

## 1.APPLICATIONS OF MATRICES AND DETERMINANTS

- ❖ If  $|A| \neq 0$ , then the square matrix A is called a **non-singular matrix**.
- ❖ If  $|A| = 0$ , then the square matrix A is called a **singular matrix**.
- ❖ **ADJOINT OF A SQUARE MATRIX:** Let A be the square matrix of order n, then the adjoint matrix of A is defined as the transpose of the matrix of cofactors of A. It is denoted by  $\text{adj } A$ .  $\text{adj } A = [A_{ij}]^T = [(-1)^{i+j} M_{ij}]^T$
- ❖  $A(\text{adj } A) = (\text{adj } A)A = |A|I_n$
- ❖ **INVERSE MATRIX:** Let A be a square matrix of order n. If there exists a square matrix B of order n such that  $AB = BA = I_n$ , then the matrix B is called an inverse of A.
- ❖  $AA^{-1} = A^{-1}A = I_n$
- ❖ A singular matrix has no inverse.
- ❖ If A is non-singular, then
  - (i)  $|A^{-1}| = \frac{1}{|A|}$  (ii)  $(A^T)^{-1} = (A^{-1})^T$  (iii)  $(\lambda A)^{-1} = \frac{1}{\lambda} A^{-1}$
- ❖ If A is non-singular, then  $(\lambda A)^{-1} = \frac{1}{\lambda} A^{-1}$ , where  $\lambda$  is a non-zero scalar.
- ❖  $(AB)^{-1} = B^{-1}A^{-1}$
- ❖ If A is non-singular, then  $A^{-1}$  is also non-singular and  $(A^{-1})^{-1} = A$
- ❖ If A is non-singular square matrix of order n, then
  - (i)  $(\text{adj } A)^{-1} = \text{adj}(A^{-1}) = \frac{1}{|A|} A \Rightarrow \text{adj } A = |A|A^{-1}$
  - (ii)  $|\text{adj } A| = |A|^{n-1}$  (iii)  $\text{adj}(\text{adj } A) = |A|^{n-2} A$
  - (iv)  $\text{adj}(\lambda A) = \lambda^{n-1} \text{adj}(A)$  (v)  $|\text{adj}(\text{adj } A)| = |A|^{(n-1)^2}$
  - (vi)  $(\text{adj } A)^T = \text{adj}(A^T)$  (vii)  $\text{adj}(AB) = (\text{adj } B)(\text{adj } A)$
- ❖  $|\text{adj } A| = |A|^2 \Rightarrow |A| = \pm \sqrt{|\text{adj } A|}$
- ❖  $A^{-1} = \pm \frac{1}{\sqrt{|\text{adj } A|}} \text{adj } A$
- ❖  $A = \pm \frac{1}{\sqrt{|\text{adj } A|}} \text{adj}(\text{adj } A)$
- ❖ If A and B are any two non-singular square matrices of order n, then  $\text{adj}(AB) = (\text{adj } B)(\text{adj } A)$
- ❖ A square matrix A is called orthogonal if  $AA^T = A^T A = I$
- ❖ A is orthogonal if and only if A is non-singular and  $A^{-1} = A^T$
- ❖ Application of matrices is in computer graphic and cryptography.

**THEOREM 1:** For every square matrix A of order n,  $A(\text{adj } A) = (\text{adj } A)A = |A|I_n$

**Proof:**

For simplicity, we prove the theorem for  $n=3$  only.

$$\text{Consider } A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \text{ and } \text{adj } A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$$

$$A(\text{adj } A) = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} = \begin{bmatrix} |A| & 0 & 0 \\ 0 & |A| & 0 \\ 0 & 0 & |A| \end{bmatrix}$$

$$= |A| \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = |A|I_3 \rightarrow (1)$$

$$(\text{adj } A)A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} |A| & 0 & 0 \\ 0 & |A| & 0 \\ 0 & 0 & |A| \end{bmatrix}$$

$$= |A| \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = |A|I_3 \rightarrow (2)$$

From (1) & (2), we get  $A(\text{adj } A) = (\text{adj } A)A = |A|I_n$

**THEOREM 2:** If a square matrix has an inverse, then it is unique.

**Proof:**

Let A be a square matrix of order n such that an inverse of A exists. If possible, let there be two inverse B and C of A.

By definition,  $AB = BA = I_n$  and  $AC = CA = I_n$

$C = CI_n = C(AB) = (CA)B = I_n B = B \Rightarrow B = C$

Hence it is proved.

**THEOREM 3:** Let A be a square matrix of order n. Then  $A^{-1}$  exists if and only if A is non-singular.

**Proof:**

Case (i) : Suppose that  $A^{-1}$  exists, then  $AA^{-1} = A^{-1}A = I_n$ .

By the product rule for determinants,

$\det(AA^{-1}) = \det(A^{-1})\det(A) = \det(I_n) = 1 \Rightarrow |A| \neq 0$ .

Hence A is non-singular.

Case (ii) : Conversely, suppose A is non-singular, then  $|A| \neq 0$ .

$$\text{W.k.T. } A(\text{adj } A) = (\text{adj } A)A = |A|I_n$$

$$\div \text{ by } |A| \Rightarrow A \left( \frac{1}{|A|} \text{adj } A \right) = \left( \frac{1}{|A|} \text{adj } A \right) A = I_n$$

$$\Rightarrow AB = BA = I_n ; B = \frac{1}{|A|} \text{adj } A$$

Hence the inverse of A exists and the inverse is  $A^{-1} = \frac{1}{|A|} \text{adj } A$

### PROPERTIES OF INVERSES OF MATRICES

**THEOREM 4: If A is non-singular, then  $|A^{-1}| = \frac{1}{|A|}$**

**Proof:**

Let A be non-singular then  $|A| \neq 0$  and  $A^{-1}$  exists.

$$\text{By definition, } AA^{-1} = A^{-1}A = I_n \Rightarrow |AA^{-1}| = |A^{-1}A| = |I_n|$$

By the product rule for determinants,

$$|AA^{-1}| = |I_n| \Rightarrow |A||A^{-1}| = 1 \Rightarrow |A^{-1}| = \frac{1}{|A|}$$

**THEOREM 5: If A is non-singular, then  $(A^T)^{-1} = (A^{-1})^T$**

**Proof:**

$$\text{By definition, } AA^{-1} = A^{-1}A = I_n$$

$$\text{Taking transpose, } (AA^{-1})^T = (A^{-1}A)^T = (I_n)^T$$

$$\text{By the reversal law of transpose, } (A^{-1})^T A^T = A^T (A^{-1})^T = |I_n|$$

$$\text{Hence } (A^T)^{-1} = (A^{-1})^T$$

**THEOREM 6: If A is non-singular, then  $(\lambda A)^{-1} = \frac{1}{\lambda} A^{-1}$ , where  $\lambda$  is a non-zero scalar.**

**Proof:**

$$\text{By definition, } AA^{-1} = A^{-1}A = I_n$$

$$\text{Multiply \& divide by } \lambda, (\lambda A) \left( \frac{1}{\lambda} A^{-1} \right) = \left( \frac{1}{\lambda} A^{-1} \right) (\lambda A) = I_n \Rightarrow (\lambda A)^{-1} = \frac{1}{\lambda} A^{-1}$$

**THEOREM 7: LEFT CANCELLATION LAW**

**Let A, B and C be square matrices of order n. If A is non-singular and  $AB=AC$  then  $B=C$**

**Proof:**

Since A is non-singular,  $A^{-1}$  exists and  $AA^{-1} = A^{-1}A = I_n$ .

Given  $AB=AC$ . Pre multiply by  $A^{-1}$  on both sides,

$$A^{-1}(AB) = A^{-1}(AC) \Rightarrow (A^{-1}A)B = (A^{-1}A)C \Rightarrow I_n B = I_n C \Rightarrow B = C$$

**THEOREM 8: RIGHT CANCELLATION LAW**

**Let A, B and C be square matrices of order n. If A is non-singular and  $BA=CA$  then  $B=C$**

**Proof:**

Since A is non-singular,  $A^{-1}$  exists and  $AA^{-1} = A^{-1}A = I_n$ .

Given  $BA=CA$ . Post multiply by  $A^{-1}$  on both sides,

$$(BA)A^{-1} = (CA)A^{-1} \Rightarrow B(AA^{-1}) = C(AA^{-1}) \Rightarrow BI_n = CI_n \Rightarrow B = C$$

**THEOREM 9: REVERSAL LAW FOR INVERSES**

**If A and B are non-singular matrices of the same order, then the product AB is also non-singular and  $(AB)^{-1} = B^{-1}A^{-1}$**

**Proof:**

Since A and B are non-singular matrices of the same order n, then  $|A| \neq 0, |B| \neq 0 \Rightarrow A^{-1}$  and  $B^{-1}$  of order n exists. The product of AB and  $B^{-1}A^{-1}$  can also found.

$$|AB| = |A||B| \neq 0 \Rightarrow AB \text{ is also non-singular and } (AB)^{-1} \text{ exists.}$$

$$(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = A(I_n)A^{-1} = AA^{-1} = I_n$$

$$(B^{-1}A^{-1})(AB) = B^{-1}(A^{-1}A)B = B^{-1}(I_n)B = B^{-1}B = I_n$$

$$\text{Hence } (AB)^{-1} = B^{-1}A^{-1}$$

**THEOREM 10: LAW OF DOUBLE INVERSE**

**If A is non-singular, then  $A^{-1}$  is also non-singular and  $(A^{-1})^{-1} = A$**

**Proof:**

Since A is non-singular, then  $|A| \neq 0$  and  $A^{-1}$  exists.

$$\text{Now } |A^{-1}| = \frac{1}{|A|} \neq 0 \Rightarrow A^{-1} \text{ is also non-singular and } AA^{-1} = A^{-1}A = I_n.$$

$$\text{Now } AA^{-1} = I_n \Rightarrow (AA^{-1})^{-1} = I \Rightarrow (A^{-1})^{-1}A^{-1} = I$$

$$\text{Post- multiply by A on both sides, } (A^{-1})^{-1} = A$$

**THEOREM 11:** If  $A$  is non-singular square matrix of order  $n$ , then  $(\text{adj}A)^{-1} = \text{adj}(A^{-1}) = \frac{1}{|A|} A$

**Proof:**

$$A^{-1} = \frac{1}{|A|} (\text{adj} A) \Rightarrow \text{adj} A = |A| A^{-1}$$

$$\Rightarrow (\text{adj} A)^{-1} = (|A| A^{-1})^{-1} = (A^{-1})^{-1} |A|^{-1} = \frac{1}{|A|} A \rightarrow (1)$$

$$A^{-1} = \frac{1}{|A|} (\text{adj} A) \Rightarrow \text{adj} A = |A| A^{-1}$$

$$\text{Replacing } A \text{ by } A^{-1}, \text{adj}(A^{-1}) = |A^{-1}| (A^{-1})^{-1} = \frac{1}{|A|} A \rightarrow (2)$$

$$\text{From (1) and (2), } (\text{adj}A)^{-1} = \text{adj}(A^{-1}) = \frac{1}{|A|} A$$

**THEOREM 12:** If  $A$  is non-singular square matrix of order  $n$ , then  $|\text{adj}A| = |A|^{n-1}$

**Proof:**

$$A(\text{adj} A) = (\text{adj} A)A = |A|I_n \Rightarrow \det[A(\text{adj} A)] = \det(A)\det(\text{adj}A) = \det[|A|I_n] \\ \Rightarrow |A||\text{adj} A| = |A|^n \Rightarrow |\text{adj}A| = |A|^{n-1}$$

**THEOREM 13:** If  $A$  is non-singular square matrix of order  $n$ , then  $\text{adj}(\text{adj}A) = |A|^{n-2} A$

**Proof:**

$$\text{For any non-singular matrix } B \text{ of order } n, B(\text{adj} B) = (\text{adj} B)B = |B|I_n$$

$$\text{Put } B = \text{adj}A \Rightarrow \text{adj}A(\text{adj}(\text{adj}A)) = |\text{adj}A|I_n$$

$$\Rightarrow \text{adj}A(\text{adj}(\text{adj}A)) = |A|^{n-1}I_n$$

$$\text{Pre-multiply by } A \text{ on both sides,}$$

$$\Rightarrow [A(\text{adj}A)](\text{adj}(\text{adj}A)) = A|A|^{n-1}I_n \Rightarrow |A|I_n(\text{adj}(\text{adj}A)) = A|A|^{n-1}I_n$$

$$\Rightarrow \text{adj}(\text{adj}A) = |A|^{n-2}A$$

**THEOREM 14:** If  $A$  is non-singular square matrix of order  $n$ , then  $\text{adj}(\lambda A) = \lambda^{n-1} \text{adj}(A)$

**Proof:**

$$\text{W.K.T, } (\text{adj}A)^{-1} = \frac{1}{|A|} A \Rightarrow \text{adj} A = |A| A^{-1}$$

$$\text{Replace } A \text{ by } \lambda A \Rightarrow \text{adj}(\lambda A) = |\lambda A|(\lambda A)^{-1} = \lambda^n |A| \frac{1}{\lambda} A^{-1} = \lambda^{n-1} |A| A^{-1} = \lambda^{n-1} \text{adj}(A)$$

**THEOREM 15:** If  $A$  is non-singular square matrix of order  $n$ , then  $|\text{adj}(\text{adj}A)| = |A|^{(n-1)^2}$

**Proof:**

$$\text{W.K.T, } \text{adj}(\text{adj}A) = |A|^{n-2} A$$

$$\Rightarrow |\text{adj}(\text{adj}A)| = ||A|^{n-2} A| = (|A|^{n-2})^n |A| = |A|^{n^2-2n+1} = |A|^{(n-1)^2}$$

**THEOREM 16:** If  $A$  is non-singular square matrix of order  $n$ , then  $(\text{adj}A)^T = \text{adj}(A^T)$

**Proof:**

$$\text{W.K.T, } A^{-1} = \frac{1}{|A|} (\text{adj} A)$$

$$\text{Replace } A \text{ by } A^T \Rightarrow (A^T)^{-1} = \frac{1}{|A^T|} (\text{adj}(A^T))$$

$$\Rightarrow \text{adj}(A^T) = |A^T| (A^T)^{-1} = (|A| A^{-1})^T = (\text{adj}A)^T$$

**THEOREM 17:** If  $A$  and  $B$  are any two non-singular matrices of order  $n$ , then  $\text{adj}(AB) = (\text{adj}B)(\text{adj}A)$

**Proof:**

$$\text{W.K.T, } \text{adj} A = |A| A^{-1}$$

$$\text{Replace } A \text{ by } AB \Rightarrow \text{adj}(AB) = |AB|(AB)^{-1} = |A||B|A^{-1}B^{-1}$$

$$= (|B|B^{-1})(|A|A^{-1}) = (\text{adj}B)(\text{adj}A)$$

### EXERCISE 1.1

1. Find the adjoint of the following matrices.

$$(i) \begin{bmatrix} -3 & 4 \\ 6 & 2 \end{bmatrix}$$

$$\text{Let } A = \begin{bmatrix} -3 & 4 \\ 6 & 2 \end{bmatrix} \Rightarrow \text{adj} A = \begin{bmatrix} 2 & -4 \\ -6 & -3 \end{bmatrix}$$

$$(ii) \begin{bmatrix} 2 & 3 & 1 \\ 3 & 4 & 1 \\ 3 & 7 & 2 \end{bmatrix}$$

$$\text{Let } A = \begin{bmatrix} 2 & 3 & 1 \\ 3 & 4 & 1 \\ 3 & 7 & 2 \\ 4 & 7 & 3 & 4 \\ 1 & 2 & 1 & 1 \\ 3 & 3 & 2 & 3 \\ 4 & 7 & 3 & 4 \end{bmatrix} \Rightarrow \text{adj } A = \begin{bmatrix} 8-7 & 7-6 & 3-4 \\ 3-6 & 4-3 & 3-2 \\ 21-12 & 9-14 & 8-9 \end{bmatrix} = \begin{bmatrix} 1 & 1 & -1 \\ -3 & 1 & 1 \\ 9 & -5 & -1 \end{bmatrix}$$

$$\text{(iii)} \frac{1}{3} \begin{bmatrix} 2 & 2 & 1 \\ -2 & 1 & 2 \\ 1 & -2 & 2 \\ 1 & -2 & 2 & 1 \\ 2 & 2 & 1 & 2 \\ -2 & 1 & 2 & -2 \\ 1 & -2 & 2 & 1 \end{bmatrix} \Rightarrow \text{adj } A = \left(\frac{1}{3}\right)^{3-1} \begin{bmatrix} 2+4 & -2-4 & 4-1 \\ 2+4 & 4-1 & -2-4 \\ 4-1 & 2+4 & 2+4 \end{bmatrix}$$

$$= \frac{1}{9} \begin{bmatrix} 6 & -6 & 3 \\ 6 & 3 & -6 \\ 3 & 6 & 6 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 2 & -2 & 1 \\ 2 & 1 & -2 \\ 1 & 2 & 2 \end{bmatrix}$$

$\text{adj}(\lambda A) = \lambda^{n-1} \text{adj}(A)$

2. Find the inverse of the following

(i)  $\begin{bmatrix} -2 & 4 \\ 1 & -3 \end{bmatrix}$

Let  $A = \begin{bmatrix} -2 & 4 \\ 1 & -3 \end{bmatrix}$  ;  $|A| = \begin{vmatrix} -2 & 4 \\ 1 & -3 \end{vmatrix} = 6 - 4 = 2 \neq 0$  ;  $\text{adj } A = \begin{bmatrix} -3 & -4 \\ -1 & -2 \end{bmatrix}$

$A^{-1} = \frac{1}{|A|} \text{adj } A \Rightarrow A^{-1} = \frac{1}{2} \begin{bmatrix} -3 & -4 \\ -1 & -2 \end{bmatrix}$

(ii)  $\begin{bmatrix} 5 & 1 & 1 \\ 1 & 5 & 1 \\ 1 & 1 & 5 \end{bmatrix}$

Let  $A = \begin{bmatrix} 5 & 1 & 1 \\ 1 & 5 & 1 \\ 1 & 1 & 5 \end{bmatrix}$

$|A| = \begin{vmatrix} 5 & 1 & 1 \\ 1 & 5 & 1 \\ 1 & 1 & 5 \end{vmatrix} = 5(25 - 1) - 1(5 - 1) + 1(1 - 5)$

$= 120 - 4 - 4$

$= 112 \neq 0$

$$\begin{bmatrix} 5 & 1 & 1 & 5 \\ 1 & 5 & 1 & 1 \\ 1 & 1 & 5 & 1 \\ 5 & 1 & 1 & 5 \end{bmatrix} \Rightarrow \text{adj } A = \begin{bmatrix} 25-1 & 1-5 & 1-5 \\ 1-5 & 25-1 & 1-5 \\ 1-5 & 1-5 & 25-1 \end{bmatrix} = \begin{bmatrix} 24 & -4 & -4 \\ -4 & 24 & -4 \\ -4 & -4 & 24 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|} \text{adj } A \Rightarrow A^{-1} = \frac{1}{112} \begin{bmatrix} 24 & -4 & -4 \\ -4 & 24 & -4 \\ -4 & -4 & 24 \end{bmatrix} = \frac{1}{28} \begin{bmatrix} 6 & -1 & -1 \\ -1 & 6 & -1 \\ -1 & -1 & 6 \end{bmatrix}$$

