S.T

(i) $\cos3\alpha+\cos3\beta+\cos3\gamma = 3\cos(\alpha+\beta+\gamma)$

(ii) $\sin3\alpha+\sin3\beta+\sin3\gamma = 3\sin(\alpha+\beta+\gamma)$

solution:

let $a = \cos\alpha + i\sin\alpha = cis\,\alpha$

$b = \cos\beta + i\sin\beta = cis\,\beta$

$c = \cos\gamma + i\sin\gamma = cis\,\gamma$

$$a+b+c = (\cos\alpha+\cos\beta+\cos\gamma) + i(\sin\alpha+\sin\beta+\sin\gamma)$$

$$= 0+i0$$

$$a+b+c = 0$$

$$\Rightarrow a^3+b^3+c^3 = 3abc$$

$$\Rightarrow (e^{i\alpha})^3 + (e^{i\beta})^3 + (e^{i\gamma})^3 = 3e^{i\alpha}\cdot e^{i\beta}\cdot e^{i\gamma}$$

$$e^{i3\alpha} + e^{i3\beta} + e^{i3\gamma} = 3e^{i(\alpha+\beta+\gamma)}$$

$$\cos3\alpha + i\sin3\alpha + \cos3\beta + i\sin3\beta + \cos3\gamma + i\sin3\gamma = 3[\cos(\alpha+\beta+\gamma) + i\sin(\alpha+\beta+\gamma)]$$

Equating Real, Imaginary part,

$$\cos3\alpha+\cos3\beta+\cos3\gamma = 3\cos(\alpha+\beta+\gamma)$$

$$\sin3\alpha+\sin3\beta+\sin3\gamma = 3\sin(\alpha+\beta+\gamma).$$

de Moivre's Theorem:

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$$

corollary:

* $(\cos\theta - i\sin\theta)^n = \cos n\theta - i\sin n\theta$
* $(\cos\theta + i\sin\theta)^{-n} = \cos n\theta - i\sin n\theta$
* $(\cos\theta - i\sin\theta)^{-n} = \cos n\theta + i\sin n\theta$
* $\sin\theta + i\cos\theta = i(\cos\theta - i\sin\theta)$

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Exercise: 2.8

② S.T $\left(\frac{\sqrt{3}}{2}+\frac{i}{2}\right)^5+\left(\frac{\sqrt{3}}{2}-\frac{i}{2}\right)^5=-\sqrt{3}$

Solution:

$\frac{\sqrt{3}}{2}+\frac{i}{2}=r(\cos\theta+i\sin\theta)$

$r=\sqrt{\frac{3}{4}+\frac{1}{4}}=\sqrt{\frac{4}{4}}=1$

$\alpha=\tan^{-1}\left|\frac{\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right|=\tan^{-1}\frac{1}{\sqrt{3}}=\frac{\pi}{6}$

$\frac{\sqrt{3}}{2}+\frac{i}{2}=\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}$

$\left(\frac{\sqrt{3}}{2}+\frac{i}{2}\right)^5=\left(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right)^5$

$=\cos\frac{5\pi}{6}+i\sin\frac{5\pi}{6}$ ——①

IIly

$\left(\frac{\sqrt{3}}{2}-\frac{i}{2}\right)^5=\cos\frac{5\pi}{6}-i\sin\frac{5\pi}{6}$ ——②

① + ② ⇒

$\left(\frac{\sqrt{3}}{2}+\frac{i}{2}\right)^5+\left(\frac{\sqrt{3}}{2}-\frac{i}{2}\right)^5=2\cos\frac{5\pi}{6}$

$=2\times-\cos 30°$

$=-2\times\frac{\sqrt{3}}{2}$

$=-\sqrt{3}$

$\cos\frac{5\pi}{6}=\cos 150°=\cos(180°-30°)$

$=-\cos 30°$

⑤ Solve the eqn $z^3+27=0$

Solution:

$z^3+27=0$

$z^3=-27$

$=27(-1)$

$=\cancel{(3^3(\cos\pi}$

$z=(27)^{\frac{1}{3}}(-1)^{\frac{1}{3}}$

$=(3^3)^{\frac{1}{3}}(\cos\pi+i\sin\pi)^{\frac{1}{3}}$

$z=3\left[\cos(2k\pi+\pi)+i\sin(2k\pi+\pi)\right]^{\frac{1}{3}}$

$=3\left[\cos\left(\frac{2k\pi+\pi}{3}\right)+i\sin\left(\frac{2k\pi+\pi}{3}\right)\right]$

