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12^{th} Standard

VOLUME - I

Based on the New Syllabus and New Textbook for the year 2019-20

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PREFACE

Sir,
An equation has no meaning, for me
unless it expresses a thought of GOD

- Ramanujam
[Statement to a friend]

Respected Principals, Correspondents, Head Masters / Head Mistresses, Teachers,

From the bottom of our heart, we at SURA Publications sincerely thank you for the support and patronage that you have extended to us for more than a decade.

It is in our sincerest effort we take the pride of releasing **SURA's Mathematics Guide** for +2 Standard – Edition 2019. This guide has been authored and edited by qualified teachers having teaching experience for over a decade in their respective subject fields. This Guide has been reviewed by reputed Professors who are currently serving as Head of the Department in esteemed Universities and Colleges.

With due respect to Teachers, I would like to mention that this guide will serve as a teaching companion to qualified teachers. Also, this guide will be an excellent learning companion to students with exhaustive exercises and in-text questions in addition to precise answers for textual questions.

In complete cognizance of the dedicated role of Teachers, I completely believe that our students will learn the subject effectively with this guide and prove their excellence in Board Examinations.

I once again sincerely thank the Teachers, Parents and Students for supporting and valuing our efforts.

God Bless all.

Subash Raj, B.E., M.S.
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All the Best

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CHAPTER 1

APPLICATION OF MATRICES AND DETERMINANTS

MUST KNOW DEFINITIONS

- If $|A| \neq 0$, then A is a non-singular matrix and if y $|A| = 0$, then A is a singular matrix.
- The adjoint matrix of A is defined as the transpose of the matrix of co-factors of A (adj A).
- If $AB = BA = I_n$, then the matrix B is called an inverse of A.
- If a square matrix has an inverse, then it is unique.
- A^{-1} exists if and only if A is non-singular.
- Singular matrix has no inverse.
- If A is non – singular and AB = AC, then B = C (left cancellation law).
- If A is non – singular and BA = CA then B = C (Right cancellation law).
- If A and B are any two non-singular square matrices of order n, then adj (AB) = (adj B) (adj A)
- A square matrix A is called orthogonal if $AA^T = A^TA = I$
- Two matrices A and B of same order are said to the **equivalent** if one can be obtained from the other by the applications of elementary transformations (A~B).
- A non – zero matrix is in a **row - echelon f**orm if all zero rows occur as bottom rows of the matrix and if the first non – zero element in any lower row occurs to the right of the first non – zero entry in the higher row.
- The **rank** of a matrix A is defined as the order of a highest order non – vanishing minor of the matrix A [ρ(A)].
- The **rank** of a non – zero matrix is equal to the number of non – zero rows in a row – echelon form of the matrix.
- An **elementary matrix** is a matrix which is obtained from an identity matrix by applying only one elementary transformation. Every non-singular matrix can be transformed to an identity matrix by a sequence of elementary row operations.
- A system of linear equations having atleast one solution is said to be **consistent**.
- A system of linear equations having no solutions is said to be **inconsistent**.

IMPORTAN T FORMULA TO REMEMBER

- Co – factor of a_{ij} is $A_{ij} = (-1)^{i+j} M_{ij}$, where M_{ij} is the minor of *aij*
- For every square matrix A of order n, $A(\text{adj } A) = (\text{adj } A)A = |A|\, I_n$
  $AA^{-1} = A^{-1}A = I_n$
- If A is non – Singular then
  (i) $|A^{-1}| = \dfrac{1}{|A|}$
  (ii) $(A^T)^{-1} = (A^{-1})^T$
  (iii) $(\lambda A^{-1}) = \dfrac{1}{\lambda} A^{-1}$ where λ is a non – zero scalar.

Reversal law for inverses :

- $(AB)^{-1} = B^{-1}A^{-1}$ where A, B are non – singular matrices of same order.

Law of double inverse :

- If A is non - singular, A^{-1} is also non – singular and $(A^{-1})^{-1} = A$.
- If A is a non - singular square matrix of order n, then
  (i) $(\text{adj } A)^{-1} = \text{adj } (A^{-1}) = \dfrac{1}{|A|} \cdot A$
  (ii) $|\text{adj } A| = |A|^{n-1}$
  (iii) $\text{adj } (\text{adj } A) = |A|^{n-2} A$
  (iv) $\text{adj } (\lambda A) = \lambda^{n-1} \text{adj } (A)$ where λ is a non – zero scalar
  (v) $|\text{adj } (\text{adj } A)| = |A|^{(n-1)^2}$
  (vi) $(\text{adj } A)^T = \text{adj } (A^T)$
- If a matrix contains at least one non – zero element, then $\rho(a) \geq 1$.
- The rank of identity matrix I_n is n.
  If A is an $m \times n$ matrix then $\rho(A) \leq \min\{m, n\}$.
- A square matrix A of order n is invertible if and only if $\rho(A) = n$.
- Transforming a non-singular matrix A to the form I_n, by applying row operations is called Gauss – Jordan method.

Matrix – Inversion method :

- The solution for $AX = B$ is $X = A^{-1}B$ where A and B are square matrices of same order and non – singular

Cramer's Rule :

- If $\Delta = 0$, Cramer's rule cannot be applied $x_1 = \dfrac{\Delta_1}{\Delta}, x_2 = \dfrac{\Delta_2}{\Delta}, x_3 = \dfrac{\Delta_3}{\Delta}$

Gaussian Elimination method :

Transform the augmented matrix of the system of linear equations into row – echelon form and then solve by back substitution method.

Rouches capelli Theorem :

A system of equations AX = B is consistent if and if $\rho(A) = \rho([A|B])$

(i) If $\rho(A) = \rho([A|B]) = n$, the number of unknowns, then the system is consistent and has a unique solution.

(ii) If $\rho(A) = \rho([A|B]) = = n - k$, $k \neq 0$ their the system is consistent and has infinitely many solutions.

(iii) If $\rho(A) \neq \rho([A|B])$, then the system is inconsistent and has no solution.

Homogeneous system of linear equations :

(i) If $\rho(A) = \rho([A|B]) = n$, their the system has a unique solution which is the trivial solution for trivial solution, $|A| \neq 0$

(ii) If $\rho(A) = \rho([A|O]) < n$, the system has a non – trivial solution.

For non – trivial solution, $|A| = 0$.

✦ $A^{-1} = \pm \dfrac{1}{\sqrt{|adj\ A|}} \cdot \text{adj } A$

✦ $A = \pm \dfrac{1}{\sqrt{|adj\ A|}} \cdot \text{adj (adj } A)$

EXERCISE 1.1

1. Find the adjoint of the following :

(i) $\begin{bmatrix} -3 & 4 \\ 6 & 2 \end{bmatrix}$ **(ii)** $\begin{bmatrix} 2 & 3 & 1 \\ 3 & 4 & 1 \\ 3 & 7 & 2 \end{bmatrix}$

(iii) $\dfrac{1}{3}\begin{bmatrix} 2 & 2 & 1 \\ -2 & 1 & 2 \\ 1 & -2 & 2 \end{bmatrix}$

Sol. **(i)** $\begin{bmatrix} -3 & 4 \\ 6 & 2 \end{bmatrix}$

$$\text{Let } A = \begin{pmatrix} -3 & 4 \\ 6 & 2 \end{pmatrix}$$

$$\text{adj } A = \begin{pmatrix} 2 & -4 \\ -6 & -3 \end{pmatrix}$$

[Interchange the elements in the leading diagonal and change the sign of the elements in off diagonal]

(ii) $\begin{bmatrix} 2 & 3 & 1 \\ 3 & 4 & 1 \\ 3 & 7 & 2 \end{bmatrix}$

$$\text{Let } A = \begin{pmatrix} 2 & 3 & 1 \\ 3 & 4 & 1 \\ 3 & 7 & 2 \end{pmatrix}$$

$$\text{adj } A = \begin{pmatrix} +\begin{vmatrix} 4 & 1 \\ 7 & 2 \end{vmatrix} & -\begin{vmatrix} 3 & 1 \\ 3 & 2 \end{vmatrix} & +\begin{vmatrix} 3 & 4 \\ 3 & 7 \end{vmatrix} \\ -\begin{vmatrix} 3 & 1 \\ 7 & 2 \end{vmatrix} & +\begin{vmatrix} 2 & 1 \\ 3 & 2 \end{vmatrix} & -\begin{vmatrix} 2 & 3 \\ 3 & 7 \end{vmatrix} \\ +\begin{vmatrix} 3 & 1 \\ 4 & 1 \end{vmatrix} & -\begin{vmatrix} 2 & 1 \\ 3 & 1 \end{vmatrix} & +\begin{vmatrix} 2 & 3 \\ 3 & 4 \end{vmatrix} \end{pmatrix}$$

$$= \begin{bmatrix} +(8-7) & -(6-3) & +(21-12) \\ -(6-7) & +(4-3) & -(14-9) \\ +(3-4) & -(2-3) & +(8-9) \end{bmatrix}^T$$

$$= \begin{bmatrix} 1 & -3 & 9 \\ 1 & 1 & -5 \\ -1 & 1 & -1 \end{bmatrix}^T$$

$$\text{adj } A = \begin{bmatrix} 1 & 1 & -1 \\ -3 & 1 & 1 \\ 9 & -5 & -1 \end{bmatrix}$$

(iii) $\dfrac{1}{3}\begin{bmatrix} 2 & 2 & 1 \\ -2 & 1 & 2 \\ 1 & -2 & 2 \end{bmatrix}$

$$\text{Let } A = \begin{bmatrix} 2 & 2 & 1 \\ -2 & 1 & 2 \\ 1 & -2 & 2 \end{bmatrix} \text{ and } \lambda = \frac{1}{3}$$

Since $\text{adj } (\lambda A) = \lambda^{n-1}(\text{adj } A)$

$$\text{we get adj}\left(\frac{1}{3}\begin{bmatrix} 2 & 2 & 1 \\ -2 & 1 & 2 \\ 1 & -2 & 2 \end{bmatrix}\right) = \left(\frac{1}{3}\right)^2$$

$$\text{adj}\begin{pmatrix} 2 & 2 & 1 \\ -2 & 1 & 2 \\ 1 & -2 & 1 \end{pmatrix}$$

$\therefore$ Required adjoint matrix

$$= \frac{1}{9}\begin{bmatrix} +\begin{vmatrix} 1 & 2 \\ -2 & 2 \end{vmatrix} & -\begin{vmatrix} -2 & 2 \\ 1 & 2 \end{vmatrix} & +\begin{vmatrix} -2 & 1 \\ 1 & -2 \end{vmatrix} \\ -\begin{vmatrix} 2 & 1 \\ -2 & 2 \end{vmatrix} & +\begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} & -\begin{vmatrix} 2 & 2 \\ 1 & -2 \end{vmatrix} \\ +\begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} & -\begin{vmatrix} 2 & 1 \\ -2 & 2 \end{vmatrix} & +\begin{vmatrix} 2 & 2 \\ -2 & 1 \end{vmatrix} \end{bmatrix}^T$$

$$= \frac{1}{9}\begin{bmatrix} (2+4) & -(-4-2) & +(4-1) \\ -(4+2) & +(4-1) & +(-4-2) \\ +(4-1) & -(4+2) & +(2+4) \end{bmatrix}^T$$

$$= \frac{1}{9}\begin{bmatrix} 6 & 6 & 3 \\ -6 & 3 & 6 \\ 3 & -6 & 6 \end{bmatrix}^T = \frac{1}{9}\begin{bmatrix} 6 & -6 & 3 \\ 6 & 3 & -6 \\ 3 & 6 & 6 \end{bmatrix}$$

$$= \frac{3}{9}\begin{bmatrix} 2 & -2 & 1 \\ 2 & 1 & -2 \\ 1 & 2 & 2 \end{bmatrix}$$

[Taking 3 common from each entry]

$$= \frac{1}{3}\begin{bmatrix} 2 & -2 & 1 \\ 2 & 1 & -2 \\ 1 & 2 & 2 \end{bmatrix}$$

2. Find the inverse (if it exists) of the following

(i) $\begin{bmatrix} -2 & 4 \\ 1 & -3 \end{bmatrix}$ **(ii)** $\begin{bmatrix} 5 & 1 & 1 \\ 1 & 5 & 1 \\ 1 & 1 & 5 \end{bmatrix}$

(iii) $\begin{bmatrix} 2 & 3 & 1 \\ 3 & 4 & 1 \\ 3 & 7 & 2 \end{bmatrix}$

Sol. **(i)** $\begin{bmatrix} -2 & 4 \\ 1 & -3 \end{bmatrix}$

$$\text{Let } A = \begin{bmatrix} -2 & 4 \\ 1 & -3 \end{bmatrix}$$

$$|A| = \begin{vmatrix} -2 & 4 \\ 1 & -3 \end{vmatrix} = 6 - 4 = 2 \neq 0$$

Since A is non – singular, A^{-1} exists

$$A^{-1} = \frac{1}{|A|} \text{adj } A$$

$$\text{Now, adj } A = \begin{bmatrix} -3 & -4 \\ -1 & -2 \end{bmatrix}$$

[Inter change the entries in leading diagonal and change the sign of elements in the off diagonal]

$$\therefore A^{-1} = \frac{1}{2}\begin{bmatrix} -3 & -4 \\ -1 & -2 \end{bmatrix}$$

(ii) $\begin{bmatrix} 5 & 1 & 1 \\ 1 & 5 & 1 \\ 1 & 1 & 5 \end{bmatrix}$

$$\text{Let } A = \begin{bmatrix} 5 & 1 & 1 \\ 1 & 5 & 1 \\ 1 & 1 & 5 \end{bmatrix}$$

Expending along R_1,

$$|A| = 5\begin{vmatrix} 5 & 1 \\ 1 & 5 \end{vmatrix} - 1\begin{vmatrix} 1 & 1 \\ 1 & 5 \end{vmatrix} + 1\begin{vmatrix} 1 & 5 \\ 1 & 1 \end{vmatrix}$$

$$= 5(25-1) - 1(5-1) + 1(1-5)$$

$$= 5(24) - 1(4) + 1(-4)$$

$$= 120 - 4 - 4 = 120 - 8 = 112 \neq 0$$

Since A is non singular, A^{-1} exists.

$$\text{adj } A = \begin{bmatrix} +\begin{vmatrix} 5 & 1 \\ 1 & 5 \end{vmatrix} & -\begin{vmatrix} 1 & 1 \\ 1 & 5 \end{vmatrix} & +\begin{vmatrix} 1 & 5 \\ 1 & 1 \end{vmatrix} \\ -\begin{vmatrix} 1 & 1 \\ 1 & 5 \end{vmatrix} & +\begin{vmatrix} 5 & 1 \\ 1 & 5 \end{vmatrix} & -\begin{vmatrix} 5 & 1 \\ 1 & 1 \end{vmatrix} \\ +\begin{vmatrix} 1 & 1 \\ 5 & 1 \end{vmatrix} & -\begin{vmatrix} 5 & 1 \\ 1 & 1 \end{vmatrix} & +\begin{vmatrix} 5 & 1 \\ 1 & 5 \end{vmatrix} \end{bmatrix}^T$$

$$= \begin{bmatrix} +(25-1) & -(5-1) & +(1-5) \\ -(5-1) & +(25-1) & -(5-1) \\ +(1-5) & +(5-1) & +(25-1) \end{bmatrix}^T$$

$$= \begin{bmatrix} 24 & -4 & -4 \\ -4 & 24 & -4 \\ -4 & -4 & 24 \end{bmatrix}^T = \begin{bmatrix} 24 & -4 & -4 \\ -4 & 24 & -4 \\ -4 & -4 & 24 \end{bmatrix}$$

Taking 4 common from every entry we get,

$$\text{adj } A = 4\begin{bmatrix} 6 & -1 & -1 \\ -1 & 6 & -1 \\ -1 & -1 & 6 \end{bmatrix}$$

$$\therefore A^{-1} = \frac{1}{|A|}\text{adj } A = \frac{1}{112}\cdot 4\begin{bmatrix} 6 & -1 & -1 \\ -1 & 6 & -1 \\ -1 & -1 & 6 \end{bmatrix}$$

$$= \frac{1}{28}\begin{bmatrix} 6 & -1 & -1 \\ -1 & 6 & -1 \\ -1 & -1 & 6 \end{bmatrix}$$

(iii) $\begin{bmatrix} \mathbf{2} & \mathbf{3} & \mathbf{1} \\ \mathbf{3} & \mathbf{4} & \mathbf{1} \\ \mathbf{3} & \mathbf{7} & \mathbf{2} \end{bmatrix}$

$$\text{Let } A = \begin{bmatrix} 2 & 3 & 1 \\ 3 & 4 & 1 \\ 3 & 7 & 2 \end{bmatrix}$$

Expanding along R_1 we get,

$$|A| = 2\begin{vmatrix} 4 & 1 \\ 7 & 2 \end{vmatrix} - 3\begin{vmatrix} 3 & 1 \\ 3 & 2 \end{vmatrix} + 1\begin{vmatrix} 3 & 4 \\ 3 & 7 \end{vmatrix}$$

$$= 2(8-7) - 3(6-3) + 1(21-12)$$

$$= 2(1) - 3(3) + 1(9)$$

$$= 2 - \cancel{9} + \cancel{9} = 2 \neq 0$$

Since A is a non-singular matrix, A^{-1} exists

$$\text{adj } A = \begin{bmatrix} +\begin{vmatrix} 4 & 1 \\ 7 & 2 \end{vmatrix} & -\begin{vmatrix} 3 & 1 \\ 3 & 2 \end{vmatrix} & +\begin{vmatrix} 3 & 4 \\ 3 & 7 \end{vmatrix} \\ -\begin{vmatrix} 3 & 1 \\ 7 & 2 \end{vmatrix} & +\begin{vmatrix} 2 & 1 \\ 3 & 2 \end{vmatrix} & -\begin{vmatrix} 2 & 3 \\ 3 & 7 \end{vmatrix} \\ +\begin{vmatrix} 3 & 1 \\ 4 & 1 \end{vmatrix} & -\begin{vmatrix} 2 & 1 \\ 3 & 1 \end{vmatrix} & +\begin{vmatrix} 2 & 3 \\ 3 & 4 \end{vmatrix} \end{bmatrix}^T$$

$$= \begin{bmatrix} +(8-7) - (6-3) + (21-12) \\ -(6-7) + (4-3) + (14-9) \\ +(3-4) + (2-3) + (8-9) \end{bmatrix}^T$$

$$= \begin{bmatrix} 1 & -3 & 9 \\ 1 & 1 & -5 \\ -1 & 1 & -1 \end{bmatrix}^T$$

$$\text{adj } A = \begin{bmatrix} 1 & 1 & -1 \\ -3 & 1 & 1 \\ 9 & -5 & -1 \end{bmatrix}$$

$$\text{Now, } A^{-1} = \frac{1}{|A|}\text{adj } A$$

$$\Rightarrow A^{-1} = \frac{1}{2}\begin{bmatrix} 1 & 1 & -1 \\ -3 & 1 & 1 \\ 9 & -5 & -1 \end{bmatrix}$$

3. If $\mathbf{F(\alpha) = \begin{bmatrix} \cos\alpha & 0 & \sin\alpha \\ 0 & 1 & 0 \\ -\sin\alpha & 0 & \cos\alpha \end{bmatrix}}$

Show that $\mathbf{= F(\alpha)^{-1} = F(-\alpha)}$

Sol. Given $F(\alpha) = \begin{bmatrix} \cos\alpha & 0 & \sin\alpha \\ 0 & 1 & 0 \\ -\sin\alpha & 0 & \cos\alpha \end{bmatrix}$.

Expanding along R_1 we get,

$$|F(\alpha)| = \cos\alpha\begin{vmatrix} 1 & 0 \\ 0 & \cos\alpha \end{vmatrix} - 0 + \sin\alpha\begin{vmatrix} 0 & 1 \\ -\sin\alpha & 0 \end{vmatrix}$$

$$= \cos\alpha(\cos - 0) + \sin\alpha(0 + \sin\alpha)$$

$$= \cos^2 + \sin^2\alpha = 1 \neq 0$$

Since $F(\alpha)$ is a non-singular matrix, $[F(\alpha)]^{-1}$ exists.

Now, adj $(F(\alpha)) =$

$$\begin{bmatrix} +\begin{vmatrix} 1 & 0 \\ 0 & \cos\alpha \end{vmatrix} & -\begin{vmatrix} 0 & 0 \\ \sin\alpha & \cos\alpha \end{vmatrix} & +\begin{vmatrix} 0 & 1 \\ -\sin\alpha & 0 \end{vmatrix} \\ -\begin{vmatrix} 0 & \sin\alpha \\ 0 & \cos\alpha \end{vmatrix} & +\begin{vmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{vmatrix} & -\begin{vmatrix} \cos\alpha & 0 \\ \sin\alpha & 0 \end{vmatrix} \\ +\begin{vmatrix} 0 & \sin\alpha \\ 1 & 0 \end{vmatrix} & -\begin{vmatrix} \cos\alpha & \sin\alpha \\ 0 & 0 \end{vmatrix} & +\begin{vmatrix} \cos\alpha & 0 \\ 0 & 1 \end{vmatrix} \end{bmatrix}^T$$

$$= \begin{bmatrix} +(\cos\alpha - 0) & -(0) & +(0 + \sin\alpha) \\ -(0) & +(\cos^2\alpha + \sin^2\alpha) & -(0) \\ +(0 - \sin\alpha) & -(0) & +(\cos - 0) \end{bmatrix}^T$$

$$= \begin{bmatrix} \cos\alpha & 0 & +\sin\alpha \\ 0 & 1 & 0 \\ -\sin\alpha & 0 & \cos\alpha \end{bmatrix}^T = \begin{bmatrix} \cos\alpha & 0 & -\sin\alpha \\ 0 & 1 & 0 \\ +\sin\alpha & 0 & \cos\alpha \end{bmatrix}$$

$$\therefore F(\alpha)^{-1} = \frac{1}{|F(\alpha)|}\text{adj } (F(\alpha))$$

$$[F(\alpha)]^{-1} = \frac{1}{1}\begin{bmatrix}\cos\alpha & 0 & -\sin\alpha\\ 0 & 1 & 0\\ +\sin\alpha & 0 & \cos\alpha\end{bmatrix}$$

$$= \begin{bmatrix}\cos\alpha & 0 & -\sin\alpha\\ 0 & 1 & 0\\ +\sin\alpha & 0 & \cos\alpha\end{bmatrix} \quad ...(1)$$

$$\text{Now, } F(-\alpha) = \begin{bmatrix}\cos(-\alpha) & 0 & \sin(-\alpha)\\ 0 & 1 & 0\\ -\sin(-\alpha) & 0 & \cos(-\alpha)\end{bmatrix}$$

$$= \begin{bmatrix}\cos\alpha & 0 & -\sin\alpha\\ 0 & 1 & 0\\ \sin\alpha & 0 & \cos\alpha\end{bmatrix}$$

[$\because$ $\cos\alpha$ is an even function, $\cos(-\alpha) = \cos\alpha$ and $\sin\alpha$ is an odd function, $\sin(-\alpha) = -\sin\alpha$]

From (1) and (2)

$$[F(\alpha)]^{-1} = F(-\alpha)$$

Hence proved.

4. If $A = \begin{bmatrix}5 & 3\\ -1 & -2\end{bmatrix}$, show that $A^2 - 3A - 7I_2 = 0_2$. Hence find A^{-1}.

Sol. Given $A = \begin{bmatrix}5 & 3\\ -1 & -2\end{bmatrix}$

$$A^2 = \begin{bmatrix}5 & 3\\ -1 & -2\end{bmatrix}\begin{bmatrix}5 & 3\\ -1 & -2\end{bmatrix} = \begin{bmatrix}25-3 & 15-6\\ -5+2 & -3+4\end{bmatrix}$$

$$= \begin{bmatrix}22 & 9\\ -3 & 1\end{bmatrix}$$

$\therefore A^2 - 3A - 7I_2$

$$= \begin{bmatrix}22 & 9\\ -3 & 1\end{bmatrix} - 3\begin{bmatrix}5 & 3\\ -1 & -2\end{bmatrix} - 7\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$$

$$= \begin{bmatrix}22-15-7 & 9-9+0\\ -3+3+0 & 1+6-7\end{bmatrix} = \begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix} = 0_2$$

Hence proved.

$\therefore A^2 - 3A - 7I_2 = 0$

Post – multiplying by A^{-1} we get,

$A^2.A^{-1} - 3AA^{-1} - 7I_2A^{-1} = 0.A^{-1}$

$\Rightarrow A(AA^{-1}) - 3(AA^{-1}) - 7(A^{-1}) = 0$

[$\because I_2A^{-1} = A^{-1}$ and $|(0)A^{-1} = 0$]

$\Rightarrow AI - 3I - 7A^{-1} = 0$ [$\because AA^{-1} = I$]

$\Rightarrow AI - 3I = 7A^{-1}$

$\Rightarrow A^{-1} = \frac{1}{7}[A - 3I]$ [$\therefore AI = A$]

$$\Rightarrow A^{-1} = \frac{1}{7} = \left[\begin{bmatrix}5 & 3\\ -1 & -2\end{bmatrix} - 3\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\right]$$

$$\Rightarrow A^{-1} = \frac{1}{7}\begin{bmatrix}5-3 & 3-0\\ -1-0 & -2-3\end{bmatrix} = \frac{1}{7}\begin{bmatrix}2 & 3\\ -1 & -5\end{bmatrix}$$

$$\therefore A^{-1} = \frac{1}{7}\begin{bmatrix}2 & 3\\ -1 & -5\end{bmatrix}$$

5. If $A = \frac{1}{9}\begin{bmatrix}-8 & 1 & 4\\ 4 & 4 & 7\\ 1 & -8 & 4\end{bmatrix}$, prove that $A^{-1} = A^T$.

Sol. Given $A = \frac{1}{9}\begin{bmatrix}-8 & 1 & 4\\ 4 & 4 & 7\\ 1 & -8 & 4\end{bmatrix}$

$$A^T = \frac{1}{9}\begin{bmatrix}-8 & 4 & 1\\ 1 & 4 & -8\\ 4 & 7 & 4\end{bmatrix} \quad ...(1)$$

We know that $(\lambda A)^{-1} = \frac{1}{\lambda}.A^{-1}$

$$A^{-1} = \left\{\frac{1}{9}\begin{bmatrix}-8 & 1 & 4\\ 4 & 4 & 7\\ 1 & -8 & 4\end{bmatrix}\right\}^{-1} = \frac{1}{\frac{1}{9}}\cdot\begin{bmatrix}-8 & 1 & 4\\ 4 & 4 & 7\\ 1 & -8 & 4\end{bmatrix}^T$$

where $\lambda = \frac{1}{9}$

$$A^{-1} = 9B^{-1} \text{ where } B = \begin{bmatrix}-8 & 1 & 4\\ 4 & 4 & 7\\ 1 & -8 & 4\end{bmatrix} \quad ...(2)$$

$$\text{Now, } |B| = -8\begin{vmatrix}4 & 7\\ -8 & 4\end{vmatrix} - 1\begin{vmatrix}4 & 7\\ 1 & 4\end{vmatrix} + 4\begin{vmatrix}4 & 4\\ 1 & -8\end{vmatrix}$$

$= -8(16 + 56) - 1(16 - 7) + 4(-32 - 4)$

$= -8(72) - 1(9) + 4(-36) = -576 - 9 - 144$

$= -729$

$$\text{adj } B = \begin{bmatrix}+\begin{vmatrix}4 & 7\\ -8 & 4\end{vmatrix} & -\begin{vmatrix}4 & 7\\ 1 & 4\end{vmatrix} & +\begin{vmatrix}4 & 4\\ 1 & -8\end{vmatrix}\\ -\begin{vmatrix}1 & 4\\ -8 & 4\end{vmatrix} & +\begin{vmatrix}-8 & 4\\ 1 & 4\end{vmatrix} & -\begin{vmatrix}-8 & 1\\ 1 & -8\end{vmatrix}\\ +\begin{vmatrix}1 & 4\\ 4 & 7\end{vmatrix} & -\begin{vmatrix}-8 & 4\\ 4 & 7\end{vmatrix} & +\begin{vmatrix}-8 & 1\\ 4 & 1\end{vmatrix}\end{bmatrix}^T$$

$$= \begin{bmatrix} +(16+56)-(16-7)+(-32-4) \\ -(4+32)+(-32-4)+(64-1) \\ +(7-16)-(-56-16)+(-32-4) \end{bmatrix}$$

$$= \begin{bmatrix} 72 & -9 & -36 \\ -36 & -36 & -63 \\ -9 & 72 & -36 \end{bmatrix}^T = \begin{bmatrix} 72 & -36 & -9 \\ -9 & -36 & 72 \\ -36 & -63 & -36 \end{bmatrix}$$

$$= 9\begin{bmatrix} 8 & -4 & -1 \\ -1 & -4 & +8 \\ -4 & -7 & -4 \end{bmatrix}$$

$$\therefore B^{-1} = \frac{1}{|B|} \text{adj } B = \frac{-9}{729}\begin{bmatrix} 8 & -4 & -1 \\ -1 & -4 & +8 \\ -4 & -7 & -4 \end{bmatrix}$$

$$= \frac{1}{81}\begin{bmatrix} -8 & 4 & 1 \\ 1 & 4 & -8 \\ 4 & 7 & 4 \end{bmatrix}$$

Substituting this in (2) we get,

$$A^{-1} = 9 \cdot \frac{1}{81}\begin{bmatrix} -8 & 4 & 1 \\ 1 & 4 & -8 \\ 4 & 7 & 4 \end{bmatrix} = \frac{1}{9}\begin{bmatrix} -8 & 4 & 1 \\ 1 & 4 & -8 \\ 4 & 7 & 4 \end{bmatrix} \quad ...(3)$$

From (1) and (3) we get,

$A^T = A^{-1}$

6. If $A = \begin{bmatrix} 8 & -4 \\ -5 & 3 \end{bmatrix}$, verify that $A(\text{adj } A) = (\text{adj } A)A = |A|\, I_2$.