(iii)  $\begin{bmatrix} 2 & 3 & 1 \\ 3 & 4 & 1 \\ 3 & 7 & 2 \end{bmatrix}$

Let  $A = \begin{bmatrix} 2 & 3 & 1 \\ 3 & 4 & 1 \\ 3 & 7 & 2 \end{bmatrix}$

$|A| = \begin{vmatrix} 2 & 3 & 1 \\ 3 & 4 & 1 \\ 3 & 7 & 2 \end{vmatrix} = 2(8 - 7) - 3(6 - 3) + 1(21 - 12) = 2 - 9 + 9 = 2 \neq 0$

$\begin{bmatrix} 4 & 7 & 3 & 4 \\ 1 & 2 & 1 & 1 \\ 3 & 3 & 2 & 3 \\ 4 & 7 & 3 & 4 \end{bmatrix} \Rightarrow \text{adj } A = \begin{bmatrix} 8-7 & 7-6 & 3-4 \\ 3-6 & 4-3 & 3-2 \\ 21-12 & 9-14 & 8-9 \end{bmatrix} = \begin{bmatrix} 1 & 1 & -1 \\ -3 & 1 & 1 \\ 9 & -5 & -1 \end{bmatrix}$

$A^{-1} = \frac{1}{|A|} \text{adj } A \Rightarrow A^{-1} = \frac{1}{2} \begin{bmatrix} 1 & 1 & -1 \\ -3 & 1 & 1 \\ 9 & -5 & -1 \end{bmatrix}$

**EXAMPLE 1.2** Find the inverse of  $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$

**EXAMPLE 1.3** Find the inverse of  $\begin{bmatrix} 2 & -1 & 3 \\ -5 & 3 & 1 \\ -3 & 2 & 3 \end{bmatrix}$

3. If  $F(\alpha) = \begin{bmatrix} \cos \alpha & 0 & \sin \alpha \\ 0 & 1 & 0 \\ -\sin \alpha & 0 & \cos \alpha \end{bmatrix}$ , show that  $[F(\alpha)]^{-1} = F(-\alpha)$

$F(-\alpha) = \begin{bmatrix} \cos(-\alpha) & 0 & \sin(-\alpha) \\ 0 & 1 & 0 \\ -\sin(-\alpha) & 0 & \cos(-\alpha) \end{bmatrix} = \begin{bmatrix} \cos \alpha & 0 & -\sin \alpha \\ 0 & 1 & 0 \\ \sin \alpha & 0 & \cos \alpha \end{bmatrix} \rightarrow (1)$

( $\because \cos(-\alpha) = \cos \alpha$   
 $\sin(-\alpha) = -\sin \alpha$ )

$$|F(\alpha)| = \begin{vmatrix} \cos \alpha & 0 & \sin \alpha \\ 0 & 1 & 0 \\ -\sin \alpha & 0 & \cos \alpha \end{vmatrix} = \cos \alpha (\cos \alpha) - 0 + \sin \alpha (\sin \alpha) = \cos^2 \alpha + \sin^2 \alpha = 1$$

$$\begin{vmatrix} 1 & 0 & 0 & 1 \\ 0 & \cos \alpha & \sin \alpha & 0 \\ 0 & -\sin \alpha & \cos \alpha & 0 \\ 1 & 0 & 0 & 1 \end{vmatrix} \Rightarrow \begin{vmatrix} \cos \alpha - 0 & 0 - 0 & 0 - \sin \alpha \\ 0 - 0 & \cos^2 \alpha + \sin^2 \alpha & 0 - 0 \\ 0 + \sin \alpha & 0 - 0 & \cos \alpha - 0 \end{vmatrix} = \begin{vmatrix} \cos \alpha & 0 & -\sin \alpha \\ 0 & 1 & 0 \\ \sin \alpha & 0 & \cos \alpha \end{vmatrix}$$

$$[F(\alpha)]^{-1} = \frac{1}{|F(\alpha)|} \text{adj } F(\alpha) \Rightarrow [F(\alpha)]^{-1} = \frac{1}{1} \begin{bmatrix} \cos \alpha & 0 & -\sin \alpha \\ 0 & 1 & 0 \\ \sin \alpha & 0 & \cos \alpha \end{bmatrix}$$

$$[F(\alpha)]^{-1} = \begin{bmatrix} \cos \alpha & 0 & -\sin \alpha \\ 0 & 1 & 0 \\ \sin \alpha & 0 & \cos \alpha \end{bmatrix} \rightarrow (2)$$

From (1) & (2),  $[F(\alpha)]^{-1} = F(-\alpha)$

4. If  $A = \begin{bmatrix} 5 & 3 \\ -1 & -2 \end{bmatrix}$ , show that  $A^2 - 3A - 7I_2 = 0_2$ . Hence find  $A^{-1}$

$$A^2 = \begin{bmatrix} 5 & 3 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} 5 & 3 \\ -1 & -2 \end{bmatrix} = \begin{bmatrix} 25 - 3 & 15 - 6 \\ -5 + 2 & -3 + 4 \end{bmatrix} = \begin{bmatrix} 22 & 9 \\ -3 & 1 \end{bmatrix}$$

$$-3A = -3 \begin{bmatrix} 5 & 3 \\ -1 & -2 \end{bmatrix} = \begin{bmatrix} -15 & -9 \\ 3 & 6 \end{bmatrix}$$

$$-7I_2 = -7 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -7 & 0 \\ 0 & -7 \end{bmatrix}$$

$$A^2 - 3A - 7I_2 = \begin{bmatrix} 22 & 9 \\ -3 & 1 \end{bmatrix} + \begin{bmatrix} -15 & -9 \\ 3 & 6 \end{bmatrix} + \begin{bmatrix} -7 & 0 \\ 0 & -7 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0_2$$

$A^2 - 3A - 7I_2 = 0_2$  pre-multiply by  $A^{-1}$  on both sides, we get

$$A - 3I_2 - 7A^{-1} = 0_2 \Rightarrow 7A^{-1} = A - 3I_2 \Rightarrow A^{-1} = \frac{1}{7}[A - 3I_2]$$

$$A^{-1} = \frac{1}{7} \left\{ \begin{bmatrix} 5 & 3 \\ -1 & -2 \end{bmatrix} - 3 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right\} = \frac{1}{7} \left\{ \begin{bmatrix} 5 & 3 \\ -1 & -2 \end{bmatrix} + \begin{bmatrix} -3 & 0 \\ 0 & -3 \end{bmatrix} \right\} = \frac{1}{7} \begin{bmatrix} 2 & 3 \\ -1 & -5 \end{bmatrix}$$

**EXAMPLE 1.10** If  $A = \begin{bmatrix} 4 & 3 \\ 2 & 5 \end{bmatrix}$ , find x and y such that  $A^2 + xA + yI_2 = 0_2$ .

Hence find  $A^{-1}$

$$A^2 = \begin{bmatrix} 4 & 3 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} 4 & 3 \\ 2 & 5 \end{bmatrix} = \begin{bmatrix} 16 + 6 & 12 + 15 \\ 8 + 10 & 6 + 25 \end{bmatrix} = \begin{bmatrix} 22 & 27 \\ 18 & 31 \end{bmatrix}$$

$$xA = x \begin{bmatrix} 4 & 3 \\ 2 & 5 \end{bmatrix} = \begin{bmatrix} 4x & 3x \\ 2x & 5x \end{bmatrix} ; yI_2 = y \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} y & 0 \\ 0 & y \end{bmatrix}$$

$$A^2 + xA + yI_2 = 0_2 \Rightarrow \begin{bmatrix} 22 & 27 \\ 18 & 31 \end{bmatrix} + \begin{bmatrix} 4x & 3x \\ 2x & 5x \end{bmatrix} + \begin{bmatrix} y & 0 \\ 0 & y \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 22 + 4x + y & 27 + 3x \\ 18 + 2x & 31 + 5x + y \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{cases} 27 + 3x = 0 \Rightarrow 3x = -27 \Rightarrow \boxed{x = -9} \\ 22 + 4x + y = 0 \Rightarrow 22 + 4(-9) + y = 0 \Rightarrow 22 - 36 + y = 0 \Rightarrow \boxed{y = 14} \end{cases}$$

$$A^2 + xA + yI_2 = 0_2 \Rightarrow A^2 - 9A + 14I_2 = 0_2$$

pre-multiply by  $A^{-1}$  on both sides, we get

$$A - 9I_2 + 14A^{-1} = 0_2 \Rightarrow 14A^{-1} = 9I_2 - A \Rightarrow A^{-1} = \frac{1}{14}[9I_2 - A]$$

$$A^{-1} = \frac{1}{14} \left\{ 9 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 4 & 3 \\ 2 & 5 \end{bmatrix} \right\} = \frac{1}{14} \left\{ \begin{bmatrix} 9 & 0 \\ 0 & 9 \end{bmatrix} - \begin{bmatrix} 4 & 3 \\ 2 & 5 \end{bmatrix} \right\} = \frac{1}{14} \begin{bmatrix} 5 & -3 \\ -2 & 4 \end{bmatrix}$$

$$5. \text{ If } A = \frac{1}{9} \begin{bmatrix} -8 & 1 & 4 \\ 4 & 4 & 7 \\ 1 & -8 & 4 \end{bmatrix}, \text{ prove that } A^{-1} = A^T$$

$$A^T = \frac{1}{9} \begin{bmatrix} -8 & 4 & 1 \\ 1 & 4 & 8 \\ 4 & 7 & 4 \end{bmatrix} \rightarrow (1)$$

$$|A| = \left(\frac{1}{9}\right)^3 [-8(16 - 56) - 1(16 - 7) + 4(7 - 16)] = \frac{1}{729} [-576 - 9 - 144] = \frac{1}{729} [-729] = -1$$

$$(\because |KA| = K^n |A|)$$

$$\begin{bmatrix} 4 & -8 & 1 & 4 \\ 7 & 4 & 4 & 7 \\ 4 & 1 & -8 & 4 \\ 4 & -8 & 1 & 4 \end{bmatrix} \Rightarrow$$

$$\text{adj}(\lambda A) = \lambda^{n-1} \text{adj}(A)$$

$$\text{adj } A = \left(\frac{1}{9}\right)^{3-1} \begin{bmatrix} 16 + 56 & -32 - 4 & 7 - 16 \\ 7 - 16 & -32 - 4 & 16 + 56 \\ -32 - 4 & 1 - 64 & -32 - 4 \end{bmatrix} = \frac{1}{81} \begin{bmatrix} 72 & -36 & -9 \\ -9 & -36 & 72 \\ -36 & -63 & -36 \end{bmatrix}$$

$$= \frac{1}{9} \begin{bmatrix} 8 & -4 & -1 \\ -1 & -4 & 8 \\ -4 & -7 & -4 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|} \text{adj } A \Rightarrow A^{-1} = \frac{1}{-1} \cdot \frac{1}{9} \begin{bmatrix} 8 & -4 & -1 \\ -1 & -4 & 8 \\ -4 & -7 & -4 \end{bmatrix} = \frac{1}{9} \begin{bmatrix} -8 & 4 & 1 \\ 1 & 4 & 8 \\ 4 & 7 & 4 \end{bmatrix} \rightarrow (2)$$

From (1) & (2),  $A^{-1} = A^T$

6. If  $A = \begin{bmatrix} 8 & -4 \\ -5 & 3 \end{bmatrix}$ , verify that  $A(adjA) = (adjA)A = |A|I_2$

$$|A| = 24 - 20 = 4$$

$$|A|I_2 = 4 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} \rightarrow (1)$$

$$adjA = \begin{bmatrix} 3 & 4 \\ 5 & 8 \end{bmatrix}$$

$$A(adjA) = \begin{bmatrix} 8 & -4 \\ -5 & 3 \end{bmatrix} \begin{bmatrix} 3 & 4 \\ 5 & 8 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} \rightarrow (2)$$

$$(adjA)A = \begin{bmatrix} 3 & 4 \\ 5 & 8 \end{bmatrix} \begin{bmatrix} 8 & -4 \\ -5 & 3 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} \rightarrow (3)$$

From (1),(2) & (3),  $A(adjA) = (adjA)A = |A|I_2$

**EXAMPLE 1.1** If  $A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$ , verify that  $A(adjA) = (adjA)A = |A|I_3$

$$|A| = 8(21 - 16) - (-6)(-18 + 8) + 2(24 - 14) = 40 - 60 + 20 = 0$$

$$|A|I_3 = 0 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow (1)$$

$$\begin{bmatrix} 7 & -4 & -6 & 7 \\ -4 & 3 & 2 & -4 \\ -6 & 2 & 8 & -6 \\ 7 & -4 & -6 & 7 \end{bmatrix} \Rightarrow$$

$$adjA = \begin{bmatrix} 21 - 16 & -8 + 18 & 24 - 14 \\ -8 + 18 & 24 - 4 & -12 + 32 \\ 24 - 14 & -12 + 32 & 56 - 36 \end{bmatrix} = \begin{bmatrix} 5 & 10 & 10 \\ 10 & 20 & 20 \\ 10 & 20 & 20 \end{bmatrix}$$

$$A(adjA) = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix} \begin{bmatrix} 5 & 10 & 10 \\ 10 & 20 & 20 \\ 10 & 20 & 20 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow (2)$$

$$(adjA)A = \begin{bmatrix} 5 & 10 & 10 \\ 10 & 20 & 20 \\ 10 & 20 & 20 \end{bmatrix} \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow (3)$$

From (1),(2) & (3),  $A(adjA) = (adjA)A = |A|I_3$

7. If  $A = \begin{bmatrix} 3 & 2 \\ 7 & 5 \end{bmatrix}$  and  $B = \begin{bmatrix} -1 & -3 \\ 5 & 2 \end{bmatrix}$  verify that  $(AB)^{-1} = B^{-1}A^{-1}$

$$AB = \begin{bmatrix} 3 & 2 \\ 7 & 5 \end{bmatrix} \begin{bmatrix} -1 & -3 \\ 5 & 2 \end{bmatrix} = \begin{bmatrix} -3 + 10 & -9 + 4 \\ -7 + 25 & -21 + 10 \end{bmatrix} = \begin{bmatrix} 7 & -5 \\ 18 & -11 \end{bmatrix}$$

$$|AB| = -77 + 90 = 13 ; adj(AB) = \begin{bmatrix} -11 & 5 \\ -18 & 7 \end{bmatrix}$$

$$(AB)^{-1} = \frac{1}{|AB|} adj(AB) = \frac{1}{13} \begin{bmatrix} -11 & 5 \\ -18 & 7 \end{bmatrix} \rightarrow (1)$$

$$|B| = -2 + 15 = 13 ; adj B = \begin{bmatrix} 2 & -2 \\ -7 & -1 \end{bmatrix} ; B^{-1} = \frac{1}{|B|} adj B = \frac{1}{13} \begin{bmatrix} 2 & 3 \\ -5 & -1 \end{bmatrix}$$

$$|A| = 15 - 14 = 1 ; adj A = \begin{bmatrix} 5 & -2 \\ -7 & 3 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|} adj A = \frac{1}{1} \begin{bmatrix} 5 & -2 \\ -7 & 3 \end{bmatrix} = \begin{bmatrix} 5 & -2 \\ -7 & 3 \end{bmatrix}$$

$$B^{-1}A^{-1} = \frac{1}{13} \begin{bmatrix} 2 & 3 \\ -5 & -1 \end{bmatrix} \begin{bmatrix} 5 & -2 \\ -7 & 3 \end{bmatrix} = \frac{1}{13} \begin{bmatrix} 10 - 21 & -4 + 9 \\ -25 + 7 & 10 - 3 \end{bmatrix}$$

$$= \frac{1}{13} \begin{bmatrix} -11 & 5 \\ -18 & 7 \end{bmatrix} \rightarrow (2)$$

From (1) & (2),  $(AB)^{-1} = B^{-1}A^{-1}$

**EXAMPLE 1.9** If  $A = \begin{bmatrix} 0 & -3 \\ 1 & 4 \end{bmatrix}$  and  $B = \begin{bmatrix} -2 & -3 \\ 0 & -1 \end{bmatrix}$  verify that  $(AB)^{-1} = B^{-1}A^{-1}$

8. If  $adj A = \begin{bmatrix} 2 & -4 & 2 \\ -3 & 12 & -7 \\ -2 & 0 & 2 \end{bmatrix}$ , find A.

$$A = \pm \frac{1}{\sqrt{|adjA|}} adj(adjA)$$

$$|adjA| = 2(24 - 0) - (-4)(-6 - 14) + 2(0 + 24) = 48 - 80 + 48 = 16$$

$$\sqrt{|adjA|} = \sqrt{16} = 4$$

$$\begin{bmatrix} 12 & 0 & -4 & 12 \\ -7 & 2 & 2 & -7 \\ -3 & -2 & 2 & -3 \\ 12 & 0 & -4 & 12 \end{bmatrix} \Rightarrow adj(adjA) = \begin{bmatrix} 24 - 0 & 0 + 8 & 28 - 24 \\ 14 + 6 & 4 + 4 & -6 + 14 \\ 0 + 24 & 8 - 0 & 24 - 12 \end{bmatrix}$$

$$= \begin{bmatrix} 24 & 8 & 4 \\ 20 & 8 & 8 \\ 24 & 8 & 12 \end{bmatrix}$$

$$A = \pm \frac{1}{\sqrt{|adjA|}} adj(adjA) = \pm \frac{1}{4} \begin{bmatrix} 24 & 8 & 4 \\ 20 & 8 & 8 \\ 24 & 8 & 12 \end{bmatrix} = \pm \begin{bmatrix} 6 & 2 & 1 \\ 5 & 2 & 2 \\ 6 & 2 & 3 \end{bmatrix}$$

**EXAMPLE 1.5** Find a matrix A if  $\text{adj } A = \begin{bmatrix} 7 & 7 & -7 \\ -1 & 11 & 7 \\ 11 & 5 & 7 \end{bmatrix}$

9. If  $\text{adj } A = \begin{bmatrix} 0 & -2 & 0 \\ 6 & 2 & -6 \\ -3 & 0 & 6 \end{bmatrix}$ , find  $A^{-1}$ .

$$A^{-1} = \pm \frac{1}{\sqrt{|\text{adj}A|}} \text{adj}A$$

$$|\text{adj}A| = 0(12 - 0) - (-2)(36 - 18) + 0(0 + 6) = 0 + 36 + 0 = 36$$

$$\sqrt{|\text{adj}A|} = \sqrt{36} = 6$$

$$A^{-1} = \pm \frac{1}{\sqrt{|\text{adj}A|}} \text{adj}A = \pm \frac{1}{6} \begin{bmatrix} 0 & -2 & 0 \\ 6 & 2 & -6 \\ -3 & 0 & 6 \end{bmatrix}$$

**EXAMPLE 1.6** If  $\text{adj } A = \begin{bmatrix} -1 & 2 & 2 \\ 1 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$ , find  $A^{-1}$ .