$=3\left[\cos(2k+1)\frac{\pi}{3}+i\sin(2k+1)\frac{\pi}{3}\right]$

$k=0,1,2$

$k=0$, $3\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)$ or $3\operatorname{cis}\frac{\pi}{3}$

$k=1$, $3(\underbrace{\cos\pi}_{-1}+i\underbrace{\sin\pi}_{0})$ or $3\operatorname{cis}\pi=-3$

$k=2$, $3\left(\cos\frac{5\pi}{3}+i\sin\frac{5\pi}{3}\right)$ or $3\operatorname{cis}\frac{5\pi}{3}$

③ Find the value of

$$\left[\frac{1+\sin\frac{\pi}{10}+i\cos\frac{\pi}{10}}{1+\sin\frac{\pi}{10}-i\cos\frac{\pi}{10}}\right]^{10}$$

Solution:

let $z=\sin\frac{\pi}{10}+i\cos\frac{\pi}{10}$

$\frac{1}{z}=\sin\frac{\pi}{10}-i\cos\frac{\pi}{10}$

$\left[\frac{1+\sin\frac{\pi}{10}+i\cos\frac{\pi}{10}}{1+\sin\frac{\pi}{10}-i\cos\frac{\pi}{10}}\right]^{10}=\left[\frac{1+z}{1+\frac{1}{z}}\right]^{10}$

$=\left[\frac{z(\cancel{1+z})}{\cancel{z+1}}\right]^{10}$

$=(z)^{10}$

$=\sin\frac{\pi}{10}+i\cos\frac{\pi}{10}$

$=\left[\cos\left(\frac{\pi}{2}-\frac{\pi}{10}\right)+i\sin\left(\frac{\pi}{2}-\frac{\pi}{10}\right)\right]^{10}$

$=\left[\cos\frac{8\pi}{20}+i\sin\frac{8\pi}{20}\right]^{10}$

$=\left[\cos\frac{4\pi}{10}+i\sin\frac{4\pi}{10}\right]^{10}$

$=\cos 4\pi+i\sin 4\pi$

$=1+i(0)$

$=1.$

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① If $w \neq 1$ is a cube root of unity then S.T

$$\frac{a+bw+cw^2}{b+cw+aw^2}+\frac{a+bw+cw^2}{c+aw+bw^2}=-1$$

Solution:

$$LHS=\frac{a+bw+cw^2}{b+cw+aw^2}+\frac{a+bw+cw^2}{c+aw+bw^2}$$

$$=\frac{aw^3+bw+cw^2}{b+cw+aw^2}+\frac{a+bw+cw^2}{cw^3+aw+bw^2} \quad [w^3=1]$$

$$=\frac{w(\cancel{aw^2+b+cw})}{\cancel{b+cw+aw^2}}+\frac{\cancel{a+bw+cw^2}}{w(\cancel{cw^2+a+bw})}$$

$$=w+\frac{1}{w} \qquad 1+w+w^2=0$$

$$=\frac{w^2+1}{w}$$

$$=\frac{-w}{w}$$

$$=-1$$

$=$ RHS.

④ If $2\cos\alpha = x+\frac{1}{x}$, $2\cos\beta = y+\frac{1}{y}$

S.T (i) $\frac{x}{y}+\frac{y}{x}=2\cos(\alpha-\beta)$

(ii) $xy-\frac{1}{xy}=2i\sin(\alpha+\beta)$

(iii) $\frac{x^m}{y^n}-\frac{y^n}{x^m}=2i\sin(m\alpha-n\beta)$

(iv) $x^my^n+\frac{1}{x^my^n}=2\cos(m\alpha+n\beta)$

Solution:

$$2\cos\alpha = x+1/x$$

$$2\cos\alpha=\frac{x^2+1}{x}$$

$$2x\cos\alpha = x^2+1$$

$$x^2-2x\cos\alpha+1=0$$

$$a=1,\ b=-2\cos\alpha,\ c=1$$

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

$$x=\frac{2\cos\alpha\pm\sqrt{4\cos^2\alpha-4}}{2}$$

$$=\frac{2\cos\alpha\pm\sqrt{4(\cos^2\alpha-1)}}{2}$$

$$=\frac{2\cos\alpha\pm2\sqrt{-\sin^2\alpha}}{2}$$

$$=\frac{\cancel{2}\cos\alpha\pm i\cancel{2}\sin\alpha}{\cancel{2}}$$

$$x=\cos\alpha\pm i\sin\alpha$$

Let $x=\cos\alpha+i\sin\alpha = e^{i\alpha}$

Illy $y=\cos\beta+i\sin\beta = e^{i\beta}$

$$\frac{x}{y}=\frac{e^{i\alpha}}{e^{i\beta}}=e^{i\alpha}\cdot e^{-i\beta}=e^{i(\alpha-\beta)}$$