Sol. Given $A = \begin{bmatrix} 8 & -4 \\ -5 & 3 \end{bmatrix}$

$$\text{adj } A = \begin{bmatrix} 3 & 4 \\ 5 & 8 \end{bmatrix}$$

[Interchange the elements in the leading diagonal and change the sign of the elements in the off diagonal]

$$|A| = 24 - 20 = 4$$

$$\therefore A(\text{adj } A) = \begin{bmatrix} 8 & -4 \\ -5 & 3 \end{bmatrix}\begin{bmatrix} 3 & 4 \\ 5 & 8 \end{bmatrix}$$

$$= \begin{bmatrix} 24-20 & 32-32 \\ -15+15 & -20+24 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} \quad ...(1)$$

$$(\text{adj } A)(A) = \begin{bmatrix} 3 & 4 \\ 5 & 8 \end{bmatrix}\begin{bmatrix} 8 & -4 \\ -5 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 24-20 & -12+12 \\ 40-40 & -20+24 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} \quad ...(2)$$

$$|A|\, I_2 = 4\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} \quad ...(3)$$

From (1), (2) and (3), it is proved that

$$A(\text{adj } A) = (\text{adj } A)A = |A|\, I_2$$

7. If $A = \begin{bmatrix} 3 & 2 \\ 7 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & -3 \\ 5 & 2 \end{bmatrix}$ verify that $(AB)^{-1} = B^{-1}A^{-1}$.

Sol. Given $A = \begin{bmatrix} 3 & 2 \\ 7 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & -3 \\ 5 & 2 \end{bmatrix}$

$$AB = \begin{bmatrix} 3 & 2 \\ 7 & 5 \end{bmatrix}\begin{bmatrix} -1 & -3 \\ 5 & 2 \end{bmatrix} = \begin{bmatrix} -3+10 & -9+4 \\ -7+25 & -21+10 \end{bmatrix}$$

$$= \begin{bmatrix} 7 & -5 \\ 18 & -11 \end{bmatrix}$$

$$|AB| = -77 + 90 = 13 \neq 0 \Rightarrow (AB)^{-1} \text{ exists}$$

$$|A| = 15 - 14 = 1 \neq 0 \Rightarrow A^{-1} \text{ exists}$$

$$|B| = -2 + 15 = 13 \neq 0 \Rightarrow B^{-1} \text{ exists}$$

$$(AB)^{-1} = \frac{1}{|AB|}\text{adj }(AB) = \frac{1}{13}\begin{pmatrix} -11 & 5 \\ -18 & 7 \end{pmatrix} \quad ...(1)$$

$$B^{-1} = \frac{1}{|B|}(\text{adj } B) = \frac{1}{13}\begin{pmatrix} 2 & 3 \\ -5 & -1 \end{pmatrix}$$

$$A^{-1} = \frac{1}{|A|}(\text{adj } A)$$

$$= \frac{1}{1}\begin{pmatrix} 5 & -2 \\ -7 & 3 \end{pmatrix} = \begin{pmatrix} 5 & -2 \\ -7 & 3 \end{pmatrix}$$

$$\therefore B^{-1}A^{-1} = \frac{1}{13}\begin{pmatrix} 2 & 3 \\ -5 & -1 \end{pmatrix}\begin{pmatrix} 5 & -2 \\ -7 & 3 \end{pmatrix}$$

$$= \frac{1}{13}\begin{pmatrix} 10-21 & -4+9 \\ -25+7 & 10-3 \end{pmatrix}$$

$$= \frac{1}{13}\begin{pmatrix} -11 & 5 \\ -18 & 7 \end{pmatrix} \quad ...(2)$$

From (1) or (2) it is prove that

$(AB)^{-1} = B^{-1}A^{-1}$.

8. If adj (A) = $\begin{bmatrix} 2 & -4 & 2 \\ -3 & 12 & -7 \\ -2 & 0 & 2 \end{bmatrix}$, find A.

Sol. Given adj A = $\begin{bmatrix} 2 & -4 & 2 \\ -3 & 12 & -7 \\ -2 & 0 & 2 \end{bmatrix}$

We know that $A = \pm \frac{1}{\sqrt{|\text{adj } A|}} . \text{adj (adj A)}$ ...(1)

$$|\text{adj } A| = 2\begin{vmatrix} 12 & -7 \\ 0 & 2 \end{vmatrix} + 4\begin{vmatrix} -3 & -7 \\ -2 & 2 \end{vmatrix} + 2\begin{vmatrix} -3 & 12 \\ -2 & 0 \end{vmatrix}$$

[Expanded along R_1]

$= 2(24 - 0) + 4(-6 - 14) + 2(0 + 24)$
$= 2(24) + 4(-20) + 2(24) = 48 - 80 + 48$
$= 96 - 80 = 16$

Now, adj (adj A)

$$= \begin{bmatrix} +\begin{vmatrix} 12 & -7 \\ 0 & 2 \end{vmatrix} & -\begin{vmatrix} -3 & -7 \\ -2 & 2 \end{vmatrix} & +\begin{vmatrix} -3 & 12 \\ -2 & 0 \end{vmatrix} \\ -\begin{vmatrix} -4 & 2 \\ 0 & 2 \end{vmatrix} & +\begin{vmatrix} 2 & 2 \\ -2 & 2 \end{vmatrix} & -\begin{vmatrix} 2 & -4 \\ -2 & 0 \end{vmatrix} \\ +\begin{vmatrix} -4 & 2 \\ 12 & -7 \end{vmatrix} & -\begin{vmatrix} 2 & 2 \\ -3 & -7 \end{vmatrix} & +\begin{vmatrix} 2 & -4 \\ -3 & 12 \end{vmatrix} \end{bmatrix}^T$$

$$= \begin{bmatrix} +(24-0)-(-6-14)+(0+24) \\ -(-8-0)+(4+4)-(0-8) \\ +(28-24)-(-14+6)+(24-12) \end{bmatrix}^T$$

$$= \begin{bmatrix} 24 & 20 & 24 \\ 8 & 8 & 8 \\ 4 & 8 & 12 \end{bmatrix}^T = \begin{bmatrix} 24 & 8 & 4 \\ 20 & 8 & 8 \\ 24 & 8 & 12 \end{bmatrix}$$

$$= 4\begin{bmatrix} 6 & 2 & 1 \\ 5 & 2 & 2 \\ 6 & 2 & 3 \end{bmatrix} \quad ...(3)$$

Substituting (2) and (3) in (1) we get,

$$A = \pm \frac{1}{\sqrt{16}} . 4\begin{bmatrix} 6 & 2 & 1 \\ 5 & 2 & 2 \\ 6 & 2 & 3 \end{bmatrix}$$

$$A = \pm \frac{4}{4}\begin{bmatrix} 6 & 2 & 1 \\ 5 & 2 & 2 \\ 6 & 2 & 3 \end{bmatrix} = \pm \begin{bmatrix} 6 & 2 & 1 \\ 5 & 2 & 2 \\ 6 & 2 & 3 \end{bmatrix}$$

9. If adj(A) = $\begin{bmatrix} 0 & -2 & 0 \\ 6 & 2 & -6 \\ -3 & 0 & 6 \end{bmatrix}$, find A^{-1}.

Sol. Given adj (A) = $\begin{bmatrix} 0 & -2 & 0 \\ 6 & 2 & -6 \\ -3 & 0 & 6 \end{bmatrix}$

We know that $A^{-1} = \pm \frac{1}{\sqrt{|\text{adj} A|}} (\text{adj } A)$ ...(1)

$$|\text{adj } A| = 0 + 2\begin{vmatrix} 6 & -6 \\ -3 & 6 \end{vmatrix} + 0$$

[Expanded along R_1]

$= 2(36 - 18) = 2(18) = 36$

$$\therefore A^{-1} = \pm \frac{1}{\sqrt{36}}\begin{bmatrix} 0 & -2 & 0 \\ 6 & 2 & -6 \\ -3 & 0 & 6 \end{bmatrix}$$

$$= \pm \frac{1}{6}\begin{bmatrix} 0 & -2 & 0 \\ 6 & 2 & -6 \\ -3 & 0 & 6 \end{bmatrix}$$

10. Find adj (adj (A)) if adj A = $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ -1 & 0 & 1 \end{bmatrix}$.

Sol. Given adj A = $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ -1 & 0 & 1 \end{bmatrix}$

$$\text{Now adj(adj A)} = \begin{bmatrix} +\begin{vmatrix} 2 & 0 \\ 0 & 1 \end{vmatrix} & -\begin{vmatrix} 0 & 0 \\ -1 & 1 \end{vmatrix} & +\begin{vmatrix} 0 & 2 \\ -1 & 0 \end{vmatrix} \\ -\begin{vmatrix} 0 & 1 \\ 0 & 1 \end{vmatrix} & +\begin{vmatrix} 1 & 1 \\ -1 & 1 \end{vmatrix} & -\begin{vmatrix} 1 & 0 \\ -1 & 0 \end{vmatrix} \\ +\begin{vmatrix} 0 & 1 \\ 2 & 0 \end{vmatrix} & -\begin{vmatrix} 1 & 1 \\ 0 & 0 \end{vmatrix} & +\begin{vmatrix} 1 & 0 \\ 0 & 2 \end{vmatrix} \end{bmatrix}^T$$

$$= \begin{bmatrix} +(2-0) & -(0) & +(0+2) \\ -(0) & +(1+1) & -(0) \\ +(0-2) & -(0) & +(2-0) \end{bmatrix}^T = \begin{bmatrix} 2 & 0 & 2 \\ 0 & 2 & 0 \\ -2 & 0 & 2 \end{bmatrix}^T$$

$$\text{adj (adj A)} = \begin{bmatrix} 2 & 0 & -2 \\ 0 & 2 & 0 \\ 2 & 0 & 2 \end{bmatrix}$$

11. $A = \begin{bmatrix} 1 & \tan x \\ -\tan x & 1 \end{bmatrix}$, **show that** $A^T A^{-1} = \begin{bmatrix} \cos 2x & -\sin 2x \\ \sin 2x & \cos 2x \end{bmatrix}$.

Sol. Given $A = \begin{bmatrix} 1 & \tan x \\ -\tan x & 1 \end{bmatrix}$

$$|A| = 1 + \tan^2 x$$

$$\therefore A^{-1} = \frac{1}{|A|}\text{adj}A = \frac{1}{1+\tan^2 x}\begin{bmatrix} 1 & -\tan x \\ \tan x & 1 \end{bmatrix}$$

[Interchange the elements in the leading diagonal and change the sign of elements in the off diagonal]

$$A^T = \begin{bmatrix} 1 & -\tan x \\ \tan x & 1 \end{bmatrix}$$

$\therefore A^T A^{-1}$

$$= \begin{bmatrix} 1 & -\tan x \\ \tan x & 1 \end{bmatrix} \cdot \frac{1}{1+\tan^2 x}\begin{bmatrix} 1 & -\tan x \\ \tan x & 1 \end{bmatrix}$$

$$= \frac{1}{1+\tan^2 x}\begin{bmatrix} 1 & -\tan x \\ \tan x & 1 \end{bmatrix}\begin{bmatrix} 1 & -\tan x \\ \tan x & 1 \end{bmatrix}$$

$$= \frac{1}{1+\tan^2 x}\begin{bmatrix} 1-\tan^2 x & -\tan x - \tan x \\ \tan x + \tan x & -\tan^2 x + 1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1-\tan^2 x}{1+\tan^2 x} & \frac{-2\tan x}{1+\tan^2 x} \\ \frac{2\tan x}{1+\tan^2 x} & \frac{1-\tan^2 x}{1+\tan^2 x} \end{bmatrix}$$

$$A^T A^{-1} = \begin{bmatrix} \cos 2x & -\sin 2x \\ \sin 2x & \cos 2x \end{bmatrix}$$

$$\left[\because \sin 2x = \frac{2\tan x}{1+\tan^2 x} \text{ and } \cos 2x = \frac{1-\tan^2 x}{1+\tan^2 x}\right]$$

Hence proved.

12. Find the matrix A for which $A\begin{bmatrix} 5 & 3 \\ -1 & -2 \end{bmatrix} = \begin{bmatrix} 14 & 7 \\ 7 & 7 \end{bmatrix}$.

Sol. Given $A\begin{bmatrix} 5 & 3 \\ -1 & -2 \end{bmatrix} = \begin{bmatrix} 14 & 7 \\ 7 & 7 \end{bmatrix}$

Let $B = \begin{bmatrix} 5 & 3 \\ -1 & -2 \end{bmatrix}$ and

$C = \begin{bmatrix} 14 & 7 \\ 7 & 7 \end{bmatrix}$

$\therefore AB = C$

Post multiply by B^{-1} we get

$A(BB^{-1}) = CB^{-1}$

$\Rightarrow A = CB^{-1}$ $[\because BB^{-1} = I]$

$$|B| = \begin{vmatrix} 5 & 3 \\ -1 & -2 \end{vmatrix} = -10 + 3 = -7 \neq 0$$

$\therefore B^{-1}$ exists

$$B^{-1} = \frac{1}{|B|}\text{ adj } B = \frac{-1}{7}\begin{bmatrix} -2 & -3 \\ 1 & 5 \end{bmatrix}$$

$A = CB^{-1}$

$$= \begin{bmatrix} 14 & 7 \\ 7 & 7 \end{bmatrix} \cdot \left(\frac{-1}{7}\right)\begin{bmatrix} -2 & -3 \\ 1 & 5 \end{bmatrix}$$

$$= 7\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}\left(\frac{-1}{7}\right)\begin{bmatrix} -2 & -3 \\ 1 & 5 \end{bmatrix}$$

$$= -\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} -2 & -3 \\ 1 & 5 \end{bmatrix}$$

$$= -\begin{bmatrix} -4+1 & -6+5 \\ -2+1 & -3+5 \end{bmatrix} = -\begin{bmatrix} -3 & -1 \\ -1 & 2 \end{bmatrix}$$

$$A = \begin{bmatrix} 3 & 1 \\ 1 & -2 \end{bmatrix}$$

13. Given $A = \begin{bmatrix} 1 & -1 \\ 2 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 3 & -2 \\ 1 & 1 \end{bmatrix}$ **and** $C = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$, **find a matrix X such that AXB = C.**

Sol. Given $A = \begin{bmatrix} 1 & -1 \\ 2 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 3 & -2 \\ 1 & 1 \end{bmatrix}$ and $C = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$

Also, $AXB = C$

Premultiply by A^{-1} we get,

$(A^{-1}A)XB = A^{-1}C$

$\Rightarrow XB = A^{-1}.C.$ $[\because A^{-1}A = I]$

Post Multiply by B^{-1} we get

$(XB)B^{-1} = (A^{-1}C)B^{-1}$

$\Rightarrow X = (A^{-1}C)B^{-1}$

$$|A| = \begin{vmatrix} 1 & -1 \\ 2 & 0 \end{vmatrix} = 0 + 2 = 2 \neq 0$$

$$\therefore A^{-1} = \frac{1}{|A|} \text{ adj } A = \frac{1}{2}\begin{bmatrix} 0 & 1 \\ -2 & 1 \end{bmatrix}$$

$$|B| = \begin{vmatrix} 3 & -2 \\ 1 & 1 \end{vmatrix} = 3 + 2 = 5 \neq 0$$

$$\therefore B^{-1} = \frac{1}{|B|} \text{ adj } B = \frac{1}{5}\begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}$$

$$A^{-1} C = \frac{1}{2}\begin{bmatrix} 0 & 1 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 0+2 & 0+2 \\ -2+2 & -2+2 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 2 & 2 \\ 0 & 0 \end{bmatrix} = \frac{1}{2}(2)\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

$$\therefore X = Z\,(A^{-1} C).\, B^{-1}$$

$$= \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}\frac{1}{5}\begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 1-1 & 2+3 \\ 0+0 & 0+0 \end{bmatrix}$$

$$= \frac{1}{5}\begin{bmatrix} 0 & 5 \\ 0 & 0 \end{bmatrix} = \frac{1}{5}(5)\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$\therefore X = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

14. **If $A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$, show that $A^{-1} = \frac{1}{2}(A^2 - 3I)$**

Sol. Given $A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$

$$|A| = 0 - 1\begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} + 1\begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix}$$

$$= -1\,(0-1) + 1\,(1-0) = 1 + 1 = 2$$

$$\text{adj } A = \begin{bmatrix} +\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} & -\begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} & +\begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} \\ -\begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} & +\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} & -\begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix} \\ +\begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} & -\begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix} & +\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} \end{bmatrix}^T$$

$$= \begin{bmatrix} (0-1) & -(0-1) & +(1-0) \\ -(0-1) & +(0-1) & -(0-1) \\ +(1-0) & -(0-1) & +(0-1) \end{bmatrix}^T$$

$$= \begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}^T = \begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$$

$$\therefore A^{-1} = \frac{1}{2}\begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix} \quad ...(1)$$

$$\text{Now } A^2 = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0+1+1 & 0+0+1 & 0+1+0 \\ 0+0+1 & 1+0+1 & 1+0+0 \\ 0+1+0 & 1+0+0 & 1+1+0 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

$$A^2 - 3I = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} - 3\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2-3 & 1-0 & 1-0 \\ 1-0 & 2-3 & 1-0 \\ 1-0 & 1-0 & 2-3 \end{bmatrix} = \begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix} \quad ...(2)$$

From (1) and (2), it is proved that $A^{-1} = \frac{1}{2}[A^2 - 3I]$

15. **Decrypt the received encoded message [2 –3] [20 4] with the encryption matrix $\begin{bmatrix} -1 & -1 \\ 2 & 1 \end{bmatrix}$ and the decryption matrix as its inverse, where the system of codes are described by the numbers 1 – 26 to the letters A – Z respectively, and the number 0 to a blank space.**

Sol. Let the encryption matrix be $A = \begin{bmatrix} -1 & -1 \\ 2 & 1 \end{bmatrix}$

$$|A| = -1 + 2 = 1 \neq 0$$

$$\therefore A^{-1} = \frac{1}{|A|} \text{ adj } A = \frac{1}{1}\begin{bmatrix} 1 & 1 \\ -2 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -2 & -1 \end{bmatrix}$$

Hence the decryption matrix is $\begin{bmatrix} 1 & 1 \\ -2 & -1 \end{bmatrix}$

Coded row matrix	Decoding matrix	Decoded row matrix
$[2 \ \ -3]$	$\begin{bmatrix} 1 & 1 \\ -2 & -1 \end{bmatrix}$	$= [2+6 \quad 2+3] = [8 \quad 5]$
$[20 \ \ 4]$	$\begin{bmatrix} 1 & 1 \\ -2 & -1 \end{bmatrix}$	$= [20-8 \ 20-4] = [12 \ \ 16]$

So, the sequence of decoded row matrices is $[8 \ \ 5]$, $[12 \ \ 16]$

Now the 8^{th} English alphabet is H.

5^{th} English alphabet is E.

12^{th} English alphabet is L.

and the 16^{th} English alphabet is P.

Thus the receiver reads the message as "HELP".

EXERCISE 1.2

1. Find the rank of the following matrices by minor method:

(i) $\begin{bmatrix} 2 & -4 \\ -1 & 2 \end{bmatrix}$ **(ii)** $\begin{bmatrix} -1 & 3 \\ 4 & -7 \\ 3 & -4 \end{bmatrix}$

(iii) $\begin{bmatrix} 1 & -2 & -1 & 0 \\ 3 & -6 & -3 & 1 \end{bmatrix}$

(iv) $\begin{bmatrix} 1 & -2 & 3 \\ 2 & 4 & -6 \\ 5 & 1 & -1 \end{bmatrix}$ **(v)** $\begin{bmatrix} 0 & 1 & 2 & 1 \\ 0 & 2 & 4 & 3 \\ 8 & 1 & 0 & 2 \end{bmatrix}$

Sol. **(i)** $\begin{bmatrix} 2 & -4 \\ -1 & 2 \end{bmatrix}$

$$\text{Let } A = \begin{bmatrix} 2 & -4 \\ -1 & 2 \end{bmatrix}$$

A is a matrix of order 2×2

$\therefore \rho(A) \leq \min(2, 2) = 2$

The highest order of minor of A is 2

It is $\begin{vmatrix} 2 & -4 \\ -1 & 2 \end{vmatrix} = 4 - 4 = 0$

So, $\rho(A) < 2$

Next consider the minor of order 1 $|2| = 2 \neq 0$

$\therefore \rho(A) = 1$

(ii) $\begin{bmatrix} -1 & 3 \\ 4 & -7 \\ 3 & -4 \end{bmatrix}$

$$\text{Let } A = \begin{bmatrix} -1 & 3 \\ 4 & -7 \\ 3 & -4 \end{bmatrix}$$

A is a matrix of order 3×2

$\therefore \rho(A) \leq \min(3, 2) = 2$

The highest order of minor of A is 2

It is $\begin{vmatrix} -1 & 3 \\ 4 & -7 \end{vmatrix} = 7 - 12 = -5 \neq 0$

$\therefore \rho(A) = 2.$

(iii) $\begin{bmatrix} 1 & -2 & -1 & 0 \\ 3 & -6 & -3 & 1 \end{bmatrix}$

$$\text{Let } A = \begin{bmatrix} 1 & -2 & -1 & 0 \\ 3 & -6 & -3 & 1 \end{bmatrix}$$

A is a matrix of order (2×4)

$\therefore \rho(A) \leq \min(2, 4) = 2$

The highest order of minor of A is 2

It is $\begin{vmatrix} 1 & -2 \\ 3 & -6 \end{vmatrix} = -6 + 6 = 0$

Also, $\begin{vmatrix} -1 & 0 \\ -3 & 1 \end{vmatrix} = -1 + 0 = -1 \neq 0.$

$\therefore \rho(A) = 2.$

(iv) $\begin{bmatrix} 1 & -2 & 3 \\ 2 & 4 & -6 \\ 5 & 1 & -1 \end{bmatrix}$

$$\text{Let } A = \begin{bmatrix} 1 & -2 & 3 \\ 2 & 4 & -6 \\ 5 & 1 & -1 \end{bmatrix}$$

A is a matrix of order 3×3

$\therefore \rho(A) \leq \min(3, 3) = 3$

The highest order of minor of A is 3.

It is $\begin{vmatrix} 1 & -2 & 3 \\ 2 & 4 & -6 \\ 5 & 1 & -1 \end{vmatrix} = 1\begin{vmatrix} 4 & -6 \\ 1 & -1 \end{vmatrix} + 2\begin{vmatrix} 2 & -6 \\ 5 & -1 \end{vmatrix} + 3\begin{vmatrix} 2 & 4 \\ 5 & 1 \end{vmatrix}$

[Expanded along R_1]

$= 1(-4+6) + 2(-2+30) + 3(2-20)$

$= 1(2) + 2(28) + 3(-18)$

$= 2 + 56 - 54 = 58 - 54 = 4 \neq 0$

$\therefore \rho(A) = 3.$

(v) $\begin{bmatrix} 0 & 1 & 2 & 1 \\ 0 & 2 & 4 & 3 \\ 8 & 1 & 0 & 2 \end{bmatrix}$

Let A $= \begin{bmatrix} 0 & 1 & 2 & 1 \\ 0 & 2 & 4 & 3 \\ 8 & 1 & 0 & 2 \end{bmatrix}$

A is a matrix of order 3 × 4

$\therefore \rho(A) \leq \min(3, 4) = 3$

The highest order of minor of A is 3

It is $\begin{vmatrix} 0 & 1 & 2 \\ 0 & 2 & 4 \\ 8 & 1 & 0 \end{vmatrix} = 0 + 0 - 8(4-4) = 0$

[Expanded along C_1]

Also, $\begin{vmatrix} 0 & 2 & 1 \\ 0 & 4 & 3 \\ 8 & 0 & 2 \end{vmatrix} = 0 + 0 - 8 \begin{vmatrix} 2 & 1 \\ 4 & 3 \end{vmatrix}$

[Expanded along C_1]

$= -8(6-4) = -8(2) = -16 \neq 0$

$\therefore \rho(A) = 3$

2. Find the rank of the following matrices by row reduction method :

(i) $\begin{bmatrix} 1 & 1 & 1 & 3 \\ 2 & -1 & 3 & 4 \\ 5 & -1 & 7 & 11 \end{bmatrix}$ **(ii)** $\begin{bmatrix} 1 & 2 & -1 \\ 3 & -1 & 2 \\ 1 & -2 & 3 \\ 1 & -1 & 1 \end{bmatrix}$

(iii) $\begin{bmatrix} 3 & -8 & 5 & 2 \\ 2 & -5 & 1 & 4 \\ -1 & 2 & 3 & -2 \end{bmatrix}$

Sol. **(i)** $\begin{bmatrix} 1 & 1 & 1 & 3 \\ 2 & -1 & 3 & 4 \\ 5 & -1 & 7 & 11 \end{bmatrix}$

Let A $= \begin{bmatrix} 1 & 1 & 1 & 3 \\ 2 & -1 & 3 & 4 \\ 5 & -1 & 7 & 11 \end{bmatrix}$

$A = \begin{bmatrix} 1 & 1 & 1 & 3 \\ 2 & -1 & 3 & 4 \\ 5 & -1 & 7 & 11 \end{bmatrix} \xrightarrow{R_2 \to R_2 - 2R_1} \begin{bmatrix} 1 & 1 & 1 & 3 \\ 0 & -3 & 1 & -2 \\ 5 & -1 & 7 & 11 \end{bmatrix}$

$\xrightarrow{R_3 \to R_3 - 5R_1} \begin{bmatrix} 1 & 1 & 1 & 3 \\ 0 & -3 & 1 & -2 \\ 0 & -6 & 2 & -4 \end{bmatrix}$

$\xrightarrow{R_3 \to R_3 - 2R_2} \begin{bmatrix} 1 & 1 & 1 & 3 \\ 0 & -3 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

The last equivalent matrix is in row echelon form it has two non-zero rows

$\therefore \rho(A) = 2.$

(ii) $\begin{bmatrix} 1 & 2 & -1 \\ 3 & -1 & 2 \\ 1 & -2 & 3 \\ 1 & -1 & 1 \end{bmatrix}$

Let A $= \begin{bmatrix} 1 & 2 & -1 \\ 3 & -1 & 2 \\ 1 & -2 & 3 \\ 1 & -1 & 1 \end{bmatrix}$

$A = \begin{bmatrix} 1 & 2 & -1 \\ 3 & -1 & 2 \\ 1 & -2 & 3 \\ 1 & -1 & 1 \end{bmatrix} \xrightarrow[R_4 \to R_4 - R_1]{R_2 \to R_2 - 3R_1,\ R_3 \to R_3 - R_1} \begin{bmatrix} 1 & 2 & -1 \\ 0 & -7 & 5 \\ 0 & -4 & 4 \\ 0 & -3 & 2 \end{bmatrix}$

$\xrightarrow{R_3 \to R_3 \div 4} \begin{bmatrix} 1 & 2 & -1 \\ 0 & -7 & 5 \\ 0 & -1 & 1 \\ 0 & -3 & 2 \end{bmatrix}$

$\xrightarrow{R_4 \to R_4 - 3R_3} \begin{bmatrix} 1 & 2 & -1 \\ 0 & -7 & 5 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix}$

$\xrightarrow{R_3 \to 7R_3 - R_2} \begin{bmatrix} 1 & 2 & -1 \\ 0 & -7 & 5 \\ 0 & 0 & 2 \\ 0 & 0 & -1 \end{bmatrix}$

$\xrightarrow{R_4 \to 2R_4 - R_3} \begin{bmatrix} 1 & 2 & -1 \\ 0 & -7 & 5 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$

The last equivalent matrix is in row echelon form It has three non-zero rows.