10. Find  $\text{adj}(\text{adj}A)$  if  $\text{adj } A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ -1 & 0 & 1 \end{bmatrix}$

$$\begin{bmatrix} 2 & 0 & 0 & 2 \\ 0 & 1 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 2 & 0 & 0 & 2 \end{bmatrix} \Rightarrow \text{adj}(\text{adj}A) = \begin{bmatrix} 2-0 & 0-0 & 0-2 \\ 0-0 & 1+1 & 0-0 \\ 0+2 & 0-0 & 2-0 \end{bmatrix} = \begin{bmatrix} 2 & 0 & -2 \\ 0 & 2 & 0 \\ 2 & 0 & 2 \end{bmatrix}$$

11.  $A = \begin{bmatrix} 1 & \tan x \\ -\tan x & 1 \end{bmatrix}$ , show that  $A^T A^{-1} = \begin{bmatrix} \cos 2x & -\sin 2x \\ \sin 2x & \cos 2x \end{bmatrix}$

$$A^T = \begin{bmatrix} 1 & -\tan x \\ \tan x & 1 \end{bmatrix}$$

$$|A| = 1 + \tan^2 x = \sec^2 x = \frac{1}{\cos^2 x}$$

$$\text{adj}A = \begin{bmatrix} 1 & -\tan x \\ \tan x & 1 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|} \text{adj } A = \frac{1}{\frac{1}{\cos^2 x}} \begin{bmatrix} 1 & -\tan x \\ \tan x & 1 \end{bmatrix} = \cos^2 x \begin{bmatrix} 1 & -\tan x \\ \tan x & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \cos^2 x & -\sin x \cos x \\ \sin x \cos x & \cos^2 x \end{bmatrix}$$

$$A^T A^{-1} = \begin{bmatrix} 1 & -\tan x \\ \tan x & 1 \end{bmatrix} \begin{bmatrix} \cos^2 x & -\sin x \cos x \\ \sin x \cos x & \cos^2 x \end{bmatrix}$$

$$= \begin{bmatrix} \cos^2 x - \sin^2 x & -\sin x \cos x - \sin x \cos x \\ \sin x \cos x + \sin x \cos x & \cos^2 x - \sin^2 x \end{bmatrix} = \begin{bmatrix} \cos 2x & -\sin 2x \\ \sin 2x & \cos 2x \end{bmatrix}$$

12. Find the matrix A for which  $A \begin{bmatrix} 5 & 3 \\ -1 & -2 \end{bmatrix} = \begin{bmatrix} 14 & 7 \\ 7 & 7 \end{bmatrix}$

$$A \begin{bmatrix} 5 & 3 \\ -1 & -2 \end{bmatrix} = \begin{bmatrix} 14 & 7 \\ 7 & 7 \end{bmatrix} \Rightarrow AB = \begin{bmatrix} 14 & 7 \\ 7 & 7 \end{bmatrix} \Rightarrow A = \begin{bmatrix} 14 & 7 \\ 7 & 7 \end{bmatrix} B^{-1} \rightarrow (1)$$

$$B = \begin{bmatrix} 5 & 3 \\ -1 & -2 \end{bmatrix} \Rightarrow |B| = -10 + 3 = -7; \text{adj } B = \begin{bmatrix} -2 & -3 \\ 1 & 5 \end{bmatrix}$$

$$B^{-1} = \frac{1}{|B|} \text{adj}B \Rightarrow B^{-1} = \frac{1}{-7} \begin{bmatrix} -2 & -3 \\ 1 & 5 \end{bmatrix}$$

$$(1) \Rightarrow A = \begin{bmatrix} 14 & 7 \\ 7 & 7 \end{bmatrix} \frac{1}{-7} \begin{bmatrix} -2 & -3 \\ 1 & 5 \end{bmatrix} = -\frac{1}{7} \begin{bmatrix} -28 + 7 & -42 + 35 \\ -14 + 7 & -21 + 35 \end{bmatrix}$$

$$= -\frac{1}{7} \begin{bmatrix} -21 & -7 \\ -7 & 14 \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 1 & -2 \end{bmatrix} \therefore A = \begin{bmatrix} 3 & 1 \\ 1 & -2 \end{bmatrix}$$

13. Given  $A = \begin{bmatrix} 1 & -1 \\ 2 & 0 \end{bmatrix}$ ,  $B = \begin{bmatrix} 3 & -2 \\ 1 & 1 \end{bmatrix}$  and  $C = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$ , find a matrix X such that  $AXB = C$ .

$$AXB = C$$

Pre multiplying by  $A^{-1}$  and post multiplying by  $B^{-1} \Rightarrow X = A^{-1}CB^{-1} \rightarrow (1)$

$$|A| = 0 + 2 = 2; \text{adj } A = \begin{bmatrix} 0 & 1 \\ -2 & 1 \end{bmatrix}; A^{-1} = \frac{1}{|A|} \text{adj } A = \frac{1}{2} \begin{bmatrix} 0 & 1 \\ -2 & 1 \end{bmatrix}$$

$$|B| = 3 + 2 = 5; \text{adj } B = \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}; A^{-1} = \frac{1}{|A|} \text{adj } A = \frac{1}{5} \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}$$

$$(1) \Rightarrow X = \frac{1}{2} \begin{bmatrix} 0 & 1 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} \frac{1}{5} \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix} = \frac{1}{10} \begin{bmatrix} 0+2 & 0+2 \\ -2+2 & -2+2 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}$$

$$= \frac{1}{10} \begin{bmatrix} 2 & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix} = \frac{1}{10} \begin{bmatrix} 2-2 & 4+6 \\ 0 & 0 \end{bmatrix} = \frac{1}{10} \begin{bmatrix} 0 & 10 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

14. If  $A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$ , show that  $A^{-1} = \frac{1}{2}(A^2 - 3I)$

$$|A| = 0 - 1(0 - 1) + 1(1 - 0) = 1 + 1 = 2$$

$$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \Rightarrow \text{adj } A = \begin{bmatrix} 0-1 & 1-0 & 1-0 \\ 1-0 & 0-1 & 1-0 \\ 1-0 & 1-0 & 0-1 \end{bmatrix} = \begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|} \text{adj } A = \frac{1}{2} \begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix} \rightarrow (1)$$

$$A^2 = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0+1+1 & 0+0+1 & 0+1+0 \\ 0+0+1 & 1+0+1 & 1+0+0 \\ 0+1+0 & 1+0+0 & 1+1+0 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

$$\frac{1}{2}(A^2 - 3I) = \frac{1}{2} \left( \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} - 3 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right)$$

$$= \frac{1}{2} \left( \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} + \begin{bmatrix} -3 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -3 \end{bmatrix} \right) = \frac{1}{2} \begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix} \rightarrow (2)$$

From (1) & (2),  $A^{-1} = \frac{1}{2}(A^2 - 3I)$

15. Decrypt the received encoded message  $[2 \ -3], [20 \ 4]$  with the encryption matrix  $\begin{bmatrix} -1 & -1 \\ 2 & 1 \end{bmatrix}$  and the decryption matrix as its inverse, where the system of code are described by the numbers 1-26 to the letters A-Z respectively, and the number 0 to a blank space.

$$A = [2 \ -3], B = [20 \ 4]$$

$$\text{Encoded matrix } C = \begin{bmatrix} -1 & -1 \\ 2 & 1 \end{bmatrix} \Rightarrow \text{Decoded matrix} = C^{-1}$$

$$|C| = -1 + 2 = 1; \text{adj } C = \begin{bmatrix} 1 & 1 \\ -2 & -1 \end{bmatrix}; C^{-1} = \frac{1}{|C|} \text{adj } C = \begin{bmatrix} 1 & 1 \\ -2 & -1 \end{bmatrix}$$

$$\text{Received message} = (AC^{-1})(BC^{-1})$$

$$= \left\{ [2 \ -3] \begin{bmatrix} 1 & 1 \\ -2 & -1 \end{bmatrix} \right\} \left\{ [20 \ 4] \begin{bmatrix} 1 & 1 \\ -2 & -1 \end{bmatrix} \right\}$$

$$= [2+6 \ 2+3][20-8 \ 20-4] = [8 \ 5][12 \ 16] = \text{HELP}$$

( $\because$  In alphabetical order, 8=H, 5=E, 12=L, 16=P)

**EXAMPLE 1.8** Verify the property  $(A^T)^{-1} = (A^{-1})^T$  with  $A = \begin{bmatrix} 2 & 9 \\ 1 & 7 \end{bmatrix}$

$$A^T = \begin{bmatrix} 2 & 1 \\ 9 & 7 \end{bmatrix}; |A^T| = 14 - 9 = 5; \text{adj}(A^T) = \begin{bmatrix} 7 & -1 \\ -9 & 2 \end{bmatrix}$$

$$(A^T)^{-1} = \frac{1}{|A^T|} \text{adj}(A^T) = \frac{1}{5} \begin{bmatrix} 7 & -1 \\ -9 & 2 \end{bmatrix} \rightarrow (1)$$

$$|A| = 14 - 9 = 5; \text{adj } A = \begin{bmatrix} 7 & -9 \\ -1 & 2 \end{bmatrix}; A^{-1} = \frac{1}{|A|} \text{adj } A = \frac{1}{5} \begin{bmatrix} 7 & -9 \\ -1 & 2 \end{bmatrix}$$

$$(A^{-1})^T = \frac{1}{5} \begin{bmatrix} 7 & -1 \\ -9 & 2 \end{bmatrix} \rightarrow (2)$$

From (1) & (2),  $(A^T)^{-1} = (A^{-1})^T$

**EXAMPLE 1.11** Prove that  $\begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$  is orthogonal.

A square matrix A is called orthogonal if  $AA^T = A^T A = I_n$

$$A = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \Rightarrow A^T = \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix}$$

$$AA^T = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos^2 \theta + \sin^2 \theta & \sin \theta \cos \theta - \sin \theta \cos \theta \\ \sin \theta \cos \theta - \sin \theta \cos \theta & \sin^2 \theta + \cos^2 \theta \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2 \rightarrow (1)$$

$$A^T A = \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos^2 \theta + \sin^2 \theta & \sin \theta \cos \theta - \sin \theta \cos \theta \\ \sin \theta \cos \theta - \sin \theta \cos \theta & \sin^2 \theta + \cos^2 \theta \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2 \rightarrow (2)$$

From (1) & (2),  $AA^T = A^T A = I_2$ , hence A is orthogonal

**EXAMPLE 1.12** If  $A = \frac{1}{7} \begin{bmatrix} 6 & -3 & a \\ b & -2 & 6 \\ 2 & c & 3 \end{bmatrix}$  is orthogonal, find a, b and c and hence

$$A^{-1}$$

A square matrix A is called orthogonal if  $AA^T = A^T A = I_n$

$$A = \frac{1}{7} \begin{bmatrix} 6 & -3 & a \\ b & -2 & 6 \\ 2 & c & 3 \end{bmatrix} \Rightarrow A^T = \frac{1}{7} \begin{bmatrix} 6 & b & 2 \\ -3 & -2 & c \\ a & 6 & 3 \end{bmatrix}$$

$$AA^T = I_3 \Rightarrow \frac{1}{7} \begin{bmatrix} 6 & -3 & a \\ b & -2 & 6 \\ 2 & c & 3 \end{bmatrix} \frac{1}{7} \begin{bmatrix} 6 & b & 2 \\ -3 & -2 & c \\ a & 6 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \frac{1}{49} \begin{bmatrix} 36 + 9 + a^2 & 6b + 6 + 6a & 12 - 3c + 3a \\ 6b + 6 + 6a & b^2 + 4 + 36 & 2b - 2c + 18 \\ 12 - 3c + 3a & 2b - 2c + 18 & 4 + c^2 + 9 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} a^2 + 45 & 6a + 6b + 6 & 3a - 3c + 12 \\ 6a + 6b + 6 & b^2 + 40 & 2b - 2c + 18 \\ 3a - 3c + 12 & 2b - 2c + 18 & c^2 + 13 \end{bmatrix} = \begin{bmatrix} 49 & 0 & 0 \\ 0 & 49 & 0 \\ 0 & 0 & 49 \end{bmatrix}$$

$$\Rightarrow \begin{cases} a^2 + 45 = 49 \Rightarrow a^2 = 4 \Rightarrow a = \pm 2 \\ b^2 + 40 = 49 \Rightarrow b^2 = 9 \Rightarrow b = \pm 3 \\ c^2 + 13 = 49 \Rightarrow c^2 = 36 \Rightarrow c = \pm 6 \\ 6a + 6b + 6 = 0 \Rightarrow a + b = -1 \\ 3a - 3c + 12 = 0 \Rightarrow a - c = -4 \\ 2b - 2c + 18 = 0 \Rightarrow b - c = -9 \end{cases} \Rightarrow \boxed{a = 2, b = -3, c = 6}$$

Since A is orthogonal,  $A^{-1} = A^T \Rightarrow A^{-1} = \frac{1}{7} \begin{bmatrix} 6 & -3 & 2 \\ -3 & -2 & 6 \\ 2 & 6 & 3 \end{bmatrix}$

### RANK OF A MATRIX

- ❖ **Rank of a matrix in minor method:** The rank of a matrix A is defined as the order of a highest order non-vanishing minor of the matrix A. It is denoted by  $\rho(A)$ .
- ❖ The rank of zero matrix is 0.
- ❖ The rank of the identity matrix  $I_n$  is n.
- ❖ If a matrix contains at-least one non-zero element then  $\rho(A) \geq 1$ .
- ❖ If A is an  $m \times n$  matrix then  $\rho(A) \leq \min\{m, n\}$
- ❖ A square matrix A order n is invertible  $\Leftrightarrow \rho(A) = n$
- ❖ If all the entries below the leading diagonal are zero, use minor method for finding the rank.
- ❖ **Rank of a matrix in row echelon form:** The rank of a matrix in row echelon form is the number of non-zero rows in it.
- ❖ The rank of the non-zero matrix is equal to the number of non-zero rows in a row-echelon form of the matrix.
- ❖ **Elementary matrix:** An elementary matrix is defined as a matrix which is obtained from an identity matrix by applying only one elementary transformation.
- ❖ **Gauss-Jordan Method:** Transforming a non-singular matrix A to the form  $I_n$  by applying elementary row operations is called Gauss-Jordan method.
- ❖ **Steps in finding  $A^{-1}$  by Gauss-Jordan method:**
  - Write matrix of the form  $[A|I_n]$
  - Apply elementary row operations and convert  $[A|I_n]$  to  $[I_n|A^{-1}]$  and find  $A^{-1}$ .

### RANK OF A MATRIX IN MINOR METHOD

#### EXERCISE 1.2

1. Find the rank of the following matrices by minor method:

(i)  $\begin{bmatrix} 2 & -4 \\ -1 & 2 \end{bmatrix}$

Let  $A = \begin{bmatrix} 2 & -4 \\ -1 & 2 \end{bmatrix}$ . Order of A is  $2 \times 2$ . So  $\rho(A) \leq \min\{2, 2\} = 2$

$|A| = 4 - 4 = 0 \Rightarrow \rho(A) < 2$

Since A contains at-least one non-zero element,  $\rho(A) = 1$ .

(ii)  $\begin{bmatrix} -1 & 3 \\ 4 & -7 \\ 3 & -4 \end{bmatrix}$

Let  $A = \begin{bmatrix} -1 & 3 \\ 4 & -7 \\ 3 & -4 \end{bmatrix}$ . Order of A is  $3 \times 2$ . So  $\rho(A) \leq \min\{3, 2\} = 2$

$2 \times 2$  Minor:

$\begin{vmatrix} -1 & 3 \\ 4 & -7 \end{vmatrix} = 7 - 12 = 5 \neq 0 \Rightarrow \rho(A) = 2$

(iii)  $\begin{bmatrix} 1 & -2 & -1 & 0 \\ 3 & -6 & -3 & 1 \end{bmatrix}$

Let  $A = \begin{bmatrix} 1 & -2 & -1 & 0 \\ 3 & -6 & -3 & 1 \end{bmatrix}$ . Order of A is  $2 \times 4$ . So  $\rho(A) \leq \min\{2, 4\} = 2$

$2 \times 2$  Minors:

$\begin{vmatrix} 1 & -2 \\ 3 & -6 \end{vmatrix} = 6 - 6 = 0$ ;  $\begin{vmatrix} 1 & -1 \\ 3 & -3 \end{vmatrix} = -3 + 3 = 0$ ;  $\begin{vmatrix} 1 & 0 \\ 3 & 1 \end{vmatrix} = 1 - 0 = 1 \neq 0$

$\therefore \rho(A) = 2$

(iv)  $\begin{bmatrix} 1 & -2 & 3 \\ 2 & 4 & -6 \\ 5 & 1 & -1 \end{bmatrix}$

Let  $A = \begin{bmatrix} 1 & -2 & 3 \\ 2 & 4 & -6 \\ 5 & 1 & -1 \end{bmatrix}$ . Order of A is  $3 \times 3$ . So  $\rho(A) \leq \min\{3, 3\} = 3$

$|A| = 1(-4 + 6) - (-2)(-2 + 30) + 3(2 - 20) = 2 + 56 - 54 = 4 \neq 0$

$\therefore \rho(A) = 3$

(v)  $\begin{bmatrix} 0 & 1 & 2 & 1 \\ 0 & 2 & 4 & 3 \\ 8 & 1 & 0 & 2 \end{bmatrix}$

Let  $A = \begin{bmatrix} 0 & 1 & 2 & 1 \\ 0 & 2 & 4 & 3 \\ 8 & 1 & 0 & 2 \end{bmatrix}$ . Order of A is  $3 \times 4$ . So  $\rho(A) \leq \min\{3, 4\} = 3$

3 × 3 Minors:

$$\begin{vmatrix} 0 & 1 & 2 \\ 0 & 2 & 4 \\ 8 & 1 & 0 \end{vmatrix} = 0 - 0 + 8(4 - 4) = 0; \quad \begin{vmatrix} 0 & 1 & 1 \\ 0 & 2 & 3 \\ 8 & 1 & 2 \end{vmatrix} = 0 - 0 + 8(3 - 2) = 8 \neq 0$$

$\therefore \rho(A) = 3$

### EXAMPLE 1.15

$$\begin{bmatrix} 3 & 2 & 5 \\ 1 & 1 & 2 \\ 3 & 3 & 6 \end{bmatrix}$$

Let  $A = \begin{bmatrix} 3 & 2 & 5 \\ 1 & 1 & 2 \\ 3 & 3 & 6 \end{bmatrix}$ . Order of A is  $3 \times 3$ . So  $\rho(A) \leq \min\{3, 3\} = 3$

$$|A| = 0 \quad (\because R_2: R_3)$$

$$\therefore \rho(A) < 3$$

2 × 2 Minor:

$$\begin{vmatrix} 3 & 2 \\ 1 & 1 \end{vmatrix} = 3 - 2 = 1 \neq 0. \text{ So } \rho(A) = 2$$

### EXAMPLE 1.15

$$\begin{bmatrix} 4 & 3 & 1 & -2 \\ -3 & -1 & -2 & 4 \\ 6 & 7 & -1 & 2 \end{bmatrix}$$

Let  $A = \begin{bmatrix} 4 & 3 & 1 & -2 \\ -3 & -1 & -2 & 4 \\ 6 & 7 & -1 & 2 \end{bmatrix}$ . Order of A is  $3 \times 4$ . So  $\rho(A) \leq \min\{3, 4\} = 3$

