

## 2.COMPLEX NUMBERS

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### POWERS OF IMAGINARY UNIT

- ❖  $i^2 = -1, i^3 = -i, i^4 = 1$
- ❖ Division algorithm :  $n = 4(q) + r$
- ❖  $i^n = (i)^{4q+r} = (i)^{4q}(i)^r = (i^4)^q(i)^r \Rightarrow$  if  $r = 0, i^n = 1$  ; if  $r = 1, i^n = i$  ; if  $r = 2, i^n = -1$  ; if  $r = 3, i^n = -i$
- ❖  $\sqrt{ab} = \sqrt{a}\sqrt{b}$  is valid only if at least one of a, b is non-negative.
- ❖ For  $y \in R, y^2 > 0$

### EXERCISE 2.1

Simplify the following:

1.  $i^{1947} + i^{1950}$

$$1947 \div 4 = 4(486) + 3 ; 1950 = 4(487) + 2$$

$$i^{1947} + i^{1950} = (i^4)^{486}(i)^3 + (i^4)^{487}(i)^2 = -i - 1 = -1 - i$$

2.  $i^{1948} - i^{-1869}$

$$1948 \div 4 = 4(487) + 0 ; 1869 \div 4 = 4(467) + 1$$

$$i^{1948} - i^{-1869} = i^{1948} - \frac{1}{i^{1869}} = (i^4)^{487}(i)^0 - \frac{1}{(i^4)^{467}i^1} = 1 - \frac{1}{i} = 1 - \left(\frac{1}{i} \times \frac{-i}{-i}\right) = 1 - \left(\frac{-i}{-i^2}\right) = 1 + i \quad (\because -i^2 = 1)$$

3.  $\sum_{n=1}^{12} i^n$

$$\begin{aligned} \sum_{n=1}^{12} i^n &= i^1 + i^2 + i^3 + i^4 + i^5 + i^6 + i^7 + i^8 + i^9 + i^{10} + i^{11} + i^{12} \\ &= i^1 + i^2 + i^3 + i^4 + i^4i + (i^2)^3 + (i^2)^3i + (i^2)^4 + (i^3)^3 + (i^2)^5 + (i^2)^5i + (i^2)^6 \\ &= i - 1 - i + 1 + i - 1 - i + 1 + i - 1 - i + 1 \\ &= 0 \end{aligned}$$

4.  $i^{59} + \frac{1}{i^{59}}$

$$59 \div 4 = 4(14) + 3$$

$$\begin{aligned} i^{59} + \frac{1}{i^{59}} &= (i^4)^{14}i^3 + \frac{1}{(i^4)^{14}i^3} \\ &= -i + \frac{1}{-i} = i + \frac{1}{-i} \cdot \frac{i}{i} \\ &= -i + \frac{i}{1} = i - i \\ &= 0 \end{aligned}$$

5.  $i \cdot i^2 \cdot i^3 \cdot i^4 \dots i^{2000}$

$$= i^{(1+2+3+\dots+2000)}$$

$$= i^{\frac{2000 \times 2001}{2}}$$

$$= i^{1000 \times 2001}$$

$$= i^{1000} \cdot i^{2001}$$

$$= (i^4)^{250}(i^4)^{500}i^1 \quad (\because 2001 \div 4 = 4(500) + 1)$$

$$= 1 \cdot 1 \cdot i$$

$$= i$$

6.  $\sum_{n=1}^{10} i^{n+50}$

$$\sum_{n=1}^{10} i^{n+50} = i^{51} + i^{52} + i^{53} + \dots + i^{60}$$

$$= i^{50}(i^1 + i^2 + i^3 + i^4 + i^5 + i^6 + i^7 + i^8 + i^9 + i^{10})$$

$$= (i^4)^{12}i^2(i - 1 - i + 1 + i - 1 - i + 1 + i - 1)$$

$$= -1(i - 1)$$

$$= 1 - i$$

**EXAMPLE 2.1** Simplify the following:

(i)  $i^7$  (ii)  $i^{1729}$  (iii)  $i^{-1924} + i^{2018}$  (iv)  $\sum_{n=1}^{102} i^n$  (v)  $i \cdot i^2 \cdot i^3 \cdot i^4 \dots i^{40}$

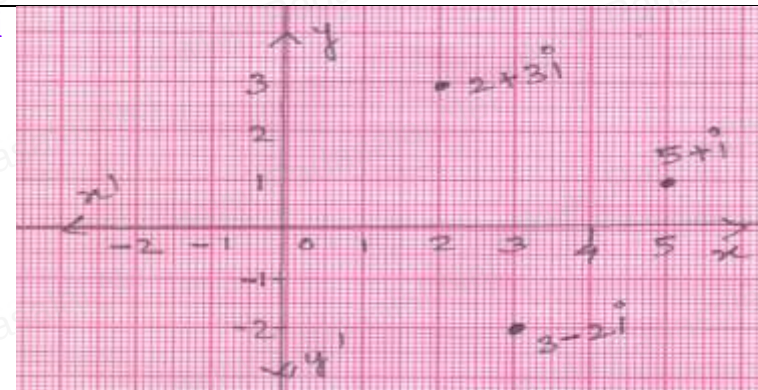
### COMPLEX NUMBERS

- ❖ **Rectangular Form Of a Complex Number:** A complex number is of the form  $z = x + iy, x, y \in R$ .  $x$  is called the real part and  $y$  is called the imaginary part of the complex number.
- ❖ Two complex numbers  $z_1 = a + ib, z_2 = c + id$  are said to be equal if and only if  $Re(z_1) = Re(z_2)$  and  $Im(z_1) = Im(z_2)$ . ie.  $a = c$  and  $b = d$
- ❖ **Scalar multiplication of complex numbers:** If  $z = x + iy$  and  $k \in R$  then  $kz = k(x + iy) = kx + kyi$
- ❖ **Addition of complex numbers:** If  $z_1 = a + ib, z_2 = c + id$ , then  $z_1 + z_2 = (a + c) + i(b + d)$
- ❖ **Subtraction of complex numbers:** If  $z_1 = a + ib, z_2 = c + id$ , then  $z_1 - z_2 = (a - c) + i(b - d)$
- ❖ **Multiplication of complex numbers:** If  $z_1 = a + ib, z_2 = c + id$ , then  $z_1 z_2 = (ac - bd) + i(ad + bc)$
- ❖ Multiplication of a complex  $z$  by  $i$  successively gives a  $90^\circ$  counter clockwise rotation successively about the origin.

### EXERCISE 2.2

1. Evaluate the following if  $z = 5 - 2i$  and  $w = -1 + 3i$

(i)  $z + w$  (ii)  $z - iw$  (iii)  $2z + 3w$  (iv)  $zw$  (v)  $z^2 + 2zw + w^2$  (vi)  $(z + w)^2$



$$\begin{aligned} (i) z + w &= 5 - 2i - 1 + 3i = 4 + i \\ (ii) z - iw &= 5 - 2i - i(-1 + 3i) = 5 - 2i + i - 3i^2 = 5 - 2i + i + 3 = 8 - i \\ &\quad \because i^2 = -1 \\ (iii) 2z + 3w &= 2(5 - 2i) + 3(-1 + 3i) = 10 - 4i - 3 + 9i = 7 + 5i \\ (iv) z_1 z_2 &= (ac - bd) + i(ad + bc); a = 5, b = -2, c = -1, d = 3 \\ zw &= (5 - 2i)(-1 + 3i) = [(5)(-1) - (-2)(3)] + i[(5)(3) + (-2)(-1)] \\ &= [-5 + 6] + i[15 + 2] = 1 + 17i \\ (v) z^2 &= (5 - 2i)^2 = 25 - 20i + 4i^2 = 25 - 20i - 4 = 21 - 20i \\ 2zw &= 2(5 - 2i)(-1 + 3i) = 2[1 + 17i] \text{ from (iv)} \\ 2zw &= 2 + 34i \\ w^2 &= (-1 + 3i)^2 = 1 - 6i + 9i^2 = 1 - 6i - 9 = -8 - 6i \\ z^2 + 2zw + w^2 &= 21 - 20i + 2 + 34i - 8 - 6i = 15 + 8i \\ (vi) (z + w)^2 &= (4 + i)^2 \text{ from (i)} \\ &= 16 + 8i + i^2 = 16 + 8i - 1 = 15 + 8i \end{aligned}$$

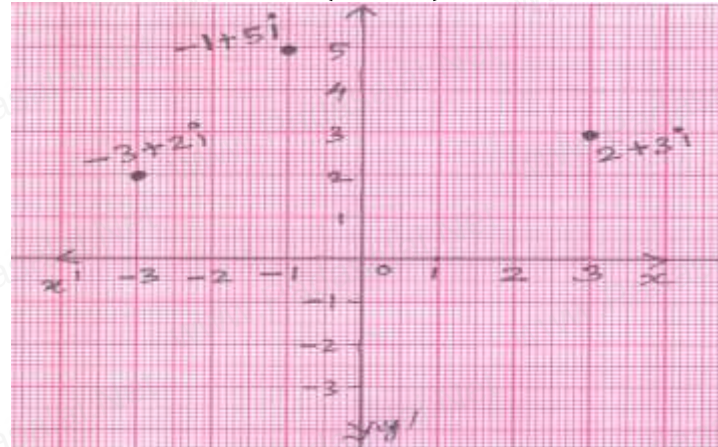
2. Given the complex number  $z = 2 + 3i$ , represent the complex numbers in Argand diagram.

(i)  $z$ ,  $iz$  and  $z + iz$  (ii)  $z$ ,  $-iz$  and  $z - iz$

$$(i) z = 2 + 3i = (2, 3)$$

$$iz = i(2 + 3i) = 2i + 3i^2 = -3 + 2i = (-3, 2)$$

$$z + iz = 2 + 3i + i(2 + 3i) = 2 + 3i - 3 + 2i = -1 + 5i = (-1, 5)$$



$$(i) z = 2 + 3i = (2, 3)$$

$$-iz = -i(2 + 3i) = -2i - 3i^2 = 3 - 2i = (3, -2)$$

$$z - iz = 2 + 3i - i(2 + 3i) = 2 + 3i + 3 - 2i = 5 + i = (5, 1)$$

3. Find the values of the real numbers  $x$  and  $y$ , if the complex numbers  $(3 - i)x - (2 - i)y + 2i + 5$  and  $2x + (-1 + 2i)y + 3 + 2i$  are equal.

$$z_1 = (3 - i)x - (2 - i)y + 2i + 5$$

$$= 3x - xi - 2y + yi + 2i + 5$$

$$= (3x - 2y + 5) + i(-x + y + 2)$$

$$z_2 = 2x + (-1 + 2i)y + 3 + 2i$$

$$= 2x - y + 2yi + 3 + 2i$$

$$= (2x - y + 3) + i(2y + 2)$$

$$\text{Given : } z_1 = z_2 \Rightarrow (3x - 2y + 5) + i(-x + y + 2) = (2x - y + 3) + i(2y + 2)$$

Equating the real and imaginary parts on both sides,

|                                |                              |
|--------------------------------|------------------------------|
| $3x - 2y + 5 = 2x - y + 3$     | $-x + y + 2 = 2y + 2$        |
| $3x - 2y + 5 - 2x + y - 3 = 0$ | $-x + y + 2 - 2y - 2 = 0$    |
| $x - y = -2 \rightarrow (1)$   | $-x - y = 0 \rightarrow (2)$ |

$$(1) + (2) \Rightarrow -2y = -2 \Rightarrow \boxed{y = 1}$$

$$\text{Sub. } y = 1 \text{ in (1)} \Rightarrow x - 1 = -2 \Rightarrow \boxed{x = -1}$$

**EXAMPLE 2.2** Find the values of the real numbers  $x$  and  $y$ , if the complex numbers  $(2 + i)x + (1 - i)y + 2i - 3$  and  $x + (-1 + 2i)y + 1 + i$  are equal.

[Ans:  $x = 2$  and  $y = 1$ ]

### PROPERTIES OF COMPLEX NUMBERS

**Properties of complex numbers under addition:**

- I. Closure property:  $\forall z_1, z_2 \in \mathbb{C}, (z_1 + z_2) \in \mathbb{C}$
- II. Commutative property:  $\forall z_1, z_2 \in \mathbb{C}, z_1 + z_2 = z_2 + z_1$
- III. Associative property:  $\forall z_1, z_2, z_3 \in \mathbb{C}, (z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$
- IV. Additive identity:  $\forall z \in \mathbb{C}, z + 0 = 0 + z = z \Rightarrow 0 = 0 + 0i$  is an additive

identity.

- V. Additive inverse:  $\forall z \in \mathbb{C}, -z \in \mathbb{C} \ni z + (-z) = (-z) + z = 0 \in \mathbb{C}$ , so  $-z$  is an additive identity.

### Properties of complex numbers under multiplication:

- I. Closure property:  $\forall z_1, z_2 \in \mathbb{C}, (z_1 \cdot z_2) \in \mathbb{C}$
- II. Commutative property:  $\forall z_1, z_2 \in \mathbb{C}, z_1 \cdot z_2 = z_2 \cdot z_1$
- III. Associative property:  $\forall z_1, z_2, z_3 \in \mathbb{C}, (z_1 \cdot z_2) \cdot z_3 = z_1 \cdot (z_2 \cdot z_3)$
- IV. Multiplicative identity:  $\forall z \in \mathbb{C}, z \cdot 1 = 1 \cdot z = z \Rightarrow 1 = 1 + 0i$  is a multiplicative identity.
- V. Multiplicative inverse:  $\forall z \in \mathbb{C} \ni \frac{1}{z} \in \mathbb{C} \ni z \cdot \frac{1}{z} = \frac{1}{z} \cdot z = 1 + i0 \in \mathbb{C}$ , so  $\frac{1}{z}$  is multiplicative identity.