$\therefore \rho(A) = 3$

(iii) $\begin{bmatrix} 3 & -8 & 5 & 2 \\ 2 & -5 & 1 & 4 \\ -1 & 2 & 3 & -2 \end{bmatrix}$

Let A $= \begin{bmatrix} 3 & -8 & 5 & 2 \\ 2 & -5 & 1 & 4 \\ -1 & 2 & 3 & -2 \end{bmatrix}$

$A \xrightarrow{R_3 \leftrightarrow R_1} \begin{bmatrix} -1 & 2 & 3 & -2 \\ 2 & -5 & 1 & 4 \\ 3 & -8 & 5 & 2 \end{bmatrix}$

$$\xrightarrow[R_3\to R_3+3R_1]{R_2\to R_2+2R_1}\begin{bmatrix}-1 & 2 & 3 & -2\\ 0 & -1 & 7 & 0\\ 0 & -2 & 14 & -4\end{bmatrix}$$

$$\xrightarrow{R_3\to R_3\div 2}\begin{bmatrix}-1 & 2 & 3 & -2\\ 0 & -1 & 7 & 0\\ 0 & -1 & 7 & -2\end{bmatrix}$$

$$\xrightarrow{R_3\to R_3-R_2}\begin{bmatrix}-1 & 2 & 3 & -2\\ 0 & -1 & 7 & 0\\ 0 & 0 & 0 & -2\end{bmatrix}$$

The last equivalent matrix is in row-echelon form. It has three non-zero rows.

$\therefore\ \rho(A) = 3$

3. Find the inverse of each of the following by Gauss – Jordan method :

(i) $\begin{bmatrix}2 & -1\\ 5 & -2\end{bmatrix}$ **(ii)** $\begin{bmatrix}1 & -1 & 0\\ 1 & 0 & -1\\ 6 & -2 & -3\end{bmatrix}$

(iii) $\begin{bmatrix}1 & 2 & 3\\ 2 & 5 & 3\\ 1 & 0 & 8\end{bmatrix}$

Sol. **(i)** $\begin{bmatrix}2 & -1\\ 5 & -2\end{bmatrix}$

Let $A = \begin{bmatrix}2 & -1\\ 5 & -2\end{bmatrix}$

Applying Gauss – Jordan method, we get

$$[A|I_2] = \left[\begin{array}{cc|cc}2 & -1 & 1 & 0\\ 5 & -2 & 0 & 1\end{array}\right]$$

$$\xrightarrow{R_1\to R_1\div 2}\left[\begin{array}{cc|cc}1 & -\frac{1}{2} & \frac{1}{2} & 0\\ 5 & -2 & 0 & 1\end{array}\right]$$

$$\xrightarrow{R_2\to R_2-5R_1}\left[\begin{array}{cc|cc}1 & -\frac{1}{2} & \frac{1}{2} & 0\\ 0 & \frac{1}{2} & -\frac{5}{2} & 1\end{array}\right]$$

$$\xrightarrow{R_2\to R_2\times 2}\left[\begin{array}{cc|cc}1 & -\frac{1}{2} & \frac{1}{2} & 0\\ 0 & 1 & -5 & 2\end{array}\right]$$

$$\xrightarrow{R_1\to R_1+\frac{1}{2}R_2}\left[\begin{array}{cc|cc}1 & 0 & -2 & 1\\ 0 & 1 & -5 & 2\end{array}\right]$$

$\therefore$ We get $A^{-1} = \begin{bmatrix}-2 & 1\\ -5 & 2\end{bmatrix}$

(ii) $\begin{bmatrix}1 & -1 & 0\\ 1 & 0 & -1\\ 6 & -2 & -3\end{bmatrix}$

Let $A = \begin{bmatrix}1 & -1 & 0\\ 1 & 0 & -1\\ 6 & -2 & -3\end{bmatrix}$

Applying Gauss – Jordan method, we get

$$[A|I_3] = \left[\begin{array}{ccc|ccc}1 & -1 & 0 & 1 & 0 & 0\\ 1 & 0 & -1 & 0 & 1 & 0\\ 6 & -2 & -3 & 0 & 0 & 1\end{array}\right]$$

$$\xrightarrow{R_2\to R_2-R_1}\left[\begin{array}{ccc|ccc}1 & -1 & 0 & 1 & 0 & 0\\ 0 & 1 & -1 & -1 & 1 & 0\\ 6 & -2 & -3 & 0 & 0 & 1\end{array}\right]$$

$$\xrightarrow{R_3\to R_3-6R_1}\left[\begin{array}{ccc|ccc}1 & -1 & 0 & 1 & 0 & 0\\ 0 & 1 & -1 & -1 & 1 & 0\\ 0 & 4 & -3 & -6 & 0 & 1\end{array}\right]$$

$$\xrightarrow{R_3\to R_3-4R_2}\left[\begin{array}{ccc|ccc}1 & -1 & 0 & 1 & 0 & 0\\ 0 & 1 & -1 & -1 & 1 & 0\\ 0 & 0 & 1 & -2 & -4 & 1\end{array}\right]$$

$$\xrightarrow{R_1\to R_1+R_2}\left[\begin{array}{ccc|ccc}1 & 0 & -1 & 0 & 1 & 0\\ 0 & 1 & -1 & -1 & 1 & 1\\ 0 & 0 & 1 & -2 & -4 & 1\end{array}\right]$$

$$\xrightarrow{R_1\to R_1+R_3}\left[\begin{array}{ccc|ccc}1 & 0 & 0 & -2 & -3 & 1\\ 0 & 1 & -1 & -1 & 1 & 1\\ 0 & 0 & 1 & -2 & -4 & 1\end{array}\right]$$

$$\xrightarrow{R_2\to R_2+R_3}\left[\begin{array}{ccc|ccc}1 & 0 & 0 & -2 & -3 & 1\\ 0 & 1 & 0 & -3 & -3 & 1\\ 0 & 0 & 1 & -2 & -4 & 1\end{array}\right]$$

So, we get $A^{-1} = \begin{bmatrix}-2 & -3 & 1\\ -3 & -3 & 1\\ -2 & -4 & 1\end{bmatrix}$

(iii) $\begin{bmatrix}1 & 2 & 3\\ 2 & 5 & 3\\ 1 & 0 & 8\end{bmatrix}$

Let $A = \begin{bmatrix}1 & 2 & 3\\ 2 & 5 & 3\\ 1 & 0 & 8\end{bmatrix}$

Applying Gauss Jordan method, we get

$$[A|I_3] = \left[\begin{array}{ccc|ccc}1 & 2 & 3 & 1 & 0 & 0\\ 2 & 5 & 3 & 0 & 1 & 0\\ 1 & 0 & 8 & 0 & 0 & 1\end{array}\right]$$

$$\xrightarrow[R_3 \to R_3 - R_1]{R_2 \to R_2 - 2R_1} \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & -2 & 5 & -1 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_1 \to R_1 + R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 8 & 0 & 0 & 1 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & -2 & 5 & -1 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_3 \to R_3 + 2R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 8 & 0 & 0 & 1 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & 0 & -1 & -5 & 2 & 1 \end{array}\right]$$

$$\xrightarrow{R_2 \to R_2 - 3R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 8 & 0 & 0 & 1 \\ 0 & 1 & 0 & 13 & -5 & -3 \\ 0 & 0 & -1 & -5 & 2 & 1 \end{array}\right]$$

$$\xrightarrow{R_1 \to R_1 + 8R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -40 & 16 & 9 \\ 0 & 1 & 0 & 13 & -5 & -3 \\ 0 & 0 & -1 & -5 & 2 & 1 \end{array}\right]$$

$$\xrightarrow{R_3 \to R_3 \times (-1)} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -40 & 16 & 9 \\ 0 & 1 & 0 & 13 & -5 & -3 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array}\right]$$

So, we get $A^{-1} = \begin{bmatrix} -40 & 16 & 9 \\ 13 & -5 & -3 \\ 5 & -2 & -1 \end{bmatrix}$

EXERCISE 1.3

1. Solve the following system of linear equations by matrix inversion method.

(i) $2x + 5y = -2,\ x + 2y = -3$

(ii) $2x - y = 8,\ 3x + 2y = -2$

(iii) $2x + 3y - z = 9,\ x + y + z = 9,\ 3x - y - z = -1$

(iv) $x + y + z - 2 = 0,\ 6x - 4y + 5z - 31 = 0,$ $5x + 2y + 2z = 13.$

Sol. **(i)** $2x + 5y = -2,\ x + 2y = -3$

The matrix form of the system is

$$= \begin{pmatrix} 2 & 5 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -2 \\ -3 \end{pmatrix}$$

$\Rightarrow$ $AX = B$ where

$$A = \begin{pmatrix} 2 & 5 \\ 1 & 2 \end{pmatrix},\ B = \begin{pmatrix} -2 \\ -3 \end{pmatrix},$$

$$X = \begin{pmatrix} x \\ y \end{pmatrix}$$

$\Rightarrow$ $X = A^{-1}B$

$$|A| = \begin{vmatrix} 2 & 5 \\ 1 & 2 \end{vmatrix} = 4 - 5 = -1 \neq 0.$$

$$\therefore A^{-1} = \frac{1}{|A|}\text{adj } A = \frac{1}{-1}\begin{bmatrix} 2 & -5 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} -2 & 5 \\ 1 & -2 \end{bmatrix}$$

$$\therefore X = A^{-1}B = \begin{bmatrix} -2 & 5 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} -2 \\ -3 \end{bmatrix} = \begin{bmatrix} 4-15 \\ -2+6 \end{bmatrix} = \begin{bmatrix} -11 \\ 4 \end{bmatrix}$$

$\therefore$ Solution set is $\{-11, 4\}$.

(ii) $2x - y = 8,\ 3x + 2y = -2$

The matrix form of the system is

$$\begin{bmatrix} 2 & -1 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 8 \\ -2 \end{bmatrix}$$

$\Rightarrow$ $AX = B$ where $A = \begin{bmatrix} 2 & -1 \\ 3 & 2 \end{bmatrix}$,

$$B = \begin{bmatrix} 8 \\ -2 \end{bmatrix}$$

$\Rightarrow$ $X = A^{-1}B.$

$$\text{Now, } |A| = \begin{vmatrix} 2 & -1 \\ 3 & 2 \end{vmatrix} = 4 + 3 = 7$$

$$\therefore A^{-1} = \frac{1}{|A|}\text{adj } A = \frac{1}{7}\begin{bmatrix} 2 & 1 \\ -3 & 2 \end{bmatrix}$$

$$\therefore X = A^{-1}B = \frac{1}{7}\begin{bmatrix} 2 & 1 \\ -3 & 2 \end{bmatrix}\begin{bmatrix} 8 \\ -2 \end{bmatrix} = \frac{1}{7}\begin{bmatrix} 16-2 \\ -24-4 \end{bmatrix}$$

$$= \frac{1}{7}\begin{bmatrix} 14 \\ -28 \end{bmatrix} = \begin{bmatrix} \frac{14}{7} \\ \frac{-28}{7} \end{bmatrix} = \begin{bmatrix} 2 \\ -4 \end{bmatrix}$$

$\therefore x = 2, y = -4$

Hence, the solution set is $\{2, -4\}$

(iii) $2x+3y-z=9, x+y+z=9, 3x-y-z=-1.$

The matrix form of the system is

$$\begin{bmatrix} 2 & 3 & -1 \\ 1 & 1 & 1 \\ 3 & -1 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 9 \\ -1 \end{bmatrix}$$

$$\Rightarrow \quad AX = B \text{ where } A = \begin{bmatrix} 2 & 3 & -1 \\ 1 & 1 & 1 \\ 3 & -1 & -1 \end{bmatrix},$$

$$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ and } B = \begin{bmatrix} 9 \\ 9 \\ -1 \end{bmatrix}$$

$$\Rightarrow \quad X = A^{-1} B$$

$$|A| = \begin{vmatrix} 2 & 3 & -1 \\ 1 & 1 & 1 \\ 3 & -1 & -1 \end{vmatrix} = 2\begin{vmatrix} 1 & 1 \\ -1 & -1 \end{vmatrix} - 3\begin{vmatrix} 1 & 1 \\ 3 & -1 \end{vmatrix} - 1\begin{vmatrix} 1 & 1 \\ 3 & -1 \end{vmatrix}$$

[Expanded along R_1]

$= 2(-1+1) - 3(-1-3) - 1(-1-3)$

$= 0 - 3(-4) = 12 + 4 = 16.$

$$\text{adj } A = \begin{bmatrix} +\begin{vmatrix} 1 & 1 \\ -1 & -1 \end{vmatrix} & -\begin{vmatrix} 1 & 1 \\ 3 & -1 \end{vmatrix} & +\begin{vmatrix} 1 & 1 \\ 3 & -1 \end{vmatrix} \\ -\begin{vmatrix} 3 & -1 \\ -1 & -1 \end{vmatrix} & +\begin{vmatrix} 2 & -1 \\ 3 & -1 \end{vmatrix} & -\begin{vmatrix} 2 & 3 \\ 3 & -1 \end{vmatrix} \\ +\begin{vmatrix} 3 & -1 \\ 1 & 1 \end{vmatrix} & -\begin{vmatrix} 2 & -1 \\ 1 & 1 \end{vmatrix} & +\begin{vmatrix} 2 & 3 \\ 1 & 1 \end{vmatrix} \end{bmatrix}^T$$

$$= \begin{bmatrix} +(-1+1) & -(-1-3) & +(-1-3) \\ -(-3-1) & +(-2+3) & -(-2-9) \\ +(3+1) & -(2+1) & +(2-3) \end{bmatrix}^T$$

$$= \begin{bmatrix} 0 & 4 & -4 \\ 4 & 1 & 11 \\ 4 & -3 & -1 \end{bmatrix}^T = \begin{bmatrix} 0 & 4 & 4 \\ 4 & 1 & -3 \\ -4 & 11 & -1 \end{bmatrix}$$

$$\therefore A^{-1} = \frac{1}{|A|} \text{ adj } A = \frac{1}{16}\begin{bmatrix} 0 & 4 & 4 \\ 4 & 1 & -3 \\ -4 & 11 & -1 \end{bmatrix}$$

$$\therefore X = A^{-1}B = \frac{1}{16}\begin{bmatrix} 0 & 4 & 4 \\ 4 & 1 & -3 \\ -4 & 11 & -1 \end{bmatrix}\begin{bmatrix} 9 \\ 9 \\ -1 \end{bmatrix}$$

$$= \frac{1}{16}\begin{bmatrix} 0+36-4 \\ 36+9+3 \\ -36+99+1 \end{bmatrix} = \frac{1}{16}\begin{bmatrix} 32 \\ 48 \\ 64 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$$

$\therefore x = 2, y = 3, z = 4$

$\therefore$ Solution set is $\{2, 3, 4\}$

(iv) $x + y + z - 2 = 0, 6x - 4y + 5z - 31 = 0,$
$5x + 2y + 2z = 13$

The matrix form of the system is

$$\begin{bmatrix} 1 & 1 & 1 \\ 6 & -4 & 5 \\ 5 & 2 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 31 \\ 13 \end{bmatrix}$$

$$AX = B \text{ where } A = \begin{bmatrix} 1 & 1 & 1 \\ 6 & -4 & 5 \\ 5 & 2 & 2 \end{bmatrix},$$

$$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, B = \begin{bmatrix} 2 \\ 31 \\ 13 \end{bmatrix}$$

$$\Rightarrow \quad X = A^{-1} B$$

$$|A| = \begin{vmatrix} 1 & 1 & 1 \\ 6 & -4 & 5 \\ 5 & 2 & 2 \end{vmatrix} = 1\begin{vmatrix} -4 & 5 \\ 2 & 2 \end{vmatrix} - 1\begin{vmatrix} 6 & 5 \\ 5 & 2 \end{vmatrix} + 1\begin{vmatrix} 6 & -4 \\ 5 & 2 \end{vmatrix}$$

$= 1(-8-10) - 1(12-25) + 1(12+20)$

$= 1(-18) - 1(-13) + 1(22) = -18 + 13 + 32 = 27$

$$\text{adj } A = \begin{bmatrix} +\begin{vmatrix} -4 & 5 \\ 2 & 2 \end{vmatrix} & -\begin{vmatrix} 6 & 5 \\ 5 & 2 \end{vmatrix} & +\begin{vmatrix} 6 & -4 \\ 5 & 2 \end{vmatrix} \\ -\begin{vmatrix} 1 & 1 \\ 2 & 2 \end{vmatrix} & +\begin{vmatrix} 1 & 1 \\ 5 & 2 \end{vmatrix} & -\begin{vmatrix} 1 & 1 \\ 5 & 2 \end{vmatrix} \\ +\begin{vmatrix} 1 & 1 \\ -4 & 5 \end{vmatrix} & -\begin{vmatrix} 1 & 1 \\ 6 & 5 \end{vmatrix} & +\begin{vmatrix} 1 & 1 \\ 6 & -4 \end{vmatrix} \end{bmatrix}^T$$

$$= \begin{bmatrix} +(-8-10) & -(12-25) & +(12+20) \\ -(2-2) & +(2-5) & -(2-5) \\ +(5+4) & -(5-6) & +(-4-6) \end{bmatrix}^T$$

$$= \begin{bmatrix} -18 & 13 & 32 \\ 0 & -3 & 3 \\ 9 & 1 & -10 \end{bmatrix}^T$$

$$= \begin{bmatrix} -18 & 0 & 9 \\ 13 & -3 & 1 \\ 32 & 3 & -10 \end{bmatrix}$$

$$\therefore A^{-1} = \frac{1}{|A|} \text{adj } A = \frac{1}{27}\begin{bmatrix} -18 & 0 & 9 \\ 13 & -3 & 1 \\ 32 & 3 & -10 \end{bmatrix}$$

$$\therefore X = A^{-1} B$$

$$= \frac{1}{27}\begin{bmatrix} -18 & 0 & 9 \\ 13 & -3 & 1 \\ 32 & 3 & -10 \end{bmatrix}\begin{bmatrix} 2 \\ 31 \\ 13 \end{bmatrix}$$

$$= \frac{1}{27}\begin{bmatrix} -36 & +0 & +117 \\ 26 & -93 & +13 \\ 64 & +93 & -130 \end{bmatrix} = \frac{1}{27}\begin{bmatrix} 81 \\ -54 \\ 27 \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}$$

$$\therefore x = 3, y = -2, z = 1$$

$\therefore$ Solution set is $\{3, -2, 1\}$

2. If $A = \begin{bmatrix} -5 & 1 & 3 \\ 7 & 1 & -5 \\ 1 & -1 & 1 \end{bmatrix}$ **and** $B = \begin{bmatrix} 1 & 1 & 2 \\ 3 & 2 & 1 \\ 2 & 1 & 3 \end{bmatrix}$**, find the products AB and BA and hence solve the system of equations $x + y + 2z = 1, 3x + 2y + z = 7, 2x + y + 3z = 2$.**

Sol. Given $A = \begin{bmatrix} -5 & 1 & 3 \\ 7 & 1 & -5 \\ 1 & -1 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 1 & 2 \\ 3 & 2 & 1 \\ 2 & 1 & 3 \end{bmatrix}$

$$AB = \begin{bmatrix} -5 & 1 & 3 \\ 7 & 1 & -5 \\ 1 & -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 2 \\ 3 & 2 & 1 \\ 2 & 1 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} -5+3+6 & -5+2+3 & -10+1+9 \\ 7+3-10 & 7+2-5 & 14+1-15 \\ 1-3+2 & 1-2+1 & 2-1+3 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix} = 4. I_3$$

$$BA = \begin{bmatrix} 1 & 1 & 2 \\ 3 & 2 & 1 \\ 2 & 1 & 3 \end{bmatrix}\begin{bmatrix} -5 & 1 & 3 \\ 7 & 1 & -5 \\ 1 & -1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -5+7+2 & 1+1-2 & 3-5+2 \\ -15+14+1 & 3+2-1 & 9-10+1 \\ -10+7+3 & 2+1-3 & 6-5+3 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix} = 4. I_3.$$

So, we get $AB = BA = 4. I_3$

$$\Rightarrow \left(\frac{1}{4}A\right) B = B\left(\frac{1}{4}A\right) = I$$

$$\Rightarrow B^{-1} = \frac{1}{4} = I$$

Writing the given set of equations in matrix form we get,

$$\begin{bmatrix} 1 & 1 & 2 \\ 3 & 2 & 1 \\ 2 & 1 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 7 \\ 2 \end{bmatrix}$$

$$\Rightarrow B\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 7 \\ 2 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = B^{-1}\begin{bmatrix} 1 \\ 7 \\ 2 \end{bmatrix} = \left[\frac{1}{4}A\right]\begin{bmatrix} 1 \\ 7 \\ 2 \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} -5 & 1 & 3 \\ 7 & 1 & -5 \\ 1 & -1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 7 \\ 2 \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} -5+7+6 \\ 7+7-10 \\ 1-7+2 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 8 \\ 4 \\ -4 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$$

$$\therefore x = 2, y = 1, z = -1$$

Hence, the solution set is $\{2, 1, -1\}$.

3. A man is appointed in a job with a monthly salary of certain amount and a fixed amount of annual increment. If his salary was ₹ 19,800 per month at the end of the first month after 3 years of service and ₹ 23,400 per month at the end of the first month after 9 years of service, find his starting salary and his annual increment. (Use matrix inversion method to solve the problem.)

Sol. Let the man's starting salary be ₹ x and his annual increment be ₹ y.

By the given data $x + 3y = 19800$ and $x + 9y = 23,400$.

The matrix form of the given system of equations is

$$\begin{bmatrix} 1 & 3 \\ 1 & 9 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 19800 \\ 23400 \end{bmatrix}$$

$$\Rightarrow AX = B \text{ where } A = \begin{bmatrix} 1 & 3 \\ 1 & 9 \end{bmatrix} \text{and } B = \begin{bmatrix} 19800 \\ 23400 \end{bmatrix}$$

$$\Rightarrow X = A^{-1} B$$

$$|A| = \begin{vmatrix} 1 & 3 \\ 1 & 9 \end{vmatrix} = 9 - 3 = 6 \neq 0$$

$$\therefore A^{-1} = \frac{1}{|A|} \text{adj} A = \frac{1}{6}\begin{bmatrix} 9 & -3 \\ -1 & 1 \end{bmatrix}$$

$$\therefore X = A^{-1} B = \frac{1}{6}\begin{bmatrix} 9 & -3 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 19800 \\ 23400 \end{bmatrix}$$

$$= \frac{1}{6}\begin{bmatrix} 178200 & -70200 \\ -19800 & +23400 \end{bmatrix} = \frac{1}{6}\begin{bmatrix} 108000 \\ 3600 \end{bmatrix}$$

$$= \begin{bmatrix} 18000 \\ 600 \end{bmatrix}$$

$\therefore x = 18000, y = 600.$

Hence the man's starting salary is ₹ 18000 and his annual increment is ₹ 600.

4. Four men and 4 women can finish a piece of work jointly in 3 days while 2 men and 5 women can finish the same work jointly in 4 days. Find the time taken by one man alone and that of one woman alone to finish the same work by using matrix inversion method.

Sol. Let the time by one man alone be x days and one woman alone be y days

$\therefore$ By the given data,

$$\frac{4}{x} + \frac{4}{y} = \frac{1}{3}$$

$$\text{and } \frac{2}{x} + \frac{5}{y} = \frac{1}{4}$$

$$\text{put } \frac{1}{x} = s \text{ and } \frac{1}{y} = t$$

$$\therefore 4s + 4t = \frac{1}{3}$$

$$\text{and } 2s + 5t = \frac{1}{4}$$

The matrix form of the system of equation is

$$\begin{bmatrix} 4 & 4 \\ 2 & 5 \end{bmatrix}\begin{bmatrix} s \\ t \end{bmatrix} = \begin{bmatrix} \frac{1}{3} \\ \frac{2}{4} \end{bmatrix} \Rightarrow AX = B \text{ where}$$

$$A = \begin{bmatrix} 4 & 4 \\ 2 & 5 \end{bmatrix} \text{ and } B = \begin{bmatrix} \frac{1}{3} \\ \frac{2}{4} \end{bmatrix}$$

$$X = A^{-1} B$$

$$\text{Now } |A| = \begin{vmatrix} 4 & 4 \\ 2 & 5 \end{vmatrix} = 20 - 8 = 12 \neq 0$$

$$\therefore A^{-1} = \frac{1}{|A|} \text{adj } A = \frac{1}{12}\begin{bmatrix} 5 & -4 \\ -2 & 4 \end{bmatrix}$$

$$\therefore X = A^{-1} B = \frac{1}{12}\begin{bmatrix} 5 & -4 \\ -2 & 4 \end{bmatrix}\begin{bmatrix} \frac{1}{3} \\ \frac{1}{4} \end{bmatrix}$$

$$= \frac{1}{12}\begin{bmatrix} \frac{5}{3} & -1 \\ \frac{-2}{3} & +1 \end{bmatrix}$$

$$= \frac{1}{12}\begin{bmatrix} \frac{2}{3} \\ \frac{1}{3} \end{bmatrix} = \begin{bmatrix} \frac{2}{3} \times \frac{1}{12} \\ \frac{1}{3} \times \frac{1}{12} \end{bmatrix} = \begin{bmatrix} \frac{1}{18} \\ \frac{1}{36} \end{bmatrix}$$

$$\therefore s = \frac{1}{18} \Rightarrow \frac{1}{x} = \frac{1}{18} \Rightarrow x = 18$$

$$t = \frac{1}{36} \Rightarrow \frac{1}{y} = \frac{1}{36} \Rightarrow y = 36.$$

Hence, the time taken by 1 man alone is 18 days and the time taken by 1 woman alone is 36 days.

5. The prices of three commodities A, B and C are ₹ x, y and z per units respectively. A person P purchases 4 units of B and sells two units of A and 5 units of C. Person Q purchases 2 units of C and sells 3 units of A and one unit of B. Person R purchases one unit of A and sells 3 unit of B and one unit of C. In the process, PQ and R earn ₹ 15,000, ₹ 1,000 and ₹ 4,000 respectively. Find the prices per unit of A, B and C. (Use matrix inversion method to solve the problem.)

Sol. Let the prices per unit for the commodities A, B and C be ₹ x, ₹ y and ₹ z.

