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chapter: 1. Applications of Matrices and Determinants

Adjoint of a square Matrix

$adj A = [A_{ij}]^T = [(-1)^{i+j} M_{ij}]^T$

① $A = \frac{1}{3}\begin{bmatrix} 2 & 2 & 1 \\ -2 & 1 & 2 \\ 1 & -2 & 2 \end{bmatrix}$ find adj A

W.K.T

$adj(KA) = K^{n-1} adj A$

② If A, such that $A^2 - 3A - 7I = 0$, find A^{-1}.

$A^2 - 3A - 7I = 0$

$A^2 - 3A = 7I_n$

pre-multiply both sides by A^{-1}

$A^{-1}A^2 - 3AA^{-1} = A^{-1} 7I_n$

$A - 3I = 7A^{-1}$

$A^{-1} = \frac{A - 3I}{7}$

③ If A, prove that $A^{-1} = A^T$

$A^{-1} = A^T$

Multiplying on both sides by A

$AA^{-1} = AA^T$

$AA^T = I \iff A^{-1} = A^T$

④ (i) If adj A, To find A.

$A = \pm \frac{1}{\sqrt{|adj A|}} adj(adj A)$

(ii) If adj A, To find A^{-1}

$A^{-1} = \pm \frac{1}{\sqrt{|adj A|}} \cdot adj A$

⑤ Find the Matrix A, for which $A\begin{bmatrix} 5 & 3 \\ -1 & -2 \end{bmatrix} = \begin{bmatrix} 14 & 7 \\ 7 & 7 \end{bmatrix}$

sol

Let $B = \begin{bmatrix} 5 & 3 \\ 1 & -2 \end{bmatrix}$ and $C = \begin{bmatrix} 14 & 7 \\ 7 & 7 \end{bmatrix}$

then, $AB = C \Rightarrow \boxed{A = CB^{-1}}$

⑥ Given $A = \begin{bmatrix} 1 & -1 \\ 2 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 3 & -2 \\ 1 & 1 \end{bmatrix}$ and $C = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$, find a Matrix X, such that AXB = C

sol

$AXB = C$

$\boxed{X = A^{-1} C B^{-1}}$

⑦ If $A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$, show that $A^{-1} = \frac{1}{2}(A^2 - 3I)$

sol

$2A^{-1} = A^2 - 3I$

Multiply on both sides by A.

$2AA^{-1} = A^2A - 3AI$

$\boxed{2I = A^3 - 3A}$

⑧ Rank (Minor Method)

(i) $A = \begin{bmatrix} 2 & -4 \\ -1 & 2 \end{bmatrix}$, $|A| = 4 - 4 = 0$

$\therefore \rho(A) \neq 2$

$\therefore \rho(A) = 1$.

(ii) $A = \begin{bmatrix} -1 & 3 \\ 4 & -7 \\ 3 & -4 \end{bmatrix}$

$|A| = \begin{vmatrix} -1 & 3 \\ 4 & -7 \end{vmatrix} = 7 - 12 = -5 \neq 0$

$\therefore \rho(A) = 2$

(iii) $A = \begin{bmatrix} 1 & -2 & 3 \\ 2 & 4 & -6 \\ 5 & 1 & -1 \end{bmatrix}$

$|A| = 4 \neq 0 \quad \therefore \rho(A) = 3$

(iv) Let $A = \begin{bmatrix} 0 & 1 & 2 & 1 \\ 0 & 2 & 4 & 3 \\ 8 & 1 & 0 & 2 \end{bmatrix}$

consider third order,

$|A| \neq 0, \quad \rho(A) = 3.$

Note:

(i) A square matrix A is called orthogonal, if $AA^T = A^TA = I$.

(ii) A is orthogonal, if and only if A is non-singular and $A^{-1} = A^T$

Gauss - Jordan Method

Step: 1.

Augment the identity Matrix I_n on the right-side of A to get the matrix $[A | I_n]$

Step: 2

obtain elementary matrices (row operations) that is $[I_n | A^{-1}]$

To write A^{-1} is called Gauss - Jordan method.

Matrix Inversion Method

consider, the matrix equation

$AX = B \longrightarrow (1)$

where A is square matrix and non-singular, A^{-1} exists and $A^{-1}A = AA^{-1} = I$.

Pre-multiplying both sides of (1) by A^{-1} we get

$A^{-1}(AX) = A^{-1}B$

ie) $(A^{-1}A)X = A^{-1}B$

$IX = A^{-1}B$

$\boxed{\therefore X = A^{-1}B.}$

Cramer's rule

$x_1 = \dfrac{\Delta_1}{\Delta}, \quad x_2 = \dfrac{\Delta_2}{\Delta}, \quad x_3 = \dfrac{\Delta_3}{\Delta}$

where,

$$\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$\Delta_1 = \begin{vmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{vmatrix}$$

$$\Delta_2 = \begin{vmatrix} a_{11} & b_1 & a_{13} \\ a_{21} & b_2 & a_{23} \\ a_{31} & b_3 & a_{33} \end{vmatrix}$$

$$\Delta_3 = \begin{vmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \\ a_{31} & a_{23} & b_3 \end{vmatrix}$$

Note:

If $\Delta = 0$, cramer's rule cannot be applied.

Row - echelon form

Let, $A = \begin{bmatrix} 2 & 0 & -7 \\ 0 & 3 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

Then A is a matrix of order 3×3 and, $\rho(A) \leq 3$

Third order minor

$|A| = \begin{vmatrix} 2 & 0 & -7 \\ 0 & 3 & 1 \\ 0 & 0 & 1 \end{vmatrix} = 2(3)(1) = 6 \neq 0$

$\therefore \rho(A) = 3$

②

(ii) $A = \begin{bmatrix} -2 & 2 & -1 \\ 0 & 5 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

Soln

$|A| = \begin{vmatrix} -2 & 2 & -1 \\ 0 & 5 & 1 \\ 0 & 0 & 0 \end{vmatrix} = (-2)(5)(0) \Rightarrow 0$

Third order minor is $= 0$

$\therefore \rho(A) \leq 2$

Next Apply, second order minor is, $|A| = \begin{vmatrix} -2 & 2 \\ 0 & 5 \end{vmatrix} = -10 \neq 0$

$\therefore \rho(A) = 2$

Gaussian - Elimination method

① $4x + 3y + 6z = 25$, $x + 5y + 7z = 13$, $2x + 9y + z = 1$.

Augmented Matrix is

$$\begin{bmatrix} 4 & 3 & 6 \\ 1 & 5 & 7 \\ 2 & 9 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 25 \\ 13 \\ 1 \end{bmatrix}$$

$$[A|B] \sim \left[\begin{array}{ccc|c} 4 & 3 & 6 & 25 \\ 1 & 5 & 7 & 13 \\ 2 & 9 & 1 & 1 \end{array}\right]$$

After, Row operations, we get

$$\left[\begin{array}{ccc|c} 1 & 5 & 7 & 13 \\ 0 & 17 & 22 & 27 \\ 0 & 0 & 199 & 398 \end{array}\right]$$

The equivalent system is written by using the row-echelon form

$x + 5y + 7z = 13$ —— ①

$17y + 22z = 27$ —— ②

$199z = 398$ —— ③

Solving, ①, ② & ③ we get the answer, $x = 4, y = -1, z = 2$

Consistency of system is linear Equations by Rank method

Non-homogeneous linear equations

Using Gaussian elimination method.

By Rouche-caplli theorem

(i) If there are n unknowns in the system of equations and $\rho(A) = \rho(A|B) = n$ then $AX = B$ is consistent and has a unique solution.

(ii) $\rho(A) = \rho[A|B] = n - k$, $k \neq 0$

Then the system is consistent and has infinitely many solution, k - parameter family

$\therefore \rho(A) = \rho[A|B] = 2$

The soln is one parameter family.

(ii) $\rho(A) = \rho[A|B] = 1$

The soln. is 2 parameter family.

(iii) If $\rho(A) \neq \rho[A|B]$

Then the system is inconsistent and no solution.