❖ If  $z = x + iy$  then its multiplicative inverse is  $z^{-1} = \left(\frac{x}{x^2+y^2}\right) + i\left(-\frac{y}{x^2+y^2}\right)$

❖ Distributive property:  $\forall z_1, z_2, z_3 \in \mathbb{C}, (z_1 + z_2) \cdot z_3 = z_1 z_3 + z_2 z_3$  (or)  
 $\circ \forall z_1, z_2, z_3 \in \mathbb{C}, z_1 \cdot (z_2 + z_3) = z_1 z_2 + z_1 z_3$

❖ Complex numbers obey the laws of indices:

$$(i) z^m z^n = z^{m+n} \quad (ii) \frac{z^m}{z^n} = z^{m-n}$$

$$(iii) (z^m)^n = z^{m \cdot n} \quad (iv) (z_1 z_2)^m = z_1^m z_2^m$$

### EXERCISE 2.3

1. If  $z_1 = 1 - 3i, z_2 = -4i$  and  $z_3 = 5$ , show that

$$(i) (z_1 + z_2) + z_3 = z_1 + (z_2 + z_3) \quad (ii) (z_1 z_2) z_3 = z_1 (z_2 z_3)$$

$$(i) (z_1 + z_2) + z_3 = (1 - 3i - 4i) + 5 = 1 - 7i + 5 = 6 - 7i \rightarrow (1)$$

$$z_1 + (z_2 + z_3) = 1 - 3i + (-4i + 5) = 1 - 3i - 4i + 5 = 6 - 7i \rightarrow (2)$$

From (1) & (2),  $(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$

$$(ii) (z_1 z_2) z_3 = [(1 - 3i)(-4i)]5 = (-4i + 12i^2)5$$

$$= (-4i - 12)5 = -60 - 20i \rightarrow (1)$$

$$z_1 (z_2 z_3) = (1 - 3i)[(-4i)(5)] = (1 - 3i)[-20i]$$

$$= -20i + 60i^2 = -60 - 20i \rightarrow (2)$$

From (1) & (2),  $(z_1 z_2) z_3 = z_1 (z_2 z_3)$

2. If  $z_1 = 3, z_2 = -7i$  and  $z_3 = 5 + 4i$ , show that

$$(i) z_1 \cdot (z_2 + z_3) = z_1 z_2 + z_1 z_3 \quad (ii) (z_1 + z_2) \cdot z_3 = z_1 z_3 + z_2 z_3$$

$$(i) z_1 \cdot (z_2 + z_3) = 3(-7i + 5 + 4i) = 3(5 - 3i) = 15 - 9i \rightarrow (1)$$

$$z_1 z_2 + z_1 z_3 = (3)(-7i) + (3)(5 + 4i) = -21i + 15 + 12i = 15 - 9i \rightarrow (2)$$

From (1) & (2),  $z_1 \cdot (z_2 + z_3) = z_1 z_2 + z_1 z_3$

$$(ii) (z_1 + z_2) \cdot z_3 = (3 - 7i)(5 + 4i)$$

$$= 15 + 12i - 35i - 28i^2 = 15 + 12i - 35i + 28 = 43 - 23i \rightarrow (1)$$

$$z_1 z_3 + z_2 z_3 = (3)(5 + 4i) + (-7i)(5 + 4i)$$

$$= 15 + 12i - 35i - 28i^2 = 15 + 12i - 35i + 28 = 43 - 23i \rightarrow (2)$$

From (1) & (2),  $(z_1 + z_2) \cdot z_3 = z_1 z_3 + z_2 z_3$

3. If  $z_1 = 2 + 5i, z_2 = -3 - 4i$  and  $z_3 = 1 + i$ , find the additive and multiplicative inverse of  $z_1, z_2$  and  $z_3$

Additive inverse of  $z$  is  $-z$ .

So, Additive inverse of  $z_1 = 2 + 5i$  is  $-z_1 = -2 - 5i$ ,

Additive inverse of  $z_2 = -3 - 4i$  is  $-z_2 = 3 + 4i$

Additive inverse of  $z_3 = 1 + i$  is  $-z_3 = -1 - i$

Multiplicative inverse of  $z = x + iy$  is  $z^{-1} = \left(\frac{x}{x^2+y^2}\right) + i\left(-\frac{y}{x^2+y^2}\right)$

Multiplicative inverse of  $z_1 = 2 + 5i$  is  $z_1^{-1} = \left(\frac{2}{2^2+5^2}\right) + i\left(-\frac{5}{2^2+5^2}\right) = \frac{2}{29} - \frac{5}{29}i$

Multiplicative inverse of  $z_2 = -3 - 4i$  is  $z_2^{-1} = \left(\frac{-3}{(-3)^2+(-4)^2}\right) + i\left(-\frac{-4}{(-3)^2+(-4)^2}\right)$

$$= -\frac{3}{25} + \frac{4}{25}i$$

Multiplicative inverse of  $z_3 = 1 + i$  is  $z^{-1} = \left(\frac{1}{1^2+1^2}\right) + i\left(-\frac{1}{1^2+1^2}\right) = \frac{1}{2} - \frac{1}{2}i$

### CONJUGATE OF A COMPLEX NUMBER

❖ The conjugate of the complex number  $z = x + iy$  is  $\bar{z} = x - iy$ .

❖  $z\bar{z} = x^2 + y^2$

❖ The conjugate is useful in division of complex numbers.

### PROPERTIES OF COMPLEX CONJUGATES:

|                                                                                       |                                                         |
|---------------------------------------------------------------------------------------|---------------------------------------------------------|
| ✓ $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$                                      | ✓ $\text{Im}(z) = \frac{z - \bar{z}}{2i}$               |
| ✓ $\overline{z_1 - z_2} = \bar{z}_1 - \bar{z}_2$                                      | ✓ $\overline{(z^n)} = (\bar{z})^n$                      |
| ✓ $\overline{z_1 \cdot z_2} = \bar{z}_1 \cdot \bar{z}_2$                              | ✓ $z$ is real if and only if $z = \bar{z}$              |
| ✓ $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z}_1}{\bar{z}_2}, z_2 \neq 0$ | ✓ $z$ is purely imaginary if and only if $z = -\bar{z}$ |
| ✓ $\text{Re}(z) = \frac{z + \bar{z}}{2}$                                              | ✓ $\bar{\bar{z}} = z$                                   |

### EXERCISE 2.4

1. Write the following in the rectangular form:

$$(i) \overline{(5+9i)} + (2-4i) \quad (ii) \frac{10-5i}{6+2i} \quad (iii) \overline{3i} + \frac{1}{2-i}$$

$$(i) \overline{(5+9i)} + (2-4i) = \overline{5+9i} + 2-4i = 5-9i+2-4i = 7-5i$$

$$(ii) \frac{10-5i}{6+2i} = \frac{10-5i}{6+2i} \times \frac{6-2i}{6-2i} = \frac{60-20i-30i+10i^2}{6^2+2^2}$$

$$= \frac{60-50i-10}{40} = \frac{50-50i}{40} = \frac{5-5i}{4} = \frac{5}{4} - \frac{5}{4}i$$

$$(iii) \overline{3i} + \frac{1}{2-i} = -3i + \frac{1}{2-i} \times \frac{2+i}{2+i} = -3i + \frac{2+i}{2^2+1^2} = -3i + \frac{2}{5} + \frac{1}{5}i$$

$$= \frac{2}{5} + \frac{-15i+i}{5} = \frac{2}{5} - \frac{14}{5}i$$

2. If  $z = x + iy$ , find the following in rectangular form:

$$(i) \operatorname{Re}\left(\frac{1}{z}\right) \quad (ii) \operatorname{Re}(i\bar{z}) \quad (iii) \operatorname{Im}(3z + 4\bar{z} - 4i)$$

$$(i) \frac{1}{z} = \frac{1}{x+iy} \times \frac{x-iy}{x-iy} = \frac{x-iy}{x^2+y^2} = \frac{x}{x^2+y^2} - \frac{y}{x^2+y^2}i$$

$$\therefore \operatorname{Re}\left(\frac{1}{z}\right) = \operatorname{Re}\left(\frac{x}{x^2+y^2} - \frac{y}{x^2+y^2}i\right) = \frac{x}{x^2+y^2}$$

$$(ii) i\bar{z} = i(\overline{x+iy}) = i(x-iy) = ix - i^2y = y + ix$$

$$\therefore \operatorname{Re}(i\bar{z}) = \operatorname{Re}(-y + ix) = y$$

$$(iii) 3z + 4\bar{z} - 4i = 3(x+iy) + 4\overline{x+iy} - 4i = 3x + 3yi + 4(x-iy) - 4i$$

$$= 3x + 3yi + 4x - 4yi - 4i = 7x - yi - 4i = 7x + i(-y-4)$$

$$\therefore \operatorname{Im}(3z + 4\bar{z} - 4i) = \operatorname{Im}(7x + i(-y-4)) = -y-4$$

3. If  $z_1 = 2 - i$  and  $z_2 = -4 + 3i$ , find the inverse of  $z_1 z_2$  and  $\frac{z_1}{z_2}$

$$z_1 z_2 = (2-i)(-4+3i) = -8 + 6i + 4i - 3i^2 = -8 + 10i + 3 = -5 + 10i$$

$$\text{Inverse of } z_1 z_2 = \left(\frac{x}{x^2+y^2}\right) + i\left(-\frac{y}{x^2+y^2}\right) = \left(\frac{-5}{(-5)^2+10^2}\right) + i\left(-\frac{10}{(-5)^2+10^2}\right)$$

$$= \frac{-5}{125} - \frac{10}{125}i = -\frac{1}{25} - \frac{2}{25}i$$

$$\frac{z_1}{z_2} = \frac{2-i}{-4+3i} = \frac{2-i}{-4+3i} \times \frac{-4-3i}{-4-3i} = \frac{-8-6i+4i+3i^2}{(-4)^2+3^2} = \frac{-8-2i-3}{25} = -\frac{11}{25} - \frac{2}{25}i$$

$$\text{Inverse of } \frac{z_1}{z_2} = \left(\frac{x}{x^2+y^2}\right) + i\left(-\frac{y}{x^2+y^2}\right)$$

$$= \left(\frac{-11/25}{(-11/25)^2 + (-2/25)^2}\right) + i\left(-\frac{-2/25}{(-11/25)^2 + (-2/25)^2}\right)$$

$$= -\frac{11}{25 \times \frac{125}{625}} + \frac{2}{25 \times \frac{125}{625}}i = -\frac{11}{5} + \frac{2}{5}i$$

4. The complex numbers  $u, v$  and  $w$  are related by  $\frac{1}{u} = \frac{1}{v} + \frac{1}{w}$ . If  $v = 3 - 4i$  and

$w = 4 + 3i$ , find  $u$  in the rectangular form.

$$\frac{1}{u} = \frac{1}{v} + \frac{1}{w} \Rightarrow \frac{1}{u} = \frac{1}{3-4i} + \frac{1}{4+3i}$$

$$= \left(\frac{1}{3-4i} \times \frac{3+4i}{3+4i}\right) + \left(\frac{1}{4+3i} \times \frac{4-3i}{4-3i}\right)$$

$$= \frac{3+4i}{3^2+4^2} + \frac{4-3i}{4^2+3^2}$$

$$= \frac{1}{25}(3+4i+4-3i)$$

$$\Rightarrow \frac{1}{u} = \frac{7+i}{25}$$

$$u = \frac{25}{7+i} = \frac{25}{7+i} \times \frac{7-i}{7-i} = \frac{25(7-i)}{7^2+1^2} = \frac{25(7-i)}{50} = \frac{7-i}{2} = \frac{7}{2} - \frac{1}{2}i$$

5. Prove the following properties:

(i)  $z$  is real if and only if  $z = \bar{z}$  (ii)  $\operatorname{Re}(z) = \frac{z+\bar{z}}{2}$  (iii)  $\operatorname{Im}(z) = \frac{z-\bar{z}}{2i}$

(i)  $z$  is real  $\Rightarrow z = x + 0i$

$\bar{z} = \overline{x+0i} = x - 0i = x \Rightarrow z$  is real if and only if  $z = \bar{z}$

(ii) Let  $z = x + iy$ ;  $\operatorname{Re}(z) = x \rightarrow (1)$

$\Rightarrow \bar{z} = x - iy$

$z + \bar{z} = x + iy + x - iy = 2x \Rightarrow \frac{z+\bar{z}}{2} = x \rightarrow (2)$

From (1) & (2),  $\operatorname{Re}(z) = \frac{z+\bar{z}}{2}$

(iii) Let  $z = x + iy$ ;  $\operatorname{Im}(z) = y \rightarrow (1)$

$\Rightarrow \bar{z} = x - iy$

$z - \bar{z} = x + iy - x + iy = 2yi \Rightarrow \frac{z-\bar{z}}{2i} = y \rightarrow (2)$

From (1) & (2),  $\operatorname{Im}(z) = \frac{z-\bar{z}}{2i}$

6. Find the least value of the positive integer  $n$  for which  $(\sqrt{3} + i)^n$

(i) real (ii) purely imaginary

$(\sqrt{3} + i)^3 = 3\sqrt{3} + 9i - 3\sqrt{3} - i = 8i$

$(\sqrt{3} + i)^6 = ((\sqrt{3} + i)^3)^2 = (8i)^2 = -64$

(i) When  $n = 6$ ,  $(\sqrt{3} + i)^n$  is real

(ii) When  $n = 3$ ,  $(\sqrt{3} + i)^n$  is purely imaginary

7. Show that (i)  $(2 + i\sqrt{3})^{10} - (2 - i\sqrt{3})^{10}$  is purely imaginary.

$$(ii) \left(\frac{19-7i}{9+i}\right)^{12} + \left(\frac{20-5i}{7-6i}\right)^{12} \text{ is real.}$$

$$(i) z = (2 + i\sqrt{3})^{10} - (2 - i\sqrt{3})^{10}$$

$$\begin{aligned}\bar{z} &= \overline{(2 + i\sqrt{3})^{10} - (2 - i\sqrt{3})^{10}} = \overline{(2 + i\sqrt{3})^{10}} - \overline{(2 - i\sqrt{3})^{10}} \\ &= \overline{(2 + i\sqrt{3})^{10}} - \overline{(2 - i\sqrt{3})^{10}} = (2 - i\sqrt{3})^{10} - (2 + i\sqrt{3})^{10} \\ &= -\{(2 + i\sqrt{3})^{10} - (2 - i\sqrt{3})^{10}\} = -z\end{aligned}$$

$$\therefore (2 + i\sqrt{3})^{10} - (2 - i\sqrt{3})^{10} \text{ is purely imaginary.}$$

$$(ii) \frac{19-7i}{9+i} = \frac{19-7i}{9+i} \times \frac{9-i}{9-i} = \frac{171-19i-63i+7i^2}{9^2+1^2} = \frac{164-82i}{82} = \frac{82(2-i)}{82} = 2-i$$

$$\frac{20-5i}{7-6i} = \frac{20-5i}{7-6i} \times \frac{7+6i}{7+6i} = \frac{140+120i-35i-30i^2}{7^2+6^2} = \frac{170+85i}{85} = 2+i$$

$$z = \left(\frac{19-7i}{9+i}\right)^{12} + \left(\frac{20-5i}{7-6i}\right)^{12} = (2-i)^{12} - (2+i)^{12}$$

$$\bar{z} = \overline{(2-i)^{12} - (2+i)^{12}} = \overline{(2-i)^{12}} - \overline{(2+i)^{12}} = (2+i)^{12} - (2-i)^{12} = z$$

$$\therefore \left(\frac{19-7i}{9+i}\right)^{12} + \left(\frac{20-5i}{7-6i}\right)^{12} \text{ is real.}$$

$$\text{EXAMPLE 2.8 Show that } (i)(2 + i\sqrt{3})^{10} + (2 - i\sqrt{3})^{10} \text{ is real and}$$

$$(ii) \left(\frac{19+9i}{5-3i}\right)^{15} - \left(\frac{8+i}{1+2i}\right)^{15} \text{ is purely imaginary.}$$

$$\text{EXAMPLE 2.3 Write } \frac{3+4i}{5-12i} \text{ in the } x + iy \text{ form, hence find its real and imaginary parts.}$$

$$\frac{3+4i}{5-12i} = \frac{3+4i}{5-12i} \times \frac{5+12i}{5+12i} = \frac{15+36i+20i-48}{5^2+12^2} = \frac{-33+56i}{169} = -\frac{33}{169} + \frac{56}{169}i$$

$$\text{Real part} = -\frac{33}{169} \text{ and imaginary part} = \frac{56}{169}$$

$$\text{EXAMPLE 2.4 Simplify } \left(\frac{1+i}{1-i}\right)^3 - \left(\frac{1-i}{1+i}\right)^3$$