By the given data,

$$2x - 4y + 5z = 15000$$
$$3x + y - 2z = 1000$$
$$-x + 3y + z = 4000$$

The matrix form of the system of equations is

$$\begin{bmatrix} 2 & -4 & 5 \\ 3 & 1 & -2 \\ -1 & 3 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 15000 \\ 1000 \\ 4000 \end{bmatrix}$$

$$AX = B \text{ where } A = \begin{bmatrix} 2 & -4 & 5 \\ 3 & 1 & -2 \\ -1 & 3 & 1 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix},$$

$$\text{and } B = \begin{bmatrix} 15000 \\ 1000 \\ 4000 \end{bmatrix}$$

$$\Rightarrow \quad X = A^{-1}B$$

$$|A| = \begin{vmatrix} 2 & -4 & 5 \\ 3 & 1 & -2 \\ -1 & 3 & 1 \end{vmatrix}$$

$$= 2\begin{vmatrix} 1 & -2 \\ 3 & 1 \end{vmatrix} + 4\begin{vmatrix} 3 & -2 \\ -1 & 1 \end{vmatrix} + 5\begin{vmatrix} 3 & 1 \\ -1 & 3 \end{vmatrix}$$

$$= 2(1+6) + 4(3-2) + 5(9+1)$$

$$= 2(7) + 4(1) + 5(10) = 14 + 4 + 50 = 68.$$

$$\text{adj } A = \begin{bmatrix} +\begin{vmatrix} 1 & -2 \\ 3 & 1 \end{vmatrix} & -\begin{vmatrix} 3 & -2 \\ -1 & 1 \end{vmatrix} & +\begin{vmatrix} 3 & 1 \\ -1 & 3 \end{vmatrix} \\ -\begin{vmatrix} -4 & 5 \\ 3 & 1 \end{vmatrix} & +\begin{vmatrix} 2 & 5 \\ -1 & 1 \end{vmatrix} & -\begin{vmatrix} 2 & -4 \\ -1 & 3 \end{vmatrix} \\ +\begin{vmatrix} -4 & 5 \\ 1 & -2 \end{vmatrix} & -\begin{vmatrix} 2 & 5 \\ 3 & -2 \end{vmatrix} & +\begin{vmatrix} 2 & -4 \\ 3 & 1 \end{vmatrix} \end{bmatrix}^T$$

$$= \begin{bmatrix} +(1+6) & -(3-2) & +(9+1) \\ -(-4-15) & +(2+5) & -(6-4) \\ +(8-5) & -(-4-15) & +(2+12) \end{bmatrix}^T$$

$$= \begin{bmatrix} 7 & -1 & 10 \\ 19 & 7 & -2 \\ 3 & 19 & 14 \end{bmatrix}^T = \begin{bmatrix} 7 & 19 & 3 \\ -1 & 7 & 19 \\ 10 & -2 & 14 \end{bmatrix}$$

$$\therefore A^{-1} = \frac{1}{|A|}\text{adj } A = \frac{1}{68}\begin{bmatrix} 7 & 19 & 3 \\ -1 & 7 & 19 \\ 10 & -2 & 14 \end{bmatrix}$$

$$\therefore X = A^{-1}B = \frac{1}{68}\begin{bmatrix} 7 & 19 & 3 \\ -1 & 7 & 19 \\ 10 & -2 & 14 \end{bmatrix}\begin{bmatrix} 15000 \\ 1000 \\ 4000 \end{bmatrix}$$

$$= \frac{1}{68}\begin{bmatrix} 105000+19000+12000 \\ -15000+7000+76000 \\ 150000-2000+56000 \end{bmatrix}$$

$$= \frac{1}{68}\begin{bmatrix} 136000 \\ 68000 \\ 204000 \end{bmatrix} = \begin{bmatrix} 2000 \\ 1000 \\ 3000 \end{bmatrix}$$

$\therefore x = 2000, y = 1000, z = 3000.$

Hence the prices per unit of the commodities A, B and C are ₹ 2000, ₹ 1000 and ₹ 3000 respectively.

EXERCISE 1.4

1. Solve the following systems of linear equation by Cramer's rule:

(i) $5x - 2y + 16 = 0, \; x + 3y - 7 = 0$

(ii) $\frac{3}{x} + 2y = 12, \; \frac{2}{x} + 3y = 13$

(iii) $3x + 3y - z = 11, 2x - y + 2z = 9,$
$4x + 3y + 2z = 25$

(iv) $\frac{3}{x} - \frac{4}{y} - \frac{2}{z} - 1 = 0, \frac{1}{x} + \frac{2}{y} + \frac{1}{z} - 2 = 0,$
$\frac{2}{x} - \frac{5}{y} - \frac{4}{z} + 1 = 0$

Sol. (i) $5x - 2y = -16, x + 3y = 7$

$$\text{Given } \Delta = \begin{vmatrix} 5 & -2 \\ 1 & 3 \end{vmatrix} = 15 + 2 = 17$$

$$\Delta_1 = \begin{vmatrix} -16 & -2 \\ 7 & 3 \end{vmatrix} = -48 + 14 = -34$$

$$\Delta_2 = \begin{vmatrix} 5 & -16 \\ 1 & 7 \end{vmatrix} = 35 + 16 = 51$$

$$\therefore x = \frac{\Delta_1}{\Delta} = \frac{-34}{17} = -2$$

$$y = \frac{\Delta_2}{\Delta} = \frac{51}{17} = 3$$

$\therefore$ Solution set is $\{-2, 3\}$

(ii) $\boldsymbol{\frac{3}{x} + 2y = 12, \frac{2}{x} + 3y = 13}$

$$\text{Let } \frac{1}{x} = z$$

$$\therefore 3z + 2y = 12, 2z + 3z = 13$$

$$\therefore \Delta = \begin{vmatrix} 3 & 2 \\ 2 & 3 \end{vmatrix} = 9 - 4 = 5$$

$$\Delta_1 = \begin{vmatrix} 12 & 2 \\ 13 & 3 \end{vmatrix} = 36 - 26 = 10$$

$$\Delta_2 = \begin{vmatrix} 3 & 12 \\ 2 & 13 \end{vmatrix} = 39 - 24 = 15$$

$$\therefore z = \frac{\Delta_1}{\Delta} = \frac{10}{5} = 2 \Rightarrow \frac{1}{x} = 2 \Rightarrow x = \frac{1}{2}$$

$$y = \frac{\Delta_2}{\Delta} = \frac{15}{5} = 3$$

$\therefore$ Solution set is $\{\frac{1}{2}, 3\}$

(iii) $\boldsymbol{3x + 3y - z = 11, 2x - y + 2z = 9, 4x + 3y + 2z = 25}$

$$\Delta = \begin{vmatrix} 3 & 3 & -1 \\ 2 & -1 & 2 \\ 4 & 3 & 2 \end{vmatrix}$$

$$= 3\begin{vmatrix} -1 & 2 \\ 3 & 2 \end{vmatrix} - 3\begin{vmatrix} 2 & 2 \\ 4 & 2 \end{vmatrix} - 1\begin{vmatrix} 2 & -1 \\ 4 & 3 \end{vmatrix}$$

$$= 3(-2-6) - 3(4-8) - 1(6+4)$$

$$= 3(-8) - 3(-4) - 1(10)$$

$$= -24 + 12 - 10 = -22$$

$$\Delta_1 = \begin{vmatrix} 11 & 3 & -1 \\ 9 & -1 & 2 \\ 25 & 3 & 2 \end{vmatrix}$$

$$= 11\begin{vmatrix} -1 & 2 \\ 3 & 2 \end{vmatrix} - 3\begin{vmatrix} 9 & 2 \\ 25 & 2 \end{vmatrix} - 1\begin{vmatrix} 9 & -1 \\ 25 & 3 \end{vmatrix}$$

$$= 11(-2-6) - 3(18-50) - 1(27+25)$$

$$= 11(-8) - 3(-32) - 1(52)$$

$$= -88 + 96 - 52 = -44$$

$$\Delta_2 = \begin{vmatrix} 3 & 11 & -1 \\ 2 & 9 & 2 \\ 4 & 25 & 2 \end{vmatrix}$$

$$= 3\begin{vmatrix} 9 & 2 \\ 25 & 2 \end{vmatrix} - 11\begin{vmatrix} 2 & 2 \\ 4 & 2 \end{vmatrix} - 1\begin{vmatrix} 2 & 9 \\ 4 & 25 \end{vmatrix}$$

$$= 3(18-50) - 11(4-8) - 1(50-36)$$

$$= 3(-32) - 11(-4) - 1(14)$$

$$= -96 + 44 - 14 = -66$$

$$\Delta_3 = \begin{matrix} 3 & 3 & 11 \\ 2 & -1 & 9 \\ 4 & 3 & 25 \end{matrix}$$

$$= 3\begin{vmatrix} -1 & 9 \\ 3 & 25 \end{vmatrix} - 3\begin{vmatrix} 2 & 9 \\ 4 & 25 \end{vmatrix} + 11\begin{vmatrix} 2 & -1 \\ 4 & 3 \end{vmatrix}$$

$$= 3(-25-27) - 3(50-36) + 11(6+4)$$

$$= 3(-52) - 3(14) + 11(10)$$

$$= -156 - 42 + 110 = -88$$

$$\therefore x = \frac{\Delta_1}{\Delta} = \frac{-44}{-22} = 2$$

$$y = \frac{\Delta_2}{\Delta} = \frac{-66}{-22} = 3$$

$$z = \frac{\Delta_3}{\Delta} = \frac{-88}{-22} = 4$$

$\therefore$ Solution set is $\{2, 3, 4\}$

(iv) $\boldsymbol{\frac{3}{x} - \frac{4}{y} - \frac{2}{z} - 1 = 0, \frac{1}{x} + \frac{2}{y} + \frac{1}{z} - 2 = 0, \frac{2}{x} - \frac{5}{y} - \frac{4}{z} + 1 = 0}$

$$\text{Put } \frac{1}{x} = u, \frac{1}{y} = v, \frac{1}{z} = w,$$

We get $3u - 4v - 2w = 1, u + 2v + w = 2,$

$$2u - 5v - 4w = -1$$

$$\therefore \Delta = \begin{vmatrix} 3 & -4 & -2 \\ 1 & 2 & 1 \\ 2 & -5 & -4 \end{vmatrix} = 3\begin{vmatrix} 2 & 1 \\ -5 & -4 \end{vmatrix} + 4\begin{vmatrix} 1 & 1 \\ 2 & -4 \end{vmatrix} - 2\begin{vmatrix} 1 & 2 \\ 2 & -5 \end{vmatrix}$$

$$= 3(-8+5) + 4(-4-2) - 2(-5-4)$$

$$= 3(-3) + 4(-6) - 2(-9)$$

$$= -9 - 24 + 18 = -15$$

$$\Delta_1 = \begin{vmatrix} 1 & -4 & -2 \\ 2 & 2 & 1 \\ -1 & -5 & -4 \end{vmatrix}$$

$$= 1\begin{vmatrix} 2 & 1 \\ -5 & -4 \end{vmatrix} + 4\begin{vmatrix} 2 & 1 \\ -1 & -4 \end{vmatrix} - 2\begin{vmatrix} 2 & 2 \\ -1 & -5 \end{vmatrix}$$

$= 1(-8+5) + 4(-8+1) - 2(-10+2)$

$= 1(-3) + 4(-7) - 2(-8)$

$= -3 - 28 + 16 = -15$

$$\Delta_2 = \begin{vmatrix} 3 & 1 & -2 \\ 1 & 2 & 1 \\ 2 & -1 & -4 \end{vmatrix}$$

$$= 3\begin{vmatrix} 2 & 1 \\ -1 & -4 \end{vmatrix} - 1\begin{vmatrix} 1 & 1 \\ 2 & -4 \end{vmatrix} - 2\begin{vmatrix} 1 & 2 \\ 2 & -1 \end{vmatrix}$$

$= 3(-8+1) - 1(-4-2) - 2(-1-4)$

$= 3(-7) - 1(-6) - 2(-5)$

$= -21 + 6 + 10 = -5$

$$\Delta_3 = \begin{vmatrix} 3 & -4 & 1 \\ 1 & 2 & 2 \\ 2 & -5 & -1 \end{vmatrix}$$

$$= 3\begin{vmatrix} 2 & 2 \\ -5 & -1 \end{vmatrix} + 4\begin{vmatrix} 1 & 2 \\ 2 & -1 \end{vmatrix} + 1\begin{vmatrix} 1 & 2 \\ 2 & -5 \end{vmatrix}$$

$= 3(-2+10) + 4(-1-4) + 1(-5-4)$

$= 3(8) + 4(-5) + 1(-9)$

$= 24 - 20 - 9 = -5$

$$\therefore u = \frac{\Delta_1}{\Delta} = \frac{-15}{-15} = 1 \Rightarrow \frac{1}{x} = 1 \Rightarrow x = 1$$

$$v = \frac{\Delta_2}{\Delta} = \frac{-5}{-15} = \frac{1}{3} \Rightarrow \frac{1}{y} = \frac{1}{3} \Rightarrow y = 3$$

$$w = \frac{\Delta_3}{\Delta} = \frac{-5}{-15} = \frac{1}{3} \Rightarrow \frac{1}{z} = \frac{1}{3} \Rightarrow z = 3$$

$\therefore$ Solution set is $\{1, 3, 3\}$

2. In a competitive examination, one mark is awarded for every correct answer while $\frac{1}{4}$ mark is deducted for every wrong answer. A student answered 100 questions and got 80 marks. How many questions did he answer correctly ? (Use Cramer's rule to solve the problem).

Sol. Let x represent the number of question with correct answer and y represent the number of questions with wrong answers.

By the given data, $x + y = 100$ and ... (1)

$$1.x - \frac{1}{4}y = 80$$

Multiplying by 4 we get,

$$4x - y = 320 \quad \text{... (2)}$$

From (1) and (2)

$$\Delta = \begin{vmatrix} 1 & 1 \\ 4 & -1 \end{vmatrix} = -1 - 4 = -5$$

$$\Delta_1 = \begin{vmatrix} 100 & 1 \\ 320 & -1 \end{vmatrix} = -100 - 320 = -420$$

$$\Delta_2 = \begin{vmatrix} 1 & 100 \\ 4 & 320 \end{vmatrix} = 320 - 400 = -80$$

$$\therefore x = \frac{\Delta_1}{\Delta} = \frac{-420}{-5} = +84$$

$$\text{and } y = \frac{\Delta_2}{\Delta} = \frac{-80}{-5} = 16$$

Hence, the number of questions with correct answer is 84.

3. A chemist has one solution which is 50% acid and another solution which is 25% acid. How much each should be mixed to make 10 litres of a 40% acid solution ? (Use Cramer's rule to solve the problem).

Sol. Let the amount of 50% acid be xl and the amount of 25% acid be y litre

By the given data, $x + y = 10$ ... (1)

$$\text{and } x\left(\frac{50}{100}\right) + y\left(\frac{25}{100}\right) = 10\left(\frac{40}{100}\right)$$

$$\Rightarrow \quad 50x + 25y = 400 \Rightarrow 2x + y = 16 \quad \text{... (2)}$$

The matrix from of the equation is $\begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 10 \\ 16 \end{bmatrix}$

$$\Rightarrow \quad AX = B \text{ where } A = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix},$$

$$X = \begin{bmatrix} x \\ y \end{bmatrix}, B = \begin{bmatrix} 10 \\ 16 \end{bmatrix}$$

$$\Rightarrow \quad X = A^{-1}B \; |A| = \begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} = 1 - 2 = -1$$

$$\Rightarrow \quad X = \frac{1}{|A|} \text{adj } A . B$$

$$\Rightarrow \quad X = -1 \begin{bmatrix} 1 & -1 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 10 \\ 16 \end{bmatrix}$$

$$= -\begin{bmatrix} 10-16 \\ -20+16 \end{bmatrix}$$

$$\Rightarrow \quad X = -\begin{bmatrix} -6 \\ -4 \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \end{bmatrix}$$

Thus, the amount of 50% acid is 6 litre and the amount of 25% acid is 4 litre.

4. A fish tank can be filled in 10 minutes using both pumps A and B simultaneously. However, pump B can pump water in or out at the same rate. If pump B is inadvertently run in reverse, then the tank will be filled in 30 minutes. How long would it take each pump to fill the tank by itself ? (Use Cramer's rule to solve the problem).

Sol. Let the pump A can fill the tank in x minutes, and the pump B can fill the tank in y minutes

In 1 minute A can fill $\frac{1}{x}$ units and in 1 minute B can fill $\frac{1}{y}$ units

$$\therefore \frac{1}{x} + \frac{1}{y} = 10$$

$$\text{and } \frac{1}{x} - \frac{1}{y} = 30$$

$$\text{Put } \frac{1}{x} = a \text{ and } \frac{1}{y} = b$$

$$\Rightarrow \quad a + b = \frac{1}{10} \quad \ldots (1)$$

$$\text{and } a - b = \frac{1}{30} \quad \ldots (2)$$

$$\Delta = \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = -1 - 1 = -2$$

$$\Delta_1 = \begin{vmatrix} \frac{1}{10} & 1 \\ \frac{1}{30} & -1 \end{vmatrix} = \frac{-1}{10} - \frac{1}{30}$$

$$= \frac{-3-1}{30} = \frac{-4}{30} = \frac{-2}{15}$$

$$\Delta_2 = \begin{vmatrix} 1 & \frac{1}{10} \\ 1 & \frac{1}{30} \end{vmatrix} = \frac{1}{30} - \frac{1}{10} = \frac{1-3}{30}$$

$$= \frac{-2}{30} = \frac{-1}{15}$$

$$\therefore a = \frac{\Delta_1}{\Delta} = \frac{\frac{-2}{15}}{-2} = \frac{1}{15} \Rightarrow \frac{1}{x} = \frac{1}{15} \Rightarrow x = 15$$

$$b = \frac{\Delta_2}{\Delta} = \frac{\frac{-1}{15}}{-2} = \frac{1}{30} \Rightarrow \frac{1}{y} = \frac{1}{30} \Rightarrow y = 30$$

Hence the pump A can fill the tank in 15 minutes and the pump B can fill the tank in 30 minutes.

5. A family of 3 people went out for dinner in a restaurant. The cost of two dosai, three idlies and two vadais is ₹ 150. The cost of the two dosai, two idlies and four vadais is ₹ 200. The cost of five dosai, four idlies and two vadais is ₹ 250. The family has ₹ 350 in hand and they ate 3 dosai and six idlies and six vadais. Will they be able to manage to pay the bill within the amount they had ?

Sol. Let the cost of one dosa be ₹ x

The cost of one idli be ₹ y

and the cost of one vadai be ₹ z

By the given data,

$$2x + 3y + 2z = 150$$
$$2x + 2y + 4z = 200$$
$$5x + 4y + 2z = 250$$

$$\therefore \Delta = \begin{vmatrix} 2 & 3 & 2 \\ 2 & 2 & 4 \\ 5 & 4 & 2 \end{vmatrix}$$

$$= 2\begin{vmatrix} 2 & 4 \\ 4 & 2 \end{vmatrix} - 3\begin{vmatrix} 2 & 4 \\ 5 & 2 \end{vmatrix} + 2\begin{vmatrix} 2 & 2 \\ 5 & 4 \end{vmatrix}$$

$$= 2(4-16) - 3(4-20) + 2(8-10)$$

$$= 2(-12) - 3(-16) + 2(-2)$$

$= -24 + 48 - 4 = 20$

$$\Delta_1 = \begin{vmatrix} 150 & 3 & 2 \\ 200 & 2 & 4 \\ 250 & 4 & 2 \end{vmatrix}$$

Taking 50 common from C_3 we get,

$$= 100\begin{vmatrix} 3 & 3 & 1 \\ 4 & 2 & 2 \\ 5 & 4 & 1 \end{vmatrix}$$

$$= 100\left[3\begin{vmatrix} 2 & 2 \\ 4 & 1 \end{vmatrix} - 3\begin{vmatrix} 4 & 2 \\ 5 & 1 \end{vmatrix} + 1\begin{vmatrix} 4 & 2 \\ 5 & 4 \end{vmatrix}\right]$$

$= 100[3(2 - 8) - 3(4 - 10) + 1(16 - 10)]$

$= 100[3(-6) - 3(-6) + 6]$

$= 100[-18 + 18 + 6] = 600.$

$$\Delta_2 = \begin{vmatrix} 2 & 150 & 2 \\ 2 & 200 & 4 \\ 5 & 250 & 2 \end{vmatrix} = 100\begin{vmatrix} 2 & 3 & 1 \\ 2 & 4 & 2 \\ 5 & 5 & 1 \end{vmatrix}$$

$$= 100\left[2\begin{vmatrix} 4 & 2 \\ 5 & 1 \end{vmatrix} - 3\begin{vmatrix} 2 & 2 \\ 5 & 1 \end{vmatrix} + 1\begin{vmatrix} 2 & 4 \\ 5 & 5 \end{vmatrix}\right]$$

$= 100[2(4 - 10) - 3(2 - 10) + 1(10 - 20)]$

$= 100[2(-6) - 3(-8) + 1(-10)]$

$= 100[-12 + 24 - 10] = 100\,[2] = 200.$

$$\Delta_3 = \begin{vmatrix} 2 & 3 & 150 \\ 2 & 2 & 200 \\ 5 & 4 & 250 \end{vmatrix} = 50\begin{vmatrix} 2 & 3 & 3 \\ 2 & 2 & 4 \\ 5 & 4 & 5 \end{vmatrix}$$

$$= 50\left[2\begin{vmatrix} 2 & 4 \\ 4 & 5 \end{vmatrix} - 3\begin{vmatrix} 2 & 4 \\ 5 & 5 \end{vmatrix} + 3\begin{vmatrix} 2 & 2 \\ 5 & 4 \end{vmatrix}\right]$$

$= 50\,[2(10 - 16) - 3(10 - 20) + 3(8 - 10)]$

$= 50[2(-6) - 3(-10) + 3(-2)]$

$= 50\,[-12 + 30 - 6] = 50\,[12] = 600.$

$$\therefore x = \frac{\Delta_1}{\Delta} = \frac{600}{20} = 30$$

$$y = \frac{\Delta_2}{\Delta} = \frac{200}{20} = 10$$

$$z = \frac{\Delta_3}{\Delta} = \frac{600}{20} = 30.$$

Hence, the price of one dosa be ₹ 30, one idli be ₹10 and the price of 1 vadai be ₹ 30.

Also the cost of 3 dosa, six idlies and six vadai is

$= 3x + 6y + 6z = 3(30) + 6(10) + 6(30)$

$= 90 + 60 + 180 =$ ₹ 330

Since the family had ₹ 350 in hand, they will be able to manage to pay the bill.

EXERCISE 1.5

1. Solve the following systems of linear equations by Gaussian elimination method.

(i) $\mathbf{2x - 3y + 3z = 2, x + 2y - z = 3, 3x - y + 2z = 1.}$

(ii) $\mathbf{2x + 4y + 6z = 22, 3x + 8y + 5z = 27,}$
$\mathbf{-x + y + 2z = 2}$

Sol. Transforming the augmented matrix to echelon form, we get

$$\left[\begin{array}{ccc|c} 2 & -2 & 3 & 2 \\ 1 & 2 & -1 & 3 \\ 3 & -1 & 2 & 1 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 2 & -2 & 3 & 2 \\ 3 & -1 & 2 & 1 \end{array}\right] \xrightarrow[R_3 \to R_3 - 3R_1]{R_2 \to R_2 - 2R_1} \left[\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 0 & -6 & 5 & -4 \\ 0 & -7 & 5 & -8 \end{array}\right]$$

$$\xrightarrow{R_3 \to R_3 - R_2} \left[\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 0 & -6 & 5 & -4 \\ 0 & -1 & 0 & -4 \end{array}\right] \xrightarrow{R_3 \to 6R_3 - 2R_2}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 0 & -6 & 5 & -4 \\ 0 & 0 & -5 & -20 \end{array}\right]$$

Writing the equivalent equations from the row-echelon matrix, we get,

$$x + 2y - z = 3 \quad \dots (1)$$

$$-6y + 5z = -4 \quad \dots (2)$$

$$-5z = -20 \Rightarrow z = \frac{-20}{-5} = 4.$$

Substituting $z = 4$ in 2 we get,

$$-6y + 5(4) = -4$$

$$\Rightarrow \quad -6y + 20 = -4 \Rightarrow -4 - 20 = -24$$

$$\Rightarrow \quad y = \frac{-24}{-6} = 4.$$

Substituting $y = z = 4$ in (1) we get

$$x + 2(4) - 4 = 3$$

$$\Rightarrow \quad x + 8 - 4 = 3$$

$\Rightarrow \quad x + 4 = 3$

$\Rightarrow \quad x = 3 - 4 = -1.$

$\therefore$ Solution set is $\{-1, 4, 4\}$

(ii) $2x + 4y + 6z = 22, 3x + 8y + 5z = 27,$ $-x + y + 2z = 2$

Reducing the augmented matrix to an equivalent row echelon form by using elementary row operations, we get

$$\left[\begin{array}{ccc|c} 2 & 4 & 6 & 22 \\ 3 & 8 & 5 & 27 \\ -1 & 1 & 2 & 2 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_3} \left[\begin{array}{ccc|c} -1 & 1 & 2 & 2 \\ 3 & 8 & 5 & 27 \\ 2 & 4 & 6 & 22 \end{array}\right]$$

$$\xrightarrow[R_3 \to R_3 + 2R_1]{R_2 \to R_2 + 3R_1} \left[\begin{array}{ccc|c} -1 & 1 & 2 & 2 \\ 0 & 11 & 11 & 33 \\ 0 & 6 & 10 & 26 \end{array}\right]$$

$$\xrightarrow[R_3 \to R_3 \div 2]{R_2 \to R_2 \div 11} \left[\begin{array}{ccc|c} -1 & 1 & 2 & 2 \\ 0 & 1 & 1 & 3 \\ 0 & 3 & 5 & 13 \end{array}\right]$$

$$\xrightarrow{R_3 \to R_3 - 3R_2} \left[\begin{array}{ccc|c} -1 & 1 & 2 & 2 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 2 & 4 \end{array}\right]$$

$$\xrightarrow{R_3 \to R_3 \div 2} \left[\begin{array}{ccc|c} -1 & 1 & 2 & 2 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

Writing the equivalent equations from the row echelon matrix we get,

$-x + y + 2z = 2 \quad \ldots(1)$

$y + z = 3 \quad \ldots(2)$

$z = 2 \quad \ldots(3)$

Substituting (3) in (2) we get, $y + 2 = 3$

$\Rightarrow y = 3 - 2 = 1$

Substituting $y = 1$ and $z = 2$ in (1) we get,

$-x + 1 + 2(2) = 2 \quad \Rightarrow -x + 1 + 4 = 2$

$\Rightarrow \quad -x + 5 = 2 \quad \Rightarrow \quad -x = 2 - 5$

$\Rightarrow \quad -x = -3 \quad \Rightarrow \quad x = 3$

$\therefore$ Solution set is $\{3, 1, 2\}$

2. If $ax^2 + bx + c$ is divided by $x + 3$, $x - 5$, and $x - 1$, the remainders are 21, 61 and 9 respectively. Find a, b and c. (Use Gaussian elimination method.)

Sol. Let $P(x) = ax^2 + bx + c$

Given $P(-3) = 21$

[$\because$ $P(x) \div x + 3$, the remainder is 21]

$\Rightarrow a(-3)^2 + b(-3) + c = 21$

$\Rightarrow \quad 9a - 3b + c = 21 \quad \ldots(1)$

Also, $P(5) = 61$

$\Rightarrow \quad a(5)^2 + b(5) + c = 61$

[using remainder theorem]

$\Rightarrow \quad 25a + 5b + c = 61 \quad \ldots(2)$

and $P(1) = 9$

$\Rightarrow \quad a(1)^2 + b(1) + c = 9$

$\Rightarrow \quad a + b + c = 9 \quad \ldots(3)$

Reducing the augment matrix to an equivalent row-echelon form using elementary row operations, we get

$$\left[\begin{array}{ccc|c} 9 & -3 & 1 & 21 \\ 25 & 5 & 1 & 61 \\ -1 & 1 & 1 & 9 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_3} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 9 \\ 25 & 5 & 1 & 61 \\ 9 & -3 & 1 & 21 \end{array}\right]$$

$$\xrightarrow[R_2 \to R_2 - 25R_1]{R_3 \to R_3 - 9R_1} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 9 \\ 0 & -20 & -24 & -164 \\ 0 & -12 & -8 & -60 \end{array}\right]$$

$$\xrightarrow[R_3 \to R_3 \div 4]{R_2 \to R_2 \div 4} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 9 \\ 0 & -5 & -6 & -41 \\ 0 & -3 & -2 & -15 \end{array}\right]$$

$$\xrightarrow{R_3 \to R_3 - \frac{3}{5}R_2} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 9 \\ 0 & -5 & -6 & -41 \\ 0 & 0 & \frac{8}{5} & \frac{48}{5} \end{array}\right]$$

$$\xrightarrow{R_3 \to 5R_3} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 9 \\ 0 & -5 & -6 & -41 \\ 0 & 0 & 8 & 48 \end{array}\right]$$

Writing the equivalent equations from the row-echelon matrix we get,

$a + b + c = 9 \quad \ldots(1)$

$-5b - 6c = -41 \quad \ldots(2)$

$$8c = 48$$

$$\Rightarrow \quad c = \frac{48}{8} = 6$$

Substituting $c = 6$ in (2) we get,

$$\Rightarrow \quad -5b - 6(6) = -41$$
$$\Rightarrow \quad -5b = 36 - 41$$
$$\Rightarrow \quad -5b = -41 + 36 = -5$$

$$\Rightarrow \quad b = \frac{-5}{-5} = 1$$

Substituting $b = 1$, $c = 6$ in (1) we get,

$$a + 1 + 6 = 9$$
$$\Rightarrow \quad a + 7 = 9$$
$$\Rightarrow \quad a = 9 - 7$$
$$\Rightarrow \quad a = 2$$

$\therefore$ $a = 2$, $b = 1$, and $c = 6$

3. An amount of ₹ 65,000 is invested in three bonds at the rates of 6%, 8% and 9% per annum respectively. The total annual income is ₹ 4,800. The income from the third bond is ₹ 600 more than that from the second bond. Determine the price of each bond. (Use Gaussian elimination method.)