Homogeneous system of linear equations

case (i)

If $\rho(A) = \rho[A|O] = n$

The system is consistent, has a unique solution (or) Trivial solution.

so for trivial soln, $|A| \neq 0$

case (ii)

$\rho(A) = \rho[A|O] < n$

The system is consistent and has a non-Trivial soln.

since $\rho(A) < n$, $|A| = 0$

case (iii)

Suppose $m > n$ then

There are more number of equations then the number of unknowns.

Reading the system by elementary transformations we get,

$\rho(A) = \rho[A|O] \leq n$

($\because$ 4 Equations, 3 unknowns)

Reducing one equation,

Summary

Homogeneous Equations

(i) $x+y+3z=0$; $x+8y+2z=0$, $3x+4y+z=0$; $3x+4y-3z=0$.

4 Equations $= 0$ உள்ளதால் homogeneous system of linear equation.

(ii) $|A| \neq 0$, so for trivial soln.

(iii) $|A| = 0$, so for Non-trivial solution.

(iv) $\rho(A) = \rho(A,B) = 3$. trivial soln is $x=0, y=0, z=0$.

Note:

No interchanging in Row and column in first element. For example,

$3x+2y+7z=0$; $4x-3y-2z=0$
$5x+9y+23z=0$

$$(A/B) \sim \left[\begin{array}{ccc|c} 3 & 2 & 7 & 0 \\ 4 & -3 & -2 & 0 \\ 5 & 9 & 23 & 0 \end{array}\right]$$

Non-homogeneous Equation

$\neq 0$ (உள்ளதால்) Non-homogeneous equation.

Note:

Interchanging possible,

Eg: $2x+2y+z=5$; $x-y+z=1$
$3x+y+2z=4$

$$[A|B] \sim \left[\begin{array}{ccc|c} 2 & 2 & 1 & 5 \\ 1 & -1 & 1 & 1 \\ 3 & 1 & 2 & 4 \end{array}\right]$$

First element is 1 (Must)
$R_1 \leftrightarrow R_2$

Summary

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① Adjoint of a square matrix A = Transpose of the cofactor Matrix of A.

② $A(\text{adj}A) = (\text{adj}A)A = |A| I_n$

③ $A^{-1} = \frac{1}{|A|}\,\text{adj}A$

④ (i) $|A^{-1}| = \frac{1}{|A|}$ (ii) $(A^T)^{-1} = (A^{-1})^T$

(iii) $(\lambda A)^{-1} = \frac{1}{\lambda}A^{-1}$, where λ is a non-zero scalar

(iv) $(AB)^{-1} = B^{-1}A^{-1}$ (v) $(A^{-1})^{-1} = A$

(vi) $(A^T)^T = A$

⑤ If A is a non-singular square matrix of order n, then

(i) $(\text{adj}A)^{-1} = \text{adj}(A^{-1}) = \frac{1}{|A|}A$

(ii) $|\text{adj}A| = |A|^{n-1}$

(iii) $\text{adj}(\text{adj}A) = |A|^{n-2} \cdot A$

(iv) $\text{adj}(\lambda A) = \lambda^{n-1}\,\text{adj}A$, λ is non-zero scalar.

(v) $|\text{adj}(\text{adj}A)| = |A|^{(n-1)^2}$

(vi) $(\text{adj}A)^T = \text{adj}(A^T)$

(vii) $\text{adj}(AB) = (\text{adj}B)(\text{adj}A)$

⑥ If adj A, To find

(i) $A^{-1} = \pm\frac{1}{\sqrt{|\text{adj}A|}}\,\text{adj}A$

(ii) $A = \pm\frac{1}{\sqrt{|\text{adj}A|}}\,\text{adj}(\text{adj}A)$

⑦ (i) A Matrix A is orthogonal if $AA^T = A^TA = I$.

(ii) A Matrix A is orthogonal if and only if A is non-singular and $A^{-1} = A^T$

⑨ Methods of solving the system of linear equations $AX = B$.

(i) By Matrix inversion Method.

$AX = B \Rightarrow X = A^{-1}B$

(ii) By cramer's rule,

$x = \frac{\Delta_1}{\Delta}, \; y = \frac{\Delta_2}{\Delta}, \; z = \frac{\Delta_3}{\Delta}$.

$\Delta \neq 0$.

(iii) By Gaussian elimination Method.

Left cancellation law of Matrices

Let A, B and C be square Matrices of order n.

If A is non-singular and $AB = AC$ then $B = C$

Right cancellation law of Matrices

Let A and B, C be square Matrices of order n.

If A is non-singular and $BA = CA \Rightarrow B = C$.

Reversal law for Inverses of Matrices

If A and B are non-singular Matrices of the same order, then the product AB is also non-singular and

$$(AB)^{-1} = B^{-1}A^{-1}$$

Law of Double Inverses of Matrices

If A is non-singular, then A^{-1} is also non-singular and $(A^{-1})^{-1} = A$.

Chapter: 2. Complex Numbers

Powers of imaginary unit i

$i^0 = 1$; $i^1 = i$; $i^2 = -1$; $i^3 = -i$

$i^4 = 1$. $(i)^{-1} = \frac{1}{i} \times \frac{i}{i} = \frac{i}{i^2} = -i$

$(i)^{-1} = -i$

$(i)^{-2} = \frac{1}{i^2} \times \frac{i^2}{i^2} = \frac{(-1)}{1} = -1$

$(i)^{-3} = \frac{1}{i^3} \times \frac{i^3}{i^3} = \frac{-i}{i^6} = \frac{-i}{(-i)(-i)}$

$= \frac{-i}{-i} = i$

$(i)^{-4} = 1.$

Properties of complex Number

(1) $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$

(2) $\overline{z_1 - z_2} = \overline{z_1} - \overline{z_2}$

(3) $\overline{z_1 z_2} = \overline{z_1}\, \overline{z_2}$

(4) $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\overline{z_1}}{\overline{z_2}}$, $\overline{z_2} \neq 0$.

(5) $Re(z) = \frac{z + \overline{z}}{2}$

(6) $Im(z) = \frac{z - \overline{z}}{2i}$

(7) $\overline{(z^n)} = (\overline{z})^n$ where n is an integer.

(8) z is real if and only if $z = \overline{z}$

(9) z is purely imaginary if and only if $z = -\overline{z}$

(10) $\overline{\overline{z}} = z$

Properties of Modulus of a complex Numbers

(1) $|z| = |\overline{z}|$

(2) $|z_1 + z_2| \leq |z_1| + |z_2|$ (Triangle inequality)

(3) $|z_1 z_2| = |z_1||z_2|$

(4) $|z_1 - z_2| \geq |z_1| - |z_2|$

(5) $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$, $z_2 \neq 0$

(6) $|z^n| = |z|^n$, where n is integer

(7) $Re(z) \le |z|$

(A)

(8) $Im(z) \le |z|$

Square roots of a complex Numbers

$$\sqrt{a+ib} = \pm\left[\sqrt{\frac{|z|+a}{2}} + i\frac{b}{|b|}\sqrt{\frac{|z|-a}{2}}\right]$$

where $z = a+ib$ and $b \neq 0$

Note:

(i) If b is negative $\frac{b}{|b|} = -1$

x and y are different sign.

(ii) If b is positive $\frac{b}{|b|} = 1$

x and y are same sign.

Some properties of argument

① $\arg(z_1 z_2) = \arg z_1 + \arg z_2$

② $\arg\left(\frac{z_1}{z_2}\right) = \arg z_1 - \arg z_2$

($\because \log ab = \log a + \log b$

$\log(a/b) = \log a - \log b$)

③ $\arg(z^n) = n \arg z$

($\because \log a^x = x \log a$.)

Find all the values [sum]

Working rule:

Step: 1 : Given condition in polar form

Step: 2 : Adding $2k\pi$ in argument.

Step: 3 : power to inside

Step: 4

$k = 0, 1, 2 \ldots n-1$

($\because (\cos\theta + i\sin\theta)^n$.