3 × 3 Minors:

$$\begin{vmatrix} 4 & 3 & 1 \\ -3 & -1 & -2 \\ 6 & 7 & -1 \end{vmatrix} = 0; \quad \begin{vmatrix} 4 & 3 & -2 \\ -3 & -1 & 4 \\ 6 & 7 & 2 \end{vmatrix} = 0;$$

$$\begin{vmatrix} 4 & 1 & -2 \\ -3 & -2 & 4 \\ 6 & -1 & 2 \end{vmatrix} = 0; \quad \begin{vmatrix} 3 & 1 & -2 \\ -1 & -2 & 4 \\ 6 & -1 & 2 \end{vmatrix} = 0$$

$$\text{So } \rho(A) < 3$$

2 × 2 Minor:

$$\begin{vmatrix} 4 & 3 \\ -3 & -1 \end{vmatrix} = -4 + 9 = 5 \neq 0. \text{ So } \rho(A) = 2$$

### EXAMPLE 1.16

Find the rank of the following matrices which are in row-echelon form:

$$(i) \begin{bmatrix} 2 & 0 & -7 \\ 0 & 3 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

Let  $A = \begin{bmatrix} 2 & 0 & -7 \\ 0 & 3 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ . Order of A is  $3 \times 3$ . So  $\rho(A) \leq \min\{3, 3\} = 3$

3 × 3 Minor:

$$\begin{vmatrix} 2 & 0 & -7 \\ 0 & 3 & 1 \\ 0 & 0 & 1 \end{vmatrix} = 2(3 - 0) - 0 + 0 = 6 \neq 0. \text{ So } \rho(A) = 3$$

$$(ii) \begin{bmatrix} -2 & 2 & -1 \\ 0 & 5 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Let  $A = \begin{bmatrix} -2 & 2 & -1 \\ 0 & 5 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ . Order of A is  $3 \times 3$ . So  $\rho(A) \leq \min\{3, 3\} = 3$

3 × 3 Minor:

$$\begin{vmatrix} -2 & 2 & -1 \\ 0 & 5 & 1 \\ 0 & 0 & 0 \end{vmatrix} = 0. \text{ So } \rho(A) < 3$$

2 × 2 Minor:

$$\begin{vmatrix} -2 & 2 \\ 0 & 5 \end{vmatrix} = -10 - 0 = -10 \neq 0. \text{ So } \rho(A) = 2$$

$$(iii) \begin{bmatrix} 6 & 0 & -9 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Let  $A = \begin{bmatrix} 6 & 0 & -9 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ . Order of A is  $4 \times 3$ . So  $\rho(A) \leq \min\{4, 3\} = 3$

3 × 3 Minor:

All the  $3 \times 3$  minors are equal to zero. So  $\rho(A) < 3$

2 × 2 Minor:

$$\begin{vmatrix} 6 & 0 \\ 0 & 2 \end{vmatrix} = 12 - 0 = 12 \neq 0.$$

$$\text{So } \rho(A) = 2$$

### RANK OF A MATRIX IN ROW REDUCTION METHOD

2. Find the rank of the following matrices by row reduction (row-echelon) method:

$$(i) \begin{bmatrix} 1 & 1 & 1 & 3 \\ 2 & -1 & 3 & 4 \\ 5 & -1 & 7 & 11 \end{bmatrix}$$

$$\text{Let } A = \begin{bmatrix} 1 & 1 & 1 & 3 \\ 2 & -1 & 3 & 4 \\ 5 & -1 & 7 & 11 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 3 \\ 0 & -3 & 1 & -2 \\ 0 & -6 & 2 & -4 \end{bmatrix} \begin{array}{l} R_2 \rightarrow R_2 - 2R_1 \\ R_3 \rightarrow R_3 - 5R_1 \end{array}$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 3 \\ 0 & -3 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix} R_3 \rightarrow R_3 - 2R_2$$

The equivalent matrix is in row-echelon form and the number of non-zero rows is 2. So  $\rho(A) = 2$

$$(ii) \begin{bmatrix} 1 & 2 & -1 \\ 3 & -1 & 2 \\ 1 & -2 & 3 \\ 1 & -1 & 1 \end{bmatrix}$$

$$\text{Let } A = \begin{bmatrix} 1 & 2 & -1 \\ 3 & -1 & 2 \\ 1 & -2 & 3 \\ 1 & -1 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & -1 \\ 0 & -7 & 5 \\ 0 & -4 & 4 \\ 0 & -3 & 2 \end{bmatrix} \begin{array}{l} R_2 \rightarrow R_2 - 3R_1 \\ R_3 \rightarrow R_3 - R_1 \\ R_4 \rightarrow R_4 - R_1 \end{array}$$

$$\sim \begin{bmatrix} 1 & 2 & -1 \\ 0 & -7 & 5 \\ 0 & 0 & 8 \\ 0 & 0 & -4 \end{bmatrix} \begin{array}{l} R_3 \rightarrow 7R_3 - 4R_2 \\ R_4 \rightarrow 4R_4 - 3R_3 \end{array}$$

$$\sim \begin{bmatrix} 1 & 2 & -1 \\ 0 & -7 & 5 \\ 0 & 0 & 8 \\ 0 & 0 & 0 \end{bmatrix} R_4 \rightarrow R_4 + 2R_3$$

The equivalent matrix is in row-echelon form and the number of non-zero rows is 3. So  $\rho(A) = 3$

$$(iii) \begin{bmatrix} 3 & -8 & 5 & 2 \\ 2 & -5 & 1 & 4 \\ -1 & 2 & 3 & -2 \end{bmatrix}$$

$$\text{Let } A = \begin{bmatrix} 3 & -8 & 5 & 2 \\ 2 & -5 & 1 & 4 \\ -1 & 2 & 3 & -2 \end{bmatrix}$$

$$\sim \begin{bmatrix} -1 & 2 & 3 & -2 \\ 2 & -5 & 1 & 4 \\ 3 & -8 & 5 & 2 \end{bmatrix} R_1 \rightarrow R_3$$

$$\sim \begin{bmatrix} -1 & 2 & 3 & -2 \\ 0 & -1 & 7 & 0 \\ 0 & -2 & 14 & -4 \end{bmatrix} \begin{array}{l} R_2 \rightarrow R_2 + 2R_1 \\ R_3 \rightarrow R_3 + 3R_1 \end{array}$$

$$\sim \begin{bmatrix} -1 & 2 & 3 & -2 \\ 0 & -1 & 7 & 0 \\ 0 & 0 & 0 & -4 \end{bmatrix} R_3 \rightarrow R_3 - 2R_2$$

The equivalent matrix is in row-echelon form and the number of non-zero rows is 3. So  $\rho(A) = 3$

**EXAMPLE 1.17** Find the rank of the matrix  $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 4 \\ 3 & 0 & 5 \end{bmatrix}$  by reducing it to a row -

echelon form.

$$\text{Let } A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 4 \\ 3 & 0 & 5 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -2 \\ 0 & -6 & -4 \end{bmatrix} \begin{array}{l} R_2 \rightarrow R_2 - 2R_1 \\ R_3 \rightarrow R_3 - 3R_1 \end{array}$$

$$\sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -2 \\ 0 & 0 & 0 \end{bmatrix} R_3 \rightarrow R_3 - 2R_2$$

The equivalent matrix is in row-echelon form and the number of non-zero rows is 2. So  $\rho(A) = 2$

**EXAMPLE 1.18** Find the rank of the matrix  $\begin{bmatrix} 2 & -2 & 4 & 3 \\ -3 & 4 & -2 & -1 \\ 6 & 2 & -1 & 7 \end{bmatrix}$  by reducing it

to a row-echelon form.

$$A = \begin{bmatrix} 2 & -2 & 4 & 3 \\ -3 & 4 & -2 & -1 \\ 6 & 2 & -1 & 7 \end{bmatrix}$$

$$\sim \begin{bmatrix} 2 & -2 & 4 & 3 \\ -6 & 8 & -4 & -2 \\ 6 & 2 & -1 & 7 \end{bmatrix} R_2 \rightarrow 2R_2$$

$$\sim \begin{bmatrix} 2 & -2 & 4 & 3 \\ 0 & 2 & 8 & 7 \\ 0 & 8 & -13 & -2 \end{bmatrix} \begin{matrix} R_2 \rightarrow R_2 + 3R_1 \\ R_3 \rightarrow R_3 - 3R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 2 & -2 & 4 & 3 \\ 0 & 2 & 8 & 7 \\ 0 & 0 & -45 & -30 \end{bmatrix} R_3 \rightarrow R_3 - 4R_2$$

The equivalent matrix is in row-echelon form and the number of non-zero rows is 3. So  $\rho(A) = 3$

#### GAUSS-JORDAN METHOD FOR FINDING INVERSE OF THE MATRIX

**EXAMPLE 1.19** Show that the matrix  $\begin{bmatrix} 3 & 1 & 4 \\ 2 & 0 & -1 \\ 5 & 2 & 1 \end{bmatrix}$  is non-singular and reduce it

to the identity matrix by elementary row transformation.

Let  $A = \begin{bmatrix} 3 & 1 & 4 \\ 2 & 0 & -1 \\ 5 & 2 & 1 \end{bmatrix}$

$$|A| = 3(0 + 2) - 1(2 + 5) + 4(4 - 0) = 6 - 7 + 16 = 15 \neq 0$$

$\therefore A$  is non-singular.

$$\begin{bmatrix} 3 & 1 & 4 \\ 2 & 0 & -1 \\ 5 & 2 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & \frac{1}{3} & \frac{4}{3} \\ 2 & 0 & -1 \\ 5 & 2 & 1 \end{bmatrix} R_1 \rightarrow \frac{1}{3}R_1$$

$$\sim \begin{bmatrix} 1 & \frac{1}{3} & \frac{4}{3} \\ 0 & -\frac{2}{3} & -\frac{11}{3} \\ 0 & \frac{1}{3} & -\frac{17}{3} \end{bmatrix} \begin{matrix} R_2 \rightarrow R_2 - 2R_1 \\ R_3 \rightarrow R_3 - 5R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & \frac{1}{3} & \frac{4}{3} \\ 0 & 1 & \frac{11}{2} \\ 0 & \frac{1}{3} & -\frac{17}{3} \end{bmatrix} R_2 \rightarrow \left(-\frac{3}{2}\right)R_2$$

$$\sim \begin{bmatrix} 1 & 0 & -\frac{1}{2} \\ 0 & 1 & \frac{11}{2} \\ 0 & 0 & -\frac{15}{2} \end{bmatrix} \begin{matrix} R_1 \rightarrow R_1 - \frac{1}{3}R_2 \\ R_3 \rightarrow R_3 - \frac{1}{3}R_2 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 0 & -\frac{1}{2} \\ 0 & 1 & \frac{11}{2} \\ 0 & 0 & 1 \end{bmatrix} R_3 \rightarrow \left(-\frac{2}{15}\right)R_3$$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{matrix} R_1 \rightarrow R_1 - \frac{1}{2}R_3 \\ R_2 \rightarrow R_2 - \frac{11}{2}R_3 \end{matrix}$$

3. Find the inverse of each of the following by Gauss-Jordan method:

(i)  $\begin{bmatrix} 2 & -1 \\ 5 & -2 \end{bmatrix}$

$$[A|I_2] = \left( \begin{array}{cc|cc} 2 & -1 & 1 & 0 \\ 5 & -2 & 0 & 1 \end{array} \right)$$

$$\sim \left( \begin{array}{cc|cc} 1 & -\frac{1}{2} & \frac{1}{2} & 0 \\ 5 & -2 & 0 & 1 \end{array} \right) R_2 \rightarrow \frac{1}{2}R_2$$

$$\sim \left( \begin{array}{cc|cc} 1 & -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & -\frac{5}{2} & 1 \end{array} \right) R_2 \rightarrow R_2 - 5R_1$$

$$\sim \left( \begin{array}{cc|cc} 1 & -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 1 & -5 & 2 \end{array} \right) R_2 \rightarrow 2R_2$$

$$\sim \left( \begin{array}{cc|cc} 1 & 0 & -2 & 1 \\ 0 & 1 & -5 & 2 \end{array} \right) R_1 \rightarrow R_1 + \frac{1}{2}R_2$$

$$= [I_2 | A^{-1}]$$

$$\Rightarrow A^{-1} = \begin{bmatrix} -2 & 1 \\ -5 & 2 \end{bmatrix}$$

$$(ii) \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ 6 & -2 & -3 \end{bmatrix}$$

$$[A | I_3] = \left( \begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 & 1 & 0 \\ 6 & -2 & -3 & 0 & 0 & 1 \end{array} \right)$$

$$\sim \left( \begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 4 & -3 & -6 & 0 & 1 \end{array} \right) \begin{array}{l} R_2 \rightarrow R_2 - R_1 \\ R_3 \rightarrow R_3 - 6R_1 \end{array}$$

$$\sim \left( \begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & 1 & -2 & -4 & 1 \end{array} \right) R_3 \rightarrow R_3 - 4R_2$$

$$\sim \left( \begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -3 & -3 & 1 \\ 0 & 0 & 1 & -2 & -4 & 1 \end{array} \right) R_2 \rightarrow R_2 + R_3$$

$$\sim \left( \begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & -3 & 1 \\ 0 & 1 & 0 & -3 & -3 & 1 \\ 0 & 0 & 1 & -2 & -4 & 1 \end{array} \right) R_1 \rightarrow R_1 + R_2$$

$$= [I_3 | A^{-1}]$$

$$\Rightarrow A^{-1} = \begin{bmatrix} -2 & -3 & 1 \\ -3 & -3 & 1 \\ -2 & -4 & 1 \end{bmatrix}$$

$$(iii) \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{bmatrix}$$

$$[A | I_3] = \left( \begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 5 & 3 & 0 & 1 & 0 \\ 1 & 0 & 8 & 0 & 0 & 1 \end{array} \right)$$

$$\sim \left( \begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & -2 & 5 & -1 & 0 & 1 \end{array} \right) \begin{array}{l} R_2 \rightarrow R_2 - 2R_1 \\ R_3 \rightarrow R_3 - R_1 \end{array}$$

$$\sim \left( \begin{array}{ccc|ccc} 1 & 0 & 9 & 5 & -2 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & 0 & -1 & -5 & 2 & 1 \end{array} \right) \begin{array}{l} R_1 \rightarrow R_1 - 2R_2 \\ R_3 \rightarrow R_3 + 2R_2 \end{array}$$

$$\sim \left( \begin{array}{ccc|ccc} 1 & 0 & 0 & -40 & 16 & 9 \\ 0 & 1 & 0 & 14 & -5 & -3 \\ 0 & 0 & -1 & -5 & 2 & 1 \end{array} \right) \begin{array}{l} R_1 \rightarrow R_1 + 9R_3 \\ R_2 \rightarrow R_2 - 3R_3 \end{array}$$

$$\sim \left( \begin{array}{ccc|ccc} 1 & 0 & 0 & -44 & 16 & 9 \\ 0 & 1 & 0 & 13 & -5 & -3 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array} \right) R_3 \rightarrow (-)R_3$$

$$= [I_3 | A^{-1}]$$

$$\Rightarrow A^{-1} = \begin{bmatrix} -40 & 16 & 9 \\ 13 & -5 & -3 \\ 5 & -2 & -1 \end{bmatrix}$$

**EXAMPLE 1.20** Find the inverse of the non-singular matrix  $A = \begin{bmatrix} 0 & 5 \\ -1 & 6 \end{bmatrix}$ , by

Gauss-Jordan method.

$$[A | I_2] = \left( \begin{array}{cc|cc} 0 & 5 & 1 & 0 \\ -1 & 6 & 0 & 1 \end{array} \right)$$

$$\sim \left( \begin{array}{cc|cc} -1 & 6 & 0 & 1 \\ 0 & 5 & 1 & 0 \end{array} \right) R_1 \leftrightarrow R_2$$

$$\sim \left( \begin{array}{cc|cc} 1 & -6 & 0 & -1 \\ 0 & 5 & 1 & 0 \end{array} \right) R_1 \rightarrow (-)R_1$$

$$\sim \left( \begin{array}{cc|cc} 1 & -6 & 0 & -1 \\ 0 & 1 & \frac{1}{5} & 0 \end{array} \right) R_2 \rightarrow \frac{1}{5}R_2$$

$$\sim \left( \begin{array}{cc|cc} 1 & 0 & \frac{6}{5} & -1 \\ 0 & 1 & \frac{1}{5} & 0 \end{array} \right) R_1 \rightarrow R_1 + 6R_2$$

$$= [I_2 | A^{-1}] \quad \Rightarrow A^{-1} = \begin{bmatrix} \frac{6}{5} & -1 \\ \frac{1}{5} & 0 \end{bmatrix} = \frac{1}{5} \begin{bmatrix} 6 & -5 \\ 1 & 0 \end{bmatrix}$$

**EXAMPLE 1.21** Find the inverse of  $A = \begin{bmatrix} 2 & 1 & 1 \\ 3 & 2 & 1 \\ 2 & 1 & 2 \end{bmatrix}$ , by Gauss-Jordan method.

$$[A | I_3] = \left( \begin{array}{ccc|ccc} 2 & 1 & 1 & 1 & 0 & 0 \\ 3 & 2 & 1 & 0 & 1 & 0 \\ 2 & 1 & 2 & 0 & 0 & 1 \end{array} \right)$$

$$\sim \left( \begin{array}{ccc|ccc} 1 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 3 & 2 & 1 & 0 & 1 & 0 \\ 2 & 1 & 2 & 0 & 0 & 1 \end{array} \right) R_1 \rightarrow \frac{1}{2}R_1$$

$$\sim \left( \begin{array}{ccc|ccc} 1 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{1}{2} & -\frac{3}{2} & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array} \right) \begin{array}{l} R_2 \rightarrow R_2 - 3R_1 \\ R_3 \rightarrow R_3 - 2R_1 \end{array}$$

$$\sim \left( \begin{array}{ccc|ccc} 1 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 1 & -1 & -3 & 2 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array} \right) R_2 \rightarrow 2R_2$$

$$\sim \left( \begin{array}{ccc|ccc} 1 & 0 & 1 & 2 & -1 & 0 \\ 0 & 1 & -1 & -3 & 2 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array} \right) R_1 \rightarrow R_1 - \frac{1}{2}R_2$$

$$\sim \left( \begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & -1 & -1 \\ 0 & 1 & 0 & -4 & 2 & 1 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array} \right) \begin{array}{l} R_1 \rightarrow R_1 - R_3 \\ R_2 \rightarrow R_2 + R_3 \end{array}$$

$$= [I_3 | A^{-1}] \Rightarrow A^{-1} = \begin{bmatrix} 3 & -1 & -1 \\ -4 & 2 & 1 \\ -1 & 0 & 1 \end{bmatrix}$$

### MATRIX INVERSION METHOD

#### EXERCISE 1.3

1. Solve the following system of linear equations by matrix inversion method:

(i)  $2x + 5y = -2, x + 2y = -3$

$$\begin{bmatrix} 2 & 5 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -2 \\ -3 \end{bmatrix} \Rightarrow AX = B \Rightarrow X = A^{-1}B$$

$$|A| = 4 - 5 = -1 ; adj A = \begin{bmatrix} 2 & -5 \\ -1 & 2 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|} adj A = \frac{1}{-1} \begin{bmatrix} 2 & -5 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} -2 & 5 \\ 1 & -2 \end{bmatrix}$$

$$X = A^{-1}B \Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -2 & 5 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} -2 \\ -3 \end{bmatrix} = \begin{bmatrix} 4 - 15 \\ -2 + 6 \end{bmatrix} = \begin{bmatrix} -11 \\ 4 \end{bmatrix}$$