Sol. Let the price of bond invested in 6%, 8% and 9% rates be let ₹ x, ₹ y and ₹ z respectively

$\therefore$ By the given data, $x + y + z = 65000$ ...(1)

$$\frac{6 \times x \times 1}{100} + \frac{8 \times y \times 1}{100} + \frac{9 \times z \times 1}{100} = 4800$$

$$\left[\because \text{Interest} = \frac{PNR}{100}\right]$$

$$\Rightarrow \quad \frac{6x}{100} + \frac{8y}{100} + \frac{9z}{100} = 4800$$

$$\Rightarrow \quad 6x + 8y + 9z = 480000 \quad \text{... (2)}$$

$$\text{Also,} \quad \frac{9z}{100} = 600 + \frac{8y}{100}$$

$$\Rightarrow \quad \frac{-8y}{100} + \frac{9z}{100} = 600$$

$$\Rightarrow \quad -8y + 9z = 60000 \quad \text{...(3)}$$

Reducing the augmented matrix to an equivalent row-echelon form by using elementary row operation, we get

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 65000 \\ 6 & 8 & 9 & 480000 \\ 0 & -8 & 9 & 60000 \end{array}\right]$$

$$\xrightarrow{R_2 \to R_2 - 6R_1} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 65000 \\ 0 & 2 & 3 & 90000 \\ 0 & -8 & 9 & 60000 \end{array}\right]$$

$$\xrightarrow{R_3 \to R_3 + 4R_2} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 65000 \\ 0 & 2 & 3 & 90000 \\ 0 & 0 & 21 & 420000 \end{array}\right]$$

Writing the equivalent from the row echelon matrix we get,

$$x + y + z = 65000 \quad \text{...(1)}$$
$$2y + 3z = 90000 \quad \text{...(2)}$$
$$21z = 42000$$
$$\Rightarrow \quad z = \frac{420000}{21} = 20000$$

Substituting z = 20,000 in (2),

$$2y + 3(20,000) = 90000$$
$$\Rightarrow \quad 2y + 60,000 = 90,000$$
$$\Rightarrow \quad 2y = 90,000 - 60,000$$
$$= 30,000$$
$$\Rightarrow \quad y = \frac{30,000}{2} = 15,000$$

Substituting $y = 15,000$ and $z = 20,000$ in (1) we get,

$$x + 15,000 + 20,000 = 65000$$
$$\Rightarrow \quad x + 35,000 = 65000$$
$$\Rightarrow \quad x = 65,000 - 35,000$$
$$\Rightarrow \quad x = 30,000$$

Thus the price of 6% bond is ₹ 30,000 the price of 8% bond is ₹ 15,000 and the price of 9% bond is ₹ 20,000.

4. A boy is walking along the path $y = ax^2 + bx + c$ through the points (–6, 8),(–2 – 12) and (3,8). He wants to meet his friend at P(7,60). Will he meet his friend? (Use Gaussian elimination method.)

Sol. Given $y = ax^2 + bx + c$ ...(1)

(–6, 8) lies on (1)

$$\Rightarrow \quad 8 = a(-6)^2 + b(-6) + c$$
$$\Rightarrow \quad 8 = 36a - 6b + c \quad \text{...(2)}$$

(–2, –12) lies on (1)

$$\Rightarrow \quad -12 = a(-2)^2 + b(-2) + c$$
$$\Rightarrow \quad -12 = 4a - 2b + c \quad \text{...(3)}$$

Also (3, 8) lies on (1)

$\Rightarrow \quad 8 = a(3)2 + b(3) + c$

$\Rightarrow \quad 8 = 9a + 3b + c \quad ...(4)$

Reducing the augment matrix to an equivalent row-echelon form by using elementary row operations, we get,

$$\left[\begin{array}{ccc|c} 36 & -6 & 1 & 8 \\ 4 & -2 & 1 & -12 \\ 0 & 3 & 1 & 8 \end{array}\right] \xrightarrow[R_3 \to 4R_3 - R_1]{R_2 \to 9R_2 - R_1} \left[\begin{array}{ccc|c} 36 & -6 & 1 & 8 \\ 0 & -12 & 8 & -116 \\ 0 & 18 & 3 & 24 \end{array}\right]$$

$$\xrightarrow[R_3 \to R_3 \div 3]{R_2 \to R_2 \div 4} \left[\begin{array}{ccc|c} 36 & -6 & 1 & 8 \\ 0 & -3 & 2 & -29 \\ 0 & 6 & 1 & 8 \end{array}\right]$$

$$\xrightarrow{R_3 \to R_3 + 2R_2} \left[\begin{array}{ccc|c} 36 & -6 & 1 & -8 \\ 0 & -3 & 2 & -29 \\ 0 & 0 & 5 & -50 \end{array}\right]$$

Writing the equivalent equation from the row echelon matrix, we get $36a - 6b + c = 8 \quad ...(1)$

$-3b + 2c = -29 \quad ...(2)$

$5c = -50$

$\Rightarrow \quad c = \frac{-50}{5} = -10$

Substituting c = –10 in (2) we get,

$-3b + 2(-10) = -29$

$\Rightarrow \quad -3b - 20 = -29$

$\Rightarrow \quad -3b = -29 + 20$

$\Rightarrow \quad -3b = -9$

$\Rightarrow \quad b = \frac{-9}{-3} = 3$

Substituting $b = 3$ and $c = -10$ in (1) we get,

$36a - 6(3) - 10 = 8$

$\Rightarrow \quad 36a - 18 - 10 = 8$

$\Rightarrow \quad 36a - 28 = 8$

$\Rightarrow \quad 36a = 8 + 28 = 36$

$\Rightarrow \quad a = \frac{36}{36} = 1$

$\therefore \; a = 1, b = 3, c = -10$

Hence the path of the boy is

$y = 1(x^2) + 3(x) - 10$

$\Rightarrow \quad y = x^2 + 3x - 10$

Since his friend is at P(7, 60),

$60 = (7)^2 + 3(7) - 10$

$\Rightarrow \quad 60 = 49 + 21 - 10$

$\Rightarrow \quad 60 = 70 - 10 = 60$

$\Rightarrow \quad 60 = 60$

Since (7, 60) satisfies his path , he can meet his friend who is at P(7, 60)

EXERCISE 1.6

1. Test for consistency and if possible solve the following system of equations by rank method.

(i) $x - y + 2z = 2, 2x + y + 4z = 7,$ $4x - y + z = 4$

(ii) $3x + y + z = 2, x - 3y + 2z = 1,$ $7x - y + 4z = 5.$

(iii) $2x + 2y + z = 5, x - y + z = 1,$ $3x + y + 2z = 4$

(iv) $2x - y + z = 2, 6x - 3y + 3z = 6,$ $4x - 2y + 2z = 4.$

Sol. (i) $x - y + 2z = 2, 2x + y + 4z = 7, 4x - y + z = 4$

The matrix form of the system is AX = B

where $A = \begin{bmatrix} 1 & -1 & 2 \\ 2 & 1 & 4 \\ 4 & -1 & 1 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $B = \begin{bmatrix} 2 \\ 7 \\ 4 \end{bmatrix}$

Applying elementary row operations on the augment matrix [A|B] we get,

$$[A|B] = \left[\begin{array}{ccc|c} 1 & -1 & 2 & 2 \\ 2 & 1 & 4 & 7 \\ 4 & -1 & 1 & 4 \end{array}\right]$$

$$\xrightarrow[R_3 \to R_3 - 4R_1]{R_2 \to R_2 - 2R_1} \left[\begin{array}{ccc|c} 1 & -1 & 2 & 2 \\ 0 & 3 & 0 & 3 \\ 0 & 3 & -7 & -4 \end{array}\right]$$

$$\xrightarrow{R_3 \to R_3 - R_2} \left[\begin{array}{ccc|c} 1 & -1 & 2 & 2 \\ 0 & 3 & 0 & 3 \\ 0 & 0 & -7 & -7 \end{array}\right]$$

Here $\rho(A) = 3$ and $\rho[A|B] = 3$

$\therefore \; \rho(A) = \rho[A|B] = 3$ = number of unknowns

Hence the system is consistent with unique solution.

Writing the equivalent equations from the row-echelon matrix, we get

$$x - y + 2z = 2 \quad ...(1)$$

$$3y = 3 \Rightarrow y = 1 \quad ...(2)$$

$$-7z = -7 \Rightarrow z = \frac{-7}{-7} = 1 \quad ...(3)$$

Solutions $y = 1$ and $z = 1$ in (1) we get,

$$x - 1 + 2(1) = 2$$

$\Rightarrow$ $x - 1 + 2 = 2$

$\Rightarrow$ $x + 1 = 2$

$\Rightarrow$ $x = 2 - 1 = 1$

$\Rightarrow$ $x = 1$

$\therefore x = 1, y = 1, z = 1$

Thus, solution set is $\{1, 1, 1\}$.

(ii) $3x + y + z = 2, x - 3y + 2z = 1,$ $7x - y + 4z = 5.$

Then matrix form of the system in AX = B

where $A = \begin{bmatrix} 3 & 1 & 1 \\ 1 & -3 & 2 \\ 7 & -1 & 4 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, B = \begin{bmatrix} 2 \\ 1 \\ 5 \end{bmatrix}$

Applying elementary row operations on the augment matrix [A|B] we get,

$$[A|B] = \left[\begin{array}{ccc|c} 3 & 1 & 1 & 2 \\ 1 & -3 & 2 & 1 \\ 7 & -1 & 4 & 5 \end{array}\right] \xrightarrow{R_2 \leftrightarrow R_1} \left[\begin{array}{ccc|c} 1 & -3 & 2 & 1 \\ 3 & 1 & 1 & 2 \\ 7 & -1 & 4 & 5 \end{array}\right]$$

$$\xrightarrow[R_3 \to R_3 - 7R_1]{R_2 \to R_2 - 3R_1} \left[\begin{array}{ccc|c} 1 & -3 & 2 & 1 \\ 0 & 10 & -5 & -1 \\ 0 & 20 & -10 & -2 \end{array}\right]$$

$$\xrightarrow{R_3 \to R_3 - 2R_2} \left[\begin{array}{ccc|c} 1 & -3 & 2 & 1 \\ 0 & 10 & -5 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Here $\rho(A) = 2$, and $\rho[A|B] = 2$ [since there only two non-zero rows]. So, $\rho(A) = \rho[A|B] = 2 < 3$, the given system is consistent with one parameter family of solutions. So, put $z = t$, $x \in \mathbb{R}$. Writing the equivalent equations from the row echelon matrix we get,

$$x - 3y + 2z = 1 \quad ...(1)$$

$$10y - 5z = -1 \quad ...(2)$$

$$z = t \quad ...(3)$$

(2) we comes $10y - 5t = -1$

$\Rightarrow$ $10y = 5t - 1$

$\Rightarrow$ $y = \frac{1}{10}[5t - 1]$

Also, from (1), $x - \frac{3}{10}[5t - 1] + 2t = 1$

$\Rightarrow x = \frac{3}{10}[5t - 1] - 2t + 1 = \frac{15t - 3 - 20t + 10}{10}$

$\Rightarrow x = \frac{1}{10}[-5t + 7]$

Hence the solution set is

$\{\frac{1}{10}(7 - 5t), \frac{1}{10}(5t - 1), t\}$ where $t \in \mathbb{R}$.

(iii) $2x + 2y + z = 5, x - y + z = 1, 3x + y + 2z = 4$

The matrix form of the given system is AX = B where

$$A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & -1 & 1 \\ 3 & 1 & 2 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, B = \begin{bmatrix} 5 \\ 1 \\ 4 \end{bmatrix}$$

Applying elementary row operations on the augmented matrix [A|B] we get,

$$[A|B] = \left[\begin{array}{ccc|c} 2 & 2 & 1 & 5 \\ 1 & -1 & 1 & 1 \\ 3 & 1 & 2 & 4 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ 2 & 2 & 1 & 5 \\ 3 & 1 & 2 & 4 \end{array}\right]$$

$$\xrightarrow[R_3 \to R_3 - 3R_1]{R_2 \to R_2 - 2R_2} \left[\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ 0 & 4 & -1 & 3 \\ 0 & 4 & -1 & -1 \end{array}\right]$$

$$\xrightarrow{R_3 \to R_3 - R_2} \left[\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ 0 & 4 & -1 & 3 \\ 0 & 0 & 0 & -4 \end{array}\right]$$

Here $\rho(A) = 2$ [$\because$ There are 2 non-Zero rows]

and $\rho[A|B] = 3$ [$\because$ There are 3 non –zero rows]

Here, $(A) \neq p[A|B]$

Hence, the given system is inconsistent and has no solution.

(iv) $2x - y + z = 2, 6x - 3y + 3z = 6,$ $4x - 2y + 2z = 4.$

The matrix form of the given system is AX = B where

$$A = \begin{bmatrix} 2 & -1 & 1 \\ 6 & -3 & 3 \\ 4 & -2 & 2 \end{bmatrix}$$

$$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad B = \begin{bmatrix} 2 \\ 6 \\ 4 \end{bmatrix}$$

Applying elementary row operations on the augment matrix[A|B], we get,

$$[A|B] = \left[\begin{array}{ccc|c} 2 & -1 & 1 & 2 \\ 6 & -3 & 3 & 6 \\ 4 & -2 & 2 & 4 \end{array}\right] \xrightarrow[R_3 \to R_3 - 2R_1]{R_2 \to R_2 - 3R_1} \left[\begin{array}{ccc|c} 2 & -1 & 1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Here $\rho(A) = 1$[$\because$ only one non zero row]

and $\rho[A|B] = 1$[$\because$ only one-zero row]

$\therefore$ $\rho(A) = \rho[A|B] = 1 < 3$, the given system is consistent and has two parameter family of solutions.

So, $z = t$ and $y = s$ where s, $t \in \mathbb{R}$.

Writing the equivalent equations from the row-echelon matrix, we get

$$2x - y + z = 2 \quad \text{...(1)}$$
$$y = s \quad \text{...(2)}$$
$$z = t \quad \text{...(3)}$$

Substituting (2) and (3) in (1) we get,

$$2x - s + t = 2$$
$$\Rightarrow \quad 2x = s - t + 2$$
$$\Rightarrow \quad x = \frac{1}{2}[s - t + 2]$$

$\therefore$ Solution set is $\{\frac{1}{2}(s - t + 2), s, t\}$

where $s,\ t \in \mathbb{R}$

2. Find the value of k for which the equations $kx - 2y + z = 1, x - 2ky + z = -2,$ $x - 2y + kz = 1$ have

(i) no solution

(ii) unique solution

(iii) infinitely many solution

Sol. $kx - 2y + z = 1, x - 2ky + z = -2, x - 2y + k = 1$

The matrix form of the system is AX = B where

$$A = \begin{bmatrix} k & -2 & 1 \\ 1 & -2k & 1 \\ 1 & -2 & k \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, B = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

Applying elementary row operation on the augment matrix [A|B] we get,

$$[A/B] = \left[\begin{array}{ccc|c} k & -2 & 1 & 1 \\ 1 & -2k & 1 & -2 \\ 1 & -2 & k & 1 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_3} \left[\begin{array}{ccc|c} 1 & -2 & k & 1 \\ 1 & -2k & 1 & -2 \\ k & -2 & 1 & 1 \end{array}\right]$$

$$\xrightarrow[R_3 \to R_3 - kR_1]{R_2 \to R_2 - R_1} \left[\begin{array}{ccc|c} 1 & -2 & k & 1 \\ 1 & -2k+2 & k & -3 \\ 0 & -2+2k & 1-k^2 & 1-k \end{array}\right]$$

$$\xrightarrow{R_3 \to R_3 + R_2} \begin{bmatrix} 1 & -2 & k & 1 \\ 0 & -2k+2 & 1-k & -3 \\ 0 & 0 & 1-k^2+1-k & -k-2 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & -2 & k & 1 \\ 0 & -2k+2 & 1-k & -3 \\ 0 & 0 & -k^2-k+2 & -k-2 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & -2 & k & 1 \\ 0 & -2k+2 & 1-k & -3 \\ 0 & 0 & (k+2)(1-k) & -k-2 \end{bmatrix} \quad \text{...(1)}$$

Case (i) when $k = 1$

$$[A|B] \to \begin{bmatrix} 1 & -2 & 1 & 1 \\ 0 & 0 & 0 & -3 \\ 0 & 0 & 0 & -3 \end{bmatrix} \xrightarrow{R_3 \to R_3 - R_2} \begin{bmatrix} 1 & -2 & 1 & 1 \\ 0 & 0 & 0 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Here $\rho(A) = 1$ and $\rho[A|B] = 2$

So, $\rho(A) \neq \rho[A|B] \Rightarrow$ The system has no solution.

Case (ii) When $k \neq 1, k \neq -2$

$$[A|B] \to \left[\begin{array}{ccc|c} 1 & -2 & k & 1 \\ 0 & -2k+2 & 1-k & -3 \\ 0 & 0 & \text{not zero} & \text{not zero} \end{array}\right]$$

$\Rightarrow \rho(A) = 3$ and $\rho[A|B] = 3$

so, $\rho(A) = \rho[A|B] = 3$ = the number of unknowns Hence, the system has unique solution.

Case (iii) when $k = -2$

$$\rho[A|B] \to \begin{bmatrix} 1 & -2 & -2 & 1 \\ 1 & 6 & 3 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Here $\rho(A) = 2$ and $\rho[A|B] = 2$

$\therefore \rho(A) = \rho[A|B] = 2 < 3$, the number of unknowns so the system is consistent with infinitely many solutions.

3. Investigate the values of λ and μ the system of linear equations $2x + 3y + 5z = 9$, $7x + 3y - 5z = 8$, $2x + 3y + \lambda z = \mu$, have

(i) no solution

(ii) a unique solution

(iii) an infinite number of solutions.

Sol. $2x + 3y = 9$, $7x + 3y - 5z = 8$, $2x + 3y + \lambda z = \mu$

The matrix form of the system is AX = B where

$$A = \begin{bmatrix} 2 & 3 & 5 \\ 7 & 3 & -5 \\ 2 & 3 & \lambda \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, B = \begin{bmatrix} 9 \\ 8 \\ \mu \end{bmatrix}$$

Applying elementary row operations on the augmented matrix [A|B] we get,

$$[A|B] = \left[\begin{array}{ccc|c} 2 & 3 & 5 & 9 \\ 7 & 3 & -5 & 8 \\ 2 & 3 & \lambda & \mu \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{ccc|c} 7 & 3 & -5 & 8 \\ 2 & 3 & 5 & 9 \\ 2 & 3 & \lambda & \mu \end{array}\right]$$

$$\xrightarrow[R_3 \to R_3 - R_2]{R_2 \to R_2 - \frac{2}{7}R_1} \left[\begin{array}{ccc|c} 7 & 3 & -5 & -8 \\ 0 & \frac{15}{7} & \frac{45}{7} & \frac{45}{7} \\ 0 & 0 & \lambda - 5 & 4 - 9 \end{array}\right]$$

$$\xrightarrow{R_2 \to R_2 \times 7} \left[\begin{array}{ccc|c} 7 & 3 & -5 & -8 \\ 0 & 15 & 45 & 47 \\ 0 & 0 & \lambda - 5 & \mu - 9 \end{array}\right]$$

Case (i) When $\lambda = 5$

$$[A|B] = \left[\begin{array}{ccc|c} 7 & 3 & -5 & -8 \\ 0 & 15 & 45 & 47 \\ 0 & 0 & 0 & -4 \end{array}\right]$$

Here $\rho(A) = 2$ and $\rho[A|B] = 3$

So, $\rho(A) \neq \rho[A|B]$

Hence the system is inconsistent and has no solution

Case (ii) When $\lambda \neq 5$, $\mu \neq 9$

$$[A|B] = \left[\begin{array}{ccc|c} 7 & 3 & -5 & -8 \\ 0 & 15 & 45 & 47 \\ 0 & 0 & \text{not zero} & \text{not zero} \end{array}\right]$$

Here $\rho(A) = 3$ and $\rho[A|B] = 3$

$\therefore$ $\rho(A) = \rho[A|B] = 3$ = number of unknowns

Hence, the system is consistent with unique solution

Case (iii) When $\lambda = 5$ and $\mu = 9$

$$[A|B] = \left[\begin{array}{ccc|c} 7 & 3 & -5 & -8 \\ 0 & 15 & 45 & 47 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Here $\rho(A) = 2$, $\rho[A|B] = 2$

$\therefore$ $\rho(A) = [A|B] = 2 <$ number of unknowns

$\therefore$ The system is consistent and has infinite number of solutions.

EXERCISE 1.7

1. Solve the following system of homogeneous equations.

(i) $3x + 2y + 7z = 0$, $4x - 3y - 2z = 0$, $5x + 9y + 23z = 0$

(ii) $2x + 3y - z = 0$, $x - y - 2z = 0$, $3x + y + 3z = 0$.

Sol. (i) $3x + 2y + 7z = 0$, $4x - 3y - 2z = 0$, $5x + 9y + 23z = 0$

Reducing the augmented matrix to row-echelon form we get,

$$[A|0] = \left[\begin{array}{ccc|c} 3 & 2 & 7 & 0 \\ 4 & -3 & -2 & 0 \\ 5 & 9 & 23 & 0 \end{array}\right] \xrightarrow{R_2 \to R_2 - \frac{4}{4}R_1} \left[\begin{array}{ccc|c} 3 & 2 & 7 & 0 \\ 0 & -\frac{17}{3} & \frac{-34}{3} & 0 \\ 5 & 9 & 23 & 0 \end{array}\right]$$

$$\xrightarrow{R_3 \to R_3 - \frac{5}{3}R_1} \left[\begin{array}{ccc|c} 2 & 2 & 7 & 0 \\ 0 & -\frac{17}{3} & \frac{-34}{3} & 0 \\ 5 & \frac{17}{3} & \frac{34}{3} & 0 \end{array}\right]$$

$$\xrightarrow{R_3 \to R_3 + R_2} \left[\begin{array}{ccc|c} 3 & 2 & 7 & 0 \\ 0 & -\frac{17}{3} & \frac{-34}{3} & 0 \\ 5 & 0 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{R_2 \to R_2 \times \frac{-3}{7}} \left[\begin{array}{ccc|c} 3 & 2 & 7 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Here $\rho(A) = 2$ and $\rho[A|0] = 2$

So, $\rho(A) = \rho([A|0]) = 2 < 3$ = number of unknowns

Here, the system is consistent and has one parameter family of solutions.

So put $z = t$ where $t \in \mathbb{R}$.
writing the equations using the echelon form, we get

$$3x + 2y + 7z = 0 \quad ...(1)$$
$$y + 2z = 0 \quad ...(2)$$
put $z = t$, (2) becomes
$$y + 2t = 0$$
$$\Rightarrow y = -2t$$
$\therefore$ (1) becomes, $3x + 2(-2t) + 7t = 0$
$$\Rightarrow 3x - 4t + 7t = 0$$
$$\Rightarrow 3x + 3t = 0$$
$$\Rightarrow 3x = -3t$$
$$\Rightarrow x = -t$$
$\therefore$ Solution set is $\{-t, -2t, t\}$ where $t \in \mathbb{R}$

(ii) $2x + 3y - z = 0, x - y - 2z = 0, 3x + y + 3z = 0.$

Reducing the augmented matrix to row – echelon form we get

$$[A|0] = \left[\begin{array}{ccc|c} 2 & 3 & -1 & 0 \\ 1 & -1 & -2 & 0 \\ 3 & 1 & 3 & 0 \end{array}\right]$$

$$\xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{ccc|c} 1 & -1 & -2 & 0 \\ 2 & 3 & -1 & 0 \\ 3 & 1 & 3 & 0 \end{array}\right]$$

$$\xrightarrow[R_3 \to R_3 - 3R_1]{R_2 \to R_2 - 2R_1} \left[\begin{array}{ccc|c} 1 & -1 & -2 & 0 \\ 0 & 5 & 3 & 0 \\ 0 & 4 & 9 & 0 \end{array}\right]$$

$$\xrightarrow{R_3 \to R_3 - \frac{4}{5}R_2} \left[\begin{array}{ccc|c} 1 & -1 & -2 & 0 \\ 0 & 5 & 3 & 0 \\ 0 & 0 & \frac{33}{5} & 0 \end{array}\right]$$

Here $\rho(A) = 3$ and $\rho([A|0]) = 3$

$\therefore \rho(A) = \rho([A|0]) = 3$ = Number of unknowns

Hence, the system is consistent with unique solutions.

Thus, the system has trivial solution only

2. Determine the values of λ for which the following system of equations $x + y + 3z = 0, 4x + 3y + \lambda z = 0, 2x + y + 2z = 0$ has

(i) a unique solution

(ii) a non-trivial solution.

Sol. $x + y + 3z = 0, 4x + 3y + \lambda z = 0, 2x + y + 2z = 0$

Reducing the augmented matrix to row – echelon form we get,

$$[A|0] = \left[\begin{array}{ccc|c} 1 & 1 & 3 & 0 \\ 4 & 3 & \lambda & 0 \\ 2 & 1 & 2 & 0 \end{array}\right]$$

$$\xrightarrow{R_1 \leftrightarrow R_3} \left[\begin{array}{ccc|c} 1 & 1 & 3 & 0 \\ 2 & 1 & 2 & 0 \\ 4 & 3 & \lambda & 0 \end{array}\right]$$

$$\xrightarrow[R_3 \to R_3 - 4R_1]{R_2 \to R_2 - 2R_1} \left[\begin{array}{ccc|c} 1 & 1 & 3 & 0 \\ 0 & -1 & -4 & 0 \\ 0 & -1 & \lambda - 2 & 0 \end{array}\right]$$

$$\xrightarrow{R_3 \to R_3 - R_2} \left[\begin{array}{ccc|c} 1 & 1 & 3 & 0 \\ 0 & -1 & -4 & 0 \\ 0 & 0 & \lambda - 8 & 0 \end{array}\right]$$

Case (i) when $\lambda \neq 8$

$$[A|0] = \left[\begin{array}{ccc|c} 1 & 1 & 3 & 0 \\ 0 & -1 & -4 & 0 \\ 0 & 0 & \text{not zero} & 0 \end{array}\right]$$

Here $\rho(A) = 3$, $\rho([A/O]) = 3$

$\therefore \rho(A) = \rho([A/0]) = 3$ = the number of unknowns

$\therefore$ The given system is consistent and has unique solution.

Case (ii) when $\lambda = 8$

$$[A|0] = \left[\begin{array}{ccc|c} 1 & 1 & 3 & 0 \\ 0 & -1 & -4 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Here $\rho(A) = 2$, $\rho([A|0]) = 2$

$\therefore \rho(A) = \rho([A|0]) = 2 < 3$, the number of unknowns,

$\therefore$ The system is consistent and has non-trivial solutions.