Properties of polar form

① If $z = r(\cos\theta + i\sin\theta)$ then,

$z^{-1} = \frac{1}{r}(\cos\theta - i\sin\theta)$

② If $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$ then

$z_1 z_2 = r_1 r_2[\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)]$

③ If $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$

$z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$ then

$\frac{z_1}{z_2} = \frac{r_1}{r_2}[\cos(\theta_1-\theta_2) + i\sin(\theta_1-\theta_2)]$

Chapter: 3 Theory of Equations

Quadratic equations:

$ax^2+bx+c=0$, b^2-4ac is called the discriminant and it is usually denoted by Δ

$\therefore \Delta = b^2 - 4ac$

(i) $\Delta = 0$ if and only if The roots are equal

(ii) $\Delta > 0$ if and only if, the roots are real and distinct.

(iii) $\Delta < 0$ if and only if, the equation no real roots.

Vieta's formulae and formation of polynomial Equations

Vieta's formula for Quadratic Equations of deg: 2

Let α and β be the roots of the quadratic equation is

$ax^2+bx+c=0 \Rightarrow a(x-\alpha)(x-\beta)$

$\Rightarrow ax^2 - a(\alpha+\beta)x + a(\alpha\beta) = 0$

$\therefore \alpha+\beta = -\frac{b}{a}$; $\alpha\beta = \frac{c}{a}$

$\therefore$ The quadratic equation is

x^2 - (sum of the roots)x + (product of the roots) $= 0$

Vieta's formula for polynomial Equation of (degree 3)

$ax^3+bx^2+cx+d=0$

Let α, β, γ be the roots

ax^3+bx^2+cx+d

$= a(x-\alpha)(x-\beta)(x-\gamma)$

Expanding Right-hand side

$ax^3 - a(\alpha+\beta+\gamma)x^2$

$+ a(\alpha\beta+\beta\gamma+\gamma\alpha)x$

$+ a(\alpha\beta\gamma)$

$\therefore \alpha+\beta+\gamma = -\frac{b}{a}$

$\alpha\beta+\beta\gamma+\gamma\alpha = \frac{c}{a}$

& $\alpha\beta\gamma = -\frac{d}{a}$

Hence,

coefficient of $x^2 = -(\alpha+\beta+\gamma)$

coefficient of x $= \alpha\beta+\beta\gamma+\gamma\alpha$.

and

constant term $= -\alpha\beta\gamma$

Rational root theorem

Let, $a_nx^n + \cdots + a_1x + a_0$ with $a_n \neq 0$ and $a_0 \neq 0$ be a polynomial with integer coefficients.

If $\frac{p}{q}$, with $(p,q)=1$ is a root of the polynomial then p is a factor of a_0 and q is a factor of a_n

(5)

Def: 3.1

A polynomial $p(x)$ of degree n is said to be reciprocal polynomial if one of the following conditions is true.

(i) $p(x) = x^n \, p\left(\frac{1}{x}\right)$

(ii) $p(x) = -x^n \, p\left(\frac{1}{x}\right)$.

Def: 3.6

A polynomial equation

$$a_n x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + \cdots + a_2 x^2 + a_1 x + a_0 = 0, \quad (a_n \neq 0)$$

is a reciprocal equation if, and only if, one of the following two statements is true.

(i) $a_n = a_0$, $a_{n-1} = a_1$, $a_{n-2} = a_2 \ldots$

(ii) $a_n = -a_0$, $a_{n-1} = -a_1$, $a_{n-2} = -a_2, \ldots$

Statement:

If $P(x) = 0$ is a polynomial equation such that, whenever α is a root, $\frac{1}{\alpha}$ is also a root then, the polynomial equation $p(x) = 0$ must be reciprocal equation is not true.

Descartes Rule

If p is the number of positive zeros of the polynomial $p(x)$ with real coefficients and s is the number of sign changes in coefficients of $p(x)$

Then, $s - p$ is a non-negative even integer.

The Fundamental theorem of Algebra

A polynomial of degree $n \geq 1$ has at least one root in $\mathbb{C}$.

Complex conjugate Root theorem

Imaginary (non real complex) roots occur as conjugate pairs, if the coefficients of the polynomial are real.

K. SAMINATHAN M.SC, B.Ed.
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Chapter: 4 Inverse Trigonometric Functions

Learning objectives:

* Define inverse Trigonometric functions.
* Evaluate the principal values of inverse trigonometric functions
* Draw the graphs of trigonometric functions and their inverses.
* Apply the properties of inverse Trigonometric functions and evaluate some expressions.

Some Fundamental concepts

A real valued fn. f is periodic if there exists a number $p>0$ such that for all x in the domain of f, $x+p$ is in the domain of f and $f(x+p)=f(x)$.

The smallest of all such numbers is called the period of the function f.

For example:

$\sin x, \cos x, \operatorname{cosec} x, \sec x$ and e^{ix} are periodic functions with period 2π radians, whereas $\tan x$ and $\cot x$ are periodic functions with period π radians.

Odd and Even fns.

(i) A real valued fn. f is an even function if for all x in the domain of f, $-x$ is also in the domain of f and

$$\boxed{f(-x)=f(x)}$$

(ii) A real valued fn. f is an odd function if for all x in the domain of f, $-x$ is also in the domain of f and $\boxed{f(-x)=-f(x)}$

For instance,

$x^3, \sin x, \operatorname{cosec} x, \tan x$, and $\cot x$ are all odd functions.

Whereas, $x^2, \cos x$ and $\sec x$ are even functions.

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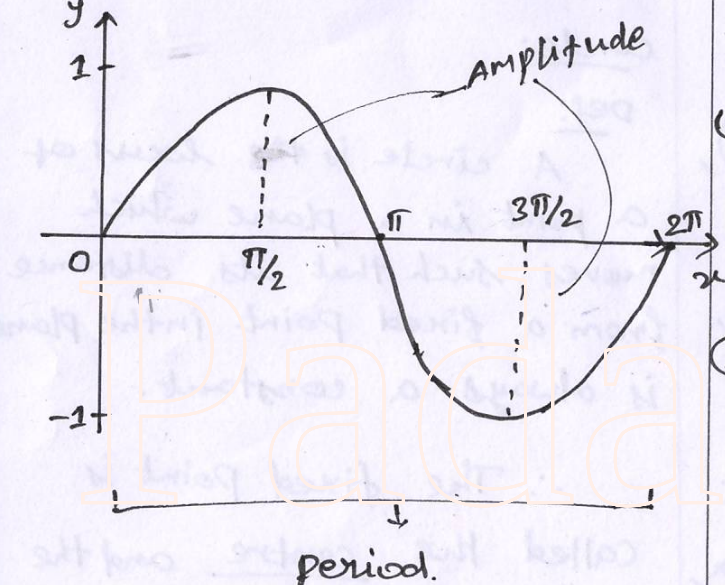
Domain and Range of Trigonometric functions

Trigonometric function	$\sin x$	$\cos x$	$\tan x$	$\operatorname{cosec} x$	$\sec x$	$\cot x$
Domain	R	R	$R \setminus \{(2n+1)\frac{\pi}{2}, n\in z\}$	$R\setminus\{n\pi, n\in z\}$	$R\setminus\{(2n+1)\frac{\pi}{2}, n\in z\}$	$R\setminus\{n\pi, n\in z\}$
Range	$[-1,1]$	$[-1,1]$	R	$R\setminus(-1,1)$	$R\setminus(-1,1)$	R.

Amplitude and period of a graph.

$y=\sin x$. in $[0, 2\pi]$



Properties of Inverse Trigonometric functions

① (i) $\sin^{-1}(\sin\theta)=\theta$, if $\theta\in[-\frac{\pi}{2},\frac{\pi}{2}]$

(ii) $\cos^{-1}(\cos\theta)=\theta$, if $\theta\in[0,\pi]$

(iii) $\tan^{-1}(\tan\theta)=\theta$, if $\theta\in[-\frac{\pi}{2},\frac{\pi}{2}]$

(iv) $\operatorname{cosec}^{-1}(\operatorname{cosec}\theta)=\theta$, if $\theta\in[-\frac{\pi}{2},\frac{\pi}{2}]\setminus\{0\}$

(v) $\sec^{-1}(\sec\theta)=\theta$; if $\theta\in[0,\pi]\setminus\{\pi/2\}$

(vi) $\cot^{-1}(\cot\theta)=\theta$; if $\theta\in(0,\pi)$

② (i) $\sin^{-1}x+\cos^{-1}x=\frac{\pi}{2}$, $x\in[-1,1]$

(ii) $\tan^{-1}x+\cot^{-1}(x)=\pi/2$, $x\in R$

(iii) $\operatorname{cosec}^{-1}x+\sec^{-1}x=\pi/2$, $x\in R\setminus(-1,1)$ (or) $|x|\geq 1$.