$$\frac{1+i}{1-i} = \frac{1+i}{1-i} \times \frac{1+i}{1+i} = \frac{1+i+i^2}{1^2+1^2} = \frac{1+2i-1}{2} = \frac{2i}{2} = i$$

$$\frac{1-i}{1+i} = \left(\frac{1+i}{1-i}\right)^{-1} = -i$$

$$\left(\frac{1+i}{1-i}\right)^3 - \left(\frac{1-i}{1+i}\right)^3 = (i)^3 - (-i)^3 = -i - i = -2i$$

$$\text{EXAMPLE 2.5 If } \frac{z+3}{z-5i} = \frac{1+4i}{2}, \text{ find the complex number } z.$$

$$\frac{z+3}{z-5i} = \frac{1+4i}{2}$$

$$\Rightarrow 2(z+3) = (1+4i)(z-5i)$$

$$\Rightarrow 2z+6 = z-5i+4zi-20i^2$$

$$\Rightarrow 2z-z-4zi = -5i+20-6$$

$$\Rightarrow z-4zi = 14-5i$$

$$\Rightarrow z(1-4i) = 14-5i$$

$$\Rightarrow z = \frac{14-5i}{1-4i}$$

$$= \frac{14-5i}{1-4i} \times \frac{1+4i}{1+4i} = \frac{14+56i-5i-20i^2}{1^2+4^2} = \frac{14+51i+20}{17} = \frac{34+51i}{17} = \frac{17(2+3i)}{17} = 2+3i$$

$$\text{EXAMPLE 2.6 If } z_1 = 3-2i \text{ and } z_2 = 6+4i, \text{ find } \frac{z_1}{z_2}.$$

$$\frac{z_1}{z_2} = \frac{3-2i}{6+4i} = \frac{3-2i}{6+4i} \times \frac{6-4i}{6-4i} = \frac{18-12i-12i-8}{6^2+4^2} = \frac{10-24i}{52} = \frac{2(5-12i)}{52} = \frac{5-12i}{26}$$

$$\text{EXAMPLE 2.7 Find } z^{-1}, \text{ if } z = (2+3i)(1-i).$$

$$z = (2+3i)(1-i) = 2-2i+3i-3i^2 = 2+i+3 = 5+i$$

$$z^{-1} = \left(\frac{x}{x^2+y^2}\right) + i\left(-\frac{y}{x^2+y^2}\right) = \left(\frac{5}{5^2+1^2}\right) + i\left(-\frac{1}{5^2+1^2}\right) = \frac{5}{26} - \frac{1}{26}i$$

## MODULUS OF A COMPLEX NUMBER

$$\diamond \text{ If } z = x + iy, \text{ then the modulus of } z \text{ is } |z| = \sqrt{x^2 + y^2}$$

$$\diamond z\bar{z} = |z|^2$$

### PROPERTIES OF MODULUS OF A COMPLEX NUMBER:

$$\checkmark |Z| = |\bar{Z}|$$

$$\checkmark |z_1 + z_2| \leq |z_1| + |z_2| \text{ (Triangle inequality)}$$

$$\checkmark |z_1 - z_2| \geq |z_1| - |z_2|$$

$$\checkmark |z_1 z_2| = |z_1| |z_2|$$

$$\checkmark \left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}, z_2 \neq 0$$

$$\checkmark |z^n| = |z|^n, n \text{ is an integer.}$$

$$\checkmark \operatorname{Re}(z) \leq |z|$$

$$\checkmark \operatorname{Im}(z) \leq |z|$$

$$\checkmark ||z_1| - |z_2|| \leq |z_1 + z_2| \leq |z_1| + |z_2|$$

$$\diamond \text{ The distance between the two points } z_1 \text{ and } z_2 \text{ in the complex plane is}$$

$$|z_1 - z_2| \text{ (or) } \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

$$\diamond \text{ Formula for finding the square root of the complex number } z = a + ib \text{ is}$$

$$\sqrt{a+ib} = \pm \left( \sqrt{\frac{|z|+a}{2}} + i \frac{b}{|b|} \sqrt{\frac{|z|-a}{2}} \right),$$

- If b is negative  $\frac{b}{|b|} = -1$ , x and y have different signs.
- If b is positive  $\frac{b}{|b|} = 1$ , x and y have same signs.

### Exercise 2.5

1. Find the modulus of the following complex numbers:

(i)  $\frac{2i}{3+4i}$  (ii)  $\frac{2-i}{1+i} + \frac{1-2i}{1-i}$  (iii)  $(1-i)^{10}$  (iv)  $2i(3-4i)(4-3i)$

$$|z| = \sqrt{x^2 + y^2}$$

(i)  $\left| \frac{2i}{3+4i} \right| = \frac{|2i|}{|3+4i|} = \frac{\sqrt{0^2+2^2}}{\sqrt{3^2+4^2}} = \frac{\sqrt{4}}{\sqrt{25}} = \frac{2}{5}$

(ii)  $\left| \frac{2-i}{1+i} + \frac{1-2i}{1-i} \right| = \left| \frac{(1-i)(2-i) + (1+i)(1-2i)}{(1+i)(1-i)} \right| = \left| \frac{2-i-2i+i^2+1-2i+i-2i^2}{1^2+1^2} \right|$   
 $= \left| \frac{2-i-2i-1+1-2i+2}{2} \right| = \left| \frac{4-4i}{2} \right| = |2-2i| = \sqrt{2^2+2^2} = \sqrt{8} = 2\sqrt{2}$

(iii)  $|(1-i)^{10}| = |1-i|^{10} = (\sqrt{1^2+1^2})^{10} = \sqrt{2}^{10} = 2^5 = 32$

(iv)  $|2i(3-4i)(4-3i)| = |2i||3-4i||4-3i| = \sqrt{0^2+2^2}\sqrt{3^2+4^2}\sqrt{4^2+3^2}$   
 $= \sqrt{4}\sqrt{25}\sqrt{25} = 2.5.5 = 50$

**EXAMPLE 2.9** If  $z_1 = 3+4i$ ,  $z_2 = 5-12i$  and  $z_3 = 6+8i$ , find  $|z_1|$ ,  $|z_2|$ ,  $|z_3|$ ,  $|z_1+z_2|$ ,  $|z_2-z_3|$  and  $|z_1+z_3|$ .

**EXAMPLE 2.10** Find the following : (i)  $\left| \frac{2+i}{-1+2i} \right|$  (ii)  $|(1+i)(2+3i)(4i-3)|$

(iii)  $\left| \frac{i(2+i)^3}{(1+i)^2} \right|$

2. For any two complex numbers  $z_1$  and  $z_2$ , such that  $|z_1| = |z_2| = 1$  and  $z_1 z_2 \neq -1$  then show that  $\frac{z_1+z_2}{1+z_1 z_2}$  is a real number.

Given  $|z_1| = |z_2| = 1$

$$|z_1| = 1 \Rightarrow |z_1|^2 = 1 \Rightarrow z_1 \bar{z}_1 = 1 \Rightarrow \bar{z}_1 = \frac{1}{z_1}$$

$$\text{Similarly } \bar{z}_2 = \frac{1}{z_2}$$

$$\text{Let } w = \frac{z_1+z_2}{1+z_1 z_2} \Rightarrow \bar{w} = \overline{\left( \frac{z_1+z_2}{1+z_1 z_2} \right)} = \frac{\bar{z}_1+\bar{z}_2}{\overline{1+z_1 z_2}} = \frac{\bar{z}_1+\bar{z}_2}{1+\bar{z}_1 \bar{z}_2} = \frac{\bar{z}_1+\bar{z}_2}{1+\frac{1}{z_1 z_2}} = \frac{\bar{z}_1+\bar{z}_2}{\frac{1+z_1 z_2}{z_1 z_2}} = \frac{z_1+z_2}{1+z_1 z_2} = w$$

Since  $\bar{w} = w$ ,  $w$  is purely real.

3. Which one of the points  $10-8i$ ,  $11+6i$  is closest to  $1+i$ .

Distance between the two points  $z_1$  and  $z_2 = \sqrt{(x_1-x_2)^2 + (y_1-y_2)^2}$

$$z = 1+i, z_1 = 10-8i$$

$$\text{Distance between the two points } z \text{ and } z_1 = \sqrt{(1-10)^2 + (1+8)^2}$$
  
 $= \sqrt{81+81} = \sqrt{162} = \sqrt{2 \times 81} = 9\sqrt{2}$

$$z = 1+i, z_2 = 11+6i$$

$$\text{Distance between the two points } z \text{ and } z_2 = \sqrt{(1-11)^2 + (1-6)^2}$$
  
 $= \sqrt{100+25} = \sqrt{125} = \sqrt{5 \times 25} = 5\sqrt{5}$

$\therefore 5\sqrt{5} < 9\sqrt{2}$ ,  $11+6i$  is the closest point to  $1+i$

**EXAMPLE 2.11** Which one of the points  $i$ ,  $-2+i$  and  $3$  is farthest from the origin?

4. If  $|z| = 3$ , show that  $7 \leq |z+6-8i| \leq 13$ .

Let  $z_1 = z$  and  $z_2 = 6-8i$

$$\text{W.K.T, } ||z_1| - |z_2|| \leq |z_1 + z_2| \leq |z_1| + |z_2|$$

$$\Rightarrow |3 - \sqrt{6^2+8^2}| \leq |z+6-8i| \leq 3 + \sqrt{6^2+8^2}$$

$$\Rightarrow |3 - \sqrt{100}| \leq |z+6-8i| \leq 3 + \sqrt{100}$$

$$\Rightarrow |3 - 10| \leq |z+6-8i| \leq 3 + 10$$

$$\Rightarrow |-7| \leq |z+6-8i| \leq 13$$

$$\Rightarrow 7 \leq |z+6-8i| \leq 13$$

**EXAMPLE 2.13** If  $|z| = 2$ , show that  $3 \leq |z+3+4i| \leq 7$ .

5. If  $|z| = 1$ , show that  $2 \leq |z^2-3| \leq 4$ .

Let  $z_1 = z^2$  and  $z_2 = -3$

$$\text{W.K.T, } ||z_1| - |z_2|| \leq |z_1 + z_2| \leq |z_1| + |z_2|$$

$$\Rightarrow ||z|^2 - |-3|| \leq |z^2-3| \leq |z|^2 + |-3|$$

$$\Rightarrow |1^2 - 3| \leq |z^2-3| \leq 1^2 + 3$$

$$\Rightarrow |-2| \leq |z^2-3| \leq 4$$

$$\Rightarrow 2 \leq |z^2-3| \leq 4$$

6. If  $\left|z - \frac{2}{z}\right| = 2$ , show that the greatest and least value of  $|z|$  are  $\sqrt{3} + 1$  and  $\sqrt{3} - 1$  respectively.

$$\left|z - \frac{2}{z}\right| = 2 \Rightarrow 2 = \left|z - \frac{2}{z}\right| \geq \left||z| - \left|\frac{2}{z}\right|\right|$$

$$\Rightarrow \left||z| - \left|\frac{2}{z}\right|\right| \leq 2 \Rightarrow -2 \leq |z| - \left|\frac{2}{z}\right| \leq 2 \quad \because |x - a| \leq r \Rightarrow -r \leq x - a \leq r$$

$$\Rightarrow -2 \leq |z| - \frac{2}{|z|} \leq 2 \Rightarrow -2 \leq \frac{|z|^2 - 2}{|z|} \leq 2 \Rightarrow -2|z| \leq |z|^2 - 2 \leq 2|z| \rightarrow (1)$$

$$\text{From (1)} \Rightarrow -2|z| \leq |z|^2 - 2$$

$$\Rightarrow 2|z| - 2|z| \leq |z|^2 - 2 + 2|z|$$

$$\Rightarrow 0 \leq |z|^2 + 2|z| - 2$$

$$\Rightarrow |z|^2 + 2|z| - 2 \geq 0$$

$$\Rightarrow [|z| - (\sqrt{3} + 1)][|z| - (\sqrt{3} - 1)] \geq 0$$

$$\Rightarrow |z| \geq (\sqrt{3} + 1); |z| \leq (\sqrt{3} - 1)$$

$$\Rightarrow |z| \geq (\sqrt{3} - 1) \rightarrow (2)$$

$$\text{From (1)} \Rightarrow |z|^2 - 2 \leq 2|z|$$

$$\Rightarrow |z|^2 - 2|z| - 2 \leq 0$$

$$\Rightarrow [|z| - (1 + \sqrt{3})][|z| - (1 - \sqrt{3})] \leq 0$$

$$\Rightarrow (1 - \sqrt{3}) \leq |z| \leq (1 + \sqrt{3})$$

$$\Rightarrow |z| \leq (1 + \sqrt{3}) \rightarrow (3)$$

$$\text{From (2) \& (3), } \sqrt{3} - 1 \leq |z| \leq \sqrt{3} + 1$$

Hence the greatest and least value of  $|z|$  are  $\sqrt{3} + 1$  and  $\sqrt{3} - 1$  respectively.

7. If  $z_1, z_2$  and  $z_3$  are three complex numbers such that  $|z_1| = 1, |z_2| = 2, |z_3| = 3$  and  $|z_1 + z_2 + z_3| = 1$ , show that  $|9z_1z_2 + 4z_1z_3 + z_2z_3| = 6$ .

$$|z_1| = 1 \Rightarrow |z_1|^2 = 1 \Rightarrow z_1\bar{z}_1 = 1 \Rightarrow z_1 = \frac{1}{\bar{z}_1}$$

$$\because z\bar{z} = |z|^2$$

$$|z_2| = 2 \Rightarrow |z_2|^2 = 4 \Rightarrow z_2\bar{z}_2 = 4 \Rightarrow z_2 = \frac{4}{\bar{z}_2}$$

$$|z_3| = 3 \Rightarrow |z_3|^2 = 9 \Rightarrow z_3\bar{z}_3 = 9 \Rightarrow z_3 = \frac{9}{\bar{z}_3}$$

$$z_1 + z_2 + z_3 = \frac{1}{\bar{z}_1} + \frac{4}{\bar{z}_2} + \frac{9}{\bar{z}_3}$$

$$\Rightarrow z_1 + z_2 + z_3 = \frac{z_2z_3 + 4z_1z_3 + 9z_1z_2}{\bar{z}_1\bar{z}_2\bar{z}_3}$$

$$\Rightarrow |z_1 + z_2 + z_3| = \left| \frac{z_2z_3 + 4z_1z_3 + 9z_1z_2}{\bar{z}_1\bar{z}_2\bar{z}_3} \right|$$

$$\Rightarrow |z_1 + z_2 + z_3| = \frac{|z_2z_3 + 4z_1z_3 + 9z_1z_2|}{|\bar{z}_1\bar{z}_2\bar{z}_3|}$$

$$\Rightarrow |z_1 + z_2 + z_3| = \frac{|9z_1z_2 + 4z_1z_3 + z_2z_3|}{|z_1z_2z_3|} \quad \because \bar{\bar{z}} = z$$

$$\Rightarrow |z_1 + z_2 + z_3| = \frac{|9z_1z_2 + 4z_1z_3 + z_2z_3|}{|z_1||z_2||z_3|} \quad \because |z_1z_2| = |z_1||z_2|$$

$$\Rightarrow 1 = \frac{|9z_1z_2 + 4z_1z_3 + z_2z_3|}{1.2.3}$$

$$\Rightarrow |9z_1z_2 + 4z_1z_3 + z_2z_3| = 6$$

**EXAMPLE 2.12** If  $z_1, z_2$  and  $z_3$  are three complex numbers such that  $|z_1| = |z_2| = |z_3| = |z_1 + z_2 + z_3| = 1$ , find the value of  $\left|\frac{1}{z_1} + \frac{1}{z_2} + \frac{1}{z_3}\right|$ .