3. By using Gaussian elimination method, balance the chemical reaction equation: $C_2H_6 + O_2 \to H_2O + CO_2$

Sol. Given $C_2H_6 + O_2 \to H_2O + CO_2$

We have to find positive integers x_1, x_2, x_3 and x_4 such that

$$x_1 C_2H_6 + x_2 O_2 \to x_3 H_2O + x_4 CO_2 \quad ...(1)$$

The number of carbon atoms on the LHS of (1) should be equal to the number of carbon atoms on the RHS of (1).

$$\therefore 2x_1 = 1x_4$$
$$\Rightarrow 2x_1 - x_4 = 0 \quad ...(2)$$

Considering hydrogen atoms we get,

$$6x_1 = 2x_3$$
$$\Rightarrow 6x_1 - 2x_3 = 0$$
$$\Rightarrow 3x_1 - x_3 = 0 \quad ...(3)$$

Also, considering oxygen atoms we get,

$$2x_2 = 1x_3 + 2x_4$$
$$\Rightarrow 2x_2 - x_3 - 2x_4 = 0 \quad ...(3)$$

Equations (2), (3) and (4) form a homogeneous system of linear equations in 4 unknowns

$\therefore$ The augmented matrix [A|0] is

$$\left[\begin{array}{cccc|c} 2 & 0 & 0 & -1 & 0 \\ 3 & 0 & -1 & 0 & 0 \\ 0 & 2 & -1 & -2 & 0 \end{array}\right]$$

By Gaussian elimination method, we get,

$$\xrightarrow{R_1 \to R_1 \div 2} \left[\begin{array}{cccc|c} 1 & 0 & 0 & \frac{-1}{2} & 0 \\ 3 & 0 & -1 & 0 & 0 \\ 0 & 2 & -1 & -2 & 0 \end{array}\right]$$

$$\xrightarrow{R_2 \to R_2 - 3R_1} \left[\begin{array}{cccc|c} 1 & 0 & 0 & \frac{-1}{2} & 0 \\ 0 & 0 & 1 & \frac{3}{2} & 0 \\ 0 & 2 & -1 & -2 & 0 \end{array}\right]$$

$$\xrightarrow{R_3 \to R_3 - R_2} \left[\begin{array}{cccc|c} 1 & 0 & 0 & \frac{-1}{2} & 0 \\ 0 & 0 & -1 & \frac{3}{2} & 0 \\ 0 & 2 & 0 & \frac{-7}{2} & 0 \end{array}\right]$$

$$\xrightarrow{R_2 \leftrightarrow R_3} \left[\begin{array}{cccc|c} 1 & 0 & 0 & \frac{-1}{2} & 0 \\ 0 & 2 & 0 & \frac{-7}{2} & 0 \\ 0 & 0 & -1 & \frac{3}{2} & 0 \end{array}\right]$$

$$\xrightarrow[R_2 \to 2R_3]{R_1 \to 2R_1 \atop R_2 \to 2R_2} \left[\begin{array}{cccc|c} 2 & 0 & 0 & -1 & 0 \\ 0 & 4 & 0 & -7 & 0 \\ 0 & 0 & -2 & 3 & 0 \end{array}\right]$$

Here $\rho(A) = \rho([A|B]) = 3 < 4$, the number of unknowns

$\therefore$ The system is consistent with one parameter family of solutions, so let $x^4 = t$

Writing the equations, from the row-echelon form we get

$$2x_1 - x_4 = 0$$
$$\Rightarrow 2x_1 = x_4$$
$$\Rightarrow 2x = t$$
$$\Rightarrow x_1 = \frac{t}{2}$$
$$4x_2 - 7x_4 = 0$$
$$\Rightarrow 4x_2 = 7t$$
$$\Rightarrow x_2 = \frac{7t}{4}$$
$$-2x_3 + 3x_4 = 0$$
$$\Rightarrow 2x_3 = 3x_4$$
$$\Rightarrow x_3 = \frac{3t}{2}$$

Since x_1, x_2, x_3 and x_4 are positive integers,

let us choose $t = 4$

$$\therefore x_1 = \frac{4}{2} = 2$$
$$\therefore x_2 = \frac{7(\cancel{4})}{\cancel{4}} = 7$$
$$\therefore x_3 = \frac{3(\cancel{4})}{\cancel{2}} = 6 \text{ and } x_4 = t = 4$$

So, the balanced equation is

$2C_2H_6 + 7O_2 \rightarrow 6H_2O + 4CO_2$.

EXERCISE 1.8

Choose the Correct or the most suitable answer from the given four alternatives :

1. **If |adj (adj A) | = $|A|^9$, then the order of the square matrix A is**

(1) 3 (2) 4 (3) 2 (4) 5

[Ans. (2) 4]

Hint : $|\text{adj (adj}A)| = |A|^{(n-1)^2}$

$$\therefore (n-1)^2 = 9$$
$$\Rightarrow (n-1)^2 = 3^2$$
$$\Rightarrow n - 1 = 3$$
$$\Rightarrow n = 4$$

2. If A is a 3 × 3 non-singular matrix such that $AA^T = A^T A$ and $B = A^{-1}A^T$, then $BB^T =$

(1) A (2) B (3) I_3 (4) B^T

[Ans. (3) I_3]

Hint :

$$\begin{aligned} BB^T &= (A^{-1}A^T)(A^{-1}A^T)^T \\ &= (A^{-1}A^T)\,(A^T)^T\cdot(A^{-1})^T \\ &= (A^{-1}A^T)\,(AA^{-1})^T \\ &= A^{-1}(A.A^T)(A^{-1})^T \\ &= (A^{-1}A).\,A^T\,(A^T)^{-1} \quad [\because (A^{-1})^T = (A^T)^{-1}] \\ &= I\cdot I = I \quad [\because A^T\cdot(A^T)^{-1} = I] \end{aligned}$$

3. $A = \begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix}$, B = adj A and C = 3A, then $\frac{|\text{adj } B|}{|C|} =$

(1) $\frac{1}{3}$ (2) $\frac{1}{9}$ (3) $\frac{1}{4}$ (4) 1

[Ans. (2) $\frac{1}{9}$]

Hint :

$$\frac{|\text{adj} B|}{|C|} = \frac{|\text{adj}(\text{adj } A)|}{|3A|} = \frac{|A|^{(n-1)^2}}{|3|^2.|A|} = \frac{|A|^{12}}{|B|^2.|A|} = \frac{|A|}{9.|A|} = \frac{1}{9}$$

4. If $A\begin{bmatrix} 1 & -2 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} 6 & 0 \\ 0 & 6 \end{bmatrix}$, then A =

(1) $\begin{bmatrix} 1 & -2 \\ 1 & 4 \end{bmatrix}$ (2) $\begin{bmatrix} 1 & 2 \\ -1 & 4 \end{bmatrix}$

(3) $\begin{bmatrix} 4 & 2 \\ -1 & 1 \end{bmatrix}$ (4) $\begin{bmatrix} 4 & -1 \\ 2 & 1 \end{bmatrix}$

[Ans. (3) $\begin{bmatrix} 4 & 2 \\ -1 & 1 \end{bmatrix}$]

Hint :

$$AX = B$$

$$\Rightarrow A = BX^{-1} \text{ where}$$

$$X = \begin{bmatrix} 1 & -2 \\ 1 & 4 \end{bmatrix}, B = \begin{bmatrix} 6 & 0 \\ 0 & 6 \end{bmatrix}$$

$$A = \begin{bmatrix} 6 & 0 \\ 0 & 6 \end{bmatrix}\cdot\frac{1}{|X|}\cdot \text{adj}(X)$$

$$= \cancel{6}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\cdot\frac{1}{\cancel{6}}\begin{bmatrix} 4 & 2 \\ -1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 4 & 2 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 2 \\ -1 & 1 \end{bmatrix}$$

5. If $A = \begin{bmatrix} 7 & 3 \\ 4 & 2 \end{bmatrix}$, then $9I_2 - A =$

(1) A^{-1} (2) $\frac{A^{-1}}{2}$ (3) $3A^{-1}$ (4) $2A^{-1}$

[Ans. (4) $2A^{-1}$]

Hint :

$$9I - A = \begin{bmatrix} 9 & 0 \\ 0 & 9 \end{bmatrix} - \begin{bmatrix} 7 & 3 \\ 4 & 2 \end{bmatrix} = \begin{bmatrix} 9-7 & 0-3 \\ 0-4 & 9-2 \end{bmatrix} = \begin{bmatrix} 2 & -3 \\ -4 & 7 \end{bmatrix} = \text{adj } A$$

$$\text{But } A^{-1} = \frac{1}{|A|}\text{adj } A = \frac{1}{2} \Rightarrow \text{adj } A = 2A^{-1}$$

6. If $A = \begin{bmatrix} 2 & 0 \\ 1 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 4 \\ 2 & 0 \end{bmatrix}$ then |adj (AB)| =

(1) –40 (2) –80 (3) –60 (4) –20

[Ans. (2) –80]

Hint :

$$AB = \begin{bmatrix} 2+0 & 8+0 \\ 1+10 & 4+0 \end{bmatrix} = \begin{bmatrix} 2 & 8 \\ 11 & 4 \end{bmatrix}$$

$$\text{adj}(AB) = \begin{bmatrix} 4 & -8 \\ -11 & 2 \end{bmatrix}$$

$$|\text{adj}(AB)| = 8 - 88 = -80$$

7. If $P = \begin{vmatrix} 1 & x & 0 \\ 1 & 3 & 0 \\ 2 & 4 & -2 \end{vmatrix}$ is the adjoint of 3 × 3 matrix A and |A| = 4, then x is

(1) 15 (2) 12 (3) 14 (4) 11

[Ans. (4) 11]

Hint : $|\text{adj } A| = |A|^{n-1}$

$$1\begin{vmatrix} 3 & 0 \\ 4 & -2 \end{vmatrix} - x\begin{vmatrix} 1 & 0 \\ 2 & -2 \end{vmatrix} + 0 = 4^{3-1}$$

$$\begin{aligned} \Rightarrow -6 - x(-2) &= 4^2 \\ \Rightarrow -6 + 2x &= 16 \\ \Rightarrow 2x &= 22 \\ \Rightarrow x &= 11 \end{aligned}$$

8. **If** $A=\begin{bmatrix}3&1&-1\\2&-2&0\\1&2&-1\end{bmatrix}$ **and** $A^{-1}=\begin{bmatrix}a_{11}&a_{12}&a_{13}\\a_{21}&a_{22}&a_{23}\\a_{31}&a_{32}&a_{33}\end{bmatrix}$

then the value of a_{23} is

(1) 0 (2) −2 (3) −3 (4) −1

[Ans. (4) −1]

Hint : $|A| = 3\begin{vmatrix}-2&0\\2&-1\end{vmatrix}-1\begin{vmatrix}2&0\\1&-1\end{vmatrix}-1\begin{vmatrix}2&-2\\1&2\end{vmatrix}$

$= 3(2)-1(-2)-1(4+2)$

$= 6+2-6=2$

$a_{23} = \frac{1}{|A|}$ co-factor of a_{32}

$= \frac{1}{2}\times-\begin{vmatrix}3&-1\\2&0\end{vmatrix}$

$= -\frac{1}{2}(0+2)=\frac{-2}{2}=-1$

9. **If A, B and C are invertible matrices of some order, then which one of the following is not true?**

(1) $\text{adj } A = |A|A^{-1}$

(2) $\text{adj }(AB) = (\text{adj } A)(\text{adj } B)$

(3) $\det A^{-1} = (\det A)^{-1}$

(4) $(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$

[Ans. (2) adj (AB) = (adj A) (adj B)]

10. **If** $(AB)^{-1}=\begin{bmatrix}12&-17\\-19&27\end{bmatrix}$ **and** $A^{-1}=\begin{bmatrix}1&-1\\-2&3\end{bmatrix}$, **then** $B^{-1}=$

(1) $\begin{bmatrix}2&-5\\-3&8\end{bmatrix}$ (2) $\begin{bmatrix}8&5\\3&2\end{bmatrix}$

(3) $\begin{bmatrix}3&1\\2&1\end{bmatrix}$ (4) $\begin{bmatrix}8&-5\\-3&2\end{bmatrix}$

[Ans. (1) $\begin{bmatrix}2&-5\\-3&8\end{bmatrix}$]

Hint : Since $(AB)^{-1}=B^{-1}A^{-1}$, we get,

$\begin{bmatrix}12&-17\\-19&27\end{bmatrix} = B^{-1}\begin{bmatrix}1&-1\\-2&3\end{bmatrix}$

Let $X = B^{-1}Y$

$B^{-1}=XY^{-1} = \begin{bmatrix}12&-17\\-19&27\end{bmatrix}\cdot\frac{1}{|Y|}.(\text{adj } Y)$

$= \begin{bmatrix}12&-17\\-19&27\end{bmatrix}\cdot\frac{1}{1}\begin{bmatrix}3&1\\2&1\end{bmatrix}$

$= \begin{bmatrix}12&-17\\-19&27\end{bmatrix}\begin{bmatrix}3&1\\2&1\end{bmatrix}$

$= \begin{bmatrix}36-34&12-17\\-57+54&-19+27\end{bmatrix}=\begin{bmatrix}2&-5\\-3&8\end{bmatrix}$

11. **If $A^T. A^{-1}$ is symmetric, then A^2 =**

(1) A^{-1} (2) $(A^T)^2$

(3) A^T (4) $(A^{-1})^2$

[Ans : (2) $(A^T)^2$]

Hint : $\Rightarrow$ $A^TA^{-1} = (A^TA^{-1})^T$

$= (A^{-1})^T(A^T)^T = (A^{-1})^TA$

$\Rightarrow$ $A = A^T \Rightarrow A$ is symmetric

$\therefore A^2 = (A^T)^2$

12. **If A is a non-singular matrix such that** $A^{-1}=\begin{bmatrix}5&3\\-2&-1\end{bmatrix}$, **then** $(A^T)^{-1}=$

(1) $\begin{bmatrix}-5&3\\2&1\end{bmatrix}$ (2) $\begin{bmatrix}5&3\\-2&-1\end{bmatrix}$

(3) $\begin{bmatrix}-1&-3\\2&5\end{bmatrix}$ (4) $\begin{bmatrix}5&-2\\3&-1\end{bmatrix}$

[Ans : (4) $\begin{bmatrix}5&-2\\3&-1\end{bmatrix}$]

Hint : $(A^T)^{-1}=(A^{-1})^T=\begin{bmatrix}5&-2\\3&-1\end{bmatrix}$

13. $A=\begin{bmatrix}\frac{3}{5}&\frac{4}{5}\\x&\frac{3}{5}\end{bmatrix}$ **and $A^T=A^{-1}$, then the value of x is**

(1) $\frac{-4}{5}$ (2) $\frac{-3}{5}$ (3) $\frac{3}{5}$ (4) $\frac{4}{5}$

[Ans: (1) $\frac{-4}{5}$]

Hint : Since $A^T=A^{-1}$, $AA^T=A^TA=I$ [$\therefore$ they are orthogonal]

$\Rightarrow \begin{bmatrix}\frac{3}{5}&\frac{4}{5}\\x&\frac{3}{5}\end{bmatrix}\begin{bmatrix}\frac{3}{5}&x\\\frac{4}{5}&\frac{3}{5}\end{bmatrix}=\begin{bmatrix}1&0\\0&1\end{bmatrix}$

$\Rightarrow \begin{bmatrix}\frac{9}{25}+\frac{16}{25}&\frac{3x}{5}+\frac{12}{25}\\\frac{3x}{5}+\frac{12}{25}&x^2+\frac{9}{25}\end{bmatrix}=\begin{bmatrix}1&0\\0&1\end{bmatrix}$

$$\Rightarrow \frac{3x}{5}+\frac{12}{25} = 0 \text{ [Equating } a_{12} \text{ both sides]}$$

$$\Rightarrow \frac{3x}{5} = \frac{-12}{25} \Rightarrow x = \frac{-12}{25} \times \frac{5}{3} = \frac{-4}{5}$$

14. **If** $A = \begin{bmatrix} 1 & \tan\frac{\theta}{2} \\ -\tan\frac{\theta}{2} & 1 \end{bmatrix}$ **and AB =** I_2**, then B =**

(1) $\left(\cos^2\frac{\theta}{2}\right)A$ (2) $\left(\cos^2\frac{\theta}{2}\right)A^T$

(3) $(\cos^2\theta)I$ (4) $\left(\sin^2\frac{\theta}{2}\right)A$

[Ans: (2) $\left(\cos^2\frac{\theta}{2}\right)A^T$ **]**

Hint : $B = A^{-1} = \frac{1}{|A|}\text{ adj } A$

$$= \frac{1}{1+\tan^2\frac{\theta}{2}}\begin{bmatrix} 1 & -\tan\frac{\theta}{2} \\ \tan\frac{\theta}{2} & 1 \end{bmatrix} = \frac{1}{\sec^2\frac{\theta}{2}}.A^T$$

$$= \left(\cos^2\frac{\theta}{2}\right)A^T$$

15. $A = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$ **and A (adj A) =** $\begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}$**, then = k**

(1) 0 (2) sin θ (3) cos θ (4) 1

[Ans: (4) 1]

Hint : We know A (adj A) = (adj A)A = |A| I

$\Rightarrow |A| = K$

$$\therefore K = \begin{vmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{vmatrix} = \cos^2\theta + \sin^2\theta = 1$$

16. **If A =** $\begin{bmatrix} 2 & 3 \\ 5 & -2 \end{bmatrix}$ **be such that** $\lambda A^{-1} = A$**, then λ is**

(1) 17 (2) 14 (3) 19 (4) 21

[Ans: (3) 19]

Hint : $\lambda.\frac{1}{|A|}\text{ adj } A = A$

$$\Rightarrow \lambda.\frac{1}{(-4-15)}.\begin{bmatrix} -2 & -3 \\ -5 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 5 & -2 \end{bmatrix}$$

$$\Rightarrow \frac{-\lambda}{19}\begin{bmatrix} -2 & -3 \\ -5 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 5 & -2 \end{bmatrix}$$

$$\Rightarrow \frac{\lambda}{19}\begin{bmatrix} 2 & 3 \\ 5 & -2 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 5 & -2 \end{bmatrix}$$

$$\Rightarrow \frac{\lambda}{19} = 1 \Rightarrow \lambda = 19$$

17. **If adj A =** $\begin{bmatrix} 2 & 3 \\ 4 & -1 \end{bmatrix}$ **and adj B =** $\begin{bmatrix} 1 & -2 \\ -3 & 1 \end{bmatrix}$ **then adj (AB) is**

(1) $\begin{bmatrix} -7 & -1 \\ 7 & -9 \end{bmatrix}$ (2) $\begin{bmatrix} -6 & 5 \\ -2 & -10 \end{bmatrix}$

(3) $\begin{bmatrix} -7 & 7 \\ -1 & -9 \end{bmatrix}$ (4) $\begin{bmatrix} -6 & -2 \\ 5 & -10 \end{bmatrix}$

[Ans: (2) $\begin{bmatrix} -6 & 5 \\ -2 & -10 \end{bmatrix}$**]**

Hint : adj (AB) = (adj B) (adj A)

$$= \begin{bmatrix} 1 & -2 \\ -3 & 1 \end{bmatrix}\begin{bmatrix} 2 & 3 \\ 4 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 2-8 & 3+2 \\ -6+4 & -9-1 \end{bmatrix} = \begin{bmatrix} -6 & 5 \\ -2 & -10 \end{bmatrix}$$

18. **The rank of the matrix** $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 6 & 8 \\ -1 & -2 & -3 & -4 \end{bmatrix}$ **is**

(1) 1 (2) 2 (3) 4 (4) 3

[Ans: (1) 1]

Hint : $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 6 & 8 \\ -1 & -2 & -3 & -4 \end{bmatrix} \xrightarrow[R_3 \to R_3 + R_1]{R_2 \to R_2 - 2R_1} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

∴ Rank is 1 [∴ only one non – zero row]

19. **If** $x^a y^b = e^m, x^c y^d = e^n, \Delta_1 = \begin{vmatrix} m & b \\ n & d \end{vmatrix}, \Delta_2 = \begin{vmatrix} a & m \\ c & n \end{vmatrix}$, $\Delta_3 = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$**, then the value of x and y are respectively,**

(1) $e^{(\Delta_2/\Delta_1)}, e^{(\Delta_3/\Delta_1)}$

(2) $\log(\Delta_1/\Delta_3), \log(\Delta_2/\Delta_3)$

(3) $\log(\Delta_2/\Delta_1), \log(\Delta_3/\Delta_1)$

(4) $e^{(\Delta_1/\Delta_3)}, e^{(\Delta_2/\Delta_3)}$

[Ans: (4) $e^{(\Delta_1/\Delta_3)}, e^{(\Delta_2/\Delta_3)}$ **]**

Hint :
$$x^a y^b = e^m$$
$$\Rightarrow a\log x + b\log y = m \quad \text{[Taking log both sides]}$$
$$x^c . y^d = e^n$$
$$\Rightarrow c\log x + d\log y = n$$
$$\text{put } \Delta_3 = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$
$$\Delta_1 = \begin{vmatrix} m & b \\ n & d \end{vmatrix}, \Delta_2 = \begin{vmatrix} a & m \\ c & n \end{vmatrix}$$
$$\log x = \frac{\Delta_1}{\Delta_3} \text{ and } \log y = \frac{\Delta_2}{\Delta_3}$$
$$\therefore x = e^{\Delta_1/\Delta_3} \text{ and } y = e^{\Delta_2/\Delta_3}$$

20. Which of the following is/are correct?

(i) Adjoint of a symmetric matrix is also a symmetric matrix.
(ii) Adjoint of a diagonal matrix is also a diagonal matrix.
(iii) If A is a square matrix of order n and λ is a scalar, then adj $(\lambda A) = \lambda^n$ adj (A).
(iv) A(adjA) = (adj A) A = |A|I

(1) Only (i) (2) (ii) and (iii)
(3) (iii) and (iv) (4) (i), (ii) and (iv)

[Ans: (4) (i) (ii) and (iv)]

21. If $\rho(A) = \rho([A|B])$, then the system AX = B of linear equations is

(1) consistent and has a unique solution
(2) consistent
(3) consistent and has infinitely many solution
(4) inconsistent **[Ans: (2) Consistent]**

22. If $0 \le \theta \le \pi$ and the system of equations $x + (\sin\theta)y - (\cos\theta)z = 0$, $(\cos\theta)x - y + z = 0$, $(\sin\theta)x + y - z = 0$ has a non-trivial solution then θ is

(1) $\frac{2\pi}{3}$ (2) $\frac{3\pi}{4}$ (3) $\frac{5\pi}{6}$ (4) $\frac{\pi}{4}$

[Ans: (4) $\frac{\pi}{4}$]

Hint : $A = \begin{bmatrix} 1 & \sin\theta & -\cos\theta \\ \cos\theta & -1 & 1 \\ \sin\theta & 1 & -1 \end{bmatrix}$

The system has non – trivial solution if |A| = 0

$$\begin{bmatrix} 1 & \sin\theta & -\cos\theta \\ \cos\theta & -1 & 1 \\ \sin\theta & 1 & 1 \end{bmatrix} = 0$$

$$\Rightarrow 1\begin{vmatrix} -1 & 1 \\ 1 & -1 \end{vmatrix} - \sin\theta\begin{vmatrix} \cos\theta & 1 \\ \sin\theta & -1 \end{vmatrix} - \cos\theta\begin{vmatrix} \cos\theta & -1 \\ \sin\theta & 1 \end{vmatrix} = 0$$

$\Rightarrow 1(1-1) - \sin\theta(-\cos\theta - \sin\theta) - \cos\theta(\cos\theta + \sin\theta) = 0$

$\Rightarrow \sin\theta\cancel{\cos\theta} + \sin^2\theta - \cos^2\theta - \cancel{\cos\theta}\sin\theta = 0$

$\Rightarrow \cos^2\theta - \sin^2\theta = 0 \Rightarrow \cos 2\theta = 0$

$\Rightarrow \cos 2\theta = \cos\frac{\pi}{2}$ $\left[\because \cos\frac{\pi}{2} = 0\right]$

$\Rightarrow 2\theta = \frac{\pi}{2} \Rightarrow \theta = \frac{\pi}{4}$

23. The augmented matrix of a system of linear equations is $\begin{bmatrix} 1 & 2 & 7 & 3 \\ 0 & 1 & 4 & 6 \\ 0 & 0 & \lambda-7 & \mu+5 \end{bmatrix}$. The system has infinitely many solutions if

(1) $\lambda = 7, \mu \ne -5$ (2) $\lambda = -7, \mu = -5$
(3) $\lambda \ne 7, \mu \ne -5$ (4) $\lambda = 7, \mu = -5$

[Ans: (4) $\lambda = 7, \mu = -5$]

Hint : When $\lambda = 7$ and $\mu = -5$,

$$[A|B] = \begin{bmatrix} 1 & 2 & 7 & 3 \\ 0 & 1 & 4 & 6 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\rho(A) = \rho([A|B]) = 2 < 3$, the number of unknowns.

$\therefore$ The system is consistent and has infinitely many solutions.