③ (i) $\sin^{-1}x+\sin^{-1}y=\sin^{-1}(x\sqrt{1-y^2}+y\sqrt{1-x^2})$

where, either $x^2+y^2\leq 1$ (or) $xy<0$

(ii) $\sin^{-1}x-\sin^{-1}y=\sin^{-1}(x\sqrt{1-y^2}-y\sqrt{1-x^2})$

where, either $x^2+y^2\leq 1$ (or) $xy>0$

(iii) $\cos^{-1}x+\cos^{-1}y=\cos^{-1}[xy-\sqrt{1-x^2}\sqrt{1-y^2}]$ if $x+y\geq 0$

(iv) $\cos^{-1}x-\cos^{-1}y=\cos^{-1}[xy+\sqrt{1-x^2}\sqrt{1-y^2}]$, if $x''y$

(v) $\tan^{-1}x + \tan^{-1}y = \tan^{-1}\left(\frac{x+y}{1-xy}\right)$
if $xy<1$

(vi) $\tan^{-1}x - \tan^{-1}y = \tan^{-1}\left(\frac{x-y}{1+xy}\right)$
if $xy>-1$

(4)
(i) $2\tan^{-1}x = \tan^{-1}\left(\frac{2x}{1-x^2}\right)$, $|x|<1$

(ii) $2\tan^{-1}x = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$, $x\geq 0$

(iii) $2\tan^{-1}x = \sin^{-1}\left(\frac{2x}{1+x^2}\right)$, $|x|\leq 1$

(5)
(i) $\sin^{-1}(2x\sqrt{1-x^2}) = 2\sin^{-1}x$, if $|x|\leq\frac{1}{\sqrt{2}}$
(or) $-\frac{1}{\sqrt{2}}\leq x\leq\frac{1}{\sqrt{2}}$

(ii) $\sin^{-1}(2x\sqrt{1-x^2}) = 2\cos^{-1}x$, if $\frac{1}{\sqrt{2}}\leq x\leq 1$

(6)
(i) $\sin^{-1}x = \cos^{-1}\sqrt{1-x^2}$, if $0\leq x\leq 1$

(ii) $\sin^{-1}x = -\cos^{-1}\sqrt{1-x^2}$, if $-1\leq x\leq 0$

(iii) $\sin^{-1}x = \tan^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right)$, if $-1<x<1$

(iv) $\cos^{-1}x = \sin^{-1}\sqrt{1-x^2}$, if $0\leq x\leq 1$

(v) $\cos^{-1}x = \pi - \sin^{-1}\sqrt{1-x^2}$, if $-1\leq x<0$

(vi) $\tan^{-1}x = \sin^{-1}\left(\frac{x}{\sqrt{1+x^2}}\right)$
$= \cos^{-1}\left(\frac{1}{\sqrt{1+x^2}}\right)$, if $x>0$.

(7)
(i) $3\sin^{-1}x = \sin^{-1}(3x-4x^3)$, $x\in\left[-\frac{1}{2},\frac{1}{2}\right]$

(ii) $3\cos^{-1}x = \cos^{-1}(4x^3-3x)$, $x\in\left[\frac{1}{2},1\right]$

Chapter: 5

Two Dimentional Analytical Geometry - II

Circle:

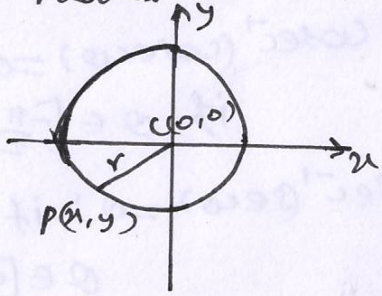
Def:

A circle is the locus of a point in a plane which moves such that its distance from a fixed point in the plane is always a constant.

$\therefore$ The fixed point is called the centre and the constant distance is called the radius of the circle.

Equation of the circle in standard form

(i) Equation of circle with centre (0,0) and radius r.



$CP = r$

$$\sqrt{(x_2-x_1)^2+(y_2-y_1)^2} = r$$

$$(x_2-x_1)^2+(y_2-y_1)^2 = r^2$$

$$\boxed{x^2+y^2=r^2.}$$

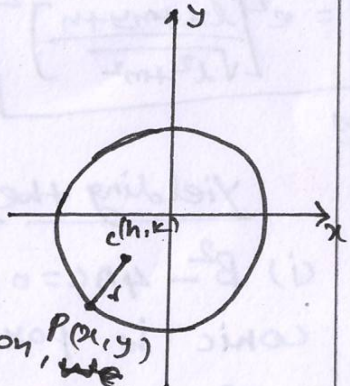
Here $(x_1,y_1)=(0,0)$
$(x_2,y_2)=(x,y)$

(ii) Equation of circle with centre (h,k) and radius r.

$CP = r$

$$\sqrt{(x-h)^2+(y-k)^2} = r$$

$$\therefore (x-h)^2+(y-k)^2 = r^2$$



Expanding equation, we get,

$$x^2+y^2-2hx-2ky+h^2+k^2-r^2=0 \quad \text{—(1)}$$

Taking,

$2g=-2h$, $2f=-2k$,

$c = h^2+k^2-r^2$

$$\boxed{\therefore x^2+y^2+2gx+2fy+c=0.}$$

is called the general form of a circle.

Note:

The equation, $x^2+y^2+2gx+2fy+c=0$ represents a circle with centre $(-g,-f)$ and radius $\sqrt{g^2+f^2-c}$

(i) It is a real circle, if $g^2+f^2-c>0$.

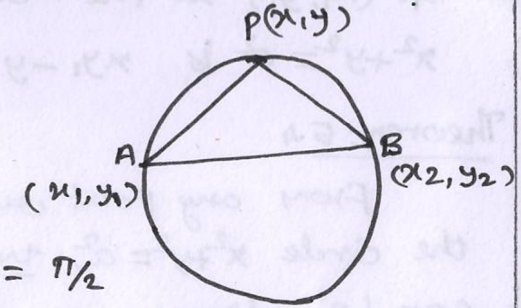
(ii) A point circle, if $g^2+f^2-c=0$

(iii) It is imaginary circle if, $g^2+f^2-c<0$ with no locus.

Theorem: 5

The equation of a circle with (x_1,y_1) and (x_2,y_2) as extremities of one of the diameters of the circle is $(x-x_1)(x-x_2)+(y-y_1)(y-y_2)=0$

proof:



$\angle APB = \pi/2$

$\therefore$ The product of slopes of AP and PB is ~~called~~ equal to -1.

(i.e) $\left(\frac{y-y_1}{x-x_1}\right)\left(\frac{y-y_2}{x-x_2}\right)=-1.$

$\therefore (x-x_1)(x-x_2)+(y-y_1)(y-y_2)=0.$

(1) Find the centre and radius of the circle $3x^2+(a+1)y^2+6x-9y+a+4=0$.

Soln.

centre : $(-g,-f)$

radius: $\sqrt{g^2+f^2-c}$

Given,

$$3x^2+(a+1)y^2+6x-9y+a+4=0 \quad \text{—(1)}$$

The general form of circle is

$$x^2+y^2+2gx+2fy+c=0 \quad \text{—(2)}$$

coefficients of x^2 = coefficient of y^2

(1) $\Rightarrow$ $3=a+1$
$\boxed{a=2}$ $\Rightarrow$ sub. (1), we get the answer.

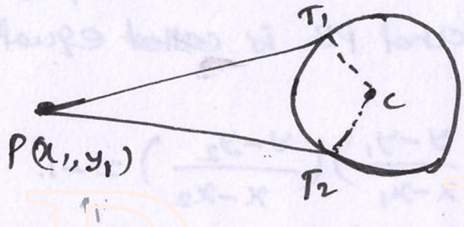
Remark:

① The equation of tangent at (x_1, y_1) to the circle $x^2+y^2=a^2$ is $xx_1+yy_1=a^2$

② The equation of normal at (x_1, y_1) to the circle $x^2+y^2=a^2$ is $xy_1-yx_1=0$.