Solution:  $(x, y) = (-11, 4)$

(ii)  $2x - y = 8, 3x + 2y = -2$

$$\begin{bmatrix} 2 & -1 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 8 \\ -2 \end{bmatrix} \Rightarrow AX = B \Rightarrow X = A^{-1}B$$

$$|A| = 4 + 3 = 7 ; adj A = \begin{bmatrix} 2 & 1 \\ -3 & 2 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|} adj A = \frac{1}{7} \begin{bmatrix} 2 & 1 \\ -3 & 2 \end{bmatrix}$$

$$X = A^{-1}B \Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{7} \begin{bmatrix} 2 & 1 \\ -3 & 2 \end{bmatrix} \begin{bmatrix} 8 \\ -2 \end{bmatrix} = \frac{1}{7} \begin{bmatrix} 16 - 2 \\ -24 - 4 \end{bmatrix} = \frac{1}{7} \begin{bmatrix} 14 \\ -28 \end{bmatrix} = \begin{bmatrix} 2 \\ -4 \end{bmatrix}$$

Solution:  $(x, y) = (2, -4)$

**EXAMPLE 1.22** Solve the following system of linear equations by matrix inversion method:  $5x + 2y = 3, 3x + 2y = 5$

(iii)  $2x + 3y - z = 9, x + y + z = 9, 3x - y - z = -1$

$$\begin{bmatrix} 2 & 3 & -1 \\ 1 & 1 & 1 \\ 3 & -1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 9 \\ -1 \end{bmatrix} \Rightarrow AX = B \Rightarrow X = A^{-1}B$$

$$|A| = 2(-1 + 1) - 3(-1 - 3) - 1(-1 - 3) = 12 + 4 = 16$$

$$\begin{bmatrix} 1 & -1 & 3 \\ 1 & -1 & -1 \\ 1 & 3 & 2 \\ 1 & -1 & 3 \end{bmatrix} \Rightarrow adj A = \begin{bmatrix} -1 + 1 & 1 + 3 & 3 + 1 \\ 3 + 1 & -2 + 3 & -1 - 2 \\ -1 - 3 & 9 + 2 & 2 - 3 \end{bmatrix} = \begin{bmatrix} 0 & 4 & 4 \\ 4 & 1 & -3 \\ -4 & 11 & -1 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|} adj A = \frac{1}{16} \begin{bmatrix} 0 & 4 & 4 \\ 4 & 1 & -3 \\ -4 & 11 & -1 \end{bmatrix}$$

$$X = A^{-1}B \Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{16} \begin{bmatrix} 0 & 4 & 4 \\ 4 & 1 & -3 \\ -4 & 11 & -1 \end{bmatrix} \begin{bmatrix} 9 \\ 9 \\ -1 \end{bmatrix} = \frac{1}{16} \begin{bmatrix} 0 + 36 - 4 \\ 36 + 9 + 3 \\ -36 + 99 + 1 \end{bmatrix} = \frac{1}{16} \begin{bmatrix} 32 \\ 48 \\ 64 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} \text{ Solution : } (x, y, z) = (2, 3, 4)$$

(iv)  $x + y + z - 2 = 0, 6x - 4y + 5z - 31 = 0, 5x + 2y + 2z = 13$

$$\begin{bmatrix} 1 & 1 & 1 \\ 6 & -4 & 5 \\ 5 & 2 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 31 \\ 13 \end{bmatrix} \Rightarrow AX = B \Rightarrow X = A^{-1}B$$

$$|A| = 1(-8 - 10) - 1(12 - 25) + 1(12 + 20) = -18 + 13 + 32 = 27$$

$$\begin{bmatrix} -4 & 2 & 1 \\ 5 & 2 & 1 \\ 6 & 5 & 1 \\ -4 & 2 & 1 \end{bmatrix} \Rightarrow adj A = \begin{bmatrix} -8 - 10 & 2 - 2 & 5 + 4 \\ 25 - 12 & 2 - 5 & 6 - 5 \\ 12 + 20 & 5 - 2 & -4 - 6 \end{bmatrix} = \begin{bmatrix} -18 & 0 & 9 \\ 13 & -3 & 1 \\ 32 & 3 & -10 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|} \text{adj}A = \frac{1}{27} \begin{bmatrix} -18 & 0 & 9 \\ 13 & -3 & 1 \\ 32 & 3 & -10 \end{bmatrix}$$

$$X = A^{-1}B \Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{27} \begin{bmatrix} -18 & 0 & 9 \\ 13 & -3 & 1 \\ 32 & 3 & -10 \end{bmatrix} \begin{bmatrix} 2 \\ 31 \\ 13 \end{bmatrix} = \frac{1}{27} \begin{bmatrix} -36 + 0 + 117 \\ 26 - 93 + 13 \\ 64 + 93 - 130 \end{bmatrix}$$

$$= \frac{1}{27} \begin{bmatrix} 81 \\ -54 \\ 27 \end{bmatrix} \Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix} \quad \text{Solution : } (x, y, z) = (3, -2, 1)$$

**EXAMPLE 1.23** Solve the following system of linear equations by matrix inversion method:  $2x_1 + 3x_2 + 3x_3 = 5, x_1 - 2x_2 + x_3 = -4, 3x_1 - x_2 - 2x_3 = 3$

$$\begin{bmatrix} 2 & 3 & 3 \\ 1 & -2 & 1 \\ 3 & -1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5 \\ -4 \\ 3 \end{bmatrix} \Rightarrow AX = B \Rightarrow X = A^{-1}B$$

$$|A| = 2(4 + 1) - 3(-2 - 3) + 3(-1 + 6) = 10 + 15 + 15 = 40$$

$$\begin{bmatrix} -2 & -1 & 3 & -2 \\ 1 & -2 & 3 & 1 \\ 1 & 3 & 2 & 1 \\ -2 & -1 & 3 & -2 \end{bmatrix} \Rightarrow \text{adj}A = \begin{bmatrix} 4 + 1 & -3 + 6 & 3 + 6 \\ 3 + 2 & -4 - 9 & 3 - 2 \\ -1 + 6 & 9 + 2 & -4 - 3 \end{bmatrix} = \begin{bmatrix} 5 & 3 & 9 \\ 5 & -13 & 1 \\ 5 & 11 & -7 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|} \text{adj}A = \frac{1}{40} \begin{bmatrix} 5 & 3 & 9 \\ 5 & -13 & 1 \\ 5 & 11 & -7 \end{bmatrix}$$

$$X = A^{-1}B \Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{40} \begin{bmatrix} 5 & 3 & 9 \\ 5 & -13 & 1 \\ 5 & 11 & -7 \end{bmatrix} \begin{bmatrix} 5 \\ -4 \\ 3 \end{bmatrix} = \frac{1}{40} \begin{bmatrix} 25 - 12 + 27 \\ 25 + 52 + 3 \\ 25 - 44 - 21 \end{bmatrix} = \frac{1}{40} \begin{bmatrix} 40 \\ 80 \\ -40 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} \quad \text{Solution : } (x, y, z) = (1, 2, -1)$$

2. If  $A = \begin{bmatrix} -5 & 1 & 3 \\ 7 & 1 & -5 \\ 1 & -1 & 1 \end{bmatrix}$  and  $B = \begin{bmatrix} 1 & 1 & 2 \\ 3 & 2 & 1 \\ 2 & 1 & 3 \end{bmatrix}$ , find the products AB and BA and hence solve the system of equations  $x + y + 2z = 1, 3x + 2y + z = 7, 2x + y + 3z = 2$

$$AB = \begin{bmatrix} -5 & 1 & 3 \\ 7 & 1 & -5 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 2 \\ 3 & 2 & 1 \\ 2 & 1 & 3 \end{bmatrix} = \begin{bmatrix} -5 + 3 + 6 & -5 + 2 + 3 & -10 + 1 + 9 \\ 7 + 3 - 10 & 7 + 2 - 5 & 14 + 1 - 15 \\ 1 - 3 + 2 & 1 - 2 + 1 & 2 - 1 + 3 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix} = 4I_3$$

$$BA = \begin{bmatrix} 1 & 1 & 2 \\ 3 & 2 & 1 \\ 2 & 1 & 3 \end{bmatrix} \begin{bmatrix} -5 & 1 & 3 \\ 7 & 1 & -5 \\ 1 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix} = 4I_3$$

$$AB = BA = 4I_3 \Rightarrow B^{-1} = \frac{1}{4}A \rightarrow (1)$$

$$\begin{bmatrix} 1 & 1 & 2 \\ 3 & 2 & 1 \\ 2 & 1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 7 \\ 2 \end{bmatrix} \Rightarrow BX = C \Rightarrow X = B^{-1}C \rightarrow (2)$$

$$\text{From (1)} \Rightarrow B^{-1} = \frac{1}{4}A = \frac{1}{4} \begin{bmatrix} -5 & 1 & 3 \\ 7 & 1 & -5 \\ 1 & -1 & 1 \end{bmatrix}$$

$$\text{From (2)} \Rightarrow X = B^{-1}C \Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{4} \begin{bmatrix} -5 & 1 & 3 \\ 7 & 1 & -5 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 7 \\ 2 \end{bmatrix} = \frac{1}{4} \begin{bmatrix} -5 + 7 + 6 \\ 7 + 7 - 10 \\ 1 - 7 + 2 \end{bmatrix}$$

$$= \frac{1}{4} \begin{bmatrix} 8 \\ 4 \\ -4 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} \quad \text{Solution : } (x, y, z) = (2, 1, -1)$$

**EXAMPLE 1.24** If  $A = \begin{bmatrix} -4 & 4 & 4 \\ -7 & 1 & 3 \\ 5 & -3 & -1 \end{bmatrix}$  and  $B = \begin{bmatrix} 1 & -1 & 1 \\ 1 & -2 & -2 \\ 2 & 1 & 3 \end{bmatrix}$ , find the products AB and BA and hence solve the system of equations  $x - y + z = 4, x - 2y - 2z = 9, 2x + y + 3z = 1$

3. A man is appointed in a job with a monthly salary of certain amount and a fixed amount of annual increment. If his salary was Rs.19800 per month at the end of the first month after 3 years of service and Rs.23400 per month at the end of the first month after 9 years of service, find his starting salary and his annual increment. ( Use matrix inversion method )

Let his starting salary be x and the annual increment be y.

From the given data,  $x + 3y = 19800 \rightarrow (1); x + 9y = 23400 \rightarrow (2)$

$$\begin{bmatrix} 1 & 3 \\ 1 & 9 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 19800 \\ 23400 \end{bmatrix} \Rightarrow AX = B \Rightarrow X = A^{-1}B$$

$$|A| = 9 - 3 = 6; \text{adj} A = \begin{bmatrix} 9 & -3 \\ -1 & 1 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|} \text{adj}A = \frac{1}{6} \begin{bmatrix} 9 & -3 \\ -1 & 1 \end{bmatrix}$$

$$X = A^{-1}B \Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{6} \begin{bmatrix} 9 & -3 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 19800 \\ 23400 \end{bmatrix} = \frac{1}{6} \begin{bmatrix} 178200 - 70200 \\ -19800 + 23400 \end{bmatrix} = \frac{1}{6} \begin{bmatrix} 108000 \\ 3600 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 18000 \\ 600 \end{bmatrix}$$

Solution: His starting salary = Rs.18000 ; Annual increment = Rs.600

4. 4 men and 4 women can finish a piece of work jointly in 3 days while 2 men and 5 women can finish the same work jointly in 4 days. Find the time taken by one man alone and that of one woman alone to finish the same work by using matrix inversion method.

Let the number of days taken by a man and woman be  $a$  and  $y$  respectively.

Work finished by a man in one day =  $\frac{1}{x}$

Work finished by a woman in one day =  $\frac{1}{y}$

From the given data,  $\frac{4}{x} + \frac{4}{y} = \frac{1}{3}$  ;  $\frac{2}{x} + \frac{5}{y} = \frac{1}{4}$

Let  $\frac{1}{x} = a$  ,  $\frac{1}{y} = b$

$$\frac{4}{x} + \frac{4}{y} = \frac{1}{3} \Rightarrow 4a + 4b = \frac{1}{3} \Rightarrow 12a + 12b = 1 \rightarrow (1)$$

$$\frac{2}{x} + \frac{5}{y} = \frac{1}{4} \Rightarrow 2a + 5b = \frac{1}{4} \Rightarrow 8a + 20b = 1 \rightarrow (2)$$

$$\begin{bmatrix} 12 & 12 \\ 8 & 20 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \Rightarrow AX = B \Rightarrow X = A^{-1}B$$

$$|A| = 240 - 96 = 144 ; \text{adj} A = \begin{bmatrix} 20 & -12 \\ -8 & 12 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|} \text{adj}A = \frac{1}{144} \begin{bmatrix} 20 & -12 \\ -8 & 12 \end{bmatrix}$$

$$X = A^{-1}B \Rightarrow \begin{bmatrix} a \\ b \end{bmatrix} = \frac{1}{144} \begin{bmatrix} 20 & -12 \\ -8 & 12 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1}{144} \begin{bmatrix} 20 - 12 \\ -8 + 12 \end{bmatrix} = \frac{1}{144} \begin{bmatrix} 8 \\ 4 \end{bmatrix} = \begin{bmatrix} \frac{1}{18} \\ \frac{1}{36} \end{bmatrix}$$

$$\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} \frac{1}{18} \\ \frac{1}{36} \end{bmatrix} \Rightarrow x = \frac{1}{a} = 18 ; y = \frac{1}{b} = 36$$

Solution: A man can finish the work in 18 days and a woman can finish the work in 36 days.

5. The prices of three commodities A, B and C are Rs.x, y and z per units respectively. A person P purchases 4 units of B and sells 2 units of A and 5 units of C. Person Q purchase 2 unit of C and sells 3 units of A and one unit of B. Person R purchases one unit of A and sells 3 unit of B and one unit of C. In the process P, Q and R earn Rs.15000, Rs.1000 and Rs.4000 respectively. Find the prices per unit of A, B and C. ( Use matrix inversion method )

Let the prices of one unit of A, B and C are Rs. x, y and z respectively.

From the given data,

$$2x - 4y + 5z = 15000 \rightarrow (1)$$

$$3x + y - 2z = 1000 \rightarrow (1)$$

$$-x + 3y + z = 4000 \rightarrow (1)$$

$$\begin{bmatrix} 2 & -4 & 5 \\ 3 & 1 & -2 \\ -1 & 3 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 15000 \\ 1000 \\ 4000 \end{bmatrix} \Rightarrow AX = B \Rightarrow X = A^{-1}B$$

$$|A| = 2(1 + 6) - (-4)(3 - 2) + 5(9 + 1) = 14 + 4 + 50 = 68$$

$$\begin{bmatrix} 1 & 3 & -4 & 1 \\ -2 & 1 & 5 & -2 \\ 3 & -1 & 2 & 3 \\ 1 & 3 & -4 & 1 \end{bmatrix} \Rightarrow \text{adj}A = \begin{bmatrix} 1+6 & 15+4 & 8-5 \\ 2-3 & 2+5 & 15+4 \\ 9+1 & 4-6 & 2+12 \end{bmatrix} = \begin{bmatrix} 7 & 19 & 3 \\ -1 & 7 & 19 \\ 10 & -2 & 14 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|} \text{adj}A = \frac{1}{68} \begin{bmatrix} 7 & 19 & 3 \\ -1 & 7 & 19 \\ 10 & -2 & 14 \end{bmatrix}$$

$$X = A^{-1}B \Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{68} \begin{bmatrix} 7 & 19 & 3 \\ -1 & 7 & 19 \\ 10 & -2 & 14 \end{bmatrix} \begin{bmatrix} 15000 \\ 1000 \\ 4000 \end{bmatrix}$$

$$= \frac{1000}{68} \begin{bmatrix} 7 & 19 & 3 \\ -1 & 7 & 19 \\ 10 & -2 & 14 \end{bmatrix} \begin{bmatrix} 15 \\ 1 \\ 4 \end{bmatrix}$$

$$= \frac{1000}{68} \begin{bmatrix} 105 + 19 + 12 \\ -15 + 7 + 76 \\ 150 - 2 + 56 \end{bmatrix}$$

$$= \frac{1000}{68} \begin{bmatrix} 136 \\ 68 \\ 204 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2000 \\ 1000 \\ 3000 \end{bmatrix}$$

Solution: The price of one unit of A, B and C are Rs.2000, 1000 and 3000 respectively.

### CRAMER'S RULE

#### EXERCISE 1.4

1. Solve the following systems of linear equations by Cramer's rule:

(i)  $5x - 2y + 16 = 0, x + 3y - 7 = 0$

$$\Delta = \begin{vmatrix} 5 & -2 \\ 1 & 3 \end{vmatrix} = 15 + 2 = 17$$

$$\Delta_x = \begin{vmatrix} -16 & -2 \\ 7 & 3 \end{vmatrix} = -48 + 14 = -34$$

$$\Delta_y = \begin{vmatrix} 5 & -16 \\ 1 & 7 \end{vmatrix} = 35 + 16 = 51$$

$$x = \frac{\Delta_x}{\Delta} = \frac{-34}{17} = -2 \quad ; \quad y = \frac{\Delta_y}{\Delta} = \frac{51}{17} = 3$$

Solution:  $(x, y) = (-2, 3)$

(ii)  $\frac{3}{x} + 2y = 12, \frac{2}{x} + 3y = 13$

Let  $\frac{1}{x} = a, y = b \Rightarrow 3a + 2b = 12; 2a + 3b = 13$

$$\Delta = \begin{vmatrix} 3 & 2 \\ 2 & 3 \end{vmatrix} = 9 - 4 = 5$$

$$\Delta_a = \begin{vmatrix} 12 & 2 \\ 13 & 3 \end{vmatrix} = 36 - 26 = 10$$

$$\Delta_y = \begin{vmatrix} 3 & 12 \\ 2 & 13 \end{vmatrix} = 39 - 24 = 15$$

$$a = \frac{\Delta_a}{\Delta} = \frac{10}{5} = 2 \Rightarrow x = \frac{1}{a} \Rightarrow x = \frac{1}{2} \quad ; \quad b = \frac{\Delta_b}{\Delta} = \frac{15}{5} = 3 \Rightarrow y = 3$$

Solution:  $(x, y) = \left(\frac{1}{2}, 3\right)$

(iii)  $3x + 3y - z = 11, 2x - y + 2z = 9, 4x + 3y + 2z = 25$

$$\Delta = \begin{vmatrix} 3 & 3 & -1 \\ 2 & -1 & 2 \\ 4 & 3 & 2 \end{vmatrix} = 3(-2 - 6) - 3(4 - 8) - 1(6 + 4) = -24 + 12 - 10 = -22$$

$$\Delta_x = \begin{vmatrix} 11 & 3 & -1 \\ 9 & -1 & 2 \\ 25 & 3 & 2 \end{vmatrix} = 11(-2 - 6) - 3(18 - 50) - 1(27 + 25) = -88 + 96 - 52 = -44$$

$$\Delta_y = \begin{vmatrix} 3 & 11 & -1 \\ 2 & 9 & 2 \\ 4 & 25 & 2 \end{vmatrix} = 3(18 - 50) - 11(50 - 36) - 1(50 - 36) = -96 + 44 - 14 = -66$$

$$\Delta_z = \begin{vmatrix} 3 & 3 & 11 \\ 2 & -1 & 9 \\ 4 & 3 & 25 \end{vmatrix} = 3(-25 - 27) - 3(50 - 36) + 11(6 + 4) = -156 - 42 + 110 = -88$$