$$=\cos(\alpha-\beta)+i\sin(\alpha-\beta) \quad \text{—①}$$

$$\frac{y}{x}=\cos(\alpha-\beta)-i\sin(\alpha-\beta) \quad \text{—②}$$

① + ② ⇒

$$\frac{x}{y}+\frac{y}{x}=2\cos(\alpha-\beta).$$

(ii)

$$xy=e^{i\alpha}\cdot e^{i\beta}=e^{i(\alpha+\beta)}$$

$$=\cos(\alpha+\beta)+i\sin(\alpha+\beta) \quad \text{—③}$$

$$\frac{1}{xy}=\cos(\alpha+\beta)-i\sin(\alpha+\beta) \quad \text{—④}$$

③ − ④ ⇒

$$xy-\frac{1}{xy}=2i\sin(\alpha+\beta)$$

(iii) $x^m=(\cos\alpha+i\sin\alpha)^m=(e^{i\alpha})^m=e^{im\alpha}$

$$y^n=(\cos\beta+i\sin\beta)^n=(e^{i\beta})^n=e^{in\beta}$$

$$\frac{x^m}{y^n}=\frac{e^{im\alpha}}{e^{in\beta}}=e^{im\alpha}\cdot e^{-in\beta}=e^{i(m\alpha-n\beta)}$$

$$=\cos(m\alpha-n\beta)+i\sin(m\alpha-n\beta) \quad \text{—①}$$

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$$\frac{y^n}{x^m} = \cos(m\alpha - n\beta) - i\sin(m\alpha - n\beta) \quad \text{---} (2)$$

$(1) - (2) \Rightarrow$

$$\frac{x^m}{y^n} - \frac{y^n}{x^m} = 2i\sin(m\alpha - n\beta).$$

(iv)

$$x^m y^n = (\cos\alpha + i\sin\alpha)^m (\cos\beta + i\sin\beta)^n$$
$$= (e^{i\alpha})^m \cdot (e^{i\beta})^n$$
$$= e^{im\alpha} \cdot e^{in\beta}$$
$$= e^{i(m\alpha + n\beta)}$$
$$= \cos(m\alpha + n\beta) + i\sin(m\alpha + n\beta) \quad \text{---} (1)$$

$$\frac{1}{x^m y^n} = \cos(m\alpha + n\beta) - i\sin(m\alpha + n\beta) \quad \text{---} (2)$$

$(1) + (2) \Rightarrow$

$$x^m y^n + \frac{1}{x^m y^n} = 2\cos(m\alpha + n\beta).$$

(8) If $w \neq 1$ is a cube root of unity, S.T

(i) $(1 - w + w^2)^6 + (1 + w - w^2)^6 = 128$

Solution:

$$(1 - w + w^2)^6 + (1 + w - w^2)^6 = (-w - w)^6 + (-w^2 - w^2)^6$$
$$= (-2w)^6 + (-2w^2)^6$$
$$= 64w^6 + 64w^{12}$$
$$= 64 + 64$$
$$= 128$$

$\left[\because w^6 = w^3 \cdot w^3 = 1 \right.$

$\left. w^{12} = w^3 \cdot w^3 \cdot w^3 \cdot w^3 = 1 \right]$

(ii) $(1+w)(1+w^2)(1+w^4)(1+w^8) \cdots (1+w^{2^{11}}) = 1$

Solution:

$$(1+w)(1+w^2) = (-w^2)(-w) = w^3 = 1$$
$$(1+w^4)(1+w^8) = (1+w)(1+w^2) = 1$$
$$(1+w^{2^{10}})(1+w^{2^{11}}) = (1+w^{1024})(1+w^{2048})$$
$$= (1+w)(1+w^2)$$
$$= 1$$

$$\therefore (1+w)(1+w^2)(1+w^4)(1+w^8) \cdots (1+w^{2^{11}}) = 1.$$

(9) If $z = 2 - 2i$, find the rotation of z by θ radians in the counter clockwise direction about the origin when