Theorem: 5.4

From any point outside the circle $x^2+y^2=a^2$, two tangents can be drawn



Conics

A conic is the locus of a point which moves in a plane.

* The fixed point is called focus

* The fixed line is called directrix

* The constant ratio is called eccentricity which is denoted by e.

(i) If this constant $e=1$ then the conic is called a parabola

(ii) If this constant $e<1$ then the conic is called ellipse

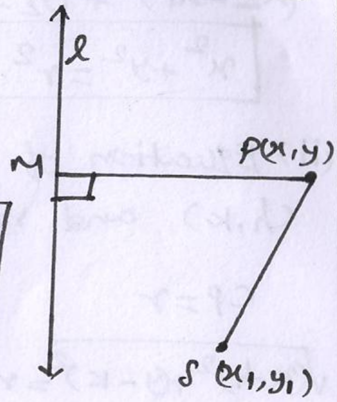
(iii) If this constant $e>1$, then the conic is called a hyperbola

The general equation of a conic

$$\frac{SP}{PM}=e$$

$$SP^2=e^2PM^2$$

$$\boxed{(x-x_1)^2+(y-y_1)^2 = e^2\left[\frac{lx+my+n}{\sqrt{l^2+m^2}}\right]^2}$$



Yielding the following cases:

(i) $B^2-4AC=0 \Leftrightarrow e=1 \Leftrightarrow$ The conic is parabola.

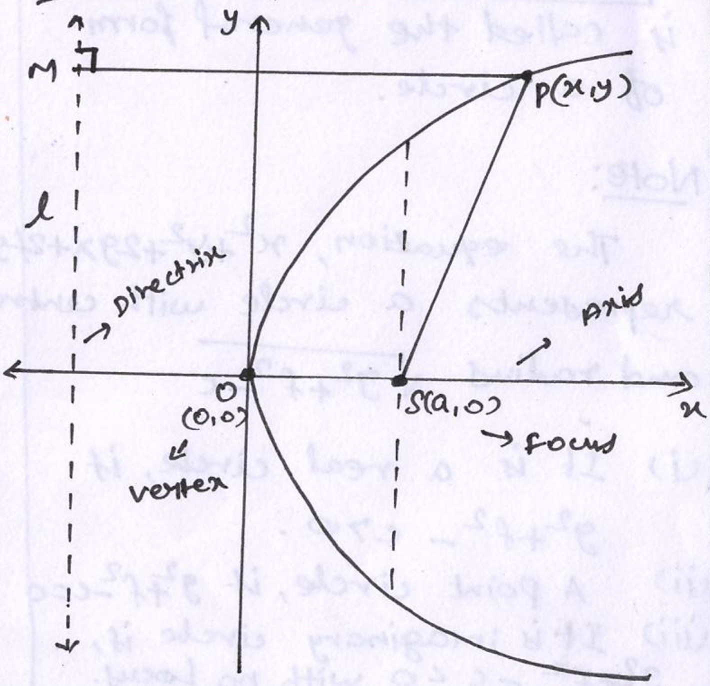
(ii) $B^2-4AC<0 \Leftrightarrow 0<e<1 \Leftrightarrow$ The conic is an ellipse.

(iii) $B^2-4AC>0 \Leftrightarrow e>1 \Leftrightarrow$ The conic is hyperbola

Parabola

since $e=1$.

(i) Equation of a parabola in standard form with vertex at $(0,0)$



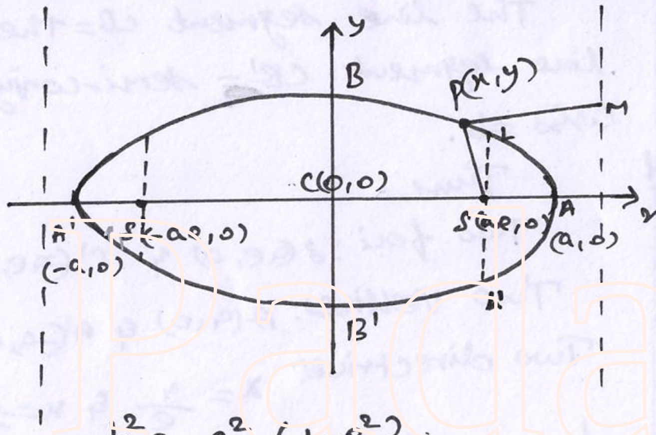
Ellipse

A constant ratio (eccentricity) less than unity

$\therefore 0 < e < 1$.

(i) Equation of an Ellipse in standard form

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, which is the equation of an ellipse in standard form, and note that it is symmetrical about x and y axis.



$b^2 = a^2(1-e^2)$

Taking, $ae = c$

$\boxed{b^2 = a^2 - c^2}$

Def:

(i) The line segment AA' is called the Major axis of the ellipse and is of length 2a.

(ii) The line segment BB' is called the minor axis of the ellipse and is of length 2b

(iii) The line segment CA = The line segment CA' = semi-major axis = a and the line segment CB = The line segment CB' = semi minor axis = b.

$\therefore$ length of the Major axis = 2a

$\therefore$ length of the minor axis = 2b.

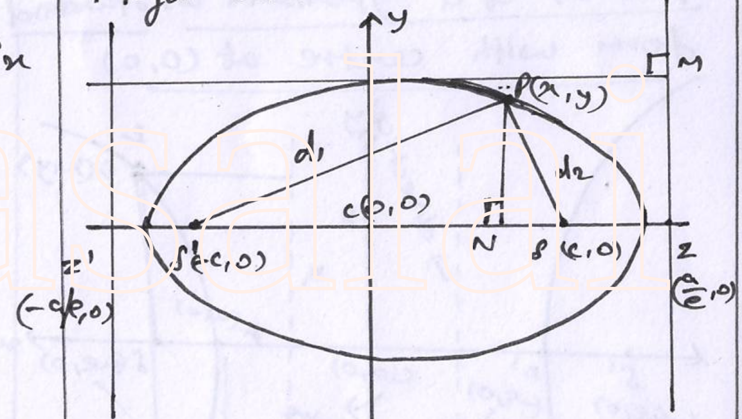
(iv) Two foci $S(ae, 0)$ and $S'(-ae, 0)$ and two vertices $A(a, 0)$ and $A'(-a, 0)$

Two directrices, $x = \frac{a}{e}$, $x = \frac{-a}{e}$

(v) Length of the latus rectum, $LL' = \frac{2b^2}{a}$

Theorem: 5.5

The sum of the focal distances of any point on the ellipse is equal to length of the Major axis.



$\boxed{c = ae}$

Hence, $SP + S'P = a - ex + a + ex$

$\therefore \boxed{SP + S'P = 2a}$

Proof:

By defn:

$SP = ePM$

$= e(NZ)$

$= e(CZ - CN)$

$= e\left(\frac{a}{e} - x\right)$

$= \cancel{e}\left(\frac{a - ex}{\cancel{e}}\right)$

$SP = a - ex$

$SP' = ePM'$

$= e[CN + CZ']$

$= e[x + \frac{a}{e}]$

$SP' = ex + a$

Hence,

$SP + SP' = a - ex + a + ex$

$\therefore \boxed{SP + SP' = 2a}$

Note:

eccentricity, $e = \sqrt{1 - \frac{b^2}{a^2}}$

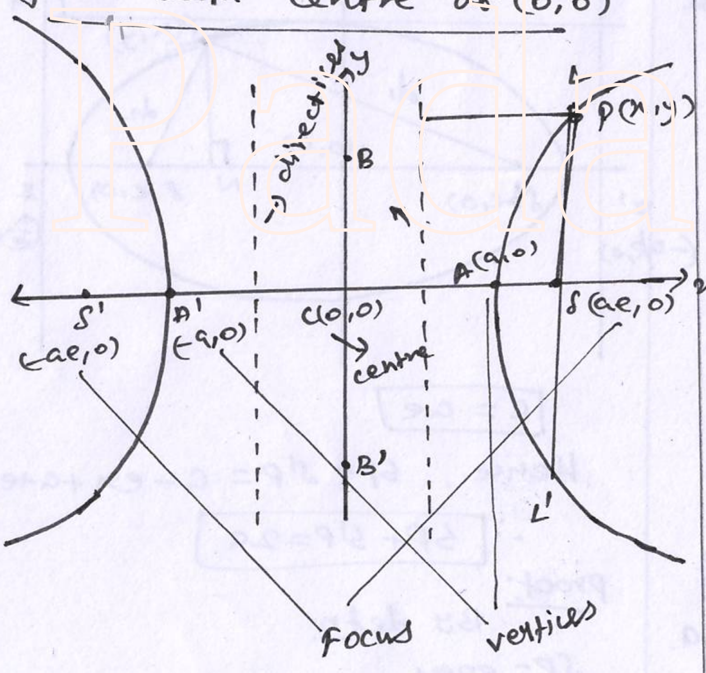
when $b = a \Rightarrow$

$\therefore e = \sqrt{1 - \frac{a^2}{a^2}}$

Hence the eccentricity of the circle is zero.