Padasalai

$$x = \frac{\Delta_x}{\Delta} = \frac{-44}{-22} = 2 \quad ; \quad y = \frac{\Delta_y}{\Delta} = \frac{-66}{-22} = 3 \quad ; \quad z = \frac{\Delta_z}{\Delta} = \frac{-88}{-22} = 4$$

Solution :  $(x, y, z) = (2, 3, 4)$

(iv)  $\frac{3}{x} - \frac{4}{y} - \frac{2}{z} - 1 = 0, \frac{1}{x} + \frac{2}{y} + \frac{1}{z} - 2 = 0, \frac{2}{x} - \frac{5}{y} - \frac{4}{z} + 1 = 0$

Let  $\frac{1}{x} = a, \frac{1}{y} = b, \frac{1}{z} = c$

$$\Rightarrow 3a - 4b - 2c = 1 \rightarrow (1), a + 2b + c = 2 \rightarrow (2), 2a - 5b - 4c = -1 \rightarrow (3)$$

$$\Delta = \begin{vmatrix} 3 & -4 & -2 \\ 1 & 2 & 1 \\ 2 & -5 & -4 \end{vmatrix} = 3(-8 + 5) - (-4)(-4 - 2) - 2(-5 - 4) = -9 - 24 + 18 = -15$$

$$\Delta_a = \begin{vmatrix} 1 & -4 & -2 \\ 2 & 2 & 1 \\ -1 & -5 & -4 \end{vmatrix} = 1(-8 + 5) - (-4)(-8 + 1) - 2(-10 + 2) = -3 - 28 + 16 = -15$$

$$\Delta_b = \begin{vmatrix} 3 & 1 & -2 \\ 1 & 2 & 1 \\ 2 & -1 & -4 \end{vmatrix} = 3(-8 + 1) - 1(-4 - 2) - 2(-1 - 4) = -21 + 6 + 10 = -5$$

$$\Delta_c = \begin{vmatrix} 3 & -4 & 1 \\ 1 & 2 & 2 \\ 2 & -5 & -1 \end{vmatrix} = 3(-2 + 10) - (-4)(-1 - 4) + 1(-5 - 4) = 24 - 20 - 9 = -5$$

$$a = \frac{\Delta_a}{\Delta} = \frac{-15}{-15} = 1 \Rightarrow x = \frac{1}{a} = 1$$

$$b = \frac{\Delta_b}{\Delta} = \frac{-5}{-15} = \frac{1}{3} \Rightarrow y = \frac{1}{b} = 3$$

$$c = \frac{\Delta_c}{\Delta} = \frac{-5}{-15} = \frac{1}{3} \Rightarrow z = \frac{1}{c} = 3$$

Solution :  $(x, y, z) = (1, 3, 3)$

**EXAMPLE 1.25** Solve by Cramer's rule, the system of equations  $x_1 - x_2 =$

$3, 2x_1 + 3x_2 + 4x_3 = 17, x_2 + 2x_3 = 7$

$$\Delta = \begin{vmatrix} 1 & -1 & 0 \\ 2 & 3 & 4 \\ 0 & 1 & 2 \end{vmatrix} = 1(6 - 4) - (-1)(4 - 0) + 0 = 2 + 4 = 6$$

$$\Delta_1 = \begin{vmatrix} 3 & -1 & 0 \\ 17 & 3 & 4 \\ 7 & 1 & 2 \end{vmatrix} = 3(6 - 4) - (-1)(34 - 28) + 0 = 6 + 6 = 12$$

$$\Delta_2 = \begin{vmatrix} 1 & 3 & 0 \\ 2 & 17 & 4 \\ 0 & 7 & 2 \end{vmatrix} = 1(34 - 28) - 3(4 - 0) + 0 = 6 - 12 = -6$$

$$\Delta_3 = \begin{vmatrix} 1 & -1 & 3 \\ 2 & 3 & 17 \\ 0 & 1 & 7 \end{vmatrix} = 1(21 - 17) - (-1)(14 - 0) + 3(2 - 0) = 4 + 14 + 6 = 24$$

$$x_1 = \frac{\Delta_1}{\Delta} = \frac{12}{6} = 2 ; x_2 = \frac{\Delta_2}{\Delta} = \frac{-6}{6} = -1 ; x_3 = \frac{\Delta_3}{\Delta} = \frac{24}{6} = 4$$

$$\text{Solution : } (x_1, x_2, x_3) = (2, -1, 4)$$

2. In a competitive examination, one mark is awarded for every correct answer while  $\frac{1}{4}$  mark is deducted for every wrong answer. A student answered 100 questions and get 80 marks. How many questions did he answer correctly? (Use Cramer's rule )

Let the number questions answered correctly be x and y be the number of questions answered wrong.

$$\text{From the given data, } x + y = 100 \rightarrow (1) ; x - \frac{1}{4}y = 80 \Rightarrow 4x - y = 320 \rightarrow (2)$$

$$\Delta = \begin{vmatrix} 1 & 1 \\ 4 & -1 \end{vmatrix} = -1 - 4 = -5$$

$$\Delta_x = \begin{vmatrix} 100 & 1 \\ 320 & -1 \end{vmatrix} = -100 - 320 = -420$$

$$\Delta_y = \begin{vmatrix} 1 & 100 \\ 4 & 320 \end{vmatrix} = 320 - 400 = -80$$

$$x = \frac{\Delta_x}{\Delta} = \frac{-420}{-5} = 84 ; y = \frac{\Delta_y}{\Delta} = \frac{-80}{-5} = 16$$

Solution:

The number questions answered correctly= 84

The number questions answered wrong= 16

3. A chemist has one solution which is 50% acid and another solution which is 25% acid. How much each should be mixed to make 10 litres of 40% acid solution? (Use Cramer's rule )

Let x and y be the amount of solution containing 50% and 25% acid respectively.

From the given data,

$$x + y = 10 \rightarrow (1)$$

$$50\% \text{ of } x + 25\% \text{ of } y = 40\% \text{ of } 10 \Rightarrow \frac{50}{100}x + \frac{25}{100}y = \frac{40}{100} \cdot 10$$

$$\Rightarrow 50x + 25y = 400 \rightarrow (2)$$

$$\Delta = \begin{vmatrix} 1 & 1 \\ 50 & 25 \end{vmatrix} = 25 - 50 = -25$$

$$\Delta_x = \begin{vmatrix} 10 & 1 \\ 400 & 25 \end{vmatrix} = 250 - 400 = -150$$

$$\Delta_y = \begin{vmatrix} 1 & 100 \\ 50 & 400 \end{vmatrix} = 400 - 500 = -100$$

$$x = \frac{\Delta_x}{\Delta} = \frac{-150}{-25} = 6 ; y = \frac{\Delta_y}{\Delta} = \frac{-100}{-25} = 4$$

Solution:

6 litres of solution containing 50% acid and 4 litres of solution containing 25% acid must be mixed to make 40% acid solution.

4. A fish tank can be filled in 10 minutes using both pumps A and B simultaneously. However, pump B can pump water in or out at the same rate. If pump B is inadvertently run in reverse, then the tank will be filled in 30 minutes. How long would it take each pump to fill the tank by itself? (Use Cramer's rule )

Let x and y be the time taken by pumps A and B respectively.

$$\text{Amount of water filled by pump A in one minute} = \frac{1}{x}$$

$$\text{Amount of water filled by pump B in one minute} = \frac{1}{y}$$

$$\text{From the given data, } \frac{1}{x} + \frac{1}{y} = \frac{1}{10} \rightarrow (1) ; \frac{1}{x} - \frac{1}{y} = \frac{1}{30} \rightarrow (2)$$

$$\text{Let } \frac{1}{x} = a, \frac{1}{y} = b$$

$$(1) \Rightarrow a + b = \frac{1}{10} \Rightarrow 10a + 10b = 1 \rightarrow (3)$$

$$(2) \Rightarrow a - b = \frac{1}{30} \Rightarrow 30a - 30b = 1 \rightarrow (4)$$

Solving (3) & (4),

$$\Delta = \begin{vmatrix} 10 & 10 \\ 30 & -30 \end{vmatrix} = -300 - 300 = -600$$

$$\Delta_a = \begin{vmatrix} 1 & 10 \\ 1 & -30 \end{vmatrix} = -30 - 10 = -40$$

$$\Delta_b = \begin{vmatrix} 10 & 1 \\ 30 & 1 \end{vmatrix} = 10 - 30 = -20$$

$$a = \frac{\Delta_a}{\Delta} = \frac{-40}{-600} = \frac{1}{15} ; b = \frac{\Delta_b}{\Delta} = \frac{-20}{-600} = \frac{1}{30} \Rightarrow x = \frac{1}{a} = 15, y = \frac{1}{b} = 30$$

Solution:

Time taken by pump A= 15 minutes ; Time taken by pump B= 30 minutes

5. A family of 3 people went out for dinner in a restaurant. The cost of two dosai, three idlies and two vadais is Rs. 150. The cost of two dosai, two idlies and four vadais is Rs. 200. The cost of five dosai, four idlies and two vadais is Rs. 250. The family has Rs. 350 in hand and they ate 3 dosai and six idlies and six vadais. Will they be able to manage to pay the bill within the amount they had? (Use Cramer's

rule )

Let Rs.x, y and z be the cost of 1 dosai, 1 idly and 1 vadai respectively.

From the given data,

$$2x + 3y + 2z = 150 \rightarrow (1)$$

$$2x + 2y + 4z = 200, \div 2 \Rightarrow x + y + 2z = 100 \rightarrow (2)$$

$$5x + 4y + 2z = 250 \rightarrow (3)$$

$$\Delta = \begin{vmatrix} 2 & 3 & 2 \\ 1 & 1 & 2 \\ 5 & 4 & 2 \end{vmatrix} = 2(2 - 8) - 3(2 - 10) + 2(4 - 5) = -12 + 24 - 2 = 10$$

$$\Delta_x = \begin{vmatrix} 150 & 3 & 2 \\ 100 & 1 & 2 \\ 250 & 4 & 2 \end{vmatrix} = 150(2 - 8) - 3(200 - 500) + 2(400 - 250) = -900 + 900 + 300 = 300$$

$$\Delta_y = \begin{vmatrix} 2 & 150 & 2 \\ 1 & 100 & 2 \\ 5 & 250 & 2 \end{vmatrix} = 2(200 - 500) - 150(2 - 10) + 2(250 - 500) = -600 + 1200 - 500 = 100$$

$$\Delta_z = \begin{vmatrix} 2 & 3 & 150 \\ 1 & 1 & 100 \\ 5 & 4 & 250 \end{vmatrix} = 2(250 - 400) - 3(250 - 500) + 150(4 - 5) = -300 + 750 - 150 = 300$$

$$x = \frac{\Delta_x}{\Delta} = \frac{300}{10} = 30 ; y = \frac{\Delta_y}{\Delta} = \frac{100}{10} = 10 ; z = \frac{\Delta_z}{\Delta} = \frac{300}{10} = 30$$

$$\text{The cost of 3 dosai, 6 idlies and 6 vadais} = 3x + 6y + 6z = 3(30) + 6(10) + 6(30) = 90 + 60 + 180 = 330 < 350$$

Hence they can manage to pay the bill.

**EXAMPLE 1.26** In a T20 match, Chennai Super Kings needed just 6 runs to win with 1 ball left to go in the last over. The last ball was bowled and the batsman at the crease hit it high up. The ball travelled along the path in a vertical plane and the equation of the path is  $y = ax^2 + bx + c$  with respect to a xy-coordinate system in the vertical plane and the ball travelled through the points (10,8), (20,16), (40,22), can you conclude that, Chennai Super Kings won the match? Justify your answer. (All the distances are measured in metres and the meeting point of the plane of the path with the farthest boundary line is (70,0).)

$$y = ax^2 + bx + c$$

$$(10,8) \Rightarrow 100a + 10b + c = 8 \rightarrow (1)$$

$$(20,16) \Rightarrow 400a + 20b + c = 16 \rightarrow (2)$$

$$(40,22) \Rightarrow 1600a + 40b + c = 22 \rightarrow (3)$$

$$\Delta = \begin{vmatrix} 100 & 10 & 1 \\ 400 & 20 & 1 \\ 1600 & 40 & 1 \end{vmatrix} = 100 \times 10 \begin{vmatrix} 1 & 1 & 1 \\ 4 & 2 & 1 \\ 16 & 4 & 1 \end{vmatrix} = 1000[1(2 - 4) - 1(4 - 16) + 1(16 - 32)] = 1000[-2 + 12 - 16] = -6000$$

$$\Delta_a = \begin{vmatrix} 8 & 10 & 1 \\ 16 & 20 & 1 \\ 22 & 40 & 1 \end{vmatrix} = 2 \times 10 \begin{vmatrix} 4 & 1 & 1 \\ 8 & 2 & 1 \\ 11 & 4 & 1 \end{vmatrix} = 20[4(2 - 4) - 1(8 - 11) + 1(32 - 22)] = 20[-8 + 3 + 10] = 100$$

$$\Delta_b = \begin{vmatrix} 100 & 8 & 1 \\ 400 & 16 & 1 \\ 1600 & 22 & 1 \end{vmatrix} = 100 \times 2 \begin{vmatrix} 1 & 4 & 1 \\ 4 & 8 & 1 \\ 16 & 11 & 1 \end{vmatrix} = 200[1(8 - 11) - 4(4 - 16) + 1(44 - 128)] = 200[-3 + 48 - 84] = -7800$$

$$\Delta_c = \begin{vmatrix} 100 & 10 & 8 \\ 400 & 20 & 16 \\ 1600 & 40 & 22 \end{vmatrix} = 100 \times 10 \times 2 \begin{vmatrix} 1 & 1 & 4 \\ 4 & 2 & 8 \\ 16 & 4 & 11 \end{vmatrix} = 2000[1(22 - 32) - 1(44 - 128) + 4(16 - 32)] = 2000[-10 + 84 - 64] = 20000$$

$$a = \frac{\Delta_a}{\Delta} = \frac{100}{-6000} = -\frac{1}{60} ; b = \frac{\Delta_b}{\Delta} = \frac{-7800}{-6000} = \frac{13}{10} ; c = \frac{\Delta_c}{\Delta} = \frac{20000}{-6000} = -\frac{10}{3}$$

$$\therefore \text{The equation of the path is } y = -\frac{1}{60}x^2 + \frac{13}{10}x - \frac{10}{3}$$

$$\text{Sub. } x = 70, \text{ we get } y = -\frac{1}{60}(70)^2 + \frac{13}{10}(70) - \frac{10}{3} = -\frac{4900}{60} + \frac{910}{10} - \frac{10}{3} = 6$$

$\Rightarrow$  The ball went 6 m high over the boundary line and it is impossible for a fielder to catch the ball. Hence the ball went for six and the Chennai Super Kings won the match.

## GAUSSIAN ELIMINATION METHOD

### EXERCISE 1.5

1. Solve the following systems of linear equations by Gaussian elimination method:

$$(i) 2x - 2y + 3z = 2, x + 2y - z = 3, 3x - y + 2z = 1$$

$$[A|B] = \left( \begin{array}{ccc|c} 2 & -2 & 3 & 2 \\ 1 & 2 & -1 & 3 \\ 3 & -1 & 2 & 1 \end{array} \right) \sim \left( \begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 2 & -2 & 3 & 2 \\ 3 & -1 & 2 & 1 \end{array} \right) R_1 \leftrightarrow R_2$$

$$\sim \left( \begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 0 & -6 & 5 & -4 \\ 0 & -7 & 5 & -8 \end{array} \right) \begin{array}{l} R_2 \rightarrow R_2 - 2R_1 \\ R_3 \rightarrow R_3 - 3R_1 \end{array}$$

$$\sim \left( \begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 0 & -6 & 5 & -4 \\ 0 & 0 & -5 & -20 \end{array} \right) R_3 \rightarrow 6R_3 - 7R_2$$

$$\Rightarrow x + 2y - z = 3 \rightarrow (1); -6y + 5z = -4 \rightarrow (2); -5z = -20 \Rightarrow z = 4$$

$$\text{Put } z = 4 \text{ in } (2) \Rightarrow -6y + 5(4) = -4 \Rightarrow -6y + 20 = -4$$

$$\Rightarrow -6y = -24 \Rightarrow y = 4$$

$$\text{Put } y = 4, z = 4 \text{ in } (1) \Rightarrow x + 2(4) - 4 = 3 \Rightarrow x + 8 - 4 = 3 \Rightarrow x = -1$$

$$\text{Solution: } (x, y, z) = (-1, 4, 4)$$

$$(ii) 2x + 4y + 6z = 22, 3x + 8y + 5z = 27, -x + y + 2z = 2$$

$$2x + 4y + 6z = 22, \div 2 \Rightarrow x + 2y + 3z = 11$$

$$[A|B] = \left( \begin{array}{ccc|c} 1 & 2 & 3 & 11 \\ 3 & 8 & 5 & 27 \\ -1 & 1 & 2 & 2 \end{array} \right)$$

$$\sim \left( \begin{array}{ccc|c} 1 & 2 & 3 & 11 \\ 0 & 2 & -4 & -6 \\ 0 & 3 & 5 & 13 \end{array} \right) \begin{array}{l} R_2 \rightarrow R_2 - 3R_1 \\ R_3 \rightarrow R_3 + R_1 \end{array}$$

$$\sim \left( \begin{array}{ccc|c} 1 & 2 & 3 & 11 \\ 0 & 2 & -4 & -6 \\ 0 & 0 & 22 & 44 \end{array} \right) R_3 \rightarrow 2R_3 - 3R_2$$

$$\Rightarrow x + 2y + 3z = 11 \rightarrow (1); 2y - 4z = -6 \rightarrow (2); 22z = 44 \Rightarrow z = 2$$

$$\text{Put } z = 2 \text{ in } (2) \Rightarrow 2y - 4(2) = -6 \Rightarrow 2y = -6 + 8 \Rightarrow 2y = 2 \Rightarrow y = 1$$

$$\text{Put } y = 1, z = 2 \text{ in } (1) \Rightarrow x + 2(1) + 3(2) = 11 \Rightarrow x + 8 = 11 \Rightarrow x = 3$$

$$\text{Solution: } (x, y, z) = (3, 1, 2)$$

**EXAMPLE 1.27** Solve the following systems of linear equations by Gaussian elimination method:  $4x + 3y + 6z = 25, x + 5 + 7z = 13, 2x + 9y + z = 1$

2. If  $ax^2 + bx + c$  is divided by  $x + 3, x - 5$  and  $x - 1$  the remainders are 21, 61 and 9 respectively. Find  $a, b$  and  $c$ . ( Use Gaussian elimination method )

$$\text{Let } p(x) = ax^2 + bx + c$$

$$p(-3) = 21 \Rightarrow a(-3)^2 + b(-3) + c = 21 \Rightarrow 9a - 3b + c = 21 \rightarrow (1)$$

$$p(5) = 61 \Rightarrow a(5)^2 + b(5) + c = 61 \Rightarrow 25a + 5b + c = 61 \rightarrow (2)$$

$$p(1) = 9 \Rightarrow a(1)^2 + b(1) + c = 9 \Rightarrow a + b + c = 9 \rightarrow (3)$$

$$[A|B] = \left( \begin{array}{ccc|c} 9 & -3 & 1 & 21 \\ 25 & 5 & 1 & 61 \\ 1 & 1 & 1 & 9 \end{array} \right)$$

$$\sim \left( \begin{array}{ccc|c} 1 & 1 & 1 & 9 \\ 25 & 5 & 1 & 61 \\ 9 & -3 & 1 & 21 \end{array} \right) R_1 \leftrightarrow R_3$$

$$\sim \left( \begin{array}{ccc|c} 1 & 1 & 1 & 9 \\ 0 & -20 & -24 & -164 \\ 0 & -12 & -8 & -60 \end{array} \right) \begin{array}{l} R_2 \rightarrow R_2 - 25R_1 \\ R_3 \rightarrow R_3 - 9R_1 \end{array}$$

$$\sim \left( \begin{array}{ccc|c} 1 & 1 & 1 & 9 \\ 0 & 5 & 6 & 41 \\ 0 & 3 & 2 & -15 \end{array} \right) \begin{array}{l} R_2 \rightarrow \left(-\frac{1}{4}\right) R_2 \\ R_3 \rightarrow \left(-\frac{1}{4}\right) R_3 \end{array}$$

$$\sim \left( \begin{array}{ccc|c} 1 & 1 & 1 & 9 \\ 0 & 5 & 6 & 41 \\ 0 & 0 & -8 & -48 \end{array} \right) R_3 \rightarrow 5R_3 - 3R_2$$

$$\Rightarrow a + b + c = 9 \rightarrow (1); 5b + 6c = 41 \rightarrow (2); -8c = -48 \Rightarrow c = 6$$

$$\text{Put } c = 6 \text{ in } (2) \Rightarrow 5b + 6(6) = 41 \Rightarrow 5b = 41 - 36 \Rightarrow 5b = 5 \Rightarrow b = 1$$

$$\text{Put } b = 1, c = 6 \text{ in } (1) \Rightarrow a + 1 + 6 = 9 \Rightarrow a + 7 = 9 \Rightarrow a = 2$$

$$\text{Solution: } (a, b, c) = (2, 1, 6)$$

3. An amount of Rs.65,000 is invested in three bonds at the rates of 6%, 8% and 10% per annum respectively. The total annual income is Rs.5,000. The income from the third bond is Rs.800 more than that from the second bond. Determine the price of each bond. ( Use Gaussian elimination method )

Let the price of three bond be  $x, y$  and  $z$  respectively.