**EXAMPLE 2.15** If  $z_1, z_2$  and  $z_3$  are three complex numbers such that  $|z_1| = |z_2| = |z_3| = r > 0$  and  $z_1 + z_2 + z_3 \neq 0$ . Prove that  $\left|\frac{z_1z_2 + z_2z_3 + z_3z_1}{z_1 + z_2 + z_3}\right| = r$

$$|z_1| = r \Rightarrow |z_1|^2 = r^2 \Rightarrow z_1\bar{z}_1 = r^2 \Rightarrow z_1 = \frac{r^2}{\bar{z}_1}$$

$$\because z\bar{z} = |z|^2$$

$$|z_2| = r \Rightarrow |z_2|^2 = r^2 \Rightarrow z_2\bar{z}_2 = r^2 \Rightarrow z_2 = \frac{r^2}{\bar{z}_2}$$

$$|z_3| = r \Rightarrow |z_3|^2 = r^2 \Rightarrow z_3\bar{z}_3 = r^2 \Rightarrow z_3 = \frac{r^2}{\bar{z}_3}$$

$$z_1 + z_2 + z_3 = \frac{r^2}{\bar{z}_1} + \frac{r^2}{\bar{z}_2} + \frac{r^2}{\bar{z}_3} = r^2 \left( \frac{z_2z_3 + z_1z_3 + z_1z_2}{\bar{z}_1\bar{z}_2\bar{z}_3} \right)$$

$$\Rightarrow |z_1 + z_2 + z_3| = |r^2| \left| \frac{z_2z_3 + z_1z_3 + z_1z_2}{\bar{z}_1\bar{z}_2\bar{z}_3} \right| = r^2 \frac{|z_2z_3 + z_1z_3 + z_1z_2|}{|\bar{z}_1\bar{z}_2\bar{z}_3|} = r^2 \frac{|z_1z_2 + z_2z_3 + z_3z_1|}{|z_1||z_2||z_3|}$$

$$\Rightarrow |z_1 + z_2 + z_3| = r^2 \frac{|z_1z_2 + z_2z_3 + z_3z_1|}{r.r.r} \Rightarrow |z_1 + z_2 + z_3| = \frac{|z_1z_2 + z_2z_3 + z_3z_1|}{r}$$

$$\Rightarrow \left| \frac{z_1z_2 + z_2z_3 + z_3z_1}{z_1 + z_2 + z_3} \right| = r$$

8. If the area of the triangle formed by the vertices  $z, iz$  and  $z + iz$  is 50 square units, find the value of  $|z|$ .

$$\text{Let } z = a + ib = (a, b) \Rightarrow |z| = \sqrt{a^2 + b^2}$$

$$\text{Vertices : } z = (a, b)$$

$$iz = i(a + ib) = ia + i^2b = -b + ia = (-b, a)$$

$$z + iz = a + ib - b + ia = (a - b) + i(a + b) = (a - b, a + b)$$

Area of the triangle = 50 sq. units

$$\Rightarrow \frac{1}{2} \begin{vmatrix} a & -b & a-b & a \\ b & a & a+b & b \end{vmatrix} = 50$$

$$\Rightarrow (a^2 - ab - b^2 + ab - b^2) - (-b^2 + a^2 - ab + a^2 + ab) = 50 \times 2$$

$$\Rightarrow a^2 - 2b^2 + b^2 - 2a^2 = 100$$

$$\Rightarrow -a^2 - b^2 = 100$$

$$\Rightarrow -(a^2 + b^2) = \pm 100$$

$$\Rightarrow a^2 + b^2 = \mp 100$$

$$\Rightarrow \sqrt{a^2 + b^2} = \mp 10$$

$$\Rightarrow |z| = 10 \quad (\because a^2 + b^2 \text{ is not negative})$$

**NOTE:** This can also done by using distance formula, find the length of base and height and use *Area of the  $\Delta = \frac{1}{2}bh$*

**EXAMPLE 2.14** Show that the points  $1, -\frac{1}{2} + i\frac{\sqrt{3}}{2}$  and  $-\frac{1}{2} - i\frac{\sqrt{3}}{2}$  are the vertices of an equilateral triangle.

$$\text{Let } z_1 = 1, z_2 = -\frac{1}{2} + i\frac{\sqrt{3}}{2} \text{ and } z_3 = -\frac{1}{2} - i\frac{\sqrt{3}}{2}$$

The length of the sides of the triangle are

$$|z_1 - z_2| = \left| 1 - \left( -\frac{1}{2} + i\frac{\sqrt{3}}{2} \right) \right| = \left| \frac{3}{2} - i\frac{\sqrt{3}}{2} \right| = \sqrt{\left( \frac{3}{2} \right)^2 + \left( \frac{\sqrt{3}}{2} \right)^2} = \sqrt{\frac{9}{4} + \frac{3}{4}} = \sqrt{\frac{12}{4}} = \sqrt{3}$$

$$|z_2 - z_3| = \left| \left( -\frac{1}{2} + i\frac{\sqrt{3}}{2} \right) - \left( -\frac{1}{2} - i\frac{\sqrt{3}}{2} \right) \right| = \left| -\frac{1}{2} + i\frac{\sqrt{3}}{2} + \frac{1}{2} + i\frac{\sqrt{3}}{2} \right| = \sqrt{\left( \frac{2\sqrt{3}}{2} \right)^2} = \sqrt{3}$$

$$|z_3 - z_1| = \left| \left( -\frac{1}{2} - i\frac{\sqrt{3}}{2} \right) - 1 \right| = \left| -\frac{3}{2} - i\frac{\sqrt{3}}{2} \right| = \sqrt{\left( \frac{3}{2} \right)^2 + \left( \frac{\sqrt{3}}{2} \right)^2} = \sqrt{\frac{9}{4} + \frac{3}{4}} = \sqrt{3}$$

Since all the three sides are equal, the given points form an equilateral triangle.

**EXAMPLE 2.18** Given the complex number  $= 3 + 2i$ , represent the complex numbers  $z, iz$  and  $z + iz$  in one Argand diagram. Show that these complex numbers form the vertices of an isosceles right triangle.

$$\text{Given : } z = 3 + 2i = A(3,2)$$

$$iz = i(3 + 2i) = -2 + 3i = B(-2,3)$$

$$z + iz = 3 + 2i - 2 + 3i = 1 + 5i = C(1,5)$$

The length of the sides of the triangle are

$$AB = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} = \sqrt{(3 + 2)^2 + (2 - 3)^2} = \sqrt{25 + 1} = \sqrt{26}$$

$$BC = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} = \sqrt{(-2 - 1)^2 + (3 - 5)^2} = \sqrt{9 + 4} = \sqrt{13}$$

$$CA = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} = \sqrt{(1 - 3)^2 + (5 - 2)^2} = \sqrt{4 + 9} = \sqrt{13}$$

$$AB^2 = 26, BC^2 = 13, CA^2 = 13$$

Since (i)  $BC = CA = \sqrt{13}$  and (ii)  $BC^2 + CA^2 = 13 + 13 = 26 = AB^2$ , So the given vertices form an isosceles right triangle.

9. Show that the equation  $z^3 + 2\bar{z} = 0$  has five solutions.

$$z^3 + 2\bar{z} = 0$$

$$\Rightarrow z^3 = -2\bar{z}$$

$$\Rightarrow |z^3| = |-2||\bar{z}|$$

$$\Rightarrow |z|^3 = 2|z|$$

$$\Rightarrow |z|^3 - 2|z| = 0$$

$$\Rightarrow |z|(|z|^2 - 2) = 0 \Rightarrow \begin{cases} |z| = 0 \Rightarrow z = 0 \text{ is one of the solution} \\ |z|^2 - 2 = 0 \Rightarrow z\bar{z} - 2 = 0 \Rightarrow \boxed{\bar{z} = \frac{2}{z}} \end{cases}$$

Put  $\bar{z} = \frac{2}{z}$  in  $z^3 + 2\bar{z} = 0 \Rightarrow z^3 + 2\left(\frac{2}{z}\right) = 0 \Rightarrow z^4 + 4 = 0$ , which provides four solution.

Hence  $z^3 + 2\bar{z} = 0$  has five solutions.

**EXAMPLE 2.16** Show that the equation  $z^2 = \bar{z}$  has four solutions.

10. Find the square root of (i)  $4 + 3i$  (ii)  $-6 + 8i$  (iii)  $-5 - 12i$ .

$$\sqrt{a + ib} = \pm \left( \sqrt{\frac{|z| + a}{2}} + i \frac{b}{|b|} \sqrt{\frac{|z| - a}{2}} \right)$$

(i)  $4 + 3i$

$$a = 4, b = 3, |z| = \sqrt{4^2 + 3^2} = \sqrt{25} = 5$$

$$\sqrt{4 + 3i} = \pm \left( \sqrt{\frac{5+4}{2}} + i \frac{3}{3} \sqrt{\frac{5-4}{2}} \right) = \pm \left( \frac{3}{\sqrt{2}} + i \frac{1}{\sqrt{2}} \right)$$

(ii)  $-6 + 8i$

$$a = -6, b = 8, |z| = \sqrt{6^2 + 8^2} = \sqrt{100} = 10$$

$$\sqrt{-6 + 8i} = \pm \left( \sqrt{\frac{10-6}{2}} + i \frac{8}{8} \sqrt{\frac{10+6}{2}} \right) = \pm (\sqrt{2} + i\sqrt{8}) = \pm (\sqrt{2} + i2\sqrt{2})$$

(iii)  $-5 - 12i$

$$a = -5, b = -12, |z| = \sqrt{5^2 + 12^2} = \sqrt{169} = 13$$

$$\sqrt{-5 - 12i} = \pm \left( \sqrt{\frac{13-5}{2}} + i \frac{-12}{12} \sqrt{\frac{13+5}{2}} \right) = \pm (\sqrt{4} - i\sqrt{9}) = \pm (2 - 3i)$$

**EXAMPLE 2.17** Find the square root of  $6 + 8i$

### GEOMETRY AND LOCUS OF COMPLEX NUMBERS

- ❖ A circle is defined as the locus of a point which moves in a plane such that its distance from a fixed point in that plane is always a constant. The fixed point is the centre and the constant distance is the radius of the circle.
- ❖ Equation of complex form of a circle is  $|z - z_0| = r$ .
- ❖  $|z - z_0| < r$  represents the points interior of the circle.
- ❖  $|z - z_0| > r$  represents the points exterior of the circle.
- ❖  $x^2 + y^2 = r^2$  represents a circle with centre at the origin and the radius  $r$  units.
- ❖  $r = \sqrt{x^2 + y^2}$
- ❖ If  $z = \frac{a+ib}{c+id}$  then  $\text{Re}(Z) = \frac{ac+bd}{c^2+d^2}$
- ❖ If  $z = \frac{a+ib}{c+id}$  then  $\text{Im}(Z) = \frac{bc-ad}{c^2+d^2}$

### EXERCISE 2.6

1. If  $z = x + iy$  is a complex number such that  $\left| \frac{z-4i}{z+4i} \right| = 1$ . Show that the locus of  $z$  is real axis.

Given:  $z = x + iy$

$$\left| \frac{z-4i}{z+4i} \right| = 1 \Rightarrow |z - 4i| = |z + 4i|$$

$$\Rightarrow |x + iy - 4i| = |x + iy + 4i|$$

$$\Rightarrow |x + i(y - 4)| = |x + i(y + 4)|$$

$$\Rightarrow \sqrt{x^2 + (y - 4)^2} = \sqrt{x^2 + (y + 4)^2}$$

Squaring on both sides,

$$\Rightarrow x^2 + (y - 4)^2 = x^2 + (y + 4)^2$$

$$\Rightarrow x^2 + y^2 - 8y + 16 = x^2 + y^2 + 8y + 16$$

$$\Rightarrow 8y = 0$$

$$\Rightarrow y = 0$$

$\therefore$  The locus of  $z$  is real axis.

2. If  $z = x + iy$  is a complex number such that  $\text{Im} \left( \frac{2z+1}{iz+1} \right) = 0$ , show that the locus of  $z$  is  $2x^2 + 2y^2 + x - 2y = 0$ .

Given:  $z = x + iy$

$$\frac{2z+1}{iz+1} = \frac{2(x+iy)+1}{i(x+iy)+1} = \frac{(2x+1)+i2y}{(1-y)+ix}$$

$$\text{Im}(Z) = \frac{bc-ad}{c^2+d^2}$$

Here  $a = 2x + 1$ ,  $b = 2y$ ,  $c = 1 - y$ ,  $d = x$

$$\text{Im} \left( \frac{2z+1}{iz+1} \right) = 0 \Rightarrow \frac{2y(1-y) - x(2x+1)}{(1-y)^2 + x^2} = 0 \Rightarrow 2y - 2y^2 - 2x^2 - x = 0$$

$$\Rightarrow 2x^2 + 2y^2 + x - 2y = 0$$

3. Obtain the Cartesian form of the locus of  $z = x + iy$  in each of the following cases:

(i)  $[\text{Re}(iz)]^2 = 3$  (ii)  $\text{Im}[(1-i)z + 1] = 0$  (iii)  $|z + i| = |z - 1|$  (iv)  $\bar{z} = z^{-1}$

Given:  $z = x + iy$

$$(i) iz = i(x + iy) = y - ix$$

$$[\text{Re}(iz)]^2 = 3$$

$$\Rightarrow [\text{Re}(y - ix)]^2 = 3$$

$$\Rightarrow y^2 = 3$$

$$(ii) (1-i)z + 1 = (1-i)(x + iy) + 1 = x + iy - ix + y + 1$$

$$= (x + y + 1) + i(y - x)$$

$$\text{Im}[(1-i)z + 1] = 0$$

$$\Rightarrow \text{Im}[(x + y + 1) + i(y - x)] = 0$$

$$\Rightarrow y - x = 0 \text{ (or) } x - y = 0$$

$$(iii) |z + i| = |z - 1|$$

$$\Rightarrow |x + iy + i| = |x + iy - 1|$$

$$\Rightarrow |x + i(y + 1)| = |(x - 1) + iy|$$

$$\Rightarrow \sqrt{(x)^2 + (y + 1)^2} = \sqrt{(x - 1)^2 + (y)^2}$$

Squaring on both sides,

$$\Rightarrow (x)^2 + (y + 1)^2 = (x - 1)^2 + (y)^2$$

$$\Rightarrow x^2 + y^2 + 2y + 1 = x^2 - 2x + 1 + y^2$$

$$\Rightarrow 2x + 2y = 0$$

$$\Rightarrow x + y = 0$$

$$(iv) \bar{z} = z^{-1} \Rightarrow \bar{z} = \frac{1}{z} \Rightarrow z\bar{z} = 1 \Rightarrow (x + iy)(x - iy) = 1 \Rightarrow x^2 + y^2 = 1$$

4. Show that the following equations represent a circle, and find its centre and

radius: (i)  $|z - 2 - i| = 3$  (ii)  $|2z + 2 - 4i| = 2$  (iii)  $|3z - 6 + 12i| = 8$

Equation of the circle is  $|z - z_0| = r$

(i)  $|z - 2 - i| = 3 \implies |z - (2 + i)| = 3$

Centre =  $2 + i = (2,1)$  ; Radius = 3 units

(ii)  $|2z + 2 - 4i| = 2 \implies 2 |z + 1 - 2i| = 2 \implies |z - (-1 + 2i)| = 1$

Centre =  $-1 + 2i = (-1,2)$  ; Radius = 1 unit

(iii)  $|3z - 6 + 12i| = 8 \implies 3|z - 2 + 4i| = 8 \implies |z - (2 - 4i)| = \frac{8}{3}$

Centre =  $2 - 4i = (2,-4)$  ; Radius =  $\frac{8}{3}$  units

**EXAMPLE 2.19** Show that  $|3z - 5 + i| = 4$  represent a circle, and find its centre and radius.

5. Obtain the Cartesian form of the locus of  $z = x + iy$  in each of the following cases: (i)  $|z - 4| = 16$  (ii)  $|z - 4|^2 - |z - 1|^2 = 16$

Given:  $z = x + iy$

(i)  $|z - 4| = 16 \implies |x + iy - 4| = 16$   
 $\implies |(x - 4) + iy| = 16$   
 $\implies \sqrt{(x - 4)^2 + y^2} = 16$

Squaring on both sides,

$\implies (x - 4)^2 + y^2 = 16^2$   
 $\implies x^2 - 8x + 16 + y^2 = 256$   
 $\implies x^2 + y^2 - 8x - 240 = 0$  is the locus of  $z$ .