Hyperbola

Equation of a hyperbola in standard form with centre at (0,0)



Since $e > 1$.

$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. which is the equation of a hyperbola in standard form.

and note that, It is symmetrical about x-axis and y axis.

Taking, $ae = c$ we get $\boxed{b^2 = c^2 - a^2}$

Def: 5.5

① The line segment AA' is the Transverse axis of length 2a.

② The line segment BB' is the conjugate axis of length 2b.

③ The line segment CA = The line segment CA' = semi transverse axis = a.

and,

The line segment CB = the line segment CB' = semi-conjugate axis = b.

Thus,

Two foci: $S(ae, 0)$ & $S'(-ae, 0)$

Two vertices: $A(a, 0)$ & $A'(-a, 0)$

Two directrices: $x = \frac{a}{e}$ & $x = -\frac{a}{e}$

Length of the latus rectum of hyperbola is $\frac{2b^2}{a}$.

eccentricity, $e = \sqrt{1 + \frac{b^2}{a^2}}$

$\boxed{b^2 = c^2 - a^2}$

Asymptotes:

Let $P(x,y)$ be a point on the curve defined by $y = f(x)$.

which moves further and further away from the origin such that the distance between P and some fixed line tends to zero.

$\therefore$ The fixed line is called an "asymptotes"

(9)

Conic sections

Identifying the conics from the general equation of the conic

$$Ax^2+Bxy+Cy^2+Dx+Ey+F=0$$

(1) $A=C=1$, $B=0$, $D=-2h$, $E=-2k$, $F=h^2+k^2-r^2$, which is a circle.

(2) $B=0$ and either A or $C=0$. The general equation yields a parabola.

(3) $A\neq C$ and A and C are of the same sign, the general equation yields an ellipse.

(4) $A\neq C$ and A and C are opposite signs, the general equation yields a hyperbola.

(5) $A=C$ and $B=D=E=F=0$. The general equation yields a point $x^2+y^2=0$.

(6) $A=C=F$ and $B=D=E=0$. The general equation yields an empty set $x^2+y^2+1=0$. as there is no solution.

(7) $A\neq 0$ or $C\neq 0$ and others are zero, the general equation yields coordinate axes.

(8) $A=-C$ and resets are zero. The general equation yields a pair of lines $x^2-y^2=0$.

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Parametric equations

Suppose that, $f(t)$ and $g(t)$ are functions of 't' then the equations $x=f(t)$, $y=g(t)$ together describe a curve in the plane.

In general 't' is simply an arbitrary variable called in this case a parameter and this method of specifying a curve is known as parametric equations.

Chapter: 6

Applications of Vector Algebra

Scalar product and vector product

Note:

(i) $\vec{a}\cdot\vec{b}$ is a scalar (or) dot Product $\Rightarrow \vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\theta$

(ii) $\vec{a}\times\vec{b}$ is a vector (or) cross Product $\Rightarrow |\vec{a}\times\vec{b}| = |\vec{a}||\vec{b}|\sin\theta$

Remark:

(i) An angle between two non-zero vectors $\vec{a}$ and $\vec{b}$ is $\theta = \cos^{-1}\left[\frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|}\right]$

(ii) $\vec{a}$ and $\vec{b}$ are said to be parallel, if the angle between them is 0 (or) π.

(iii) $\vec{a}$ and $\vec{b}$ are said to be perpendicular, if the angle between them is $\frac{\pi}{2}$ (or) $\frac{3\pi}{2}$

Cosine formula

(i) $a^2 = b^2 + c^2 - 2bc\cos A$

(ii) $b^2 = c^2 + a^2 - 2ca\cos B$

(iii) $c^2 = a^2 + b^2 - 2ab\cos C$

Prove that conditions:

(i) $\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$

(ii) $\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$

(iii) $\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$

(iv) $\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$

Note:

Work done by the force on the particle is $\boxed{w = \vec{F}\cdot\vec{d}}$

Defn:

If a force $\vec{F}$ is applied on a particle at a point with position vector $\vec{r}$, then the torque or Moment on the particle is given by $\boxed{\vec{t} = \vec{r}\times\vec{F}}$

The torque is also called the rotational force.

Scalar Triple Product

Def:

For a given set of three vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$, The scalar $(\vec{a}\times\vec{b})\cdot\vec{c}$ is called a scalar triple product of $\vec{a}, \vec{b}, \vec{c}$.

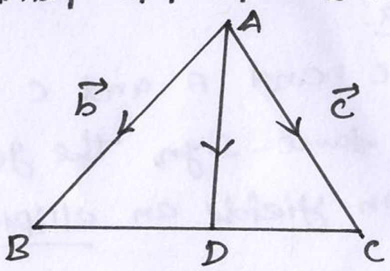
Remark:

$\vec{a}\cdot\vec{b}$ is a scalar and so $(\vec{a}\cdot\vec{b})\times\vec{c}$ has no meaning.

Apollonius Theorem

If D is the midpoint of the side BC of a triangle ABC, show by vector method that

$$|\vec{AB}|^2 + |\vec{AC}|^2 = 2(|\vec{AD}|^2 + |\vec{BD}|^2)$$



Properties of the scalar Triple product

For any three vectors $\vec{a}, \vec{b}$ and $\vec{c}$.

$$(\vec{a}\times\vec{b})\cdot\vec{c} = \vec{a}\cdot(\vec{b}\times\vec{c})$$

Note:

(i) $[\vec{a},\vec{b},\vec{c}] = (\vec{a}\times\vec{b})\cdot\vec{c} = \vec{a}\cdot(\vec{b}\times\vec{c})$
$= (\vec{b}\times\vec{c})\cdot\vec{a} = \vec{b}\cdot(\vec{c}\times\vec{a})$

$[\vec{a},\vec{b},\vec{c}] = [\vec{b},\vec{c},\vec{a}]$

(ii) $[\vec{b},\vec{c},\vec{a}] = (\vec{b}\times\vec{c})\cdot\vec{a}$
$= \vec{b}\cdot(\vec{c}\times\vec{a})$
$= (\vec{c}\times\vec{a})\cdot\vec{b}$
$= \vec{c}\cdot(\vec{a}\times\vec{b})$

$[\vec{b},\vec{c},\vec{a}] = [\vec{c},\vec{a},\vec{b}]$

(iii) $[\vec{a},\vec{b},\vec{c}] = [\vec{b},\vec{c},\vec{a}] = [\vec{c},\vec{a},\vec{b}]$

$-[\vec{a},\vec{c},\vec{b}] = -[\vec{c},\vec{b},\vec{a}] = -[\vec{b},\vec{a},\vec{c}]$

Summary

10

① The volume of the parallelpiped formed by using the three vectors $\vec{a}, \vec{b}$ and $\vec{c}$ as co-terminus edges is given by,
$|(\vec{a}\times\vec{b})\cdot\vec{c}|$

② The scalar triple product of three non-zero if and only if the three vectors are coplanar

③ Three vectors $\vec{a}, \vec{b}, \vec{c}$ are coplanar, if and only if there exist scalars $r, s, t \in R$ such that atleast one of them is non-zero and $r\vec{a}+s\vec{b}+t\vec{c}=\vec{0}$