From the given data,

$$x + y + z = 65000 \rightarrow (1)$$

$$\text{Total annual income} = \text{Rs.5,000} \Rightarrow 6\% \text{ of } x + 8\% \text{ of } y + 10\% \text{ of } z = 5000$$

$$\Rightarrow \frac{6}{100}x + \frac{8}{100}y + \frac{10}{100}z = 5000 \Rightarrow 6x + 8y + 10z = 500000$$

$$\Rightarrow \div 2, 3x + 4y + 5z = 250000 \rightarrow (2)$$

$$\text{Income from third bond} = \text{income from second bond} + \text{Rs.800}$$

$$\Rightarrow 10\% \text{ of } z = 8\% \text{ of } y + 800 \Rightarrow \frac{10}{100}z = \frac{8}{100}y + 800$$

$$\Rightarrow \frac{10}{100}z = \frac{8y + 80000}{100} \Rightarrow 10z = 8y + 80000 \Rightarrow -8y + 10z = 80000$$

$$\Rightarrow \div 2, -4y + 5z = 40000 \rightarrow (3)$$

$$[A|B] = \left( \begin{array}{ccc|c} 1 & 1 & 1 & 65000 \\ 3 & 4 & 5 & 250000 \\ 0 & -4 & 5 & 40000 \end{array} \right)$$

$$\sim \left( \begin{array}{ccc|c} 1 & 1 & 1 & 65000 \\ 0 & 1 & 2 & 55000 \\ 0 & -4 & 5 & 40000 \end{array} \right) R_2 \rightarrow R_2 - 3R_1$$

$$\sim \left( \begin{array}{ccc|c} 1 & 1 & 1 & 65000 \\ 0 & 1 & 2 & 55000 \\ 0 & 0 & 13 & 260000 \end{array} \right) R_3 \rightarrow R_3 + 4R_2$$

$$\Rightarrow x + y + z = 65000 \rightarrow (1); y + 2z = 55000 \rightarrow (2);$$

$$13z = 260000 \Rightarrow z = 20000$$

$$\text{Put } z = 20000 \text{ in } (2) \Rightarrow y + 2(20000) = 55000 \Rightarrow y = 55000 - 40000 \Rightarrow y = 15000$$

$$\text{Put } y = 15000, z = 20000 \text{ in } (1) \Rightarrow x + 15000 + 20000 = 65000$$

$$\Rightarrow x + 35000 = 65000 \Rightarrow x = 30000$$

Solution:

The price of 6%, 8% and 10% are Rs.30000, Rs.15000 and Rs.20000 respectively.

4. A boy is walking along the path  $y = ax^2 + bx + c$  through the points  $(-6, 8)$ ,  $(-2, -12)$ ,  $(3, 8)$ . He wants to meet his friend at  $P(7, 60)$ . Will he meet his friend? ( Use Gaussian elimination method )

$$y = ax^2 + bx + c$$

$$(-6, 8) \Rightarrow 8 = a(-6)^2 + b(-6) + c \Rightarrow 36a - 6b + c = 8 \rightarrow (1)$$

$$(-2, -12) \Rightarrow -12 = a(-2)^2 + b(-2) + c \Rightarrow 4a - 2b + c = -12 \rightarrow (2)$$

$$(3, 8) \Rightarrow 8 = a(3)^2 + b(3) + c \Rightarrow 9a + 3b + c = 8 \rightarrow (3)$$

$$[A|B] = \left( \begin{array}{ccc|c} 36 & -6 & 1 & 8 \\ 4 & -2 & 1 & -12 \\ 9 & 3 & 1 & 8 \end{array} \right)$$

$$\sim \left( \begin{array}{ccc|c} 36 & -6 & 1 & 8 \\ 0 & -12 & 8 & -116 \\ 0 & 18 & 3 & 24 \end{array} \right) R_2 \rightarrow 9R_2 - R_1$$

$$\sim \left( \begin{array}{ccc|c} 36 & -6 & 1 & 8 \\ 0 & -6 & 4 & -58 \\ 0 & 6 & 1 & 8 \end{array} \right) R_3 \rightarrow \left( \frac{1}{2} \right) R_2$$

$$\sim \left( \begin{array}{ccc|c} 36 & -6 & 1 & 8 \\ 0 & -6 & 4 & -58 \\ 0 & 0 & 5 & -50 \end{array} \right) R_3 \rightarrow R_3 + R_2$$

$$\Rightarrow 36a - 6b + c = 8 \rightarrow (1); -6b + 4c = -58 \rightarrow (2); 5c = -50 \Rightarrow c = -10$$

$$\text{Put } c = -10 \text{ in } (2) \Rightarrow -6b + 4(-10) = -58 \Rightarrow -6b - 40 = -58$$

$$\Rightarrow -6b = -58 + 40 \Rightarrow -6b = -18 \Rightarrow b = 3$$

$$\text{Put } b = 3, c = -10 \text{ in } (1) \Rightarrow 36a - 6(3) - 10 = 8 \Rightarrow 36a - 18 - 10 = 8$$

$$\Rightarrow 36a = 8 + 28 \Rightarrow 36a = 36 \Rightarrow a = 1$$

$$\text{Solution: } (a, b, c) = (1, 3, -10) \Rightarrow \boxed{y = x^2 + 3x - 10}$$

$$\text{Put } x = 7 \Rightarrow y = (7)^2 + 3(7) - 10 = 49 + 21 - 10 = 60 \Rightarrow y = 60$$

The point  $P(7, 60)$  satisfies the equation  $y = x^2 + 3x - 10$ , hence the boy will meet friend at  $P(7, 60)$ .

**EXAMPLE 1.28** The upward speed  $v(t)$  a rocket at time  $t$  is approximated by  $v(t) = at^2 + bt + c$ ,  $0 \leq t \leq 100$  where  $a$ ,  $b$  and  $c$  are constant. It has been found that the speed at times  $t = 3$ ,  $t = 6$  and  $t = 9$  seconds are respectively 64, 133 and 208 miles per second respectively. Find the speed at time  $t = 15$  seconds. ( Use Gaussian elimination method )

### RANK METHOD – NON-HOMOGENEOUS LINEAR EQUATIONS

- ❖ If  $\rho(A) = \rho([A|B]) = 3$ , then the system of equations is consistent and has unique solution.
- ❖ If  $\rho(A) = \rho([A|B]) = 2$ , then the system of equations is consistent and has infinitely many solutions. ( Put  $z = k$  and find  $x, y$  )
- ❖ If  $\rho(A) = \rho([A|B]) = 1$ , then the system of equations is consistent and has infinitely many solutions. ( Put  $y = s$  and  $z = t$  and find  $x$  )
- ❖ If  $\rho(A) \neq \rho([A|B])$ , then the system of equations is inconsistent and has no solution.

### EXERCISE 1.6

1. Test for consistency and if possible, solve the following systems of equations by rank method:

$$(i) x - y + 2z = 2, 2x + y + 4z = 7, 4x - y + z = 4$$

$$AX = B \Rightarrow \begin{bmatrix} 1 & -1 & 2 \\ 2 & 1 & 4 \\ 4 & -1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 7 \\ 4 \end{bmatrix}$$

$$\text{Augmented matrix } [A|B] = \left( \begin{array}{ccc|c} 1 & -1 & 2 & 2 \\ 2 & 1 & 4 & 7 \\ 4 & -1 & 1 & 4 \end{array} \right)$$

$$\sim \left( \begin{array}{ccc|c} 1 & -1 & 2 & 2 \\ 0 & 3 & 0 & 3 \\ 0 & 3 & -7 & -4 \end{array} \right) \begin{array}{l} R_2 \rightarrow R_2 - 2R_1 \\ R_3 \rightarrow R_3 - 4R_1 \end{array}$$

$$\sim \left( \begin{array}{ccc|c} 1 & -1 & 2 & 2 \\ 0 & 3 & 0 & 3 \\ 0 & 0 & -7 & -7 \end{array} \right) R_3 \rightarrow R_3 - R_2$$

The last equivalent matrix is in echelon form.

$\rho([A|B]) = 3 = \rho(A) \Rightarrow$  The system of equations is consistent and has unique solution

$$AX = B \Rightarrow \left( \begin{array}{ccc} 1 & -1 & 2 \\ 0 & 3 & 0 \\ 0 & 0 & -7 \end{array} \right) \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ -7 \end{pmatrix}$$

$$x - y + 2z = 2 \rightarrow (1)$$

$$\Rightarrow 3y = 3 \Rightarrow y = 1$$

$$-7z = -7 \Rightarrow z = 1$$

Put  $y = 1, z = 1$  in (1)  $\Rightarrow x - 1 + 2 = 2 \Rightarrow x = 1$

Solution:  $(x, y, z) = (1, 1, 1)$

**EXAMPLE 1.29** Test for consistency of the following systems of equations and if possible solve:  $x + 2y - z = 3, 3x - y + 2z = 1, x - 2y + 3z = 3, x - y + z + 1 = 0$

(ii)  $3x + y + z = 2, x - 3y + 2z = 1, 7x - y + 4z = 5$

$$AX = B \Rightarrow \begin{bmatrix} 3 & 1 & 1 \\ 1 & -3 & 2 \\ 7 & -1 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 5 \end{bmatrix}$$

$$\text{Augmented matrix } [A|B] = \left( \begin{array}{ccc|c} 3 & 1 & 1 & 2 \\ 1 & -3 & 2 & 1 \\ 7 & -1 & 4 & 5 \end{array} \right)$$

$$\sim \left( \begin{array}{ccc|c} 1 & -3 & 2 & 1 \\ 3 & 1 & 1 & 2 \\ 7 & -1 & 4 & 5 \end{array} \right) R_1 \leftrightarrow R_2$$

$$\sim \left( \begin{array}{ccc|c} 1 & -3 & 2 & 1 \\ 0 & 10 & -5 & -1 \\ 0 & 20 & -10 & -2 \end{array} \right) \begin{array}{l} R_2 \rightarrow R_2 - 3R_1 \\ R_3 \rightarrow R_3 - 7R_1 \end{array}$$

$$\sim \left( \begin{array}{ccc|c} 1 & -3 & 2 & 1 \\ 0 & 10 & -5 & -1 \\ 0 & 0 & 0 & 0 \end{array} \right) R_3 \rightarrow R_3 - 2R_2$$

The last equivalent matrix is in echelon form.

$\rho([A|B]) = 2 = \rho(A) \Rightarrow$  The system of equations is consistent and has unique solution

$$AX = B \Rightarrow \left( \begin{array}{ccc} 1 & -3 & 2 \\ 0 & 10 & -5 \\ 0 & 0 & 0 \end{array} \right) \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$$

$$\Rightarrow x - 3y + 2z = 1 \rightarrow (1)$$

$$\Rightarrow 10y - 5z = -1 \rightarrow (2)$$

$$\text{Put } z = t \text{ in (2)} \Rightarrow 10y - 5t = -1 \Rightarrow 10y = 5t - 1 \Rightarrow y = \frac{1}{10}(5t - 1)$$

$$\text{Put } y = \frac{1}{10}(5t - 1), z = t \text{ in (1)} \Rightarrow x - 3\frac{1}{10}(5t - 1) + 2t = 1$$

$$\Rightarrow x = 1 - 2t + 3\frac{1}{10}(5t - 1) = \frac{10 - 20t + 15t - 3}{10} = \frac{1}{10}(7 - 5t)$$

$$\text{Solution: } (x, y, z) = \left( \frac{1}{10}(7 - 5t), \frac{1}{10}(5t - 1), t \right); t \in R$$

**EXAMPLE 1.30** Test for consistency of the following systems of equations and if possible solve:  $4x - 2y + 6z = 8, x + y - 3z = -1, 15x - 3y + 9z = 21$

(iii)  $2x + 2y + z = 5, x - y + z = 1, 3x + y + 2z = 4$

$$AX = B \Rightarrow \begin{bmatrix} 2 & 2 & 1 \\ 1 & -1 & 1 \\ 3 & 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \\ 4 \end{bmatrix}$$

$$\text{Augmented matrix } [A|B] = \left( \begin{array}{ccc|c} 2 & 2 & 1 & 5 \\ 1 & -1 & 1 & 1 \\ 3 & 1 & 2 & 4 \end{array} \right)$$

$$\sim \left( \begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ 2 & 2 & 1 & 5 \\ 3 & 1 & 2 & 4 \end{array} \right) R_1 \leftrightarrow R_2$$

$$\sim \left( \begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ 0 & 4 & -1 & 3 \\ 0 & 4 & -1 & 1 \end{array} \right) \begin{array}{l} R_2 \rightarrow R_2 - 2R_1 \\ R_3 \rightarrow R_3 - 3R_1 \end{array}$$

$$\sim \left( \begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ 0 & 4 & -1 & 3 \\ 0 & 0 & 0 & -2 \end{array} \right) R_3 \rightarrow R_3 - R_2$$

The last equivalent matrix is in echelon form.

$\rho([A|B]) = 3, \rho(A) = 2 \Rightarrow \rho([A|B]) \neq \rho(A)$ , The system of equations is inconsistent and has no solution

**EXAMPLE 1.32** Test for consistency of the following systems of equations and if possible solve:  $x - y + z = -9, 2x - y + z = 4, 3x - y + z = 6, 4x - y + 2z = 7$

(iv)  $2x - y + z = 2, 6x - 3y + 3z = 6, 4x - 2y + 2z = 4$

$$AX = B \Rightarrow \begin{bmatrix} 2 & -1 & 1 \\ 6 & -3 & 3 \\ 4 & -2 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 6 \\ 4 \end{bmatrix}$$

$$\text{Augmented matrix } [A|B] = \left( \begin{array}{ccc|c} 2 & -1 & 1 & 2 \\ 6 & -3 & 3 & 6 \\ 4 & -2 & 2 & 4 \end{array} \right) \sim \left( \begin{array}{ccc|c} 2 & -1 & 1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right) \begin{array}{l} R_2 \rightarrow R_2 - 3R_1 \\ R_3 \rightarrow R_3 - 2R_1 \end{array}$$

The last equivalent matrix is in echelon form.

$\rho([A|B]) = 1, \rho(A) = 1 \Rightarrow$  The system of equations is consistent and has infinitely many solutions.

$$AX = B \Rightarrow \begin{pmatrix} 2 & -1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow 2x - y + z = 2$$

Put  $y = s$  and  $z = t$  in the above equation,  $2x - s + t = 2 \Rightarrow 2x = s - t + 2$

$$\Rightarrow x = \frac{1}{2}(s - t + 2)$$

$$\text{Solution: } (x, y, z) = \left( \frac{1}{2}(s - t + 2), s, t \right); s, t \in R$$

**EXAMPLE 1.31** Test for consistency of the following systems of equation and if possible solve:  $x - y + z = -9, 2x - 2y + 2z = -18, 3x - 3y + 3z + 27 = 0$

2. Find the value of  $k$  for which the equations  $kx - 2y + z = 1, x - 2ky + z = -2, x - 2y + kz = 1$  have (i) no solution (ii) unique solution (iii) infinitely many solution

$$AX = B \Rightarrow \begin{bmatrix} k & -2 & 1 \\ 1 & -2k & 1 \\ 1 & -2 & k \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

$$\begin{aligned} \text{Augmented matrix } [A|B] &= \left( \begin{array}{ccc|c} k & -2 & 1 & 1 \\ 1 & -2k & 1 & -2 \\ 1 & -2 & k & 1 \end{array} \right) \\ &\sim \left( \begin{array}{ccc|c} 1 & -2k & 1 & -2 \\ k & -2 & 1 & 1 \end{array} \right) R_1 \leftrightarrow R_3 \\ &\sim \left( \begin{array}{ccc|c} 1 & -2 & k & 1 \\ 0 & 2-2k & 1-k & -3 \\ 0 & 2k-2 & 1-k^2 & 1-k \end{array} \right) \begin{array}{l} R_2 \rightarrow R_2 - R_1 \\ R_3 \rightarrow R_3 - kR_1 \end{array} \\ &\sim \left( \begin{array}{ccc|c} 1 & -2 & k & 1 \\ 0 & 2(1-k) & 1-k & -3 \\ 0 & 0 & 2-k-k^2 & -k-2 \end{array} \right) R_3 \rightarrow R_3 + R_2 \\ &\sim \left( \begin{array}{ccc|c} 1 & -2 & k & 1 \\ 0 & 2(1-k) & 1-k & -3 \\ 0 & 0 & (k+2)(1-k) & -(k+2) \end{array} \right) \end{aligned}$$

- (i) When  $k = 1, k \neq -2, \rho([A|B]) = 3, \rho(A) = 1 \Rightarrow \rho([A|B]) \neq \rho(A)$ , the system of equations is inconsistent and has no solution.
- (ii) When  $k \neq 1, k \neq -2, \rho([A|B]) = 3 = \rho(A)$ , the system of equations is consistent and has unique solution.
- (iii) When  $k = -2, k \neq 1, \rho([A|B]) = 2 = \rho(A)$ , the system of equations is consistent and has infinitely many solutions.