24. Let $A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$ and $4B = \begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & x \\ -1 & 1 & 3 \end{bmatrix}$. If B is the inverse of A, then the value of x is

(1) 2 (2) 4 (3) 3 (4) 1

[Ans: (4) 1]

Hint : $A = B^{-1} \Rightarrow A . B = B^{-1}.B \Rightarrow AB = I$

$$\therefore \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} \frac{1}{4}\begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & x \\ -1 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\frac{1}{4}a_{13} = 0$$
$$\Rightarrow \frac{1}{4}.[-2 - x + 3] = 0$$
$$\Rightarrow -x + 1 = 0 \times 4 = 0$$
$$\Rightarrow x = 1$$

25. **If** $A = \begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}$**, then adj (adj A) is**

(1) $\begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}$ (2) $\begin{bmatrix} 6 & -6 & 8 \\ 4 & -6 & 8 \\ 0 & -2 & 2 \end{bmatrix}$

(3) $\begin{bmatrix} -3 & 3 & -4 \\ -2 & 3 & -4 \\ 0 & 1 & -1 \end{bmatrix}$ (4) $\begin{bmatrix} 3 & -3 & 4 \\ 0 & -1 & 1 \\ 2 & -3 & 4 \end{bmatrix}$

[Ans: (1) $\begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}$**]**

Hint :
$$\text{adj (adj A)} = |A|^{n-2} A = |A|.A$$
$$|A| = \begin{vmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{vmatrix}$$
$$= 3\begin{vmatrix} -3 & 4 \\ -1 & 1 \end{vmatrix} + 3\begin{vmatrix} 2 & 4 \\ 0 & 1 \end{vmatrix} + 4\begin{vmatrix} 2 & -3 \\ 0 & -1 \end{vmatrix}$$
$$= 3(-3+4)+3(2-0)+4(-2-0)$$
$$= 3(1)+6-8=1$$
$$\therefore \text{adj (adj A)} = 1-A = A = \begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}$$

ADDITIONAL QUESTIONS

1 MARK

I. Choose the Correct or the most suitable answer from the given four alternatives :

1. **The system of linear equations** $x + y + z = 6$, $x + 2y + 3z = 14$ **and** $2x + 5y + \lambda z = \mu$ $(\lambda, \mu \in \mathbb{R})$ **is consistent with unique solution if**

(1) $\lambda = 8$ (2) $\lambda = 8, \mu \neq 36$

(3) $\lambda \neq 8$ (4) none **[Ans:** (3) $\lambda \neq 8$**]**

Hint : $\left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 1 & 2 & 3 & 14 \\ 2 & 5 & \lambda & \mu \end{array}\right] \xrightarrow[R_3 \to R_3 - 2R_1]{R_1 \to R_2 - R_1} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 8 \\ 0 & 3 & \lambda-2 & \mu-12 \end{array}\right]$

$$\xrightarrow{R_3 \to R_3 - 3R_2} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 8 \\ 0 & 0 & \lambda-8 & \mu-36 \end{array}\right]$$

2. **If the system of equations** $x = cy + bz$, $y = az + cx$ **and** $z = bx + ay$ **has a non – trivial solution then**

(1) $a^2 + b^2 + c^2 = 1$ (2) $abc \neq 1$
(3) $a + b + c = 0$ (4) $a^2 + b^2 + c^2 + 2abc = 1$

[Ans: (4) $a^2 + b^2 + c^2 + 2abc = 1$**]**

Hint : $\begin{vmatrix} 1 & -c & -b \\ c & -1 & a \\ b & a & -1 \end{vmatrix} = 0$

$\Rightarrow 1[1 - a^2] + c[-c - ab] - b[ac + b] = 0$
$\Rightarrow 1 - a^2 - c^2 - abc - abc - b^2 = 0$

3. **Let A be a** 3×3 **matrix and B its adjoint matrix If** $|B| = 64$**, then** $|A| =$

(1) ± 2 (2) ± 4 (3) ± 8 (4) ± 12

[Ans: (3) ± 8**]**

Hint : $A^{-1} = \frac{1}{|A|}\text{adj } A = \frac{1}{|A|}B$

$$\Rightarrow |A|A^{-1} = B$$
$$\Rightarrow |A|^3\,|A^{-1}| = |B| = 64$$
$$\Rightarrow |A|^2 = 64$$
$$\Rightarrow |A| = \pm 8$$

4. **If** A^T **is the transpose of a square matrix A, then**

(1) $|A| \neq |A^T|$ (2) $|A| = |A^T|$
(3) $|A| + |A^T| = 0$ (4) $|A| = |A^T|$ only
When A is symmetric **[Ans:** (2) $|A| = |A^T|$**]**

5. **The number of solutions of the system of equations** $2x + y = 4$, $x - 2y = 2$, $3x + 5y = 6$ **is**

(1) 0 (2) 1
(3) 2 (4) infinitely many

[Ans: (2) 1**]**

Hint : $\left[\begin{array}{ccc|c} 1 & -2 & 2 \\ 2 & 1 & 4 \\ 3 & 5 & 6 \end{array}\right] \xrightarrow[R_3 \to R_3 - 3R_2]{R_2 \to R_2 - 2R_1} \left[\begin{array}{cc|c} 1 & -2 & 2 \\ 0 & 5 & 0 \\ 0 & 11 & 0 \end{array}\right]$

$\xrightarrow{R_3 \to R_3 - \frac{11}{5}R_2} \left[\begin{array}{cc|c} 1 & -2 & 2 \\ 0 & 5 & 0 \\ 0 & 0 & 0 \end{array}\right]$

6. **If A is a square matrix that |A| = 2, than for any positive integer n, $|A^n|$ =**

(1) 0 (2) $2n$ (3) 2^n (4) n^2

[Ans: (3) 2^n**]**

Hint : $|A^n| = |A|\,|A| \ldots$ times
$= (2)\,(2) \ldots$ times $= 2^n$

7. **The system of linear equations $x + y + z = 2$, $2x + y - z = 3$, $3x + 2y + kz =$ has a unique solution if**

(1) $k \neq 0$ (2) $-1 < k < 1$
(3) $-2 < k < 2$ (4) $k = 0$ **[Ans:** (1) $k \neq 0$**]**

Hint : $\begin{vmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ 3 & 2 & k \end{vmatrix} \neq 0$

$\Rightarrow 1[k + 2] - 1[2k + 3] + 1\,[4 - 3] \neq 0$
$\Rightarrow k + 2 - 2k - 3 + 1 \neq 0$
$\Rightarrow -k + 0 \neq 0 \Rightarrow k \neq 0$

8. **If A is a square matrix of order n, then |adj A| =**

(1) $|A|^{n-1}$ (2) $|A|^{n-2}$ (3) $|A|^n$ (4) None

[Ans: (1) $|A|^{n-1}$**]**

Hint : adj A $= |A|\, A^{-1}$
$\Rightarrow$ |adj A| $= |A|^n\, |A^{-1}| = |A|^{n-1}$

9. **If the system of equations $x + 2y - 3x = 2$, $(k + 3)\, z = 3$, $(2k + 1)\, y + z = 2$ is inconsistent then k is**

(1) $-3, -\frac{1}{2}$ (2) $-\frac{1}{2}$
(3) 1 (4) 2 **[Ans:** (1) $-3, -\frac{1}{2}$**]**

10. **If A = $\begin{pmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{pmatrix}$ and A (adj A) = λ $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ then λ is**

(1) $\sin x \cos x$ (2) 1
(3) 2 (4) none **[Ans:** (2) 1**]**

Hint : A (adj A) $= \begin{pmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{pmatrix} \begin{pmatrix} \cos x & -\sin x \\ \sin x & \cos x \end{pmatrix}$

$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

11. **If A is a matrix of order $m \times n$, then ρ(A) is**

(1) m (2) n
(3) $\leq \min(m, n)$ (4) $\geq \min(m, n)$

[Ans: (3) $\leq \min(m, n)$**]**

12. **The system of equations $x + 2y + 3z = 1$, $x - y + 4z = 0$, $2x + y + 7z = 1$ has**

(1) One solution (2) Two solution
(3) No solution
(4) Infinitely many solution

[Ans: (3) Infinitely many solution**]**

Hint :
$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 1 & -1 & 4 & 0 \\ 2 & 1 & 7 & 1 \end{array}\right] \xrightarrow[R_3 \to R_3 - 2R_1]{R_2 \to R_2 - R_1} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & -3 & 1 & -1 \\ 0 & -3 & 1 & -1 \end{array}\right]$

$\xrightarrow{R_3 \to R_3 - R_3} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & -3 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right]$

13. **If ρ(A) = ρ([A/B]) = number of unknowns, then the system is**

(1) consistent and has infinitely many solutions
(2) consistent
(3) inconsistent
(4) consistent and has unique solution.

[Ans: (4) consistent and has unique solution**]**

14. **Which of the following is not an elementary transformation?**

(1) $R_i \leftrightarrow R_j$ (2) $R_i \to 2R_i + R_j$
(3) $C_j \to C_j + C_i$ (4) $R_i \to R_i + C_j$

[Ans: (4) $R_i \to R_i + C_j$**]**

15. If $\rho(A) = r$ then which of the following is correct?

(1) all the minors of order n which do not vanish

(2) 'A' has at least one minor of order r which does not vanish and all higher order minors vanish

(3) 'A' has at least one $(r + 1)$ order minor which vanish

(4) all $(r + 1)$ and higher order minors should not vanish

[Ans: (b) 'A' has at least one minor of order r which does not vanish and all higher order minors vanish]

II. Fill in the blanks :

1. Every homogeneous system...................

(1) Is always consistent

(2) Has only trivial solution

(3) Has infinitely many solution

(4) Need not be consistent

[Ans: (1) Is always consistent]

2. If $\rho(A) \neq \rho([A|B])$, then the system is............

(1) consistent and has infinitely many solutions

(2) consistent and has a unique solution

(3) consistent

(4) inconsistent [Ans: (4) inconsistent]

3. In the non – homogeneous system of equations with 3 unknowns if $\rho(A) = \rho([A|B]) = 2$, then the system has..............

(1) unique solution

(2) one parameter family of solution

(3) two parameter family of solutions

(4) in consistent

[Ans: (2) one parameter family of solution]

4. Cramer's rule is applicable only when..........

(1) $\Delta \neq 0$ (2) $\Delta = 0$

(3) $\Delta = 0, \Delta_x = 0$ (4) $\Delta_x = \Delta_y = \Delta_z = 0$

[Ans: (1) $\Delta \neq 0$]

5. In a homogeneous system if $\rho(A) = \rho([A|0]) <$ the number of unknouns then the system has.............

(1) trivial solution

(2) only non – trivial solution

(3) no solution

(4) trivial solution and infinitely many non – trivial solutions

[Ans: (4) trivial solution and infinitely many non-trivial solutions]

6. In the system of equations with 3 unknowns, if $\Delta = 0$, and one of Δ_x, Δ_y of Δ_z is non zero then the system is...................

(1) Consistent

(2) inconsistent

(3) consistent with one parameter family of solutions

(4) consistent with two parameter family of solutions [Ans: (b) inconsistent]

7. In the system of liner equations with 3 unknowns if $\rho(A) = \rho([A|B]) = 1$, the system has...................

(a) unique solution

(b) inconsistent

(c) consistent with 2 parameter family of solution

(d) consistent with one parameter family of solution.

[Ans: (c) consistent with 2 parameter family of solution]

8. If $A = [2\ 0\ 1]$ then the rank of AA^T is ....

(1) 1 (2) 2 (3) 3 (4) 0

[Ans: (1) 1]

Hint : $$AA^T = [2\ 0\ 1]\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} = [5]$$

$$\therefore (AA^T) = 1$$

9. If A is a non-singular matrix then $|A^{-1}|$=

(1) $\left|\dfrac{1}{A^2}\right|$ (2) $\dfrac{1}{|A|^2}$ (3) $\left|\dfrac{1}{A}\right|$ (4) $\dfrac{1}{|A|}$

[Ans: (4) $\dfrac{1}{|A|}$]

10. In a square matrix the minor M_{ij} and the co-factor A_{ij} of and element a_{ij} are related by

(1) $A_{ij} = -M_{ij}$

(2) $A_{ij} = M_{ij}$

(3) $A_{ij} = (-1)^{i+j} M_{ij}$

(4) $A_{ij} = (-1)^{i-j} M_{ij}$

[Ans: (3) $A_{ij} = (-1)^{i+j} m_{ij}$]

III. Match the following :

1.

	List - I		List - II
i.	Trivial solution of $AX = 0$	a)	$\lvert A\rvert = 0$
ii.	Non – trivial solution of $AX = 0$	b)	Non – trivial solution
iii.	$\rho(A) = \rho[(A/0)] < n$	c)	Trivial solution
iv.	$\rho(A) = \rho([A/0]) = n$	d)	$\lvert A\rvert \neq 0$

The Correct match is

	(i)	(ii)	(iii)	(iv)
(1)	a	b	c	d
(2)	b	c	d	a
(3)	a	d	b	a
(4)	c	d	a	b

[Ans : (2) i – d ii – a iii – b iv – c]

2.

	List - I		List - II
i.	$\rho(A) = \rho([A\|B]) = 3 =$ number of unknowns	a)	Consistent with one parameter family of solution
ii.	$\rho(A) = \rho[A\|B]) = 2 <$number of unknowns	b)	In consistent and has no solution
iii.	$\rho(A) = \rho[A\|B]) = 1 <$number of unknowns	c)	Unique solution
iv.	$\rho(A) \neq \rho([A\|B])$	d)	Consistent with two parameter family of solution

The Correct match is

	(i)	(ii)	(iii)	(iv)
(1)	c	d	a	b
(2)	a	b	d	c
(3)	c	a	d	b
(4)	c	d	a	b

[Ans : (3) i – c ii – a iii – d iv – b]

3. If A is a non - singular matrix of order n, then

	List - I		List - II
i.	$\lvert\text{adj } A\rvert$	a)	$\lvert A\rvert^{(n-1)^2}$
ii.	$(\text{adj } A)^T$	b)	$\lvert A\rvert^{n-1}$
iii.	adj (adj A)	c)	$\text{adj}(A^T)$
iv.	$\lvert\text{adj}(\text{adj } A)\rvert$	d)	$\lvert A\rvert^{n-2} \cdot A$

The Correct match is

	(i)	(ii)	(iii)	(iv)
(1)	a	b	c	d
(2)	c	d	a	b
(3)	d	a	b	c
(4)	b	c	d	a

[Ans : (4) i – b ii – c iii – d iv – a]

4. If A is a non - singular matrix of order n, then

	List - I		List - II
i.	$(\text{adj } A)^{-1}$	a)	(adj B) (adj A)
ii.	$(\lambda A)^{-1}$	b)	λ^{n-1} adj (A)
iii.	adj (λA)	c)	$\text{adj}(A^{-1})$
iv.	adj (AB)	d)	$\frac{1}{\lambda} A^{-1}$

The Correct match is

	(i)	(ii)	(iii)	(iv)
(1)	c	d	b	a
(2)	a	b	c	d
(3)	b	c	d	a
(4)	c	a	b	d

[Ans : (1) i – c ii – d iii – b 4 – a]

5.

	List - I		List - II
i.	$(A^T)^{-1}$	(a)	A
ii.	A (adj A)	(b)	$(A^{-1})^T$
iii.	$(AB)^{-1}$	(c)	$\lvert A\rvert. I_n$
iv.	$(A^{-1})^{-1}$	(d)	$B^{-1} A^{-1}$

The Correct match is

	(i)	(ii)	(iii)	(iv)
(1)	b	c	d	a
(2)	a	b	c	d
(3)	c	d	a	b
(4)	b	a	c	d

[Ans : (1) i – b ii – c iii – d iv – a]

IV. Choose the odd man out :

1. **The rank of any 3 × 4 matrix is**
(1) May be 1 (2) May be 2
(3) May be 3 (4) May be 4
[Ans: (4) May be 4]

Hint: [$\rho(A) \leq \min(3, 4) \Rightarrow \rho(A) \leq 3$ and it cannot be 4]

2. **If A is symmetric then**
(1) $A^T = A$ (2) adj A is symmetric
(3) $\text{adj}(A^T) = (\text{adj } A)^T$
(4) A is orthogonal

Hint :[1, 2, 3 are properties of symmetric matrix]
[Ans: (4) A is orthogonal]

3. **If A is a non-singular matrix of odd order them**
(1) Order of A is $2m + 1$
(2) Order of A is $2m + 2$
(3) |adj A| is positive (4) $|A| \neq 0$
[Ans: (2) Order of A is $2m + 2$]

Hint: $(2m + 2)$ is even

4. **If A is a orthogonal matrix, then**
(1) $AA^T = A^TA = I$ (2) A is non-singular
(3) $|A| = 0$ (4) $A^{-1} = A^T$
[Ans: (3) $|A| = 0$]

Hint: $|A| = 0$ which is not true for orthogonal matrix

5. **A matrix which is obtained from an identity matrix by applying only one elementary transformation is**
(1) Identity matrix (2) Elementary matrix
(3) Square matrix
(4) Equivalent to identify matrix
[Ans: (1) Identity matrix]

V. Choose the incorrect answer :

1. **In an echelon form which of the following is incorrect ?**
(1) Every row of A which has all its entries 0 occurs below every row which has a non-zero entry.
(2) The first non-zero entry in each non-zero row is 1
(3) The number of zeros before the first non-zero element in a row is less than the number of such zeros in the next row
(4) Two row can have same number of zeros before the first non-zero entry
[Ans: (4) Two row can have same number of zeros before the first non-zero entry]

2. **Which of the following elementary transformation is not correct?**
(1) $R_i \to R_i + 2R_j$ (2) $C_i \to C_i - C_j$
(3) $R_i \to 7R_i + \frac{5}{3}R_j$ (4) $C_i \to C_i - R_j$
[Ans: (4) $C_i \to C_i - R_j$]

3. **It A is an invertible matrix, then which of the following is not true.**
(1) $(A^2)^{-1} = (A^{-1})^2$ (2) $|A^{-1}| = |A|^{-1}$
(3) $(A^T)^{-1} = (A^{-1})^T$ (4) $|A| \neq 0$
[Ans: (1) $(A^2)^{-1} = (A^{-1})^2$]

4. **The matrix $\begin{bmatrix} 5 & 10 & 3 \\ -2 & -4 & 6 \\ -1 & -2 & x \end{bmatrix}$ is a singular matrix if the value of x is**
(1) 3 (2) non- existent
(3) All values of x (4) Any value of x
[Ans: (1) 3]

5. **The number of solutions of the system of equations $2x + y - z = 7$, $x - 3y + 2z = 1$, $x + 3y - 3z = 5$ is**
(1) 0 (2) 3
(3) No-solution (4) Inconsistent
[Ans:(2) 3]

2 MARKS

1. **For any 2 × 2 matrix, if A (adj A) = $\begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix}$ then find |A|.**

Sol. Given $A(\text{adj } A) = \begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix} = 10\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ ...(1)

We know $A(\text{adj } A) = (\text{adj } A)A = |A|.I_2$ ...(2)

Comparing (1) and (2), we get $|A| = 10$.

2. **For the matrix A, if $A^3 = I$, then find A^{-1}.**

Sol. Given $A^3 = I$

Pre multiply by A^{-1} we get,

$A^{-1}.A^3 = A^{-1}.I$

$\Rightarrow (A^{-1}.A)A^2 = A^{-1}$ $[\because A^{-1}I = A^{-1}]$

$\Rightarrow I.A^2 = A^{-1}$ $[\because A^{-1}A = I]$

$\Rightarrow A^2 = A^{-1}$ $[\because I.A^2 = A^2]$

$\therefore A^{-1} = A^2$

3. If A is a square matrix such that $A^3 = I$, then prove that A is non-singular

Sol. Given $A^3 = I \Rightarrow |A^3| = |I|$

$\Rightarrow \quad |A.A.A| = 1$

$\Rightarrow \quad |A|.|A|.|A| = 1$

$\Rightarrow \quad |A|^3 = 1$

$\therefore |A| \neq 0$

Hence, A is non-singular.

4. Show that the system of equations is inconsistent. $2x + 5y = 7$, $6x + 15y = 13$.

Sol. Augmented matrix

$$[A|B] \left[\begin{array}{cc|c} 2 & 5 & 7 \\ 6 & 15 & 13 \end{array}\right] \xrightarrow{R_2 \to R_2 - 3R_1} \left[\begin{array}{cc|c} 2 & 5 & 7 \\ 0 & 0 & -8 \end{array}\right]$$

Here $\rho(A) = 2$ and $\rho([A|B]) = 3$

$\therefore \rho(a) \neq \rho([A|B])$

Hence the system is inconsistent,

5. Find the rank of the matrix $\begin{bmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{bmatrix}$.

Sol. Let $A = \begin{bmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{bmatrix}$

Now $|A| = \begin{vmatrix} 6 & -5 \\ -2 & 2 \end{vmatrix} + 1\begin{vmatrix} -15 & -5 \\ 5 & 2 \end{vmatrix} + 1\begin{vmatrix} -15 & 6 \\ 5 & -2 \end{vmatrix}$

$= 3(12 - 10) + 1(-30 + 25) + 1(30 - 30)$

$= 3(2) + 1(-5) + 0 = 6 - 5 = 1 \neq 0$

$\therefore$ Rank of A is 3.

6. Find the rank of the matrix $A = \begin{bmatrix} 4 & 5 & -6 & 1 \\ 7 & -3 & 0 & 8 \end{bmatrix}$.

Sol. $A = \begin{bmatrix} 4 & 5 & -6 & 1 \\ 7 & -3 & 0 & 8 \end{bmatrix}$

$\xrightarrow{R_2 \to R_2 - 2R_1} \begin{bmatrix} 4 & 5 & -6 & 1 \\ -1 & -3 & 12 & 6 \end{bmatrix}$

$\xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} -1 & -3 & 12 & 6 \\ 4 & 5 & -6 & 1 \end{bmatrix}$

$\xrightarrow{R_1 + (-1)R_1} \begin{bmatrix} 1 & 13 & -12 & 6 \\ 4 & 5 & -6 & 1 \end{bmatrix}$

$\xrightarrow{R_2 \to R_2 - 4R_1} \begin{bmatrix} 1 & 13 & -12 & -6 \\ 0 & -47 & 42 & 25 \end{bmatrix}$

The equivalent row-echelon matrix has two non-zero rows

$\therefore \rho(A) = 2$

7. Show that the equations $3x + y + 9z = 0$, $3x + 2y + 12z = 0$ and $2x + y + 7z = 0$ have non-trivial solutions also.

Sol. The matrix form of the system is

$$\begin{bmatrix} 3 & 1 & 9 \\ 3 & 2 & 12 \\ 2 & 1 & 7 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

AX = B where

$$A = \begin{bmatrix} 3 & 1 & 9 \\ 3 & 2 & 12 \\ 2 & 1 & 7 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, B = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 3 & 1 & 9 \\ 3 & 2 & 12 \\ 2 & 1 & 7 \end{vmatrix} = 3\begin{vmatrix} 2 & 12 \\ 1 & 7 \end{vmatrix} - 1\begin{vmatrix} 3 & 12 \\ 2 & 7 \end{vmatrix} + 9\begin{vmatrix} 3 & 2 \\ 2 & 1 \end{vmatrix}$$

$= 3(14 - 12) - 1(21 - 24) + 9(3 - 4)$

$= 3(2) - 1(-3) + 9(-1)$

$= 6 + 3 - 9 = 9 - 9 = 0$

Since $|A| = 0$, the homogeneous system of equations have non-trivial solutions also.

8. Find k if the equations $x + 2y + 2z = 0$, $x - 3y - 3z = 0$, $2x + y + kz = 0$ have only the trivial solution.

Sol. Matrix form of the given system of equations is

$$\begin{bmatrix} 1 & 2 & 2 \\ 1 & -3 & -3 \\ 2 & 1 & k \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

AX = B where $A = \begin{bmatrix} 1 & 2 & 2 \\ 1 & -3 & -3 \\ 2 & 1 & k \end{bmatrix}$

Homogeneous system of equations has trivial solution only if $|A| \neq 0$.

$$\therefore \begin{bmatrix} 1 & 2 & 2 \\ 1 & -3 & -3 \\ 2 & 1 & k \end{bmatrix} \neq 0.$$

Expanding along R_1,

$$1\begin{vmatrix} -3 & -3 \\ 1 & k \end{vmatrix} - 2\begin{vmatrix} 1 & -3 \\ 2 & k \end{vmatrix} + 2\begin{vmatrix} 1 & -3 \\ 2 & 1 \end{vmatrix} \neq 0.$$

$\Rightarrow 1(-3k + 3) - 2(k + 6) + 2(1 + 6) \neq 0$

$\Rightarrow -3k + 3 - 2k - 12 + 14 \neq 0$

$\Rightarrow -5k + 5 \neq 0$

$\Rightarrow -5k \neq -5 \Rightarrow k \neq \dfrac{-5}{-5} = 1$

$\Rightarrow k \neq 1$

9. Solve : $2x - y = 3$, $5x + y = 4$ using matrices.

Sol. The equations can be written in matrix form as

$$\begin{pmatrix} 2 & -1 \\ 5 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \end{pmatrix} \Rightarrow AX = B \text{ where}$$

$$A = \begin{pmatrix} 2 & -1 \\ 5 & 1 \end{pmatrix}, X = \begin{pmatrix} x \\ y \end{pmatrix}, B = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$$

$$\therefore X = A^{-1} B$$

$$|A| = \begin{vmatrix} 2 & -1 \\ 5 & 1 \end{vmatrix} = 2 + 5 = 7 \neq 0$$

$$\therefore A^{-1} = \frac{1}{|A|} \text{ adj } A = \frac{1}{7}\begin{bmatrix} 1 & 1 \\ -5 & 2 \end{bmatrix}$$

$$\therefore X = \frac{1}{7}\begin{bmatrix} 1 & 1 \\ -5 & 2 \end{bmatrix}\begin{bmatrix} 3 \\ 4 \end{bmatrix} = \frac{1}{7}\begin{bmatrix} 3+4 \\ -15+8 \end{bmatrix}$$

$$= \frac{1}{7}\begin{bmatrix} 7 \\ -7 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$\therefore$ Solution set is $\{1, -1\}$

10. Solve $6x - 7y = 16$, $9x - 5y = 35$ using (Cramer's rule).

Sol.
$$\Delta = \begin{vmatrix} 6 & -7 \\ 9 & -5 \end{vmatrix} = -30 + 63 = 33$$

$$\Delta_1 = \begin{vmatrix} 16 & -7 \\ 35 & -5 \end{vmatrix} = -80 + 245 = 165$$

$$\Delta_2 = \begin{vmatrix} 6 & 16 \\ 9 & 35 \end{vmatrix} = 210 - 144 = 66$$

$$\therefore x = \frac{\Delta_1}{\Delta} = \frac{165}{33} = 5$$

$$y = \frac{\Delta_2}{\Delta} = \frac{66}{33} = 2$$

$\therefore$ Solution set is $\{5, 2\}$

3 MARKS

1. Solve : $2x + 3y = 10$, $x + 6y = 4$ using Cramer's rule.

Sol.
$$\Delta = \begin{vmatrix} 2 & 3 \\ 1 & 6 \end{vmatrix} = 12 - 3 = 9 \neq 0$$

$$\Delta_1 = \begin{vmatrix} 10 & 3 \\ 4 & 6 \end{vmatrix} = 60 - 12 = 48$$

$$\Delta_2 = \begin{vmatrix} 2 & 10 \\ 1 & 4 \end{vmatrix} = 8 - 10 = -2$$

$$\therefore x = \frac{\Delta_1}{\Delta} = \frac{48}{9} = \frac{16}{3}$$

$$y = \frac{\Delta_2}{\Delta} = \frac{-2}{9}$$

$\therefore$ Solution set is $\left\{\frac{16}{3}, \frac{-2}{9}\right\}$

2. For what value of t will the system $tx + 3y - z = 1$, $x + 2y + z = 2$, $-tx + y + 2z = -1$ fail to have unique solution?

Sol.
$$\Delta = \begin{vmatrix} t & 3 & -1 \\ 1 & 2 & 1 \\ -t & 1 & 2 \end{vmatrix} = t\begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} - 3\begin{vmatrix} 1 & 1 \\ -t & 2 \end{vmatrix} - \begin{vmatrix} 1 & 2 \\ -t & 1 \end{vmatrix}$$

$$= t(4 - 1) - 3(2 + t) - 1(1 + 2t)$$
$$= 3t - 6 - 3t - 1 - 2t = -7 - 2t$$

The system will fail to have unique solution if

$$\Delta = 0 \Rightarrow -7 - 2t = 0 \Rightarrow -2t = 7 \Rightarrow t = \frac{-7}{2}$$

$$\therefore t = \frac{-7}{2}.$$

3. Solve: $3x + ay = 4$, $2x + ay = 2$, $a \neq 0$ by Cramer's rule.

Sol.
$$\Delta = \begin{vmatrix} 3 & a \\ 2 & a \end{vmatrix} = 3a - 2a = a$$

$$\Delta_1 = \begin{vmatrix} 4 & a \\ 2 & a \end{vmatrix} = 4a - 2a = 2a$$

$$\Delta_2 = \begin{vmatrix} 3 & 4 \\ 2 & 2 \end{vmatrix} = 6 - 8 = -2$$

$$\therefore x = \frac{\Delta_1}{\Delta} = \frac{2\cancel{a}}{\cancel{a}} = 2 \Rightarrow y = \frac{\Delta_2}{\Delta} = \frac{-2}{a}$$

$\therefore$ Solution set is $\left\{2, \frac{-2}{a}\right\}$

4. **Verify $(AB)^{-1} = B^{-1}A^{-1}$ for $A = \begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & 5 \\ 3 & 4 \end{bmatrix}$.**

Sol.
$$AB = \begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix}\begin{bmatrix} 4 & 5 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 8+3 & 10+4 \\ 20+9 & 25+12 \end{bmatrix} = \begin{bmatrix} 11 & 14 \\ 29 & 37 \end{bmatrix}$$
$$|AB| = \begin{bmatrix} 11 & 14 \\ 29 & 37 \end{bmatrix} = 407 - 406 = 1 \neq 0$$
$$(AB)^{-1} = \frac{1}{|AB|}\text{adj}(AB) = \frac{1}{1}\begin{bmatrix} 11 & 14 \\ 29 & 37 \end{bmatrix} = \begin{bmatrix} 37 & -14 \\ -29 & 11 \end{bmatrix} \quad ...(1)$$
$$|A| = \begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix} = 6 - 5 = 1$$
$$|B| = \begin{bmatrix} 4 & 5 \\ 3 & 4 \end{bmatrix} = 16 - 15 = 1$$
$$B^{-1} = \frac{1}{|B|}\text{adj } B = \begin{bmatrix} 4 & -5 \\ -3 & 4 \end{bmatrix}$$
$$A^{-1} = \frac{1}{|A|}\text{adj } A = \begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix}$$
$$\therefore B^{-1}A^{-1} = \begin{bmatrix} 4 & -5 \\ -3 & 4 \end{bmatrix}\begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix} = \begin{bmatrix} 12+25 & -4-10 \\ -9-20 & 3+8 \end{bmatrix} = \begin{bmatrix} 37 & -14 \\ -29 & 11 \end{bmatrix} \quad ...(2)$$

From (1) and (2), $(AB)^{-1} = B^{-1}A^{-1}$

5. **Under what conditions will the rank of the matrix $\begin{bmatrix} 1 & 0 & 0 \\ 0 & h-2 & 2 \\ 0 & 0 & h+2 \\ 0 & 0 & 3 \end{bmatrix}$ be less than 3?**

Sol.
$$\text{Let } A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & h-2 & 2 \\ 0 & 0 & h+2 \\ 0 & 0 & 3 \end{bmatrix}$$

The rank of A will be less than 3 if every minor of order 3 vanishes
$$\therefore \begin{vmatrix} 1 & 0 & 0 \\ 0 & h-2 & 0 \\ 0 & 0 & 3 \end{vmatrix} = 0$$
$$\Rightarrow 1\begin{vmatrix} h-2 & 0 \\ 0 & 3 \end{vmatrix} + 0 + 0 = 0 \Rightarrow 3(h-2) = 0$$
$$\Rightarrow h - 2 = 0 \Rightarrow h = 2.$$

6. **Find the rank of the matrix math $\begin{bmatrix} 4 & 4 & 0 & 3 \\ -2 & 3 & -1 & 5 \\ 1 & 4 & 8 & 7 \end{bmatrix}$**

Sol.
$$\text{Let } A = \begin{bmatrix} 4 & 4 & 0 & 3 \\ -2 & 3 & -1 & 5 \\ 1 & 4 & 8 & 7 \end{bmatrix}$$
$$A \xrightarrow{R_1 \leftrightarrow R_3} \begin{bmatrix} 1 & 4 & 8 & 7 \\ -2 & 3 & -1 & 5 \\ 4 & 4 & 0 & 3 \end{bmatrix}$$
$$\xrightarrow[R_3 \to R_3 - 4R_1]{R_2 \to R_2 + 2R_1} \begin{bmatrix} 1 & 4 & 8 & 7 \\ 0 & 11 & 15 & 19 \\ 0 & -12 & -32 & -25 \end{bmatrix}$$

A is in row – echelon form and it has 3 non-zero rows.