④ If $\vec{a}, \vec{b}, \vec{c}$ and $\vec{p}, \vec{q}, \vec{r}$ are any two systems of three vectors and if $\vec{p}=x_1\vec{a}+y_1\vec{b}+z_1\vec{c}$, $\vec{q}=x_2\vec{a}+y_2\vec{b}+z_2\vec{c}$ and $\vec{r}=x_3\vec{a}+y_3\vec{b}+z_3\vec{c}$, then

$$[\vec{p},\vec{q},\vec{r}]=\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}[\vec{a},\vec{b},\vec{c}]$$

⑤ For a given set of three vectors $\vec{a}, \vec{b}, \vec{c}$ the vector $\vec{a}\times(\vec{b}\times\vec{c})$ is called vector triple product

⑥ For any three vectors $\vec{a}, \vec{b}, \vec{c}$ we have, $\vec{a}\times(\vec{b}\times\vec{c})=(\vec{a}\cdot\vec{c})\vec{b}-(\vec{a}\cdot\vec{b})\vec{c}$

⑦ Jacobi's identity

For any three vectors $\vec{a}, \vec{b}, \vec{c}$ we have $\vec{a}\times(\vec{b}\times\vec{c})+\vec{b}\times(\vec{c}\times\vec{a})+\vec{c}\times(\vec{a}\times\vec{b})=\vec{0}$

⑧ Lagrange's identity:

For any four vectors $\vec{a}, \vec{b}, \vec{c}, \vec{d}$ we have

$$(\vec{a}\times\vec{b})\cdot(\vec{c}\times\vec{d})=\begin{vmatrix} \vec{a}\cdot\vec{c} & \vec{a}\cdot\vec{d} \\ \vec{b}\cdot\vec{c} & \vec{b}\cdot\vec{d} \end{vmatrix}$$

⑨ $[\vec{a}\times\vec{b}, \vec{b}\times\vec{c}, \vec{c}\times\vec{a}]=[\vec{a},\vec{b},\vec{c}]^2$

⑩ $(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d})$
$=[\vec{a},\vec{b},\vec{d}]\vec{c}-[\vec{a},\vec{b},\vec{c}]\vec{d}$

⑪ $(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d})=[\vec{a},\vec{c},\vec{d}]\vec{b}-[\vec{b},\vec{c},\vec{d}]\vec{a}$

A point on the straight line and the direction of the straight line are given

ⓐ Parametric form

If a vector equation involves parameters, then it is called a vector equation in parametric form.

Eg: $\vec{r}=\vec{a}+t\vec{b}, \ t\in R$

ⓑ Non-parametric form

If no parameter is involved, then the equation is called a vector equation in non-parametric form

Eg: $(\vec{r}-\vec{a})\times\vec{b}=0$

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(c) cartesian form (or) equation

$$\frac{x-x_1}{b_1}=\frac{y-y_1}{b_2}=\frac{z-z_1}{b_3}$$

(or)

$$\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n}$$

Straight line passing through two given points

(a) Parametric form of vector equation

The parametric form of vector equation of a line passing through two given points whose position vectors are $\vec{a}$ and $\vec{b}$ respectively is

$$\vec{r}=\vec{a}+t(\vec{b}-\vec{a}),\quad t\in R$$

(b) Non-parametric form of vector equation

$$(\vec{r}-\vec{a})\times(\vec{b}-\vec{a})=\vec{0}$$

(c) cartesian form of equation

$$\frac{x-x_1}{x_2-x_1}=\frac{y-y_1}{y_2-y_1}=\frac{z-z_1}{z_2-z_1}$$

Angle between two straight lines

(a) Vector Form:

The acute angle between two given straight lines $\vec{r}=\vec{a}+s\vec{b}$ and $\vec{r}=\vec{c}+t\vec{d}$ is

$$\cos\theta=\frac{|\vec{b}\cdot\vec{d}|}{|\vec{b}||\vec{d}|}$$

$$\therefore \theta=\cos^{-1}\left[\frac{|\vec{b}\cdot\vec{d}|}{|\vec{b}||\vec{d}|}\right]$$

Remark:

(i) The two given lines $\vec{r}=\vec{a}+s\vec{b}$; $\vec{r}=\vec{c}+t\vec{d}$ are parallel.

$\Leftrightarrow \theta=0 \Leftrightarrow \cos\theta=1$

$\Leftrightarrow |\vec{b}\cdot\vec{d}|=|\vec{b}||\vec{d}|$

(ii) The two given lines $\vec{r}=\vec{a}+s\vec{b}$ and $\vec{r}=\vec{c}+t\vec{d}$ are parallel if, and only if $\vec{b}=\lambda\vec{d}$, For some scalar λ.

(iii) The two given lines $\vec{r}=\vec{a}+s\vec{b}$ and $\vec{r}=\vec{c}+t\vec{d}$ are perpendicular if, and only if $\vec{b}\cdot\vec{d}=0$.

(b) cartesian form:

If two lines are given in cartesian form as

$$\frac{x-x_1}{b_1}=\frac{y-y_1}{b_2}=\frac{z-z_1}{b_3}$$

and, $\frac{x-x_2}{d_1}=\frac{y-y_2}{d_2}=\frac{z-z_2}{d_3}$,

then the acute angle θ between the two given lines given by,

$$\theta=\cos^{-1}\left[\frac{|b_1d_1+b_2d_2+b_3d_3|}{\sqrt{b_1^2+b_2^2+b_3^2}+\sqrt{d_1^2+d_2^2+d_3^2}}\right]$$

Remark:-

(i) The two given lines with direction ratios b_1, b_2, b_3 and d_1, d_2, d_3 are parallel if, and only if

$$\frac{b_1}{d_1} = \frac{b_2}{d_2} = \frac{b_3}{d_3}$$

(ii) The two given lines with direction ratios b_1, b_2, b_3 and d_1, d_2, d_3 are perpendicular if and only if, $b_1d_1 + b_2d_2 + b_3d_3 = 0$.

(iii) If the direction cosines of two given straight lines are l_1, m_1, n_1 and l_2, m_2, n_2 then the angle between the two given straight lines,

$$\cos\theta = |l_1 l_2 + m_1 m_2 + n_1 n_2|$$

Shortest distance between two straight lines:

Def:

① Two lines are said to be coplanar if they lie in the same plane.

② Two lines in space are called skew lines if they are not parallel and do not intersect.

Note:

If two lines are either parallel or intersecting, then they are coplanar.

① The shortest distance between the two parallel lines $\vec{r} = \vec{a} + s\vec{b}$ and $\vec{r} = \vec{c} + t\vec{b}$ is given by,

$$d = \frac{|(\vec{c} - \vec{a}) \times \vec{b}|}{|\vec{b}|}, \text{ where } \vec{b} \neq 0.$$

② The shortest distance between the two skew lines $\vec{r} = \vec{a} + s\vec{b}$ and $\vec{r} = \vec{c} + t\vec{d}$ is given by

$$\delta = \frac{|(\vec{c} - \vec{a}) \cdot (\vec{b} \times \vec{d})|}{|\vec{b} \times \vec{d}|},$$

where $|\vec{b} \times \vec{d}| \neq 0$.

Remark:

① Two straight lines $\vec{r} = \vec{a} + s\vec{b}$ and $\vec{r} = \vec{c} + t\vec{d}$ intersect each other (ie coplanar) if $(\vec{c} - \vec{a}) \cdot (\vec{b} \times \vec{d}) = 0$.

② If two lines,

$$\frac{x - x_1}{b_1} = \frac{y - y_1}{b_2} = \frac{z - z_1}{b_3}$$

and,

$$\frac{x - x_2}{d_1} = \frac{y - y_2}{d_2} = \frac{z - z_2}{d_3}$$

intersect each other (ie coplanar) then we have,

$$\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ b_1 & b_2 & b_3 \\ d_1 & d_2 & d_3 \end{vmatrix} = 0$$

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* A straight line which is perpendicular to a plane is called a Normal to the plane.

* The equation of the plane of a distance p from the origin and perpendicular to the unit normal vector $\vec{d}$ is $\vec{r}.\vec{d} = p$ (Normal form)

* Cartesian equation of the plane in normal form is $lx + my + nz = p$.