(ii)  $|z - 4|^2 - |z - 1|^2 = 16 \implies |x + iy - 4|^2 - |x + iy - 1|^2 = 16$   
 $\implies |(x - 4) + iy|^2 - |(x - 1) + iy|^2 = 16$   
 $\implies (\sqrt{(x - 4)^2 + y^2})^2 - \sqrt{(x - 1)^2 + y^2}^2 = 16$   
 $\implies (x - 4)^2 + y^2 - [(x - 1)^2 + y^2] = 16$   
 $\implies x^2 - 8x + 16 + y^2 - x^2 + 2x - 1 - y^2 = 16$   
 $\implies -6x - 1 = 0$   
 $\implies 6x + 1 = 0$  is the locus of  $z$ .

**EXAMPLE 2.21** Obtain the Cartesian form of the locus of  $z$  in each of the following cases: (i)  $|z| = |z - i|$  (ii)  $|2z - 3 - i| = 3$

**EXAMPLE 2.20** Show that  $|z + 2 - i| < 2$  represents interior points of a circle. Find its centre and radius.

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Consider  $|z + 2 - i| = 2 \implies |z - (-2 + i)| = 2$ .

Centre =  $-2 + i = (-2,1)$  ; Radius = 2 units

$\therefore |z + 2 - i| < 2$  represents all points inside the circle.

### POLAR AND EULER FORM OF A COMPLEX NUMBER

❖ Polar form of  $z = x + iy$  is  $z = r(\cos \theta + i \sin \theta)$  (or)  $z = r \operatorname{cis} \theta$

❖ Modulus  $r = \sqrt{x^2 + y^2}$

❖  $\alpha = \tan^{-1} \left( \frac{y}{x} \right)$

❖  $\arg z = \theta$  ;  $\begin{cases} \theta = \alpha ; \text{if}(+,+) \theta \text{ lies in the 1st quadrant} \\ \theta = \pi - \alpha ; \text{if}(-,+) \theta \text{ lies in the 2nd quadrant} \\ \theta = -\pi + \alpha ; \text{if}(-,-) \theta \text{ lies in the 3rd quadrant} \\ \theta = -\alpha ; \text{if}(+,-) \theta \text{ lies in the 4th quadrant} \end{cases}$

❖ In general  $\arg z = \operatorname{Arg} z + 2n\pi, n \in \mathbb{Z}$ .

❖ Properties of arguments:

✓  $\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$

✓  $\arg \left( \frac{z_1}{z_2} \right) = \arg(z_1) - \arg(z_2)$

✓  $\arg(z^n) = n \arg z$

❖ Some of the principal argument and argument:

|                        |         |                         |               |                         |
|------------------------|---------|-------------------------|---------------|-------------------------|
| $z$                    | 1       | $i$                     | $-1$          | $-i$                    |
| $\operatorname{Arg} z$ | 0       | $\frac{\pi}{2}$         | $\pi$         | $-\frac{\pi}{2}$        |
| $\arg z$               | $2n\pi$ | $2n\pi + \frac{\pi}{2}$ | $2n\pi + \pi$ | $2n\pi - \frac{\pi}{2}$ |

❖ Euler's form of the complex number:  $z = re^{i\theta}$  ;  $e^{i\theta} = \cos \theta + i \sin \theta$

❖ Properties of polar form:

•  $z = r(\cos \theta + i \sin \theta) \implies z^{-1} = \frac{1}{r}(\cos \theta - i \sin \theta)$

• If  $z_1 = r_1(\cos \theta_1 + i \sin \theta_1), z_2 = r_2(\cos \theta_2 + i \sin \theta_2)$  then  
 $z_1 z_2 = r_1 r_2 [\cos(\theta_1 + \theta_2) + i \sin(\theta_1 + \theta_2)]$

• If  $z_1 = r_1(\cos \theta_1 + i \sin \theta_1), z_2 = r_2(\cos \theta_2 + i \sin \theta_2)$  then  
 $\frac{z_1}{z_2} = \frac{r_1}{r_2} [\cos(\theta_1 - \theta_2) + i \sin(\theta_1 - \theta_2)]$

❖ Some of the useful polar forms :

$$1 = \text{cis}(0) ; -1 = \text{cis}(\pi) ; i = \text{cis}\left(\frac{\pi}{2}\right) ; -i = \text{cis}\left(-\frac{\pi}{2}\right)$$

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**EXAMPLE 2.22** Find the modulus and principal argument of the following complex numbers: (i)  $\sqrt{3} + i$  (ii)  $-\sqrt{3} + i$  (iii)  $-\sqrt{3} - i$  (iv)  $\sqrt{3} - i$

(i)  $\sqrt{3} + i$

$$\text{Modulus } r = \sqrt{x^2 + y^2} = \sqrt{\sqrt{3}^2 + 1^2} = \sqrt{4} = 2$$

$$\alpha = \tan^{-1} \left| \frac{y}{x} \right| = \tan^{-1} \left| \frac{1}{\sqrt{3}} \right| = \frac{\pi}{6}$$

Since the complex number  $\sqrt{3} + i$  lies in the first quadrant, principal argument  $\theta = \alpha \Rightarrow \theta = \frac{\pi}{6}$

(ii)  $-\sqrt{3} + i$

$$\text{Modulus } r = \sqrt{x^2 + y^2} = \sqrt{(-\sqrt{3})^2 + 1^2} = \sqrt{4} = 2$$

$$\alpha = \tan^{-1} \left( \frac{y}{x} \right) = \tan^{-1} \left| \frac{1}{-\sqrt{3}} \right| = \frac{\pi}{6}$$

Since the complex number  $-\sqrt{3} + i$  lies in the second quadrant, principal argument  $\theta = \pi - \alpha \Rightarrow \theta = \pi - \frac{\pi}{6} = \frac{5\pi}{6}$

(iii)  $-\sqrt{3} - i$

$$\text{Modulus } r = \sqrt{x^2 + y^2} = \sqrt{(-\sqrt{3})^2 + (-1)^2} = \sqrt{4} = 2$$

$$\alpha = \tan^{-1} \left( \frac{y}{x} \right) = \tan^{-1} \left| \frac{-1}{-\sqrt{3}} \right| = \frac{\pi}{6}$$

Since the complex number  $-\sqrt{3} - i$  lies in the third quadrant, principal argument  $\theta = -\pi + \alpha \Rightarrow \theta = -\pi + \frac{\pi}{6} = -\frac{5\pi}{6}$

(iv)  $\sqrt{3} - i$

$$\text{Modulus } r = \sqrt{x^2 + y^2} = \sqrt{\sqrt{3}^2 + (-1)^2} = \sqrt{4} = 2$$

$$\alpha = \tan^{-1} \left( \frac{y}{x} \right) = \tan^{-1} \left| \frac{-1}{\sqrt{3}} \right| = \frac{\pi}{6}$$

Since the complex number  $\sqrt{3} - i$  lies in the fourth quadrant, principal argument  $\theta = -\alpha \Rightarrow \theta = -\frac{\pi}{6}$

**EXAMPLE 2.24** Find the principal argument  $\text{Arg } z$ , when  $z = \frac{-2}{1+i\sqrt{3}}$

$$\arg z = \arg \left( \frac{-2}{1+i\sqrt{3}} \right) = \arg(-2) - \arg(1+i\sqrt{3}) \rightarrow (1)$$

$$\arg(-2) = \tan^{-1} \left| \frac{y}{x} \right| = \tan^{-1} \left| \frac{0}{-2} \right| = 0$$

Since the complex number  $-2$  lies in the second quadrant, principal argument  $\theta = \pi - \alpha \Rightarrow \theta = \pi - 0 = \pi$

$$\arg(1+i\sqrt{3}) = \tan^{-1} \left| \frac{y}{x} \right| = \tan^{-1} \left| \frac{\sqrt{3}}{1} \right| = \frac{\pi}{3}$$

Since the complex number  $1+i\sqrt{3}$  lies in the first quadrant, principal argument  $\theta = \alpha \Rightarrow \theta = \frac{\pi}{3}$

$$(1) \Rightarrow \arg \left( \frac{-2}{1+i\sqrt{3}} \right) = \pi - \frac{\pi}{3} = \frac{2\pi}{3}$$

### EXERCISE 2.7

1. Write in polar form of the following complex numbers:

(i)  $2 + i2\sqrt{3}$  (ii)  $3 - i\sqrt{3}$  (iii)  $-2 - i2$  (iv)  $\frac{i-1}{\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}}$

(i)  $2 + i2\sqrt{3} = r\text{cis}(\theta) \rightarrow (1)$

$$r = \sqrt{x^2 + y^2} = \sqrt{(2)^2 + (2\sqrt{3})^2} = \sqrt{4 + 12} = \sqrt{16} = 4$$

$$\alpha = \tan^{-1} \left| \frac{y}{x} \right| = \tan^{-1} \left| \frac{2\sqrt{3}}{2} \right| = \frac{\pi}{3}$$

Principal argument lies in the first quadrant  $\theta = \alpha = \frac{\pi}{3}$

$$(1) \Rightarrow 2 + i2\sqrt{3} = 4\text{cis}\left(\frac{\pi}{3}\right) = 4\text{cis}\left(2k\pi + \frac{\pi}{3}\right), k \in \mathbb{Z}$$

(ii)  $3 - i\sqrt{3} = r\text{cis}(\theta) \rightarrow (1)$

$$r = \sqrt{x^2 + y^2} = \sqrt{(3)^2 + (-\sqrt{3})^2} = \sqrt{9 + 3} = \sqrt{4 \times 3} = 2\sqrt{3}$$

$$\alpha = \tan^{-1} \left| \frac{y}{x} \right| = \tan^{-1} \left| \frac{-\sqrt{3}}{3} \right| = \tan^{-1} \left| \frac{1}{\sqrt{3}} \right| = \frac{\pi}{6}$$

Principal argument lies in the 4th quadrant  $\theta = -\alpha = -\frac{\pi}{6}$

$$(1) \Rightarrow 3 - i\sqrt{3} = 2\sqrt{3}\text{cis}\left(-\frac{\pi}{6}\right) = 2\sqrt{3}\text{cis}\left(2k\pi - \frac{\pi}{6}\right), k \in \mathbb{Z}$$

(iii)  $-2 - i2 = r\text{cis}(\theta) \rightarrow (1)$

$$r = \sqrt{x^2 + y^2} = \sqrt{(-2)^2 + (-2)^2} = \sqrt{4 + 4} = \sqrt{2 \times 4} = 2\sqrt{2}$$

$$\alpha = \tan^{-1} \left| \frac{y}{x} \right| = \tan^{-1} \left| -\frac{-2}{2} \right| = \frac{\pi}{4}$$

Principal argument lies in the 3rd quadrant  $\theta = -\pi + \alpha = -\pi + \frac{\pi}{4} = -\frac{3\pi}{4}$

$$(1) \Rightarrow -2 - i2 = 2\sqrt{2} \operatorname{cis} \left( -\frac{3\pi}{4} \right) = 2\sqrt{2} \operatorname{cis} \left( 2k\pi - \frac{3\pi}{4} \right), k \in \mathbb{Z}$$

$$(iv) \frac{i-1}{\cos \frac{\pi}{3} + i \sin \frac{\pi}{3}}$$

$$i - 1 = -1 + i = r \operatorname{cis}(\theta) \rightarrow (1)$$

$$r = \sqrt{x^2 + y^2} = \sqrt{(-1)^2 + (1)^2} = \sqrt{2}$$

$$\alpha = \tan^{-1} \left| \frac{y}{x} \right| = \tan^{-1} \left| \frac{1}{-1} \right| = \frac{\pi}{4}$$

Principal argument lies in the 2<sup>nd</sup> quadrant  $\theta = \pi - \alpha \Rightarrow \theta = \pi - \frac{\pi}{4} = \frac{3\pi}{4}$

$$(1) \Rightarrow i - 1 = \sqrt{2} \operatorname{cis} \left( \frac{3\pi}{4} \right)$$

$$\frac{i-1}{\cos \frac{\pi}{3} + i \sin \frac{\pi}{3}} = \frac{\sqrt{2} \operatorname{cis} \left( \frac{3\pi}{4} \right)}{\operatorname{cis} \left( \frac{\pi}{3} \right)} = \sqrt{2} \operatorname{cis} \left( \frac{3\pi}{4} - \frac{\pi}{3} \right) = \sqrt{2} \operatorname{cis} \left( \frac{5\pi}{12} \right) = \sqrt{2} \operatorname{cis} \left( 2k\pi + \frac{5\pi}{12} \right), k \in \mathbb{Z}$$

**EXAMPLE 2.23** Represent the complex number (i)  $-1 - i$  (ii)  $1 + i\sqrt{3}$  in polar form.

2. Find the rectangular form of the complex numbers:

$$(i) \left( \cos \frac{\pi}{6} + i \sin \frac{\pi}{6} \right) \left( \cos \frac{\pi}{12} + i \sin \frac{\pi}{12} \right) \quad (ii) \frac{\cos \frac{\pi}{6} - i \sin \frac{\pi}{6}}{2 \left( \cos \frac{\pi}{3} + i \sin \frac{\pi}{3} \right)}$$

$$(i) \left( \cos \frac{\pi}{6} + i \sin \frac{\pi}{6} \right) \left( \cos \frac{\pi}{12} + i \sin \frac{\pi}{12} \right)$$

$$= \operatorname{cis} \left( \frac{\pi}{6} + \frac{\pi}{12} \right)$$

$$= \operatorname{cis} \left( \frac{3\pi}{12} \right) = \operatorname{cis} \left( \frac{\pi}{4} \right)$$

$$= \cos \frac{\pi}{4} + i \sin \frac{\pi}{4}$$

$$= \frac{1}{\sqrt{2}} + i \frac{1}{\sqrt{2}}$$

$$(ii) \frac{\cos \frac{\pi}{6} - i \sin \frac{\pi}{6}}{2 \left( \cos \frac{\pi}{3} + i \sin \frac{\pi}{3} \right)}$$

$$= \frac{1}{2} \operatorname{cis} \left( -\frac{\pi}{6} - \frac{\pi}{3} \right) = \frac{1}{2} \operatorname{cis} \left( -\frac{3\pi}{6} \right) = \frac{1}{2} \operatorname{cis} \left( -\frac{\pi}{2} \right) = \frac{1}{2} \left[ \cos \left( -\frac{\pi}{2} \right) + i \sin \left( -\frac{\pi}{2} \right) \right]$$

$$= \frac{1}{2} \left( \cos \frac{\pi}{2} - i \sin \frac{\pi}{2} \right)$$

$$= \frac{1}{2} (0 - i)$$

$$= -\frac{1}{2} i$$

**EXAMPLE 2.25** Find the product  $\frac{3}{2} \left( \cos \frac{\pi}{3} + i \sin \frac{\pi}{3} \right) \cdot 6 \left( \cos \frac{5\pi}{6} + i \sin \frac{5\pi}{6} \right)$  in rectangular form.