3. Investigate the values of  $\lambda$  and  $\mu$  the system of linear equations  $2x + 3y + 5z = 9, 7x + 3y - 5z = 8, 2x + 3y + \lambda z = \mu$ , have (i) no solution (ii) unique solution (iii) infinitely many solution

$$AX = B \Rightarrow \begin{bmatrix} 2 & 3 & 5 \\ 7 & 3 & -5 \\ 2 & 3 & \lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 8 \\ \mu \end{bmatrix}$$

$$\begin{aligned} \text{Augmented matrix } [A|B] &= \left( \begin{array}{ccc|c} 2 & 3 & 5 & 9 \\ 7 & 3 & -5 & 8 \\ 2 & 3 & \lambda & \mu \end{array} \right) \\ &\sim \left( \begin{array}{ccc|c} 2 & 3 & 5 & 9 \\ 0 & -15 & -25 & -9 \\ 0 & 0 & \lambda - 5 & \mu - 9 \end{array} \right) \begin{array}{l} R_2 \rightarrow 2R_2 - 7R_1 \\ R_3 \rightarrow R_3 - R_1 \end{array} \end{aligned}$$

- (i) When  $\lambda = 5, \mu \neq 9, \rho([A|B]) = 3, \rho(A) = 2 \Rightarrow \rho([A|B]) \neq \rho(A)$ , the system of equations is inconsistent and has no solution.

(ii) When  $\lambda \neq 5, \mu \neq 9, \rho([A|B]) = 3 = \rho(A)$ , the system of equations is consistent and has unique solution.

(iii) When  $\lambda = 5, \mu = 9, \rho([A|B]) = 2 = \rho(A)$ , the system of equations is consistent and has infinitely many solutions.

**EXAMPLE 1.34** Investigate for what values of  $\lambda$  and  $\mu$  the system of linear equations  $x + 2y + z = 7, x + y + \lambda z = \mu, x + 3y - 5z = 5$ , have (i) no solution (ii) unique solution (iii) infinitely many solution

$$AX = B \Rightarrow \begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & \lambda \\ 1 & 3 & -5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 7 \\ \mu \\ 5 \end{bmatrix}$$

$$\begin{aligned} \text{Augmented matrix } [A|B] &= \left( \begin{array}{ccc|c} 1 & 2 & 1 & 7 \\ 1 & 1 & \lambda & \mu \\ 1 & 3 & -5 & 5 \end{array} \right) \\ &\sim \left( \begin{array}{ccc|c} 1 & 2 & 1 & 7 \\ 1 & 3 & -5 & 5 \\ 1 & 1 & \lambda & \mu \end{array} \right) R_2 \leftrightarrow R_3 \\ &\sim \left( \begin{array}{ccc|c} 1 & 2 & 1 & 7 \\ 0 & 1 & -6 & -2 \\ 0 & -1 & \lambda - 1 & \mu - 7 \end{array} \right) R_2 \rightarrow R_2 - R_1 \\ &\quad R_3 \rightarrow R_3 - R_1 \\ &\sim \left( \begin{array}{ccc|c} 1 & 2 & 1 & 7 \\ 0 & 1 & -6 & -2 \\ 0 & 0 & \lambda - 7 & \mu - 9 \end{array} \right) R_3 \rightarrow R_3 + R_2 \end{aligned}$$

(i) When  $\lambda = 7, \mu \neq 9, \rho([A|B]) = 3, \rho(A) = 2 \Rightarrow \rho([A|B]) \neq \rho(A)$ , the system of equations is inconsistent and has no solution.

(ii) When  $\lambda \neq 7, \mu \neq 9, \rho([A|B]) = 3 = \rho(A)$ , the system of equations is consistent and has unique solution.

(iii) When  $\lambda = 7, \mu = 9, \rho([A|B]) = 2 = \rho(A)$ , the system of equations is consistent and has infinitely many solutions.

**EXAMPLE 1.33** Find the condition on a, b and c so that the following system of linear equations has one parameter family of solutions:  $x + y + z = a, x + 2y + 3z = b, 3x + 5y + 7z = c$

$$AX = B \Rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 3 & 5 & 7 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

$$\begin{aligned} \text{Augmented matrix } [A|B] &= \left( \begin{array}{ccc|c} 1 & 1 & 1 & a \\ 1 & 2 & 3 & b \\ 3 & 5 & 7 & c \end{array} \right) \\ &\sim \left( \begin{array}{ccc|c} 1 & 1 & 1 & a \\ 0 & 1 & 2 & b - a \\ 0 & 2 & 4 & c - 3a \end{array} \right) R_2 \rightarrow R_2 - R_1 \\ &\quad R_3 \rightarrow R_3 - 3R_1 \\ &\sim \left( \begin{array}{ccc|c} 1 & 1 & 1 & a \\ 0 & 1 & 2 & b - a \\ 0 & 0 & 0 & (c - 3a) - 2(b - a) \end{array} \right) R_3 \rightarrow R_3 - 2R_2 \\ &\sim \left( \begin{array}{ccc|c} 1 & 1 & 1 & a \\ 0 & 1 & 2 & b - a \\ 0 & 0 & 0 & c - 2b - a \end{array} \right) \end{aligned}$$

Since given that the system have one parameter family of solutions,  $\rho([A|B]) = \rho(A) = 2$ . So the third row must be a zero row. So  $c - 2b - a = 0 \Rightarrow c = a + 2b$

#### RANK METHOD – HOMOGENEOUS LINEAR EQUATIONS

- ❖ If  $\rho(A) = \rho([A|O]) = 3$ , then the system of equations is consistent and has trivial solution. i.e.,  $(x, y, z) = (0, 0, 0)$
- ❖ If  $\rho(A) = \rho([A|O]) = 2$  or  $1 < n$ , then the system of equations is consistent and has non-trivial solutions.
- ❖ If the system of equations has non-trivial solution,  $\rho([A|O]) < n$ , so the determinant of the coefficient matrix is 0.

#### EXERCISE 1.7

1. Solve the following system of homogenous equations:

(i)  $3x + 2y + 7z = 0, 4x - 3y - 2z = 0, 5x + 9y + 23z = 0$

$$AX = O \Rightarrow \begin{bmatrix} 3 & 2 & 7 \\ 4 & -3 & -2 \\ 5 & 9 & 23 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{aligned} \text{Augmented matrix } [A|O] &= \left( \begin{array}{ccc|c} 3 & 2 & 7 & 0 \\ 4 & -3 & -2 & 0 \\ 5 & 9 & 23 & 0 \end{array} \right) \\ &\sim \left( \begin{array}{ccc|c} 3 & 2 & 7 & 0 \\ 0 & -17 & -34 & 0 \\ 0 & 17 & 34 & 0 \end{array} \right) R_2 \rightarrow 3R_2 - 4R_1 \\ &\quad R_3 \rightarrow 3R_3 - 5R_1 \end{aligned}$$

$$\sim \left( \begin{array}{ccc|c} 3 & 2 & 7 & 0 \\ 0 & -17 & -34 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right) R_3 \rightarrow R_3 + R_2$$

$\Rightarrow \rho([A|B]) = 2 = \rho(A)$ , the system of equations is consistent and has non-trivial solution.

$$AX = O \Rightarrow \begin{pmatrix} 3 & 2 & 7 \\ 0 & -17 & -34 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{aligned} 3x + 2y + 7z &= 0 \rightarrow (1) \\ -17y - 34z &= 0 \rightarrow (2) \end{aligned}$$

Put  $z=t$  in (2)  $\Rightarrow -17y - 34t = 0 \Rightarrow y = -2t$

Put  $y = -2t, z = t$  in (1)  $\Rightarrow 3x + 2(-2t) + 7t = 0 \Rightarrow 3x = -3t \Rightarrow x = -t$

Solution :  $(x, y, z) = (-t, -2t, t); t \in R$

**EXAMPLE 1.36** Solve the system :  $x + 3y - 2z = 0, 2x - y + 4z = 0, x - 11y + 14z = 0$

1(ii)  $2x + 3y - z = 0, x - y - 2z = 0, 3x + y + 3z = 0$

$$AX = O \Rightarrow \begin{bmatrix} 2 & 3 & -1 \\ 1 & -1 & -2 \\ 3 & 1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\text{Augmented matrix } [A|O] = \left( \begin{array}{ccc|c} 2 & 3 & -1 & 0 \\ 1 & -1 & -2 & 0 \\ 3 & 1 & 3 & 0 \end{array} \right)$$

$$\sim \left( \begin{array}{ccc|c} 1 & -1 & -2 & 0 \\ 2 & 3 & -1 & 0 \\ 3 & 1 & 3 & 0 \end{array} \right) R_1 \leftrightarrow R_2$$

$$\sim \left( \begin{array}{ccc|c} 1 & -1 & -2 & 0 \\ 0 & 5 & 3 & 0 \\ 0 & 4 & 9 & 0 \end{array} \right) \begin{aligned} R_2 &\rightarrow R_2 - 2R_1 \\ R_3 &\rightarrow R_3 - 3R_1 \end{aligned}$$

$$\sim \left( \begin{array}{ccc|c} 1 & -1 & -2 & 0 \\ 0 & 5 & 3 & 0 \\ 0 & 0 & 33 & 0 \end{array} \right) R_3 \rightarrow 5R_3 - 4R_2$$

$\Rightarrow \rho([A|B]) = 3 = \rho(A)$ , the system of equations is consistent and has trivial solution.  $(x, y, z) = (0, 0, 0)$

**EXAMPLE 1.35** Solve the system :  $x + 2y + 3z = 0, 3x + 4y + 4z = 0, 7x + 10y + 12z = 0$

**EXAMPLE 1.37** Solve the system :  $x + y - 2z = 0, 2x - 3y + z = 0, 3x - 7y + 10z = 0, 6x - 9y + 10z = 0$

2. Determine the values of  $\lambda$  for which the following system of equations  $x + y + 3z = 0, 4x + 3y + \lambda z = 0, 2x + y + 2z = 0$  has (i) a unique solution (ii) a non-trivial solution

$$AX = O \Rightarrow \begin{bmatrix} 1 & 1 & 3 \\ 4 & 3 & \lambda \\ 2 & 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\text{Augmented matrix } [A|O] = \left( \begin{array}{ccc|c} 1 & 1 & 3 & 0 \\ 4 & 3 & \lambda & 0 \\ 2 & 1 & 2 & 0 \end{array} \right)$$

$$\sim \left( \begin{array}{ccc|c} 1 & 1 & 3 & 0 \\ 2 & 1 & 2 & 0 \\ 4 & 3 & \lambda & 0 \end{array} \right) R_2 \leftrightarrow R_3$$

$$\sim \left( \begin{array}{ccc|c} 1 & 1 & 3 & 0 \\ 0 & -1 & -4 & 0 \\ 0 & -1 & \lambda - 12 & 0 \end{array} \right) \begin{aligned} R_2 &\rightarrow R_2 - 2R_1 \\ R_3 &\rightarrow R_3 - 4R_1 \end{aligned}$$

$$\sim \left( \begin{array}{ccc|c} 1 & 1 & 3 & 0 \\ 0 & -1 & -4 & 0 \\ 0 & 0 & \lambda - 8 & 0 \end{array} \right) R_3 \rightarrow R_3 - R_2$$

(i) When  $\lambda \neq 8 \Rightarrow \rho(A) = \rho([A|O]) = 3$ , then the system of equations is consistent and has trivial solution.  $(x, y, z) = (0, 0, 0)$

(ii) When  $\lambda = 8 \Rightarrow \rho(A) = \rho([A|O]) = 2$ , then the system of equations is consistent and has non-trivial solution.

**EXAMPLE 1.38** Determine the values of  $\lambda$  for which the following system of equations  $(3\lambda - 8)x + 3y + 3z = 0, 3x + (3\lambda - 8)y + 3z = 0, 3x + 3y + (3\lambda - 8)z = 0$  has a non-trivial solution.

Here the number of unknowns is 3.

Since the system of equations has non-trivial solution,  $\rho([A|O]) < 3$ , so the determinant of the coefficient matrix is 0.

$$\begin{vmatrix} 3\lambda - 8 & 3 & 3 \\ 3 & 3\lambda - 8 & 3 \\ 3 & 3 & 3\lambda - 8 \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} 3\lambda - 2 & 3\lambda - 2 & 3\lambda - 2 \\ 3 & 3\lambda - 8 & 3 \\ 3 & 3 & 3\lambda - 8 \end{vmatrix} = 0 \quad (R_1 \rightarrow R_1 + R_2 + R_3)$$

$$\Rightarrow (3\lambda - 2) \begin{vmatrix} 1 & 1 & 1 \\ 3 & 3\lambda - 8 & 3 \\ 1 & 1 & 1 \end{vmatrix} = 0$$

$$\Rightarrow (3\lambda - 2) \begin{vmatrix} 3 & 3 & 3\lambda - 8 \\ 1 & 1 & 1 \\ 0 & 0 & 3\lambda - 11 \end{vmatrix} = 0 \quad \left( \begin{array}{l} R_2 \rightarrow R_2 - 3R_1 \\ R_3 \rightarrow R_3 - 3R_1 \end{array} \right)$$

Expanding through  $C_1$ ,

$$\Rightarrow (3\lambda - 2)(3\lambda - 11)^2 = 0 \Rightarrow \lambda = \frac{2}{3}, \lambda = \frac{11}{3}$$

**EXAMPLE 1.40** If the system of equations  $px + by + cz = 0, ax + qy + cz = 0, ax + by + rz = 0$  has a non-trivial solution and  $p \neq a, q \neq b, r \neq c$ , prove that

$$\frac{p}{p-a} + \frac{q}{q-b} + \frac{r}{r-c} = 2$$

Since the system of equations has non-trivial solution,  $\rho([A|O]) < 3$ , so the determinant of the coefficient matrix is 0.

$$\begin{vmatrix} p & b & c \\ a & q & c \\ a & b & r \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} p & b & c \\ a-p & q-b & 0 \\ a-p & 0 & r-c \end{vmatrix} = 0 \quad \left( \begin{array}{l} R_2 \rightarrow R_2 - R_1 \\ R_3 \rightarrow R_3 - R_1 \end{array} \right)$$

$$\Rightarrow \begin{vmatrix} p & b & c \\ -(p-a) & q-b & 0 \\ -(p-a) & 0 & r-c \end{vmatrix} = 0$$

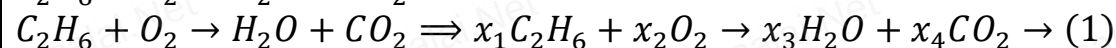
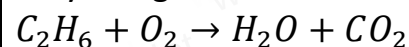
$$\Rightarrow (p-a)(q-b)(r-c) \begin{vmatrix} \frac{p}{p-a} & \frac{q}{q-b} & \frac{r}{r-c} \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{vmatrix} = 0 \Rightarrow \begin{vmatrix} \frac{p}{p-a} & \frac{q}{q-b} & \frac{r}{r-c} \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{vmatrix} = 0$$

$$\Rightarrow \frac{p}{p-a} + \frac{q}{q-b} + \frac{r}{r-c} = 0$$

$$\Rightarrow \frac{p}{p-a} + \frac{q-(q-b)}{q-b} + \frac{r-(r-c)}{r-c} = 0$$

$$\Rightarrow \frac{p}{p-a} + \frac{q}{q-b} + \frac{r}{r-c} - 1 - 1 = 0 \Rightarrow \frac{p}{p-a} + \frac{q}{q-b} + \frac{r}{r-c} = 2$$

3. By using Gaussian elimination method, balance the chemical reaction equation :



| $x_1$ |   |   | $x_2$ |   |   | $\rightarrow$ | $x_3$ |   |   | $x_4$ |   |   |
|-------|---|---|-------|---|---|---------------|-------|---|---|-------|---|---|
| C     | H | O | C     | H | O |               | C     | H | O | C     | H | O |
| 2     | 6 | 0 | 0     | 0 | 2 |               | 0     | 2 | 1 | 1     | 0 | 2 |

Equating the number of atom of C on both sides with  $x_1, x_2, x_3, x_4$ , we get

$$2x_1 + 0x_2 = 0x_3 + 1x_4 \Rightarrow 2x_1 + 0x_2 - 0x_3 - 1x_4 = 0 \rightarrow (2)$$

Equating the number of atom of H on both sides with  $x_1, x_2, x_3, x_4$ , we get

$$6x_1 + 0x_2 = 2x_3 + 0x_4 \Rightarrow 6x_1 + 0x_2 - 2x_3 - 0x_4 = 0$$

$$\div 2 \Rightarrow 3x_1 + 0x_2 - x_3 - 0x_4 = 0 \rightarrow (3)$$

Equating the number of atom of O on both sides with  $x_1, x_2, x_3, x_4$ , we get

$$0x_1 + 2x_2 = 1x_3 + 2x_4 \Rightarrow 0x_1 + 2x_2 - 1x_3 - 2x_4 = 0 \rightarrow (4)$$

$$\text{From (2), (3), (4)} \Rightarrow \text{Augmented matrix } [A|O] = \left( \begin{array}{cccc|c} 2 & 0 & 0 & -1 & 0 \\ 3 & 0 & -1 & 0 & 0 \\ 0 & 2 & -1 & -2 & 0 \end{array} \right)$$

$$\sim \left( \begin{array}{cccc|c} 2 & 0 & 0 & -1 & 0 \\ 0 & 0 & -2 & 3 & 0 \\ 0 & 2 & -1 & -2 & 0 \end{array} \right) \sim R_2 \rightarrow 2R_2 - 3R_1$$

$$\sim \left( \begin{array}{cccc|c} 2 & 0 & 0 & -1 & 0 \\ 0 & 2 & -1 & -2 & 0 \\ 0 & 2 & -1 & -2 & 0 \end{array} \right) R_2 \leftrightarrow R_3$$

$\rho(A) = \rho([A|O]) = 3 < 4$ , the systems of equations is consistent and has infinite solution.

$$2x_1 - 1x_4 = 0 \rightarrow (5); 2x_2 - x_3 - 2x_4 = 0 \rightarrow (6); -2x_3 + 3x_4 = 0 \rightarrow (7)$$

$$\text{Put } x_4 = t \text{ in (5)} \Rightarrow 2x_1 - t = 0 \Rightarrow x_1 = \frac{t}{2}$$

$$\text{Put } x_4 = t \text{ in (7)} \Rightarrow -2x_3 + 3x_4 = 0 \Rightarrow -2x_3 + 3t = 0 \Rightarrow x_3 = \frac{3t}{2}$$

$$\text{Put } x_3 = \frac{3t}{2}, x_4 = t \text{ in (6)} \Rightarrow 2x_2 - \frac{3t}{2} - 2t = 0 \Rightarrow x_2 = \frac{7t}{4}$$

$$(x_1, x_2, x_3, x_4) = \left( \frac{t}{2}, \frac{7t}{4}, \frac{3t}{2}, t \right)$$

$$\text{Put } t = 4 \Rightarrow (x_1, x_2, x_3, x_4) = (2, 7, 6, 4)$$

$$\text{Sub. The above values in (1)} \Rightarrow 2C_2H_6 + 7O_2 \rightarrow 6H_2O + 4CO_2$$

**EXAMPLE 1.38** By using Gaussian elimination method, balance the chemical reaction equation :  $C_5H_8 + O_2 \rightarrow CO_2 + H_2O$