(i) $\theta = \frac{\pi}{3}$ (ii) $\theta = \frac{2\pi}{3}$ (iii) $\theta = \frac{3\pi}{2}$.

$$z = 2 - 2i = r(\cos\theta + i\sin\theta)$$
$$r = \sqrt{4+4} = \sqrt{8} = 2\sqrt{2}$$

$(+, -)$ θ lies on the 4^{th} quadrant

$\theta = -\alpha$ $\quad \alpha = \tan^{-1}\left|\frac{-2}{2}\right|$

$= -\frac{\pi}{4}$ $\quad = \tan^{-1} 1$

$\quad = \pi/4.$

$$z = 2\sqrt{2}\left(\cos -\frac{\pi}{4} + i\sin\left(-\frac{\pi}{4}\right)\right)$$
$$= 2\sqrt{2}\, e^{-i\frac{\pi}{4}}$$

(i) $\theta = \frac{\pi}{3}$

$$z_1 = z\, e^{i\frac{\pi}{3}}$$
$$= 2\sqrt{2}\, e^{-i\frac{\pi}{4}} e^{i\frac{\pi}{3}}$$
$$= 2\sqrt{2}\, e^{i\left(-\frac{\pi}{4} + \frac{\pi}{3}\right)}$$
$$= 2\sqrt{2}\, e^{i\frac{\pi}{12}}$$

(ii) $\theta = \frac{2\pi}{3}$

$$z_2 = z\, e^{i\frac{2\pi}{3}}$$
$$= 2\sqrt{2}\, e^{-i\frac{\pi}{4}} e^{i\frac{2\pi}{3}}$$
$$= 2\sqrt{2}\, e^{i\left(-\frac{\pi}{4} + \frac{2\pi}{3}\right)}$$
$$= 2\sqrt{2}\, e^{i\frac{5\pi}{12}}$$

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(iii) $\theta = \frac{3\pi}{2}$

$$z_3 = z e^{i\frac{3\pi}{2}}$$
$$= 2\sqrt{2}\, e^{-i\frac{\pi}{4}} e^{i\frac{3\pi}{2}}$$
$$= 2\sqrt{2}\, e^{i\left(-\frac{\pi}{4}+\frac{3\pi}{2}\right)}$$
$$= 2\sqrt{2}\, e^{i\left(\frac{5\pi}{4}\right)}$$
$$= 2\sqrt{2}\, e^{i\frac{5\pi}{4}}$$

(10) P.T the values of $\sqrt[4]{-1}$ are $\pm\frac{1}{\sqrt{2}}(1\pm i)$

solution:

$$x = \sqrt[4]{-1}$$
$$= (-1)^{1/4}$$
$$= (\cos\pi + i\sin\pi)^{1/4}$$
$$= [\cos(2k\pi+\pi) + i\sin(2k\pi+\pi)]^{1/4}$$
$$= \cos\frac{(2k+1)\pi}{4} + i\sin\frac{(2k+1)\pi}{4}$$

$k = 0, 1, 2, 3$

$k=0$, $\cos\frac{\pi}{4} + i\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}$

$k=1$, $\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4} = \frac{-1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}$

$k=2$, $\cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4} = \frac{-1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}$

$k=3$, $\cos\frac{7\pi}{4} + i\sin\frac{7\pi}{4} = \frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}$

$$\therefore x = \pm\left(\frac{1}{\sqrt{2}} \pm i\frac{1}{\sqrt{2}}\right)$$
$$= \pm\frac{1}{\sqrt{2}}(1\pm i)$$

(6) If $\omega \neq 1$ is a cube root of unity S.T the roots of the equ $(z-1)^3+8=0$ are $-1, 1-2\omega, 1-2\omega^2$

solution:

$$(z-1)^3 + 8 = 0$$
$$(z-1)^3 = -8$$
$$(z-1)^3 = (-2)^3$$
$$(z-1)^3 = (-2)^3 (1)^{1/3}$$
$$z-1 = -2(1)^{1/3}$$

$z-1=-2$	$z-1=-2\omega$	$z-1=-2\omega^2$
$z=-2+1$	$z=1-2\omega$	$z=1-2\omega^2$
$z=-1$		

The roots are $-1, 1-2\omega, 1-2\omega^2$.

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