**EXAMPLE 2.26** Find the quotient  $\frac{2 \left( \cos \frac{9\pi}{4} + i \sin \frac{9\pi}{4} \right)}{4 \left( \cos \left( \frac{-3\pi}{2} \right) + i \sin \left( \frac{-3\pi}{2} \right) \right)}$  in rectangular form.

3. If  $(x_1 + iy_1)(x_2 + iy_2) \dots (x_n + iy_n) = a + ib$ , show that (i)  $(x_1^2 + y_1^2)(x_2^2 + y_2^2) \dots (x_n^2 + y_n^2) = a^2 + b^2$  (ii)  $\sum_{r=1}^n \tan^{-1} \left( \frac{y_r}{x_r} \right) = \tan^{-1} \left( \frac{b}{a} \right) + 2k\pi, k \in \mathbb{Z}$

$$(i) (x_1 + iy_1)(x_2 + iy_2) \dots (x_n + iy_n) = a + ib$$

Taking modulus on both sides,

$$\Rightarrow |(x_1 + iy_1)(x_2 + iy_2) \dots (x_n + iy_n)| = |a + ib|$$

$$\Rightarrow |x_1 + iy_1| |x_2 + iy_2| \dots |x_n + iy_n| = |a + ib|$$

$$\Rightarrow \sqrt{(x_1^2 + y_1^2)} \sqrt{(x_2^2 + y_2^2)} \dots \sqrt{(x_n^2 + y_n^2)} = \sqrt{a^2 + b^2}$$

Squaring on both sides,

$$\Rightarrow (x_1^2 + y_1^2)(x_2^2 + y_2^2) \dots (x_n^2 + y_n^2) = a^2 + b^2$$

$$(ii) (x_1 + iy_1)(x_2 + iy_2) \dots (x_n + iy_n) = a + ib$$

Taking argument on both sides,

$$\Rightarrow \arg[(x_1 + iy_1)(x_2 + iy_2) \dots (x_n + iy_n)] = \arg(a + ib)$$

$$\Rightarrow \tan^{-1} \left( \frac{y_1}{x_1} \right) + \tan^{-1} \left( \frac{y_2}{x_2} \right) + \dots + \tan^{-1} \left( \frac{y_n}{x_n} \right) = \tan^{-1} \left( \frac{b}{a} \right)$$

$$\Rightarrow \sum_{r=1}^n \tan^{-1} \left( \frac{y_r}{x_r} \right) = \tan^{-1} \left( \frac{b}{a} \right) + 2k\pi, k \in \mathbb{Z}$$

4. If  $\frac{1+z}{1-z} = \cos 2\theta + i \sin 2\theta$ , show that  $z = i \tan \theta$ .

$$\frac{1+z}{1-z} = \cos 2\theta + i \sin 2\theta$$

Add 1 on both sides,  $\frac{1+z}{1-z} + 1 = (1 + \cos 2\theta) + i \sin 2\theta$

$$\Rightarrow \frac{1+z+1-z}{1-z} = 2\cos^2\theta + i2\sin\theta\cos\theta$$

$$\Rightarrow \frac{2}{1-z} = 2\cos\theta [\cos\theta + i\sin\theta]$$

$$\div \text{by } 2 \Rightarrow \frac{1}{1-z} = \cos\theta [\cos\theta + i\sin\theta]$$

$$\Rightarrow \frac{1}{\cos\theta [\cos\theta + i\sin\theta]} = 1 - z$$

$$\Rightarrow z = 1 - \frac{1}{\cos \theta [\cos \theta + i \sin \theta]}$$

$$\Rightarrow z = 1 - \frac{1}{\cos \theta [\cos \theta + i \sin \theta]} \times \frac{\cos \theta - i \sin \theta}{\cos \theta - i \sin \theta}$$

$$\Rightarrow z = 1 - \frac{\cos \theta - i \sin \theta}{\cos \theta (\cos^2 \theta + \sin^2 \theta)}$$

$$\Rightarrow z = 1 - \left[ \frac{\cos \theta}{\cos \theta} - i \frac{\sin \theta}{\cos \theta} \right]$$

$$\Rightarrow z = 1 - 1 + i \tan \theta$$

$$\Rightarrow z = i \tan \theta$$

5. If  $\cos \alpha + \cos \beta + \cos \gamma = \sin \alpha + \sin \beta + \sin \gamma = 0$ , then show that

(i)  $\cos 3\alpha + \cos 3\beta + \cos 3\gamma = 3 \cos(\alpha + \beta + \gamma)$

(ii)  $\sin 3\alpha + \sin 3\beta + \sin 3\gamma = 3 \sin(\alpha + \beta + \gamma)$

Let  $a = \text{cis} \alpha, b = \text{cis} \beta, c = \text{cis} \gamma$

W.K.T, If  $a + b + c = 0$  then  $a^3 + b^3 + c^3 = 3abc$

$$a + b + c = (\cos \alpha + \cos \beta + \cos \gamma) + i(\sin \alpha + \sin \beta + \sin \gamma) = 0$$

$$a^3 + b^3 + c^3 = 3abc$$

$$\Rightarrow (\text{cis} \alpha)^3 + (\text{cis} \beta)^3 + (\text{cis} \gamma)^3 = 3 \text{cis} \alpha \cdot \text{cis} \beta \cdot \text{cis} \gamma$$

$$\Rightarrow \text{cis}(3\alpha) + \text{cis}(3\beta) + \text{cis}(3\gamma) = 3 \text{cis}(\alpha + \beta + \gamma)$$

$$\Rightarrow (\cos 3\alpha + \cos 3\beta + \cos 3\gamma) + i(\sin 3\alpha + \sin 3\beta + \sin 3\gamma) = 3(\cos(\alpha + \beta + \gamma) + i \sin(\alpha + \beta + \gamma))$$

Equating the real and imaginary parts on both sides,

$$\cos 3\alpha + \cos 3\beta + \cos 3\gamma = 3 \cos(\alpha + \beta + \gamma)$$

$$\sin 3\alpha + \sin 3\beta + \sin 3\gamma = 3 \sin(\alpha + \beta + \gamma)$$

6. If  $z = x + iy$  and  $\arg \left( \frac{z-i}{z+2} \right) = \frac{\pi}{4}$ , then show that  $x^2 + y^2 + 3x - 3y + 2 = 0$ .

Given:  $Z = x + iy$

$$\arg \left( \frac{z-i}{z+2} \right) = \frac{\pi}{4}$$

$$\Rightarrow \arg(z - i) - \arg(z + 2) = \frac{\pi}{4}$$

$$\Rightarrow \arg(x + iy - i) - \arg(x + iy + 2) = \frac{\pi}{4}$$

$$\Rightarrow \arg(x + i(y - 1)) - \arg(x + 2 + iy) = \frac{\pi}{4}$$

$$\Rightarrow \tan^{-1} \frac{y-1}{x} - \tan^{-1} \frac{y}{x+2} = \frac{\pi}{4}$$

$$\Rightarrow \tan^{-1} \frac{\frac{y-1}{x} - \frac{y}{x+2}}{1 + \left( \frac{y-1}{x} \times \frac{y}{x+2} \right)} = \frac{\pi}{4}$$

$$\Rightarrow \frac{\frac{((y-1)(x+2) - xy)}{x(x+2)}}{1 + \frac{y^2 - y}{x(x+2)}} = \tan \frac{\pi}{4}$$

$$\Rightarrow \frac{\frac{(y-1)(x+2) - xy}{x(x+2)}}{\frac{xy + 2y - x - 2 - xy}{x(x+2)}} = 1$$

$$\Rightarrow \frac{-x + 2y - 2}{x^2 + y^2 + 2x - y} = 1$$

$$\Rightarrow x^2 + y^2 + 2x - y + x - 2y + 2 = 0$$

$$\Rightarrow x^2 + y^2 + 3x - 3y + 2 = 0$$

**EXAMPLE 2.27** If  $z = x + iy$  and  $\arg \left( \frac{z-1}{z+1} \right) = \frac{\pi}{2}$ , then show that  $x^2 + y^2 = 1$ .

### De MOIVRE'S THEOREM AND ITS APPLICATIONS

- ❖ De Moivre's theorem: Given any complex number  $\cos \theta + i \sin \theta$  and any integer  $n$ ,  $(\cos \theta + i \sin \theta)^n = \cos n\theta + i \sin n\theta$
- ❖ Some results:
  - ✓  $(\cos \theta - i \sin \theta)^n = \cos n\theta - i \sin n\theta$
  - ✓  $(\cos \theta + i \sin \theta)^{-n} = \cos n\theta - i \sin n\theta$
  - ✓  $(\cos \theta - i \sin \theta)^{-n} = \cos n\theta + i \sin n\theta$
  - ✓  $\sin \theta + i \cos \theta = i(\cos \theta - i \sin \theta)$
- ❖ Formula for finding  $n^{\text{th}}$  root of a complex number:
 
$$z^{1/n} = r^{1/n} \text{cis} \left( \frac{\theta + 2k\pi}{n} \right), k = 0, 1, \dots, (n-1)$$
- ❖  $\omega^3 = 1 \Rightarrow 1 + \omega + \omega^2 = 0$
- ❖ The sum of all the  $n^{\text{th}}$  roots of unity is 0. (ie.,  $1 + \omega + \omega^2 + \dots + \omega^{n-1} = 0$ )
- ❖ The product of all the  $n^{\text{th}}$  roots of unity is  $(-1)^{n-1}$ . (ie.,  $1 + \omega + \omega^2 + \dots + \omega^{n-1} = (-1)^{n-1}$ )
- ❖ All the  $n$  roots of  $n^{\text{th}}$  roots of unity are in geometrical progression.
- ❖ All the  $n$  roots of  $n^{\text{th}}$  roots of unity lie on the circumference of a circle whose

centre is at the origin and radius equal to 1 and these roots divide the circle into n equal parts and form a polygon of n sides.

### EXERCISE 2.8

1.If  $\omega \neq 1$  is a cube root of unity, then show that  $\frac{a+b\omega+c\omega^2}{b+c\omega+a\omega^2} + \frac{a+b\omega+c\omega^2}{c+a\omega+b\omega^2} = -1$ .

$$\begin{aligned} & \frac{a+b\omega+c\omega^2}{b+c\omega+a\omega^2} + \frac{a+b\omega+c\omega^2}{c+a\omega+b\omega^2} \\ &= \left( \frac{a+b\omega+c\omega^2}{b+c\omega+a\omega^2} \times \frac{\omega}{\omega} \right) + \left( \frac{a+b\omega+c\omega^2}{c+a\omega+b\omega^2} \times \frac{\omega^2}{\omega^2} \right) \\ &= \frac{\omega(a+b\omega+c\omega^2)}{a\omega^3+c\omega^2+b\omega} + \frac{\omega^2(a+b\omega+c\omega^2)}{a\omega^3+b\omega^4+c\omega^2} \\ &= \frac{\omega(a+b\omega+c\omega^2)}{a+b\omega+c\omega^2} + \frac{\omega^2(a+b\omega+c\omega^2)}{a+b\omega+c\omega^2} \\ &= \omega + \omega^2 \\ &= -1 \end{aligned}$$

$$\boxed{\omega^3 = 1 ; 1 + \omega + \omega^2 = 0}$$

2.Show that  $\left(\frac{\sqrt{3}}{2} + \frac{i}{2}\right)^5 + \left(\frac{\sqrt{3}}{2} - \frac{i}{2}\right)^5 = -\sqrt{3}$

$$\frac{\sqrt{3}}{2} + \frac{i}{2} = r \operatorname{cis}(\theta) \rightarrow (1)$$

$$r = \sqrt{x^2 + y^2} = \sqrt{\left(\frac{\sqrt{3}}{2}\right)^2 + \left(\frac{1}{2}\right)^2} = \sqrt{\frac{3}{4} + \frac{1}{4}} = \sqrt{1} = 1$$

$$\alpha = \tan^{-1} \left| \frac{y}{x} \right| = \tan^{-1} \left| \frac{1/2}{\sqrt{3}/2} \right| = \frac{\pi}{6}$$

Principal argument lies in the first quadrant  $\theta = \alpha = \frac{\pi}{6}$

$$(1) \Rightarrow \frac{\sqrt{3}}{2} + \frac{i}{2} = \operatorname{cis} \left( \frac{\pi}{6} \right)$$

$$\Rightarrow \left( \frac{\sqrt{3}}{2} + \frac{i}{2} \right)^5 = \left[ \operatorname{cis} \left( \frac{\pi}{6} \right) \right]^5 = \operatorname{cis} \left( \frac{5\pi}{6} \right) = \cos \left( \frac{5\pi}{6} \right) + i \sin \left( \frac{5\pi}{6} \right)$$

$$\text{Similarly, } \left( \frac{\sqrt{3}}{2} - \frac{i}{2} \right)^5 = \cos \left( \frac{5\pi}{6} \right) - i \sin \left( \frac{5\pi}{6} \right)$$

$$\begin{aligned} & \left( \frac{\sqrt{3}}{2} + \frac{i}{2} \right)^5 + \left( \frac{\sqrt{3}}{2} - \frac{i}{2} \right)^5 = \cos \left( \frac{5\pi}{6} \right) + i \sin \left( \frac{5\pi}{6} \right) + \cos \left( \frac{5\pi}{6} \right) - i \sin \left( \frac{5\pi}{6} \right) \\ &= 2 \cos \left( \frac{5\pi}{6} \right) = 2 \cos \left( \frac{5 \times 180}{6} \right) = 2 \cos 150 = -2 \cos 30 = -2 \times \frac{\sqrt{3}}{2} = -\sqrt{3} \end{aligned}$$