$\therefore \rho(A) = 3.$

7. **Verify that $(A^{-1})^T = (A^T)^{-1}$ for $A = \begin{bmatrix} -2 & -3 \\ 5 & -6 \end{bmatrix}$.**

Sol.
$$|A| = \begin{bmatrix} -2 & -3 \\ 5 & -6 \end{bmatrix} = 12 + 15 = 27$$
$$\therefore A^{-1} = \frac{1}{|A|}\text{adj } A = \frac{1}{27}\begin{bmatrix} -6 & 3 \\ -5 & 2 \end{bmatrix}$$
$$(A^{-1})^T = \frac{1}{27}\begin{bmatrix} -6 & -5 \\ 3 & 2 \end{bmatrix} \quad ...(1)$$
$$A^T = \begin{bmatrix} -2 & 5 \\ -3 & -6 \end{bmatrix}$$

$$|A^T| = \begin{vmatrix} -2 & 5 \\ -3 & -6 \end{vmatrix} = 12 + 15 = 27$$

$$\therefore (A^T)^{-1} = \frac{1}{|A^T|} \text{adj}(A^T) = \frac{1}{27}\begin{bmatrix} -6 & -5 \\ 3 & -2 \end{bmatrix} \quad ...(2)$$

From (1) and (2), $(A^{-1})^T = (A^T)^{-1}$

8. Solve $2x - 3y = 7$, $4x - 6y = 14$ by Gaussian Jordan method.

Sol. The matrix from of the system of equations is

$$\begin{bmatrix} 2 & -3 \\ 4 & -6 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 7 \\ 14 \end{bmatrix} \Rightarrow AX = B \text{ where}$$

$$A = \begin{bmatrix} 2 & -3 \\ 4 & -6 \end{bmatrix}, X = \begin{bmatrix} x \\ y \end{bmatrix} \text{ and } B = \begin{bmatrix} 7 \\ 14 \end{bmatrix}$$

Transforming augmented matrix to row-echelon form we get

$$[A|B] = \left[\begin{array}{cc|c} 2 & -3 & 7 \\ 4 & -6 & 14 \end{array}\right] \xrightarrow{R_2 \to R_2 - 2R_1} \left[\begin{array}{cc|c} 2 & -3 & 7 \\ 0 & 0 & 0 \end{array}\right]$$

Here $\rho(A) = 1$, $\rho[A|B] = 1$

$\therefore \rho(A) = \rho[A|B] = 1 <$ the number of unknowns, the system is consistent and has one parameter family of solutions

$\therefore$ Put $y = t$, where $t \in \mathbb{R}$

Writing the row-echelon form to equations we get

$$2x - 3y = 7$$
$$\therefore 2x - 3t = 7$$
$$\Rightarrow \quad 2x = 7 + 3t$$
$$\Rightarrow \quad x = \frac{1}{2}(7 + 3t)$$

$\therefore$ Solution set is $\{\frac{1}{2}(7 + 3t), t\}$ where $t \in \mathbb{R}$.

9. Solve: $x + y + 3z = 4$, $2x + 2y + 6z = 7$, $2x + y + z = 10$.

Sol. Augmented matrix $[A|B] = \left[\begin{array}{ccc|c} 1 & 1 & 3 & 4 \\ 2 & 2 & 6 & 7 \\ 2 & 1 & 1 & 10 \end{array}\right]$

$$[A\,|\,B] \xrightarrow[R_3 \to R_3 - 2R_1]{R_2 \to R_2 - 2R_1} \left[\begin{array}{ccc|c} 1 & 1 & 3 & 4 \\ 0 & 0 & 0 & -1 \\ 0 & -1 & -5 & 2 \end{array}\right]$$

$$\xrightarrow{R_2 \leftrightarrow R_3} \left[\begin{array}{ccc|c} 1 & 1 & 3 & 4 \\ 0 & -1 & -5 & 2 \\ 0 & 0 & 0 & -1 \end{array}\right]$$

Here $\rho(A) = 2$ [only 2 two-non zero rows]

And $\rho([A|B]) = 3$ [There are 3 non-zero rows]

$\therefore \rho(A) \neq \rho([A|B])$

$\therefore$ The system is inconsistent.

10. If the rank of the matrix $\begin{bmatrix} \lambda & -1 & 0 \\ 0 & \lambda & -1 \\ -1 & 0 & \lambda \end{bmatrix}$ is 2, then find λ.

Sol. Given rank of $\begin{bmatrix} \lambda & -1 & 0 \\ 0 & \lambda & -1 \\ -1 & 0 & \lambda \end{bmatrix}$ is 2

$\Rightarrow$ The value of the third order determinant is zero

$$\Rightarrow \begin{vmatrix} \lambda & -1 & 0 \\ 0 & \lambda & -1 \\ -1 & 0 & \lambda \end{vmatrix} = 0$$

$$\Rightarrow \lambda\begin{vmatrix} \lambda & -1 \\ 0 & \lambda \end{vmatrix} + 1\begin{vmatrix} 0 & -1 \\ -1 & \lambda \end{vmatrix} + 0 = 0$$

$$\Rightarrow \quad \lambda(\lambda^2 - 0) + 1(0 - 1) = 0$$

$$\Rightarrow \lambda^3 - 1 = 0 \Rightarrow \lambda^3 = 1 \Rightarrow \lambda = 1$$

$$\therefore \lambda = 1$$

5 MARKS

1. Using determinants, find the quadratic defined by $f(x) = ax^2 + bx + c$, if $f(1) = 0$, $f(2) = -2$ and $f(3) = -6$.

Sol. Given $f(x) = ax^2 + bx + c$

$$f(1) = 0$$
$$\Rightarrow a(1)^2 + b(1) + c = 0$$
$$\Rightarrow \quad a + b + c = 0 \quad ...(1)$$
$$f(2) = -2$$
$$\Rightarrow a(2^2) + b(2) + c = -2$$
$$\Rightarrow \quad 4a + 2b + c = -2 \quad ...(2)$$
$$f(3) = -6$$
$$\Rightarrow a(3^2) + b(3) + c = -6$$
$$\Rightarrow \quad 9a + 3b + c = -6 \quad ...(3)$$

$$\Delta = \begin{vmatrix} 1 & 1 & 1 \\ 4 & 2 & 1 \\ 9 & 3 & 1 \end{vmatrix} = 1\begin{vmatrix} 2 & 1 \\ 3 & 1 \end{vmatrix} - 1\begin{vmatrix} 4 & 1 \\ 9 & 1 \end{vmatrix} + 1\begin{vmatrix} 4 & 2 \\ 9 & 3 \end{vmatrix}$$

$= 1\,(2 - 3) - 1\,(4 - 9) + 1\,(12 - 18)$

$= -1 + 5 - 6 = -2 \neq 0$

$$\Delta_1 = \begin{vmatrix} 0 & 1 & 1 \\ -2 & 2 & 1 \\ -6 & 3 & 1 \end{vmatrix} = 0 - 1\begin{vmatrix} -2 & 1 \\ -6 & 1 \end{vmatrix} + 1\begin{vmatrix} -2 & 2 \\ -6 & 3 \end{vmatrix}$$

$= -1\,(-2 + 6) + 1(-6 + 12) = -1\,(4) + 1\,(6) = 2$

$$\Delta_2 = \begin{vmatrix} 1 & 0 & 1 \\ 4 & -2 & 1 \\ 9 & -6 & 1 \end{vmatrix} = 1\begin{vmatrix} -2 & 1 \\ -6 & 1 \end{vmatrix} + 0 + 1\begin{vmatrix} 4 & -2 \\ 9 & -6 \end{vmatrix}$$

$-1\,(-2 + 6) + 1(-24 + 18) = 4 - 6 = -2$

$$\Delta_3 = \begin{vmatrix} 1 & 1 & 0 \\ 4 & 2 & -2 \\ 9 & 3 & -6 \end{vmatrix} = 1\begin{vmatrix} 2 & -2 \\ 3 & -6 \end{vmatrix} - 1\begin{vmatrix} 4 & -2 \\ 9 & -6 \end{vmatrix}$$

$= 1\,(-12 + 6) - 1\,(-24 + 18) = -6 + 6 = 0$

$\therefore$ By Cramer's rule,

$$a = \frac{\Delta_1}{\Delta} = \frac{2}{-2} = -1$$

$$b = \frac{\Delta_2}{\Delta} = \frac{-2}{-2} = 1$$

$$c = \frac{\Delta_3}{\Delta} = \frac{0}{-2} = 0$$

$$\therefore f(x) = (-1)\,x^2 + 1x + 0$$

$$\Rightarrow \quad f(x) = x^2 + x$$

2. Solve:

$$\frac{2}{x} + \frac{3}{y} + \frac{10}{z} = 4,\ \frac{4}{x} - \frac{6}{y} + \frac{5}{z} = 1,\ \frac{6}{x} + \frac{9}{y} - \frac{20}{z} = 2$$

Sol. Put $\frac{1}{x} = a,\ \frac{1}{y} = b,\ \frac{1}{z} = c$

$\therefore 2a + 3b + 10c = 4$ ...(1)

$4a - 6b + 5c = 1$ ...(2)

$6a + 9b - 20c = 2$ ...(3)

$$\Delta = \begin{vmatrix} 2 & 3 & 10 \\ 4 & -6 & 5 \\ 6 & 9 & -20 \end{vmatrix}$$

$$= 2\begin{vmatrix} -6 & 5 \\ 9 & -20 \end{vmatrix} - 3\begin{vmatrix} 4 & 5 \\ 6 & -20 \end{vmatrix} + 10\begin{vmatrix} 4 & -6 \\ 6 & 9 \end{vmatrix}$$

$= 2\,(120 - 45) - 3\,(-80 - 30) + 10\,(36 + 36)$

$= 150 + 330 + 720 = 1200$

$$\Delta_1 = \begin{vmatrix} 4 & 3 & 10 \\ 1 & -6 & 5 \\ 2 & 9 & -20 \end{vmatrix}$$

$$= 4\begin{vmatrix} -6 & 5 \\ 9 & -20 \end{vmatrix} - 3\begin{vmatrix} 1 & 5 \\ 2 & -20 \end{vmatrix} + 10\begin{vmatrix} 1 & -6 \\ 2 & 9 \end{vmatrix}$$

$= 4\,(120 - 45) - 3\,(-20 - 10) + 10\,(9 + 12)$

$= 300 + 90 + 120 = 600$

$$\Delta_2 = \begin{vmatrix} 2 & 4 & 10 \\ 4 & 1 & 5 \\ 6 & 2 & -20 \end{vmatrix} = 2\begin{vmatrix} 1 & 5 \\ 2 & -20 \end{vmatrix} - 4\begin{vmatrix} 4 & 5 \\ 6 & -20 \end{vmatrix} + 10\begin{vmatrix} 4 & 1 \\ 6 & 2 \end{vmatrix}$$

$= 2\,(-2 - 10) - 4\,(-80 - 30) + 10\,(8 - 6)$

$= -60 + 440 + 20 = 400$

$$\Delta_3 = \begin{vmatrix} 2 & 3 & 4 \\ 4 & -6 & 1 \\ 6 & 9 & 2 \end{vmatrix} = 2\begin{vmatrix} -6 & 1 \\ 9 & 2 \end{vmatrix} - 3\begin{vmatrix} 4 & 1 \\ 6 & 2 \end{vmatrix} + 4\begin{vmatrix} 4 & -6 \\ 6 & 9 \end{vmatrix}$$

$= 2\,(-12 - 9) - 3\,(8 - 6) + 4\,(36 + 36)$

$= -42 - 6 + 288 = 240$

$$\therefore a = \frac{\Delta_1}{\Delta} = \frac{600}{1200} = \frac{1}{2} \Rightarrow \frac{1}{x} = \frac{1}{2} \Rightarrow x = 2$$

$$\therefore b = \frac{\Delta_2}{\Delta} = \frac{400}{1200} = \frac{1}{3} \Rightarrow \frac{1}{y} = \frac{1}{3} \Rightarrow y = 3$$

$$\therefore c = \frac{\Delta_3}{\Delta} = \frac{240}{1200} = \frac{1}{5} \Rightarrow \frac{1}{z} = \frac{1}{5} \Rightarrow z = 5$$

$\therefore$ Solution set is $\{2, 3, 5\}$

3. The sum of three numbers is 20. If we multiply the third number by 2 and add the first number to the result we get 23. By adding second and third numbers to 3 times the first number we get 46. Find the numbers using Cramer's rule.

Sol. Let the required numbers be x, y and z

By the given data,

$x + y + z = 20$ ...(1)

$2z + x = 23 \Rightarrow x + 2z = 23$ ...(2)

$y + z + 3x = 46 \Rightarrow 3x + y + z = 46$ ...(3)

$$\Delta = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 0 & 2 \\ 3 & 1 & 1 \end{vmatrix} = 1\begin{vmatrix} 0 & 2 \\ 0 & 1 \end{vmatrix} - 1\begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix} + 1\begin{vmatrix} 1 & 0 \\ 3 & 1 \end{vmatrix}$$

$= -2 + 5 + 1 = 4$

$$\Delta_1 = \begin{vmatrix} 20 & 1 & 1 \\ 23 & 0 & 2 \\ 46 & 1 & 1 \end{vmatrix} = 20\begin{vmatrix} 0 & 2 \\ 1 & 1 \end{vmatrix} - 1\begin{vmatrix} 23 & 2 \\ 46 & 1 \end{vmatrix} + 1\begin{vmatrix} 23 & 0 \\ 46 & 1 \end{vmatrix}$$

$$= -40 + 69 + 23 = 52$$

$$\Delta_2 = \begin{vmatrix} 1 & 20 & 1 \\ 1 & 23 & 2 \\ 3 & 46 & 1 \end{vmatrix}$$

$$= 1\begin{vmatrix} 23 & 2 \\ 46 & 1 \end{vmatrix} - 20\begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix} + 1\begin{vmatrix} 1 & 23 \\ 3 & 46 \end{vmatrix}$$

$$= -69 + 100 - 23 = 8$$

$$\Delta_3 = \begin{vmatrix} 1 & 1 & 20 \\ 1 & 0 & 23 \\ 3 & 1 & 46 \end{vmatrix}$$

$$= 1\begin{vmatrix} 0 & 23 \\ 1 & 46 \end{vmatrix} - 1\begin{vmatrix} 1 & 23 \\ 3 & 46 \end{vmatrix} + 20\begin{vmatrix} 1 & 0 \\ 3 & 1 \end{vmatrix}$$

$$= -23 + 23 + 20 = 20$$

$$\therefore x = \frac{\Delta_1}{\Delta} = \frac{52}{4} = 13$$

$$y = \frac{\Delta_2}{\Delta} = \frac{8}{4} = 2 \text{ and } z = \frac{\Delta_3}{\Delta} = \frac{20}{4} = 5.$$

Hence the required numbers are 13, 2 and 5.

4. For what value of λ, the system of equations $x + y + z = 1$, $x + 2y + 4z = \lambda$, $x + 4y + 10z = \lambda^2$ is consistent

Sol. Augmented matrix = $[A|B] = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & \lambda \\ 1 & 4 & 10 & \lambda^2 \end{array}\right]$

$$[A|B] \xrightarrow[R_3 \to R_3 - R_1]{R_2 \to R_2 - R_1} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & \lambda - 1 \\ 0 & 3 & 9 & \lambda^2 - 1 \end{array}\right]$$

$$\xrightarrow{R_3 \to R_3 - 3R_2} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & \lambda - 1 \\ 0 & 0 & 0 & \lambda^2 - 1 - 3\lambda + 3 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & \lambda - 1 \\ 0 & 0 & 0 & \lambda^2 - 3\lambda + 2 \end{array}\right]$$

Here $\rho(A) = 2$

The given system of equations is consistent only when $\rho([A|B]) = 2$

$\rho([A|B]) = 2$ only when $\lambda^2 - 3\lambda + 2 = 0$

$\Rightarrow (\lambda - 1)(\lambda - 2) = 0 \Rightarrow \lambda = 1$ or $\lambda = 2$

$\therefore$ The given system is consistent when the values of λ are 1 and 2.

5. Show that the equations $-2x + y + z = a$, $x - 2y + z = b$, $x + y - 2z = c$ are consistent only if $a + b + c = 0$.

Sol. Augmented matrix [A|B] is $\left[\begin{array}{ccc|c} -2 & 1 & 1 & a \\ 1 & -2 & 1 & b \\ 1 & 1 & -2 & c \end{array}\right]$

$$[A|B] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{ccc|c} 1 & -2 & 1 & b \\ -2 & 1 & 1 & a \\ 1 & 1 & -2 & c \end{array}\right]$$

$$\xrightarrow[R_3 \to R_3 - R_1]{R_2 \to R_2 + 2R_1} \left[\begin{array}{ccc|c} 1 & -2 & 1 & b \\ 0 & -3 & 3 & a + 2b \\ 0 & 3 & -3 & c - b \end{array}\right]$$

$$\xrightarrow{R_3 \to R_3 + R_2} \left[\begin{array}{ccc|c} 1 & -2 & 1 & b \\ 0 & -3 & 3 & c + 2b \\ 0 & 0 & 0 & a + b + c \end{array}\right]$$

Here $\rho(A) = 2$

The given system is consistent only when $\rho([A|B]) = 2$ $\rho([A|B]) = 2$ only if $a + b + c = 0$

Hence proved.

6. Using Gaussian Jordan method, find the values of λ and μ so that the system of equations $2x - 3y + 5z = 12$, $3x + y + \lambda z = \mu$, $x - 7y + 8z = 17$ has (i) unique solution (ii) infinite solutions and (iii) no solution.

Sol. The augmented matrix [A/B] is $\left[\begin{array}{ccc|c} 2 & -3 & 5 & 12 \\ 3 & 1 & \lambda & \mu \\ 1 & -7 & 8 & 17 \end{array}\right]$

$$\xrightarrow{R_1 \leftrightarrow R_3} \left[\begin{array}{ccc|c} 1 & -7 & 8 & 17 \\ 3 & 1 & \lambda & \mu \\ 2 & -3 & 5 & 12 \end{array}\right]$$

$$\xrightarrow[R_3 \to R_3 - 2R_1]{R_2 \to R_2 - 3R_1} \left[\begin{array}{ccc|c} 1 & -7 & 8 & 17 \\ 0 & 22 & \lambda - 51 & \mu - 51 \\ 0 & 11 & -11 & -22 \end{array}\right]$$

$$\xrightarrow[R_3 \to R_3 \div 11]{R_2 \to R_2 - 2R_3} \left[\begin{array}{ccc|c} 1 & -7 & 8 & 17 \\ 0 & 0 & \lambda - 2 & \mu - 7 \\ 0 & 1 & -1 & -2 \end{array}\right]$$

$$\xrightarrow{R_2 \leftrightarrow R_3} \left[\begin{array}{ccc|c} 1 & -7 & 8 & 17 \\ 0 & 1 & -1 & -2 \\ 0 & 0 & \lambda - 2 & \mu - 7 \end{array}\right]$$

Case (i) when $\lambda \neq 2$,

$\rho([A|B]) = 3$ and $\rho(A) = 3$

$\therefore \rho([A|B]) = \rho(A) = 3 =$ the number of unknowns

$\therefore$ The system has unique solution

Case (ii) When $\lambda = 2, \mu = 7$,

$$[A|B] = \left[\begin{array}{ccc|c} 1 & -7 & 8 & 17 \\ 0 & 1 & -1 & -2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Here $\rho(A) = 2$ and $\rho([A|B]) = 2$

$\therefore \rho(A) = \rho([A|B]) = 2 <$ number of unknowns

Thus the system is consistent with infinitely many solutions.

Case (iii) When $\lambda = 2$ and $\mu \neq 7$

$\rho(A) = 2$ and $\rho([A|B]) = 3$

$\therefore \rho(A) \neq \rho([A|B])$

Thus, the given system of equations is inconsistent.

CHAPTER 2

COMPLEX NUMBERS

MUST KNOW DEFINITIONS

- If |A Complex number is of the form $x + iy$ where x is a real part and y is the imaginary part of the complex number.
- $z_1 = z_2$ off Re (z_1) = Re (z_2) and Im (z_1) = Im (z_2)

Properties of complex numbers:

Under Additions

- Let z_1, z_2 and z_3 are complex numbers.
  - (i) Closure property ($z_1 + z_2$ is a complex number)
  - (ii) Commutative property ($z_1 + z_2 = z_2 + z_1$)
  - (iii) Associative property $(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$
  - (iv) Additive identity ($z + 0 = 0 + 2 = z$)
  - (v) Additive inverse $(z + - z) = (- z + z) = 0$
- **Under Multiplication:**
  - (i) Closure property ($z_1\, z_2$ is also a complex number)
  - (ii) Commutative property ($z_1\, z_2 = z_2\, z_1$)
  - (iii) Associative property $(z_1\, z_2)\, z_3 = z_1\, (z_2\, z_3)$
  - (iv) Multiplicative identity ($z_1 . 1 = 1. z_1 = z_1$)
  - (v) Multiplicative inverse $z.w = w.z = 1 \Rightarrow w - z^{-1}$
- **Distributive property (Multiplication distributes over addition)**

  $z_1\,(z_2 + z_3) = z_1\, z_2 + z_1\, z_3$

  Also, $(z_1 + z_2)\, z_3 = z_1\, z_3 + z_2\, z_3$
- Conjugate of $x + iy$ is $x - iy$

- If $z = x + iy$ then $|z| = \sqrt{x^2 + y^2}$
- $|z - z_0| = r$ is the equation of circle where z_0 is a fixed complex number and r is the distance form z_0 to z.
- Polar form of $z = x + iy$ is $z = r(\cos\theta + i\sin\theta)$

  where $r = \sqrt{x^2 + y^2}$ and $\tan\theta = \dfrac{y}{x}$.

De Moivre's theorems

- Given any complex number $\cos\theta + i\sin\theta$ and any integer n, then $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$.
- $z^{1/n} = r^{1/n}\left[\cos\left(\dfrac{\theta + 2k\pi}{n}\right) + i\sin\left(\dfrac{\theta + 2k\pi}{n}\right)\right]$, $k = 0, 1, 2, \ldots\ldots n-1$.

IMPORTANT FORMULA TO REMEMBER

- $i^0 = 1, i^1 = i, i^4 = 1, i^{-4} = 1, (i)^{-1} = -i, (i)^{-2} = -1, (i)^{-3} = i$
- $\sqrt{ab} = \sqrt{a}.\sqrt{b}$ is valid only if atleast one of a, b is non-negative.
- When $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ then

  $z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2)$

  $z_1 - z_2 = (x_1 - x_2) + i(y_1 - y_2)$

  $z_1 z_2 = (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + x_2 y_1)$

Properties of complex conjugates

(1) $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$

(2) $\overline{z_1 - z_2} = \overline{z_1} - \overline{z_2}$

(3) $\overline{z_1 z_2} = \overline{z_1}\,\overline{z_2}$

(4) $\overline{\left(\dfrac{z_1}{z_2}\right)} = \dfrac{\overline{z_1}}{\overline{z_2}}$, $z_2 \neq 0$

(5) Re $(z) = \dfrac{z + \overline{z}}{2}$

(6) Im $(z) = \dfrac{z - \overline{z}}{2i}$

(7) $\left(\overline{z^n}\right) = \left(\overline{z}\right)^n$, where n is an integer.

(8) z is real iff $z = \overline{z}$

(9) z is purely imaginary iff $z = -\overline{z}$

(10) $\overline{\overline{z}} = z$

Properties of modulus of a complex number

(1) $|z| = |\overline{z}|$

(2) $|z_1 + z_2| \leq |z_1| + |z_2|$ (Triangle inequality)

(3) $|z_1 z_2| = |z_1|\,|z_2|$

(4) $|z_1 - z_2| \geq \big||z_1| - |z_2|\big|$

(5) $\left|\dfrac{z_1}{z_2}\right| = \dfrac{|z_1|}{|z_2|}$, $z_2 \neq 0$

(6) $|z^n| = |z|^n$, where n is an integer

(7) Re (z) $\leq |z|$

(8) Im (z) $\leq |z|$

- $|z_1 - z_2| = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$
- $|z_1 - z_2| \le |z_1| + |z_2|$
- $||z_1| - |z_2|| \le |z_1 + z_2| \le |z_1| + |z_2|$
- $||z_1| - |z_2|| \le |z_1 - z_2| \le |z_1| + |z_2|$
- $|z_1 z_2| = |z_1||z_2|$
- Let $a + ib = \sqrt{x + iy}$ then

  $x = \pm\sqrt{\frac{\sqrt{a^2 + b^2} + a}{2}}$,

  $y = \pm\sqrt{\frac{\sqrt{a^2 + b^2} - a}{2}}$
- $|z - z_0| < r$ represents the points interior of the circle.
- $|z - z_0| > r$ represents the points exterior of the circle.
- $\arg (z_1 z_2) = \arg z_1 + \arg z_2$
- $\arg \left(\frac{z_1}{z_2}\right) = \arg z_1 - \arg z_2$
- $\arg (z^n) = n \arg z$
- Alternate form of $\cos \theta + i \sin \theta$ is $\cos (2k\pi + \theta) + i \sin (2k\pi + \theta)$, $k \in \mathbb{Z}$

Euler's formula

- $e^{i\theta} = \cos \theta + i \sin \theta$ or $z = re^{i\theta}$
- If $z = r (\cos \theta + i \sin \theta)$ then

  $z^{-1} = \frac{1}{r}(\cos\theta - i\sin\theta)$
- $z_1 z_2 = r_1 r_2 [\cos (\theta_1 + \theta_2) + i \sin (\theta_1 + \theta_2)]$
- $\frac{z_1}{z_2} = \frac{r_1}{r_2}[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)]$
- $(\cos \theta - i \sin \theta)^n = \cos n\theta - i \sin n\theta$ A $(\cos \theta + i \sin \theta)^{-n} = \cos n\theta - i \sin n\theta$
- $(\cos \theta - i \sin \theta)^{-n} = \cos n\theta + i \sin n\theta$ A $\sin \theta + i \cos \theta = i (\cos \theta - i \sin \theta)$
- $1 + \omega + \omega^2 + \ldots\ldots. + \omega^{n-1} = 0$

  where w is the n^{th}--- root of unity.

  $1 - \omega.\omega^2 \ldots.\omega^{n-1} = (-1)^{n-1}$

  $\omega^{n-k} = \omega^{-k} = (\overline{\omega})^k \quad 0 \le k \le n - 1$