* Vector form of the equation of a plane passing through a point with position vector $\vec{a}$ and perpendicular to $\vec{n}$ is $(\vec{r} - \vec{a}).\vec{n} = 0$.

* Cartesian equation of a plane normal to a vector with direction ratios a, b, c and passing through a given point (x_1, y_1, z_1) is,

$$a(x - x_1) + b(y - y_1) + c(z - z_1) = 0.$$

* Intercept form of the equation of the plane $\vec{r}.\vec{n} = q$ having intercepts a, b, c on x, y, z axes respectively is,

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1.$$

Equation of a plane passing through three given non-collinear points

Vector equation in parametric form:-

$$\vec{r} = \vec{a} + s(\vec{b} - \vec{a}) + t(\vec{c} - \vec{a})$$

where $\vec{b} \neq 0$, $\vec{c} \neq 0$, and $s, t \in R$.

Vector equation in Non-parametric form

$$(\vec{r} - \vec{a}).((\vec{b} - \vec{a}) \times (\vec{c} - \vec{a})) = 0$$

(or)

$$[\vec{r} - \vec{a}, \vec{b} - \vec{a}, \vec{c} - \vec{a}] = 0.$$

Cartesian form:

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \end{vmatrix} = 0$$

Equation of a plane passing through a given point and parallel to two given non-parallel vectors [1-point, 2-vectors]

(a) Parametric form of vector equation

$$\vec{r} = \vec{a} + s\vec{b} + t\vec{c}, \text{ where } s, t \in R$$

(b) Non-parametric form of vector equation

$$(\vec{r} - \vec{a}).(\vec{b} \times \vec{c}) = 0$$

(c) Cartesian form of equation

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ b_1 & b_2 & b_3 \\ d_1 & d_2 & d_3 \end{vmatrix} = 0$$

Equation of a plane passing through two given distinct points and is parallel to a non-zero vector [2 points 1 vector]

(a) **Parametric form of vector equation**

$$\vec{r} = \vec{a} + s(\vec{b} - \vec{a}) + t\vec{c}$$

(or)

$$\vec{r} = (1-s)\vec{a} + s\vec{b} + t\vec{c}$$

(b) **Non-parametric form of vector equation**

$$(\vec{r} - \vec{a}) \cdot ((\vec{b} - \vec{a}) \times \vec{c}) = 0$$

(c) **Cartesian form**

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ x_2-x_1 & y_2-y_1 & z_2-z_1 \\ c_1 & c_2 & c_3 \end{vmatrix} = 0$$

Angle between two planes

(1) The acute angle θ between the two planes, $\vec{r} \cdot \vec{n_1} = p_1$ and $\vec{r} \cdot \vec{n_2} = p_2$ is given by,

$$\theta = \cos^{-1}\left[\frac{|\vec{n_1} \cdot \vec{n_2}|}{|\vec{n_1}||\vec{n_2}|}\right]$$

Remark:

(i) If two planes $\vec{r} \cdot \vec{n_1} = p_1$ and $\vec{r} \cdot \vec{n_2} = p_2$ are perpendicular then $\vec{n_1} \cdot \vec{n_2} = 0$

(ii) If two planes $\vec{r} \cdot \vec{n_1} = p_1$ and $\vec{r} \cdot \vec{n_2} = p_2$ are parallel, then $\vec{n_1} = \lambda \vec{n_2}$ where λ is a scalar.

(iii) Equation of a plane parallel to the plane $\vec{r} \cdot \vec{n} = p$ is $\vec{r} \cdot \vec{n} = k$, $k \in R$.

(2) The acute angle θ between the planes, $a_1x + b_1y + c_1z + d_1 = 0$ & $a_2x + b_2y + c_2z + d_2 = 0$ is given by,

$$\theta = \cos^{-1}\left[\frac{|a_1a_2 + b_1b_2 + c_1c_2|}{\sqrt{a_1^2+b_1^2+c_1^2}\sqrt{a_2^2+b_2^2+c_2^2}}\right]$$

Remark:

(i) The planes, $a_1x + b_1y + c_1z + d_1 = 0$ and, $a_2x + b_2y + c_2z + d_2 = 0$ are perpendicular if,

$$a_1a_2 + b_1b_2 + c_1c_2 = 0$$

(ii) The planes, $a_1x + b_1y + c_1z + d_1 = 0$ and, $a_2x + b_2y + c_2z + d_2 = 0$ are parallel if $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$

(iii) Equation of a plane parallel to the plane, $ax + by + cz = p$ is $ax + by + cz = k$. $k \in R$

Angle between a line and a plane

Let, $\vec{r} = \vec{a} + t\vec{b}$, $\vec{r} \cdot \vec{n} = p$

(1) $$\theta = \sin^{-1}\left[\frac{|\vec{b} \cdot \vec{n}|}{|\vec{b}||\vec{n}|}\right]$$

(2) $\vec{b} = a_1\vec{i} + b_1\vec{j} + c_1\vec{k}$ and $\vec{n} = a\vec{i} + b\vec{j} + c\vec{k}$

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$$\theta = \sin^{-1}\left[\frac{|aa_1 + bb_1 + cc_1|}{\sqrt{a^2+b^2+c^2}\sqrt{a_1^2+b_1^2+c_1^2}}\right]$$

Remark:

(i) If the line is perpendicular to the plane, then the line is parallel to the normal to the plane, so $\vec{b}$ is $\perp^r$ to $\vec{n}$, then, we have $\vec{b} = \lambda\vec{n}$, $\lambda \in R$ which gives $\frac{a_1}{a} = \frac{b_1}{b} = \frac{c_1}{c}$

(ii) If the line is parallel to the plane, then the line is $\perp^r$ to the normal to the plane, $\therefore \vec{b}\cdot\vec{n} = 0 \Rightarrow aa_1 + bb_1 + cc_1 = 0$.

Theorem:

① The $\perp^r$ to distance from a point with position vector $\vec{u}$ to the plane $\vec{r}\cdot\vec{n} = p$ is given by

$$\delta = \frac{|\vec{u}\cdot\vec{n} - p|}{|\vec{n}|}$$

② The $\perp^r$ to distance from a point (x_1, y_1, z_1) to the plane $ax + by + cz = p$ is,

$$\delta = \frac{|ax_1 + by_1 + cz_1 - p|}{\sqrt{a^2+b^2+c^2}}$$

Remark:

The $\perp^r$ to distance from the origin to the plane, $ax+by+cz+d=0$ is given by, $\delta = \frac{|d|}{\sqrt{a^2+b^2+c^2}}$

Theorem:

① The distance between two parallel planes, $ax+by+cz+d_1=0$ & $ax+by+cz+d_2=0$ is given by, $\frac{|d_1 - d_2|}{\sqrt{a^2+b^2+c^2}}$

② The vector equation of a plane which passes through the line of intersection of the planes, $\vec{r}\cdot\vec{n_1} = d_1$ & $\vec{r}\cdot\vec{n_2} = d_2$ is given by,

$$(\vec{r}\cdot\vec{n_1} - d_1) + \lambda(\vec{r}\cdot\vec{n_2} - d_2) = 0$$

where $\lambda \in R$

③ The equation of a plane passing through the line of intersection of the planes $a_1x + b_1y + c_1z = d_1$ and $a_2x + b_2y + c_2z = d_2$ is given by

$$(a_1x + b_1y + c_1z - d_1) + \lambda(a_2x + b_2y + c_2z - d_2) = 0.$$

④ The position vector of the point of intersection of the line, $\vec{r} = \vec{a} + t\vec{b}$ and the plane $\vec{r}\cdot\vec{n} = p$ is

$$\vec{u} = \vec{a} + \left(\frac{p - (\vec{a}\cdot\vec{n})}{\vec{b}\cdot\vec{n}}\right)\vec{b},$$

where $\vec{b}\cdot\vec{n} \neq 0$

⑤ If $\vec{v}$ is the position vector of the image of $\vec{u}$ in the plane $\vec{r}\cdot\vec{n} = p$, then

$$\vec{v} = \vec{u} + 2\left[\frac{p - (\vec{u}\cdot\vec{n})}{|\vec{n}|^2}\right]\vec{n}$$

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