**EXAMPLE 2.31** Simplify (i)  $(1+i)^{18}$  (ii)  $(-\sqrt{3}+3i)^{31}$

3.Find the value of  $\left( \frac{1+\sin \frac{\pi}{10} + i \cos \frac{\pi}{10}}{1+\sin \frac{\pi}{10} - i \cos \frac{\pi}{10}} \right)^{10}$

$$\text{Let } z = \sin \frac{\pi}{10} + i \cos \frac{\pi}{10} \Rightarrow \frac{1}{z} = \sin \frac{\pi}{10} - i \cos \frac{\pi}{10}$$

$$\left( \frac{1+\sin \frac{\pi}{10} + i \cos \frac{\pi}{10}}{1+\sin \frac{\pi}{10} - i \cos \frac{\pi}{10}} \right)^{10} = \left( \frac{1+z}{1+\frac{1}{z}} \right)^{10} = \left( \frac{1+z}{1+z} \times z \right)^{10} = z^{10}$$

$$\begin{aligned} z^{10} &= \left( \sin \frac{\pi}{10} + i \cos \frac{\pi}{10} \right)^{10} = \left[ i \left( \cos \frac{\pi}{10} - i \sin \frac{\pi}{10} \right) \right]^{10} \\ &= i^{10} \left[ \cos \left( 10 \times \frac{\pi}{10} \right) - i \sin \left( 10 \times \frac{\pi}{10} \right) \right] \\ &= -1 [\cos \pi - i \sin \pi] \\ &= -1(-1 - 0) \\ &= 1 \end{aligned}$$

**EXAMPLE 2.29** Simplify  $\left( \sin \frac{\pi}{6} + i \cos \frac{\pi}{6} \right)^{18}$

**EXAMPLE 2.30** Simplify  $\left( \frac{1+\cos 2\theta + i \sin 2\theta}{1+\cos 2\theta - i \sin 2\theta} \right)^{30}$

4.If  $2 \cos \alpha = x + \frac{1}{x}$  and  $2 \cos \beta = y + \frac{1}{y}$ , show that

$$(i) \frac{x}{y} + \frac{y}{x} = 2 \cos(\alpha - \beta) \quad (ii) xy - \frac{1}{xy} = 2i \sin(\alpha + \beta)$$

$$(iii) \frac{x^m}{y^n} - \frac{y^n}{x^m} = 2i \sin(m\alpha - n\beta) \quad (iv) x^m y^n + \frac{1}{x^m y^n} = 2 \cos(m\alpha + n\beta)$$

$$x + \frac{1}{x} = 2 \cos \alpha \Rightarrow x = \cos \alpha + i \sin \alpha$$

$$y + \frac{1}{y} = 2 \cos \beta \Rightarrow y = \cos \beta + i \sin \beta$$

$$(i) \frac{x}{y} = \frac{\cos \alpha + i \sin \alpha}{\cos \beta + i \sin \beta} = \cos(\alpha - \beta) + i \sin(\alpha - \beta)$$

$$\frac{y}{x} = \left( \frac{x}{y} \right)^{-1} = \cos(\alpha - \beta) - i \sin(\alpha - \beta)$$

$$\frac{x}{y} + \frac{y}{x} = 2 \cos(\alpha - \beta)$$

$$(ii) xy = (\cos \alpha + i \sin \alpha)(\cos \beta + i \sin \beta) = \cos(\alpha + \beta) + i \sin(\alpha + \beta)$$

$$\frac{1}{xy} = (xy)^{-1} = \cos(\alpha + \beta) - i \sin(\alpha + \beta)$$

$$xy - \frac{1}{xy} = 2i \sin(\alpha + \beta)$$

$$(iii) x^m = (\cos \alpha + i \sin \alpha)^m = \cos m\alpha + i \sin m\alpha$$

$$y^n = (\cos \beta + i \sin \beta)^n = \cos n\beta + i \sin n\beta$$

$$\frac{x^m}{y^n} = \frac{\cos m\alpha + i \sin m\alpha}{\cos n\beta + i \sin n\beta} = \cos(m\alpha - n\beta) + i \sin(m\alpha - n\beta)$$

$$\frac{y}{x^m} = \left(\frac{x^m}{y^n}\right)^{-1} = \cos(m\alpha - n\beta) - i \sin(m\alpha - n\beta)$$

$$\frac{x^m}{y^n} - \frac{y}{x^m} = 2i \sin(m\alpha - n\beta)$$

$$(iv) x^m = (\cos \alpha + i \sin \alpha)^m = \cos m\alpha + i \sin m\alpha$$

$$y^n = (\cos \beta + i \sin \beta)^n = \cos n\beta + i \sin n\beta$$

$$x^m y^n = (\cos m\alpha + i \sin m\alpha)(\cos n\beta + i \sin n\beta)$$

$$= \cos(m\alpha + n\beta) + i \sin(m\alpha + n\beta)$$

$$\frac{1}{x^m y^n} = \cos(m\alpha + n\beta) - i \sin(m\alpha + n\beta)$$

$$x^m y^n + \frac{1}{x^m y^n} = 2 \cos(m\alpha + n\beta)$$

**EXAMPLE 2.28** If  $z = \cos \theta + i \sin \theta$ , show that  $z^n + \frac{1}{z^n} = 2 \cos n\theta$  and  $z^n - \frac{1}{z^n} = 2i \sin n\theta$ .

5. Solve the equation  $z^3 + 27 = 0$

$$z^3 + 27 = 0 \Rightarrow z^3 = -27 \Rightarrow z = (-27)^{1/3} = (27 \times -1)^{1/3}$$

$$z = 3(-1)^{1/3} = 3[cis(\pi)]^{1/3} = 3[cis(2k\pi + \pi)]^{1/3}, k = 0, 1, 2$$

$$= 3cis(2k + 1)\frac{\pi}{3}, k = 0, 1, 2$$

$$= 3cis\left(\frac{\pi}{3}\right), 3cis(\pi), 3cis\left(\frac{5\pi}{3}\right)$$

6. If  $\omega \neq 1$  is a cube root of unity, show that the roots of the equation  $(z - 1)^3 + 8 = 0$  are  $-1, 1 - 2\omega, 1 - 2\omega^2$ .

$$(z - 1)^3 + 8 = 0$$

$$\Rightarrow (z - 1)^3 = -8$$

$$\Rightarrow (z - 1)^3 = (-2)^3 \Rightarrow \frac{(z-1)^3}{(-2)^3} = 1$$

$$\Rightarrow \left(\frac{z-1}{-2}\right)^3 = 1$$

$$\Rightarrow \frac{z-1}{-2} = (1)^{1/3}$$

$$\Rightarrow z - 1 = -2(1)^{1/3} \Rightarrow z - 1 = -2[1 \text{ (or) } \omega \text{ (or) } \omega^2]$$

Padasalai

$$\Rightarrow \begin{cases} z - 1 = -2(1) \Rightarrow z - 1 = -2 \Rightarrow z = -1 \\ z - 1 = -2\omega \Rightarrow z = 1 - 2\omega \\ z - 1 = -2\omega^2 \Rightarrow z = 1 - 2\omega^2 \end{cases}$$

Thus the roots of the equation  $(z - 1)^3 + 8 = 0$  are  $-1, 1 - 2\omega, 1 - 2\omega^2$ .

**EXAMPLE 2.34** Solve the equation  $z^3 + 8i = 0$ , where  $z \in \mathbb{C}$ .

Note:  $z^3 + 8i = 0 \Rightarrow z = (8i)^{1/3} \Rightarrow z = 2(i)^{1/3}$

find polar form for  $i$  and use de Moivre's theorem

**EXAMPLE 2.35** Find all cube roots of  $\sqrt{3} + i$ .

Note:  $(\sqrt{3} + i)^{1/3}$ ; find polar form for  $\sqrt{3} + i$  and use de Moivre's theorem

7. Find the value of  $\sum_{k=1}^8 \left( \cos \frac{2k\pi}{9} + i \sin \frac{2k\pi}{9} \right)$

$$\text{Let } x = (1)^{1/9} \Rightarrow x = [cis(0)]^{1/9} = [cis(2k\pi + 0)]^{1/9}, k = 0, 1, 2, \dots, 8$$

$$\Rightarrow x = cis\left(\frac{2k\pi}{9}\right), k = 0, 1, 2, \dots, 8$$

Thus the roots are  $cis(0), cis\left(\frac{2\pi}{9}\right), cis\left(\frac{4\pi}{9}\right), cis\left(\frac{6\pi}{9}\right), cis\left(\frac{8\pi}{9}\right), cis\left(\frac{10\pi}{9}\right), cis\left(\frac{12\pi}{9}\right), cis\left(\frac{14\pi}{9}\right), cis\left(\frac{16\pi}{9}\right)$ .

W.K.T, sum of all the roots is equal to zero.

$$cis(0) + cis\left(\frac{2\pi}{9}\right) + cis\left(\frac{4\pi}{9}\right) + cis\left(\frac{6\pi}{9}\right) + cis\left(\frac{8\pi}{9}\right) + cis\left(\frac{10\pi}{9}\right) + cis\left(\frac{12\pi}{9}\right) + cis\left(\frac{14\pi}{9}\right) + cis\left(\frac{16\pi}{9}\right) = 0$$

$$\Rightarrow 1 + cis\left(\frac{2\pi}{9}\right) + cis\left(\frac{4\pi}{9}\right) + cis\left(\frac{6\pi}{9}\right) + cis\left(\frac{8\pi}{9}\right) + cis\left(\frac{10\pi}{9}\right) + cis\left(\frac{12\pi}{9}\right) + cis\left(\frac{14\pi}{9}\right) + cis\left(\frac{16\pi}{9}\right) = 0$$

$$\Rightarrow cis\left(\frac{2\pi}{9}\right) + cis\left(\frac{4\pi}{9}\right) + cis\left(\frac{6\pi}{9}\right) + cis\left(\frac{8\pi}{9}\right) + cis\left(\frac{10\pi}{9}\right) + cis\left(\frac{12\pi}{9}\right) + cis\left(\frac{14\pi}{9}\right) + cis\left(\frac{16\pi}{9}\right) = -1$$

$$\Rightarrow \sum_{k=1}^8 cis\left(\frac{2k\pi}{9}\right) = -1$$

$$\Rightarrow \sum_{k=1}^8 \left( \cos \frac{2k\pi}{9} + i \sin \frac{2k\pi}{9} \right) = -1$$

8. If  $\omega \neq 1$  is a cube root of unity, show that

$$(i)(1 - \omega + \omega^2)^6 + (1 + \omega - \omega^2)^6 = 128$$

$$(ii)(1 + \omega)(1 + \omega^2)(1 + \omega^4)(1 + \omega^8) \dots (1 + \omega^{2^{11}}) = 1$$

$$\text{W.K.T, } \boxed{\omega^3 = 1 ; 1 + \omega + \omega^2 = 0}$$

$$\begin{aligned} (i)(1 - \omega + \omega^2)^6 + (1 + \omega - \omega^2)^6 \\ = (-\omega - \omega)^6 + (-\omega^2 - \omega^2)^6 \\ = (-2\omega)^6 + (-2\omega^2)^6 = 2^6\omega^6 + 2^6\omega^{12} = 64(\omega^3)^2 + 64(\omega^3)^4 = 64 + 64 = 128 \end{aligned}$$

$$\begin{aligned} (ii)(1 + \omega)(1 + \omega^2)(1 + \omega^4)(1 + \omega^8) \dots (1 + \omega^{2^{11}}) \\ = (1 + \omega)(1 + \omega^2)(1 + \omega^{2^2})(1 + \omega^{2^3}) \dots (1 + \omega^{2^{11}}) \end{aligned}$$

Clearly the above expression contains 12 terms,

$$\begin{aligned} &= (1 + \omega)(1 + \omega^2)(1 + \omega^4)(1 + \omega^8) \dots 12 \text{ terms} \\ &= [(1 + \omega)(1 + \omega^2)][(1 + \omega)(1 + \omega^2)] \dots 6 \text{ terms} \\ &= [(1 + \omega)(1 + \omega^2)]^6 \\ &= (1 + \omega + \omega^2 + \omega^3)^6 \\ &= (0 + 1)^6 \\ &= 1 \end{aligned}$$

9.If  $z = 2 - 2i$ , find the rotation of  $z$  by  $\theta$  radians in the counter clockwise direction about the origin when (i) $\theta = \frac{\pi}{3}$  (ii) $\theta = \frac{2\pi}{3}$  (iii) $\theta = \frac{3\pi}{2}$

$$z = 2 - 2i = r \text{cis}(\theta) \rightarrow (1)$$

$$r = \sqrt{x^2 + y^2} = \sqrt{(2)^2 + (-2)^2} = \sqrt{4 + 4} = \sqrt{4 \times 2} = 2\sqrt{2}$$

$$\alpha = \tan^{-1} \left| \frac{y}{x} \right| = \tan^{-1} \left| \frac{-2}{2} \right| = \frac{\pi}{4}$$

Principal argument lies in the 4th quadrant  $\theta = -\alpha = -\frac{\pi}{4}$

$$(1) \Rightarrow z = 2 - 2i = 2\sqrt{2} \text{cis} \left( -\frac{\pi}{4} \right)$$

$$\begin{aligned} (i) \text{when } \theta = \frac{\pi}{3} \Rightarrow z &= 2\sqrt{2} \text{cis} \left( -\frac{\pi}{4} \right) \text{cis} \left( \frac{\pi}{3} \right) \\ &= 2\sqrt{2} \text{cis} \left( -\frac{\pi}{4} + \frac{\pi}{3} \right) \\ &= 2\sqrt{2} \text{cis} \left( \frac{-3\pi + 4\pi}{12} \right) \\ &= 2\sqrt{2} \text{cis} \left( \frac{\pi}{12} \right) \end{aligned}$$

$$\begin{aligned} (ii) \text{when } \theta = \frac{2\pi}{3} \Rightarrow z &= 2\sqrt{2} \text{cis} \left( -\frac{\pi}{4} \right) \text{cis} \left( \frac{2\pi}{3} \right) \\ &= 2\sqrt{2} \text{cis} \left( -\frac{\pi}{4} + \frac{2\pi}{3} \right) \end{aligned}$$

$$\begin{aligned} &= 2\sqrt{2} \text{cis} \left( \frac{-3\pi + 8\pi}{12} \right) \\ &= 2\sqrt{2} \text{cis} \left( \frac{5\pi}{12} \right) \\ (ii) \text{when } \theta = \frac{3\pi}{2} \Rightarrow z &= 2\sqrt{2} \text{cis} \left( -\frac{\pi}{4} \right) \text{cis} \left( \frac{3\pi}{2} \right) \\ &= 2\sqrt{2} \text{cis} \left( -\frac{\pi}{4} + \frac{3\pi}{2} \right) \\ &= 2\sqrt{2} \text{cis} \left( \frac{-\pi + 6\pi}{4} \right) \\ &= 2\sqrt{2} \text{cis} \left( \frac{5\pi}{4} \right) \end{aligned}$$

**EXAMPLE 2.36** Suppose  $z_1, z_2$  and  $z_3$  are the vertices of an equilateral triangle inscribed in the circle  $|z| = 2$ . If  $z_1 = 1 + i\sqrt{3}$ , then find  $z_2$  and  $z_3$ .

$|z| = 2$  represents the circle with centre (0,0) and radius 2.

Let  $A, B$  and  $C$  be the vertices of the given triangle. Since the vertices  $z_1, z_2$  and  $z_3$  form an equilateral triangle inscribed in the circle  $|z| = 2$ , the sides of this triangle  $AB, BC$  and  $CA$  subtend  $\frac{2\pi}{3}$  radian at the origin.

We obtain  $z_2$  and  $z_3$  by the rotation of  $z_1$  by  $\frac{2\pi}{3}$  and  $\frac{4\pi}{3}$  respectively.

$$\text{Given: } z_1 = 1 + i\sqrt{3}$$

$$\begin{aligned} z_2 &= z_1 \text{cis} \left( \frac{2\pi}{3} \right) = (1 + i\sqrt{3}) \left[ \cos \left( \frac{2\pi}{3} \right) + i \sin \left( \frac{2\pi}{3} \right) \right] \\ &= (1 + i\sqrt{3}) [\cos(180 - 60) + i \sin(180 - 60)] \\ &= (1 + i\sqrt{3}) (-\cos 60 + i \sin 60) \\ &= (1 + i\sqrt{3}) \left( -\frac{1}{2} + i \frac{\sqrt{3}}{2} \right) \\ &= -2 \end{aligned}$$

$$\begin{aligned} z_3 &= z_2 \text{cis} \left( \frac{2\pi}{3} \right) = -2 \left[ \cos \left( \frac{2\pi}{3} \right) + i \sin \left( \frac{2\pi}{3} \right) \right] \\ &= -2 [\cos(180 - 60) + i \sin(180 - 60)] \\ &= -2 (-\cos 60 + i \sin 60) \\ &= -2 \left( -\frac{1}{2} + i \frac{\sqrt{3}}{2} \right) \\ &= 1 - i\sqrt{3} \end{aligned}$$

10.Prove that the values of  $\sqrt[4]{-1}$  are  $\pm \frac{1}{\sqrt{2}} (1 \pm i)$

$$\text{Let } x = \sqrt[4]{-1}$$

$$\Rightarrow x = (-1)^{1/4} \Rightarrow x^4 = -1 \Rightarrow x^4 + 1 = 0$$

$$\Rightarrow (x^4 + 2x^2 + 1) - (2) = 0$$

$$\Rightarrow (x^2 + 1)^2 - (\sqrt{2}x)^2 = 0$$

$$\Rightarrow (x^2 + 1 + \sqrt{2}x)(x^2 + 1 - \sqrt{2}x) = 0$$

$$\Rightarrow (x^2 + 1 + \sqrt{2}x) = 0 ; (x^2 + 1 - \sqrt{2}x) = 0$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\Rightarrow x = \frac{-\sqrt{2} \pm \sqrt{(\sqrt{2})^2 - 4(1)(1)}}{2(1)} ; \frac{-(-\sqrt{2}) \pm \sqrt{(-\sqrt{2})^2 - 4(1)(1)}}{2(1)}$$

$$\Rightarrow x = \frac{-\sqrt{2} \pm \sqrt{2-4}}{2} ; \frac{\sqrt{2} \pm \sqrt{2-4}}{2}$$

$$\Rightarrow x = \frac{-\sqrt{2} \pm \sqrt{-2}}{2} ; \frac{\sqrt{2} \pm \sqrt{-2}}{2}$$

$$\Rightarrow x = \frac{-\sqrt{2} \pm \sqrt{2}i^2}{2} ; \frac{\sqrt{2} \pm \sqrt{2}i^2}{2}$$

$$\Rightarrow x = \frac{-\sqrt{2} \pm \sqrt{2}i}{2} ; \frac{\sqrt{2} \pm \sqrt{2}i}{2}$$

$$\Rightarrow x = \frac{-\sqrt{2}}{2}(1 \pm i) ; \frac{\sqrt{2}}{2}(1 \pm i)$$

$$\Rightarrow x = \pm \frac{1}{\sqrt{2}}(1 \pm i)$$

**EXAMPLE 2.32** Find the cube roots of unity.

$$\text{Let } z^3 = 1 \Rightarrow z = (1)^{1/3} = [cis(0)]^{1/3}$$

$$\Rightarrow z = [cis(2k\pi + 0)]^{1/3}, k = 0,1,2$$

$$\Rightarrow z = cis\left(\frac{2k\pi}{3}\right), k = 0,1,2$$

$$\Rightarrow z = cis(0), cis\left(\frac{2\pi}{3}\right), cis\left(\frac{4\pi}{3}\right)$$

$$\Rightarrow z = cis(0), cis\left(\pi - \frac{\pi}{3}\right), cis\left(\pi + \frac{\pi}{3}\right)$$

$$\Rightarrow z = \cos 0 + i \sin 0, \cos\left(\pi - \frac{\pi}{3}\right) + i \sin\left(\pi - \frac{\pi}{3}\right), \cos\left(\pi + \frac{\pi}{3}\right) + i \sin\left(\pi + \frac{\pi}{3}\right)$$

$$\Rightarrow z = 1, -\cos\frac{\pi}{3} + i \sin\frac{\pi}{3}, -\cos\frac{\pi}{3} - i \sin\frac{\pi}{3}$$

$$\Rightarrow z = 1, -\frac{1}{2} + i\frac{\sqrt{3}}{2}, -\frac{1}{2} - i\frac{\sqrt{3}}{2} \Rightarrow z = 1, \frac{-1+i\sqrt{3}}{2}, \frac{-1-i\sqrt{3}}{2}$$

$\therefore$  The cube roots of unity 1,  $\omega$ ,  $\omega^2$  are  $1, \frac{-1+i\sqrt{3}}{2}, \frac{-1-i\sqrt{3}}{2}$

**EXAMPLE 2.33** Find the fourth roots of unity.

$$\text{Let } z^4 = 1 \Rightarrow z = (1)^{1/4} = [cis(0)]^{1/4}$$

$$\Rightarrow z = [cis(2k\pi + 0)]^{1/4}, k = 0,1,2,3$$

$$\Rightarrow z = cis\left(\frac{2k\pi}{4}\right), k = 0,1,2,3$$

$$\Rightarrow z = cis(0), cis\left(\frac{2\pi}{4}\right), cis\left(\frac{4\pi}{4}\right), cis\left(\frac{6\pi}{4}\right)$$

$$\Rightarrow z = cis(0), cis\left(\frac{\pi}{2}\right), cis(\pi), cis\left(\frac{3\pi}{2}\right)$$

$$\Rightarrow z = \cos 0 + i \sin 0, \cos\left(\frac{\pi}{2}\right) + i \sin\left(\frac{\pi}{2}\right), \cos(\pi) + i \sin(\pi), \cos\left(\frac{3\pi}{2}\right) + i \sin\left(\frac{3\pi}{2}\right)$$

$$\Rightarrow z = 1, i, -1, -i$$

$\therefore$  The fourth roots of unity 1,  $\omega$ ,  $\omega^2$ ,  $\omega^3$  are 1,  $i$ ,  $-1$ ,  $-i$ .

**1. Prove that the multiplicative inverse of a non-zero complex number  $z = x + iy$**

$$\text{is } z^{-1} = \left(\frac{x}{x^2+y^2}\right) + i\left(-\frac{y}{x^2+y^2}\right)$$

PROOF:

$$\text{Let } z^{-1} = u + iv \text{ be the inverse of } z = x + iy$$

$$\text{W.K.T, } zz^{-1} = 1 \Rightarrow (x + iy)(u + iv) = 1$$

$$\Rightarrow (xu - yv) + i(xv + yu) = 1 + i0$$

Equating the Real and Imaginary parts on both sides,

$$xu - yv = 1 \rightarrow (1) ; xv + yu = 0 \rightarrow (2)$$

$$\Delta = \begin{vmatrix} x & -y \\ y & x \end{vmatrix} = x^2 + y^2 ; \Delta_u = \begin{vmatrix} 1 & -y \\ 0 & x \end{vmatrix} = x ; \Delta_v = \begin{vmatrix} x & 1 \\ y & 0 \end{vmatrix} = -y$$

$$u = \frac{\Delta_u}{\Delta} = \frac{x}{x^2+y^2} ; v = \frac{\Delta_v}{\Delta} = \frac{-y}{x^2+y^2}$$

$$z^{-1} = u + iv \Rightarrow z^{-1} = \left(\frac{x}{x^2+y^2}\right) + i\left(-\frac{y}{x^2+y^2}\right)$$

**2. For any two complex numbers  $z_1$  and  $z_2$  prove that  $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$**

PROOF:

$$\text{Let } z_1 = a + ib \text{ and } z_2 = c + id, a, b, c, d \in R$$

$$\overline{z_1 + z_2} = \overline{(a + ib) + (c + id)}$$

$$= \overline{(a + c) + i(b + d)}$$

$$= (a + c) - i(b + d)$$

$$= (a - ib) + (c - id)$$

|                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                  |
|--------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| $= \bar{z}_1 + \bar{z}_2$                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                        |
| <b>3. Prove that <math>\overline{z_1 \cdot z_2} = \bar{z}_1 \cdot \bar{z}_2</math> where <math>a, b, c, d \in R</math>.</b>                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                      |
| PROOF:<br>Let $z_1 = a + ib$ and $z_2 = c + id$ , $a, b, c, d \in R$<br>$\overline{z_1 \cdot z_2} = \overline{(a + ib)(c + id)}$ $= \overline{(ac - bd) + i(ad + bc)}$ $= (ac - bd) - i(ad + bc) \rightarrow (1)$ $\bar{z}_1 \cdot \bar{z}_2 = \overline{a + ib} \cdot \overline{c + id}$ $= (a - ib)(c - id)$ $= (ac - bd) - i(ad + bc) \rightarrow (2)$ From (1) and (2), $\overline{z_1 \cdot z_2} = \bar{z}_1 \cdot \bar{z}_2$                                                                                                                                                                                                                                                                                                                                                               |
| <b>4. Prove that <math>z</math> is purely imaginary if and only if <math>z = -\bar{z}</math></b>                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                 |
| PROOF:<br>Let $z = x + iy \Rightarrow \bar{z} = x - iy$<br>$z = -\bar{z} \Leftrightarrow x + iy = -(x - iy)$<br>$\Leftrightarrow x + iy = -x + iy$<br>$\Leftrightarrow 2x = 0$<br>$\Leftrightarrow x = 0$<br>$\Leftrightarrow z$ is purely imaginary                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                             |
| <b>5. State and prove triangle inequality of two complex numbers (or)</b><br><b>For any two complex numbers <math>z_1</math> and <math>z_2</math> prove that <math> z_1 + z_2  \leq  z_1  +  z_2 </math></b>                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                     |
| PROOF:<br>$ z_1 + z_2 ^2 = (z_1 + z_2)(\overline{z_1 + z_2}) \quad (\because z\bar{z} =  z ^2)$ $= (z_1 + z_2)(\bar{z}_1 + \bar{z}_2) \quad (\because \overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2)$ $= z_1\bar{z}_1 + (z_1\bar{z}_2 + \bar{z}_1z_2) + z_2\bar{z}_2$ $= z_1\bar{z}_1 + (z_1\bar{z}_2 + \overline{z_1\bar{z}_2}) + z_2\bar{z}_2 \quad (\because \bar{\bar{z}} = z)$ $=  z_1 ^2 + 2\text{Re}(z_1\bar{z}_2) +  z_2 ^2 \quad (\because z + \bar{z} = 2\text{Re}(z))$ $\leq  z_1 ^2 + 2 z_1  z_2  +  z_2 ^2 \quad (\because \text{Re}(z) \leq  z )$ $=  z_1 ^2 + 2 z_1  z_2  +  z_2 ^2 \quad (\because \overline{z_1 \cdot z_2} = \bar{z}_1 \cdot \bar{z}_2 \text{ \& }  z  =  \bar{z} )$ $= ( z_1  +  z_2 )^2$ Taking square root on both sides,<br>$ z_1 + z_2  \leq  z_1  +  z_2 $ |
| <b>6. For any two complex numbers <math>z_1</math> and <math>z_2</math> prove that <math> z_1 z_2  =  z_1   z_2 </math></b>                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                      |

|                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                 |
|---------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| PROOF:<br>$ z_1 z_2 ^2 = (z_1 z_2)(\overline{z_1 z_2}) \quad (\because z\bar{z} =  z ^2)$ $= (z_1)(z_2)(\bar{z}_1)(\bar{z}_2) \quad (\because \overline{z_1 \cdot z_2} = \bar{z}_1 \cdot \bar{z}_2)$ $= (z_1 \bar{z}_1)(z_2 \bar{z}_2)$ $=  z_1 ^2  z_2 ^2$ Taking square root on both sides,<br>$ z_1 z_2  =  z_1   z_2 $                                                                                                                                                                                                                                                      |
| <b>7. If <math>z = r(\cos \theta + i \sin \theta)</math> then prove that <math>z^{-1} = \frac{1}{r}(\cos \theta - i \sin \theta)</math></b>                                                                                                                                                                                                                                                                                                                                                                                                                                     |
| PROOF:<br>$z = r(\cos \theta + i \sin \theta)$<br>$z^{-1} = \frac{1}{z} = \frac{1}{r(\cos \theta + i \sin \theta)}$ $= \frac{1}{r} \cdot \frac{1}{\cos \theta + i \sin \theta} \times \frac{\cos \theta - i \sin \theta}{\cos \theta - i \sin \theta}$ $= \frac{1}{r} \cdot \frac{\cos \theta - i \sin \theta}{\cos^2 \theta + \sin^2 \theta}$ $= \frac{1}{r}(\cos \theta - i \sin \theta)$                                                                                                                                                                                     |
| <b>8. If <math>z_1 = r_1(\cos \theta_1 + i \sin \theta_1)</math>, <math>z_2 = r_2(\cos \theta_2 + i \sin \theta_2)</math> then <math>z_1 z_2 = r_1 r_2 [\cos(\theta_1 + \theta_2) + i \sin(\theta_1 + \theta_2)]</math></b>                                                                                                                                                                                                                                                                                                                                                     |
| PROOF:<br>$z_1 z_2 = [r_1(\cos \theta_1 + i \sin \theta_1)][r_2(\cos \theta_2 + i \sin \theta_2)]$ $= r_1 r_2 [(\cos \theta_1 \cos \theta_2 - \sin \theta_1 \sin \theta_2) + i(\sin \theta_1 \cos \theta_2 + \cos \theta_1 \sin \theta_2)]$ $= r_1 r_2 [\cos(\theta_1 + \theta_2) + i \sin(\theta_1 + \theta_2)]$                                                                                                                                                                                                                                                               |
| <b>9. If <math>z_1 = r_1(\cos \theta_1 + i \sin \theta_1)</math>, <math>z_2 = r_2(\cos \theta_2 + i \sin \theta_2)</math> then <math>\frac{z_1}{z_2} = \frac{r_1}{r_2} [\cos(\theta_1 - \theta_2) + i \sin(\theta_1 - \theta_2)]</math></b>                                                                                                                                                                                                                                                                                                                                     |
| PROOF:<br>$\frac{z_1}{z_2} = \frac{r_1(\cos \theta_1 + i \sin \theta_1)}{r_2(\cos \theta_2 + i \sin \theta_2)}$ $= \frac{r_1}{r_2} \cdot \frac{\cos \theta_1 + i \sin \theta_1}{\cos \theta_2 + i \sin \theta_2} \times \frac{\cos \theta_2 - i \sin \theta_2}{\cos \theta_2 - i \sin \theta_2}$ $= \frac{r_1}{r_2} \cdot \frac{(\cos \theta_1 \cos \theta_2 + \sin \theta_1 \sin \theta_2) + i(\sin \theta_1 \cos \theta_2 - \cos \theta_1 \sin \theta_2)}{\cos^2(\theta_2) + \sin^2(\theta_2)}$ $= \frac{r_1}{r_2} [\cos(\theta_1 - \theta_2) + i \sin(\theta_1 - \theta_2)]$ |